A new approach to solving number theoretic problems

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Abstract

For a positive integer x, let $\Gamma(x)$ denote $(x-1)!$. Let $\Gamma^{-1}\colon \{1,2,6,24,\ldots\} \to \mathbb{N} \setminus \{0\}$ denote the inverse function that satisfies $\Gamma^{-1}(1)=2$. For a positive integer n, by a Γ-computation of length n we understand any sequence of terms $x_1,\ldots,x_n$ such that x_1 is identical to the variable x and for every integer $i \in \{2,\ldots,n\}$ there exist integers $j,k \in \{1,\ldots,i-1\}$ such that x_i is identical to $x_j \cdot x_k$, or $\Gamma(x_j)$, or $\Gamma^{-1}(x_j)$. Let $f(6)=15!$, and let $f(n+1)=\Gamma(f(n))$ for every integer $n \geqslant 6$. For an integer $n \geqslant 6$, let Ψ_n denote the following statement: if a Γ-computation of length n produces positive integers $x_1,\ldots,x_n$ for at most finitely many positive integers x, then $\max(x_1,\ldots,x_n) \leqslant f(n)$ for every such x. For every integer $n \geqslant 6$, we formulate the statements Φ_n and Θ_n. We prove: (1) the statement Ψ_6 implies that if the equation $x(x+1)=y!$ has at most finitely many solutions in positive integers, then each such solution (x,y) belongs to the set $\{(1,2),(2,3)\}$; (2) if $y!+1$ is a square for at most finitely many positive integers y, then the statement Ψ_8 implies that every such y is smaller than $f(7)$; (3) the statement Φ_7 implies that the set of Wilson primes is infinite; (4) the statement Θ_6 implies that there are infinitely many primes of the form n^2+1; (5) the statement Θ_6 implies that there are infinitely many primes of the form $n!+1$; (6) the statement Θ_6 implies that there are infinitely many primes of the form $n!-1$; (7) the statement Θ_8 implies that any twin prime that is greater than $\Gamma(\Gamma(120))$ proves that the set of twin primes is infinite.

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For a positive integer x, let $\Gamma(x)$ denote $(x-1)!$. Let $\Gamma^{-1}\colon \{1,2,6,24,\ldots\} \to \mathbb{N} \setminus \{0\}$ denote the inverse function that satisfies $\Gamma^{-1}(1)=2$. For positive integers x and y, let $\mathrm{rest}(x,y)$ denote the rest from dividing x by y.

Definition 1. *For a positive integer n, by a Γ-computation of length n we understand any sequence of terms $x_1,\ldots,x_n$ such that x_1 is identical to the variable x and for every integer $i \in \{2,\ldots,n\}$ there exist integers $j,k \in \{1,\ldots,i-1\}$ such that x_i is identical to $x_j \cdot x_k$, or $\Gamma(x_j)$, or $\Gamma^{-1}(x_j)$.*

Definition 2. *For a positive integer n, by a Q-computation of length n we understand any sequence of terms $x_1,\ldots,x_n$ such that x_1 is identical to the variable x and for every integer $i \in \{2,\ldots,n\}$ there exist integers $j,k \in \{1,\ldots,i-1\}$ such that x_i is identical to $x_j \cdot x_k$, or $\frac{x_j}{x_k}$, or $\Gamma(x_j)$, or $\Gamma^{-1}(x_j)$.*

Definition 3. *For a positive integer n, by a R-computation of length n we understand any sequence of terms $x_1, \ldots, x_n$ such that x_1 is identical to the variable x and for every integer $i \in \{2, \ldots, n\}$ there exist integers $j, k \in \{1, \ldots, i-1\}$ such that x_i is identical to $x_j \cdot x_k$, or* $\mathrm{rest}(x_j, x_k)$*, or* $\Gamma(x_j)$*, or* $\Gamma^{-1}(x_j)$.

Let $f(6) = 15!$, and let $f(n+1) = \Gamma(f(n))$ for every integer $n \geqslant 6$. For an integer $n \geqslant 6$, let Ψ_n denote the following statement: if a Γ-computation of length n produces positive integers $x_1, \ldots, x_n$ for at most finitely many positive integers x, then $\max(x_1, \ldots, x_n) \leqslant f(n)$ for every such x.

Theorem 1. *For every integer $n \geqslant 6$ and for every positive integer x, the following Γ-computation*

$$\begin{cases} x_1 := x \\ x_2 := x_1 \cdot x_1 \\ x_3 := \Gamma^{-1}(x_2) \\ x_4 := x_3 \cdot x_3 \\ x_5 := x_4 \cdot x_4 \\ \forall i \in \{6, \ldots, n\}\ x_i := \Gamma(x_{i-1}) \end{cases}$$

produces positive integers $x_1, \ldots, x_n$ if and only if $x = 1$. If $x = 1$, then $\max(x_1, \ldots, x_n) = f(n)$.

Proof. If $x = 1$, then $x_1 = x_2 = 1$, $x_3 = 2$, $x_4 = 4$, $x_5 = 16$, and $x_i = f(i)$ for every integer $i \in \{6, \ldots, n\}$. Hence, $\max(x_1, \ldots, x_n) = f(n)$. If an integer x is greater than 1, then the term x_3 (that is identical to $\Gamma^{-1}(x^2)$) is not a positive integer ([4, p. 46]), see also [5], where a more general problem is solved. □

Theorem 2. *For every integer $n \geqslant 6$, the bound $f(n)$ in the statement Ψ_n cannot be decreased.*

Proof. It follows from Theorem 1. □

Let $g(6) = 24!$, and let $g(n+1) = \Gamma(g(n))$ for every integer $n \geqslant 6$. For an integer $n \geqslant 6$, let Φ_n denote the following statement: if a Q-computation of length n produces positive integers $x_1, \ldots, x_n$ for at most finitely many positive integers x, then $\max(x_1, \ldots, x_n) \leqslant g(n)$ for every such x.

Theorem 3. *For every integer $n \geqslant 6$ and for every positive integer x, the following Q-computation*

$$\begin{cases} x_1 := x \\ x_2 := x_1 \cdot x_1 \\ x_3 := \Gamma(x_2) \\ x_4 := \Gamma(x_1) \\ x_5 := \Gamma(x_4) \\ x_6 := \frac{x_3}{x_5} \\ x_7 := \Gamma(x_3) \quad (\text{if } n \geqslant 7) \\ \forall i \in \{8, \ldots, n\}\ x_i := \Gamma(x_{i-1}) \ (\text{if } n \geqslant 8) \end{cases}$$

produces positive integers $x_1, \ldots, x_n$ if and only if $x \in \{1, 2, 3, 4, 5\}$. If $x \in \{1, 2, 3, 4\}$, then $\max(x_1, \ldots, x_n) < g(n)$*. If $x = 5$, then* $\max(x_1, \ldots, x_n) = g(n)$.

Proof. If $x = 1$, then $x_1 = \ldots = x_6 = 1$. Since x_3 is a positive integer, we obtain that $x_7, \ldots, x_n$ are positive integers, if $n \geqslant 7$. Since $\max(x_1, \ldots, x_6) < 24!$, we obtain that $\max(x_1, \ldots, x_n) < g(n)$.

If $x = 2$, then $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, $x_4 = 1$, $x_5 = 1$, $x_6 = 6$. Since x_3 is a positive integer, we obtain that $x_7, \dots, x_n$ are positive integers, if $n \geqslant 7$. Since $\max(x_1, \dots, x_6) < 24!$, we obtain that $\max(x_1, \dots, x_n) < g(n)$.

If $x = 3$, then $x_1 = 3$, $x_2 = 9$, $x_3 = 8!$, $x_4 = 2$, $x_5 = 1$, $x_6 = 8!$. Since x_3 is a positive integer, we obtain that $x_7, \dots, x_n$ are positive integers, if $n \geqslant 7$. Since $\max(x_1, \dots, x_6) < 24!$, we obtain that $\max(x_1, \dots, x_n) < g(n)$.

If $x = 4$, then $x_1 = 4$, $x_2 = 16$, $x_3 = 15!$, $x_4 = 6$, $x_5 = 120$, $x_6 = \frac{15!}{120} = 10897286400$. Since x_3 is a positive integer, we obtain that $x_7, \dots, x_n$ are positive integers, if $n \geqslant 7$. Since $\max(x_1, \dots, x_6) < 24!$, we obtain that $\max(x_1, \dots, x_n) < g(n)$.

If $x = 5$, then

$$\begin{array}{rcl} x_1 &=& 5 \\ x_2 &=& x_1 \cdot x_1 = 25 \\ x_3 &=& \Gamma(x_2) = 24! \\ x_4 &=& \Gamma(x_1) = 24 \\ x_5 &=& \Gamma(x_4) = 23! \\ x_6 &=& \frac{x_3}{x_5} = \frac{24!}{23!} = 24 \end{array}$$

Since x_3 is a positive integer, we obtain that $x_7, \dots, x_n$ are positive integers, if $n \geqslant 7$. Since $x_3 = \max(x_1, \dots, x_6) = 24!$, we obtain that $\max(x_1, \dots, x_n) = g(n)$.

If an integer x is greater than 5, then

$$x_6 = \frac{x_3}{x_5} = \frac{\Gamma(x^2)}{\Gamma(\Gamma(x))} < 1$$

□

Theorem 4. *For every integer $n \geqslant 6$, the bound $g(n)$ in the statement Φ_n cannot be decreased.*

Proof. It follows from Theorem 3. □

Let $h(6) = 119!$, and let $h(n+1) = \Gamma(h(n))$ for every integer $n \geqslant 6$. For an integer $n \geqslant 6$, let Θ_n denote the following statement: if a R-computation of length n produces positive integers $x_1, \dots, x_n$ for at most finitely many positive integers x, then $\max(x_1, \dots, x_n) \leqslant h(n)$ for every such x.

Lemma 1. *([10, pp. 214–215]) . For every positive integer x, x does not divide $\Gamma(x)$ if and only if $x = 4$ or x is prime.*

Theorem 5. *For every integer $n \geqslant 6$ and for every positive integer x, the following R-computation*

$$\left\{ \begin{array}{rcl} x_1 &:=& x \\ x_2 &:=& x_1 \cdot x_1 \\ x_3 &:=& \Gamma(x_2) \\ x_4 &:=& \mathrm{rest}(x_3, x_2) \\ x_5 &:=& \Gamma(x_3) \\ \forall i \in \{6, \dots, n\}\ x_i &:=& \Gamma(x_{i-1}) \end{array} \right.$$

produces positive integers $x_1, \dots, x_n$ if and only if $x = 2$. If $x = 2$, then $\max(x_1, \dots, x_n) = h(n)$.

Proof. If $x = 1$, then $x_1 = x_2 = x_3 = 1$ and $x_4 = 0$. If $x = 2$, then $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, $x_4 = 2$, $x_5 = 120$, and $x_i = h(i)$ for every integer $i \in \{6, \ldots, n\}$. Therefore, $\max(x_1, \ldots, x_n) = h(n)$. If an integer x is greater than 2, then x^2 is composite and greater than 4. By Lemma 1,

$$x_4 = \mathrm{rest}(x_3, x_2) = \mathrm{rest}(\Gamma(x_2), x_2) = \mathrm{rest}\left(\Gamma(x^2), x^2\right) = 0$$

□

Theorem 6. *For every integer $n \geqslant 6$, the bound $h(n)$ in the statement Θ_n cannot be decreased.*

Proof. It follows from Theorem 5. □

Lemma 2. *For every positive integer n, there are only finitely many Γ-computations of length n. For every positive integer n, there are only finitely many Q-computations of length n. For every positive integer n, there are only finitely many R-computations of length n.*

Theorem 7. *For every integer $n \geqslant 6$, the statement Ψ_n is true with an unknown integer bound that depends on n. For every integer $n \geqslant 6$, the statement Φ_n is true with an unknown integer bound that depends on n. For every integer $n \geqslant 6$, the statement Θ_n is true with an unknown integer bound that depends on n.*

Proof. It follows from Lemma 2. □

Theorem 8. *For every integer $n \geqslant 6$, the statement Ψ_{n+1} implies the statement Ψ_n. For every integer $n \geqslant 6$, the statement Φ_{n+1} implies the statement Φ_n. For every integer $n \geqslant 6$, the statement Θ_{n+1} implies the statement Θ_n.*

Proof. We present only the proof for the statement Ψ_{n+1} as the proofs for the statements Φ_{n+1} and Θ_{n+1} are essentially the same. Let $n \in \{6, 7, 8, \ldots\}$. Let us assume that a Γ-computation $\mathcal{W}$ of length n produces positive integers $x_1, \ldots, x_n$ for at most finitely many positive integers x. This implies that for every integer $i \in \{1, \ldots, n\}$ the Γ-computation $\mathcal{W}$ with added instruction $x_{n+1} := \Gamma(x_i)$ produces positive integers $x_1, \ldots, x_{n+1}$ for at most finitely many positive integers x. The statement Ψ_{n+1} implies that

$$\forall i \in \{1, \ldots, n\} \;\; \Gamma(x_i) = x_{n+1} \leqslant f(n+1) = \Gamma(f(n))$$

Since $f(n) > 1$, we obtain that $x_i \leqslant f(n)$ for every integer $i \in \{1, \ldots, n\}$. □

Hypothesis. *The statements Ψ_8, Φ_7, and Θ_8 are true.*

Lemma 3. *For every positive integer x, the term $\Gamma^{-1}(x \cdot \Gamma(x))$ represents $x + 1$.*

Lemma 4. *For every positive integer x, $x(x+1)$ is a factorial of a positive integer if and only if the following Γ-computation $\mathcal{A}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & x_1 \cdot x_2 \\ x_4 & := & \Gamma^{-1}(x_3) \\ x_5 & := & x_1 \cdot x_4 \\ x_6 & := & \Gamma^{-1}(x_5) \end{cases}$$

produces positive integers $x_1, \ldots, x_6$.

Proof. By Lemma 3, for every positive integer x the terms $x_1, \ldots, x_5$ represent positive integers and $x_5 = x(x+1)$. Hence, x_6 (that is identical to $\Gamma^{-1}(x_5)$) represents a positive integer if and only if $\Gamma^{-1}(x(x+1))$ represents a positive integer. The last means that $x(x+1)$ equals $y!$ for some positive integer y. $\square$

Theorem 9. *The statement Ψ_6 implies that if the equation $x(x+1) = y!$ has at most finitely many solutions in positive integers, then each such solution (x, y) belongs to the set $\{(1,2),(2,3)\}$.*

Proof. Let us assume that the equation $x(x+1) = y!$ has at most finitely many solutions in positive integers. By Lemma 4, the Γ-computation $\mathcal{A}$ produces positive integers $x_1, \ldots, x_6$ for at most finitely many positive integers x. We take positive integers n and m that satisfy $n(n+1) = m!$. By Lemma 4, the Γ-computation $\mathcal{A}$ for $x = n$ produces positive integers $x_1, \ldots, x_6$. The statement Ψ_6 implies that

$$x_3 = n \cdot \Gamma(n) = \Gamma(n+1) \leqslant f(6) = \Gamma(16)$$

Since $16 > 1$, we obtain that $n + 1 \leqslant 16$. Consequently, $n \leqslant 15$. For every integer $n \in \{1, \ldots, 15\}$, $n(n+1)$ is a factorial of a positive integer if and only if $n \in \{1, 2\}$. $\square$

The question of solving the equation $x(x+1) = y!$ was posed by P. Erdős, see [1]. F. Luca proved that the *abc* conjecture implies that the equation $x(x+1) = y!$ has only finitely many solutions in positive integers, see [7].

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $y! + 1 = x^2$, see [9]. Let

$$F_1 = \left\{y \in \mathbb{N} \setminus \{0\} : \exists x \in \mathbb{N} \setminus \{0\} \;\; y! + 1 = x^2\right\}$$

It is conjectured that $F_1 = \{4, 5, 7\}$, see [14, p. 297].

Lemma 5. *The set F_1 is is finite if and only if the set*

$$F_2 = \{x \in \mathbb{N} \setminus \{0\} : \exists y \in \mathbb{N} \setminus \{0\} \;\; x(x+2) = y!\}$$

is finite.

Proof. If $y! + 1 = x^2$, then $x \geqslant 5$ and $(x-1)((x-1)+2) = y!$. If $x(x+2) = y!$, then $y! + 1 = (x+1)^2$. $\square$

Lemma 6. *For every positive integer x, the following Γ-computation $\mathcal{B}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & x_1 \cdot x_2 \\ x_4 & := & \Gamma^{-1}(x_3) \\ x_5 & := & x_4 \cdot x_3 \\ x_6 & := & \Gamma^{-1}(x_5) \\ x_7 & := & x_1 \cdot x_6 \\ x_8 & := & \Gamma^{-1}(x_7) \end{cases}$$

produces positive integers $x_1, \ldots, x_8$ if and only if $x(x+2)$ is a factorial of a positive integer.

Proof. By Lemma 3, for every positive integer x, the terms $x_1, \ldots, x_7$ represent positive integers and $x_7 = x \cdot (x+2)$. The term x_8 (that is identical to $\Gamma^{-1}(x(x+2))$) represents a positive integer if and only if $x(x+2)$ is a factorial of a positive integer. $\square$

Theorem 10. *If $y! + 1$ is a square for at most finitely many positive integers y, then the statement Ψ_8 implies that every such y is smaller than $f(7)$.*

Proof. If positive integers n and m satisfy $n! + 1 = m^2$, then $m \geqslant 5$ and

$$(m-1) \cdot ((m-1)+2) = \Gamma(n+1)$$

By this and Lemma 6, the Γ-computation $\mathcal{B}$ produces for $x = m - 1$ positive integers $x_1, \ldots, x_8$. The antecedent and Lemma 5 imply that the set F_2 is finite. Therefore, the statement Ψ_8 guarantees that $\Gamma(n+1) = x_7 \leqslant f(8) = \Gamma(f(7))$. Since $f(7) > 1$, we obtain that $n + 1 \leqslant f(7)$. Thus, $n < f(7)$. □

Lemma 7. *(Wilson's theorem, [6, p. 89]). For every positive integer x, x divides $\Gamma(x) + 1$ if and only if $x = 1$ or x is prime.*

A Wilson prime is a prime number p such that p^2 divides $(p-1)! + 1$. It is conjectured that the set of Wilson primes is infinite, [2] and [13].

Lemma 8. *For every positive integer x, the following Q-computation C*

$$\begin{cases} x_1 := x \\ x_2 := \Gamma(x_1) \\ x_3 := \Gamma(x_2) \\ x_4 := x_2 \cdot x_3 \\ x_5 := \Gamma^{-1}(x_4) \\ x_6 := \frac{x_5}{x_1} \\ x_7 := \frac{x_6}{x_1} \end{cases}$$

produces positive integers $x_1, \ldots, x_7$ if and only if $x = 1$ or x is a Wilson prime.

Proof. By Lemma 3, for every positive integer x, the terms $x_1, \ldots, x_5$ represent positive integers and $x_5 = \Gamma(x) + 1$. By Lemma 7, the term x_6 (that is identical to $\frac{\Gamma(x)+1}{x}$) and the term x_7 (that is identical to $\frac{\Gamma(x)+1}{x^2}$) represent positive integers if and only if $x = 1$ or x is a Wilson prime. □

Theorem 11. *The statement Φ_7 implies that the set of Wilson primes is infinite.*

Proof. The number 563 is a Wilson prime, see [2] and [13]. By Lemma 8, for $x = 563$ the Q-computation C produces positive integers $x_1, \ldots, x_7$. We have:

$$\begin{aligned} x_1 &= 563 \\ x_2 &= \Gamma(563) \\ x_3 &= \Gamma(\Gamma(563)) \\ x_4 &= \Gamma(563) \cdot \Gamma(\Gamma(563)) = \Gamma(\Gamma(563)+1) \\ x_5 &= \Gamma(563) + 1 \\ x_6 &= \frac{\Gamma(563)+1}{563} \\ x_7 &= \frac{\Gamma(563)+1}{563^2} \end{aligned}$$

Since $\max(x_1, \ldots, x_7) = x_4 = \Gamma(\Gamma(563)+1) > \Gamma(24!) = \Gamma(g(6)) = g(7)$, the statement Φ_7 implies that the Q-computation C produces positive integers $x_1, \ldots, x_7$ for infinitely many positive integers x. By Lemma 8, we obtain that the set of Wilson primes is infinite. □

Lemma 9. *For every positive integer x, the following R-computation $\mathcal{D}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & x_1 \cdot x_1 \\ x_3 & := & \Gamma(x_2) \\ x_4 & := & x_2 \cdot x_3 \\ x_5 & := & \Gamma^{-1}(x_4) \\ x_6 & := & \mathrm{rest}(x_4, x_5) \end{cases}$$

produces positive integers $x_1, \dots, x_6$ if and only if $x^2 + 1$ is prime.

Proof. It follows from Lemma 1 because $x^2 + 1 \neq 4$. □

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [8, pp. 37–38].

Theorem 12. *The statement Θ_6 implies that there are infinitely many primes of the form $n^2 + 1$.*

Proof. The number $14^2 + 1$ is prime. By Lemma 9, for $x = 14$ the R-computation $\mathcal{D}$ produces positive integers $x_1, \dots, x_6$. Since $x_4 = \Gamma(14^2 + 1) > \Gamma(120) = h(6)$, the statement Θ_6 guarantees that the R-computation $\mathcal{D}$ produces positive integers $x_1, \dots, x_6$ for infinitely many positive integers x. By Lemma 9, we obtain that there are infinitely many primes of the form $n^2 + 1$. □

Lemma 10. *For every positive integer x, the following R-computation $\mathcal{E}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & \Gamma(x_2) \\ x_4 & := & x_2 \cdot x_3 \\ x_5 & := & \Gamma^{-1}(x_4) \\ x_6 & := & \mathrm{rest}(x_4, x_5) \end{cases}$$

produces positive integers $x_1, \dots, x_6$ if and only if $\Gamma(x) + 1$ is prime.

Proof. It follows from Lemma 1 because $\Gamma(x) + 1 \neq 4$. □

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [3, p. 443] and [11].

Theorem 13. *The statement Θ_6 implies that there are infinitely many primes of the form $n! + 1$.*

Proof. The number $\Gamma(12) + 1$ is prime, see [3, p. 441] and [11]. By Lemma 10, for $x = 12$ the R-computation $\mathcal{E}$ produces positive integers $x_1, \dots, x_6$. Since $x_4 = \Gamma(\Gamma(12) + 1) > \Gamma(120) = h(6)$, the statement Θ_6 guarantees that the R-computation $\mathcal{E}$ produces positive integers $x_1, \dots, x_6$ for infinitely many positive integers x. By Lemma 10, we obtain that there are infinitely many primes of the form $\Gamma(x) + 1$. □

Let $\mathcal{P}$ denote the set of prime numbers, and let $U = \{\Gamma(n) - 1 : n \in \mathbb{N} \setminus \{0\}\}$.

Lemma 11. *For every positive integer x, the following R-computation $\mathcal{F}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & x_1 \cdot x_2 \\ x_4 & := & \Gamma^{-1}(x_3) \\ x_5 & := & \Gamma^{-1}(x_4) \\ x_6 & := & \mathrm{rest}(x_2, x_1) \end{cases}$$

produces positive integers $x_1, \dots, x_6$ if and only if $x \in \mathcal{P} \cap U$.

Proof. By Lemma 1, for every positive integer x,

$$x_6 = \mathrm{rest}(x_2, x_1) = \mathrm{rest}(\Gamma(x), x) \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x \in \{4\} \cup \mathcal{P}$$

By Lemma 3, $x_4 = x + 1$. Hence, for every positive integer x,

$$x_5 = \Gamma^{-1}(x_4) = \Gamma^{-1}(x+1) \in \mathbb{N} \setminus \{0\} \Longleftrightarrow x + 1 \in \{\Gamma(n) :\ n \in \mathbb{N} \setminus \{0\}\} \Longleftrightarrow x \in U$$

Since $4 \notin U$, we get $(\{4\} \cup \mathcal{P}) \cap U = \mathcal{P} \cap U$, which completes the proof. □

It is conjectured that there are infinitely many primes of the form $n! - 1$, see [3, p. 443] and [12].

Theorem 14. *The statement Θ_6 implies that there are infinitely many primes of the form $n! - 1$.*

Proof. The number $719 = \Gamma(7) - 1$ belongs to $\mathcal{P} \cap U$. By Lemma 11, for $x = 719$ the R-computation $\mathcal{F}$ produces positive integers $x_1, \ldots, x_6$. Since

$$x_2 = \Gamma(719) > 119! = h(6)$$

the statement Θ_6 guarantees that the R-computation $\mathcal{F}$ produces positive integers $x_1, \ldots, x_6$ for infinitely many positive integers x. By Lemma 11, we obtain that the set $\mathcal{P} \cap U$ is infinite. □

Lemma 12. *For every positive integer x, the following R-computation $\mathcal{H}$*

$$\begin{cases} x_1 & := & x \\ x_2 & := & \Gamma(x_1) \\ x_3 & := & x_1 \cdot x_2 \\ x_4 & := & \Gamma^{-1}(x_3) \\ x_5 & := & x_4 \cdot x_3 \\ x_6 & := & \Gamma^{-1}(x_5) \\ x_7 & := & \mathrm{rest}(x_2, x_1) \\ x_8 & := & \mathrm{rest}(x_5, x_6) \end{cases}$$

produces positive integers $x_1, \ldots, x_8$ if and only if $x = 2$ or both x and $x + 2$ are prime.

Proof. It follows from Lemma 1. □

A twin prime is a prime number that is either 2 less or 2 more than another prime number. The twin prime conjecture states that there are infinitely many twin primes, see [8, p. 39].

Theorem 15. *The statement Θ_8 implies that any twin prime that is greater than $h(7)$ proves that the set of twin primes is infinite.*

Proof. Let us assume that there exists a twin prime that is greater than $h(7)$. Then, there exists a positive integer n such that both n and $n + 2$ are prime and $n + 2 > h(7)$. By Lemma 12, for $x = n$ the R-computation $\mathcal{H}$ produces positive integers $x_1, \ldots, x_8$. Since

$$x_5 = \Gamma(n + 2) > \Gamma(h(7)) = h(8)$$

the statement Θ_8 guarantees that the R-computation $\mathcal{H}$ produces positive integers $x_1, \ldots, x_8$ for infinitely many positive integers x. By Lemma 12, we obtain that there are infinitely many twin primes. □

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