On *ZFC*-formulae $\varphi(x)$ for which we know a non-negative integer n such that $\{x \in \mathbb{N}: \varphi(x)\} \subseteq \{x \in \mathbb{N}: x \leqslant n - 1\}$ if the set $\{x \in \mathbb{N}: \varphi(x)\}$ is finite

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Abstract

Let $\mathcal{P}_{\text{twin}}$ denote the set of twin primes, and let $\mathcal{M}$ denote the set of multiples of twin primes greater than $9^{9^{9^{9^9}}}$. The set $\mathcal{X} = \left(\left[2, 9^{9^{9^{9^9}}}\right] \cap \mathcal{P}_{\text{twin}}\right) \cup \mathcal{M}$ satisfies the following conditions: (1) a known and simple algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$, (2) a known and simple algorithm returns an integer n such that $\mathcal{X}$ is infinite if and only if $\mathcal{X}$ contains an element greater than n, (3) new elements of $\mathcal{X}$ are still discovered, (4) it is conjectured that $\mathcal{X}$ is infinite although we do not know any algorithm deciding the infiniteness of $\mathcal{X}$. The following problem is open: *define a set $\mathcal{X} \subseteq \mathbb{N}$ such that $\mathcal{X}$ satisfies conditions* (1)-(4) *and* **some simple formula $\phi(x)$ of Peano arithmetic satisfies** $\{n \in \mathbb{N} : \phi(n)\} = \mathcal{X}$ **and** $\phi(n)$ **has the same intuitive meaning for every** $n \in \mathbb{N}$ (5). The problem remains open if condition (2) states that a known and simple algorithm returns an integer n such that $\mathcal{X}$ is infinite if and only if $\text{card}(\mathcal{X}) > n$. Let $g(3) = 4$, and let $g(n + 1) = g(n)!$ for every integer $n \geqslant 3$. For an integer $n \in \{3, \ldots, 16\}$, let Ψ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq \left\{x_i! = x_k : (i, k \in \{1, \ldots, n\}) \wedge (i \neq k)\right\} \cup \left\{x_i \cdot x_j = x_k : i, j, k \in \{1, \ldots, n\}\right\}$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant g(n)$. For every statement Ψ_n, the bound $g(n)$ cannot be decreased. The author's guess is that the statements $\Psi_3, \ldots, \Psi_{16}$ are true. The statement Ψ_9 implies that the set of primes of the form $n^2 + 1$ and the set of primes of the form $n! + 1$ satisfy conditions (1)-(5). The statement Ψ_{16} implies that the set of twin primes satisfies conditions (1)-(5).

Key words and phrases: finiteness of a set, incompleteness of *ZFC*, infiniteness of a set, prime numbers of the form $n^2 + 1$, prime numbers of the form $n! + 1$, twin primes.

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1 Introduction and basic lemmas

The phrase "we know a non-negative integer n" in the title means that we know an algorithm which returns n. The title cannot be formalised in *ZFC* because the phrase "we know a non-negative integer n" refers to currently known non-negative integers n with some property. A formally stated title may look like this: On *ZFC*-formulae $\varphi(x)$ for which there exists a non-negative integer n such that *ZFC* proves that

$$\text{card}(\{x \in \mathbb{N}: \varphi(x)\}) < \infty \implies \{x \in \mathbb{N}: \varphi(x)\} \subseteq \{x \in \mathbb{N}: x \leqslant n - 1\}$$

Unfortunately, this formulation admits formulae $\varphi(x)$ without any known non-negative integer n such that *ZFC* proves the above implication.

Lemma 1. *For every non-negative integer n,* $\mathrm{card}(\{x \in \mathbb{N}: x \leqslant n - 1\}) = n$.

Corollary 1. *The title altered to* `"On ZFC-formulae` $\varphi(x)$ `for which we know a non-negative integer` n `such that` $\mathrm{card}(\{x \in \mathbb{N}: \varphi(x)\}) \leqslant n$ `if the set` $\{x \in \mathbb{N}: \varphi(x)\}$ `is finite"` *involves a weaker assumption on* $\varphi(x)$.

Lemma 2. *For every positive integers x and y,* $x! \cdot y = y!$ *if and only if*

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 3. *For every non-negative integers b and c,* $b + 1 = c$ *if and only if*

$$2^{2^b} \cdot 2^{2^b} = 2^{2^c}$$

Lemma 4. *(Wilson's theorem, [8, p. 89]). For every positive integer x, x divides* $(x - 1)! + 1$ *if and only if* $x = 1$ *or x is prime.*

2 Subsets of $\mathbb{N}$ and their threshold numbers

Definition 1. *We say that an integer* $m \in [-1, \infty)$ *is a threshold number of a set* $X \subseteq \mathbb{N}$, *if X is infinite if and only if X contains an element greater than m, cf. [22] and [23].*

If a set $X \subseteq \mathbb{N}$ is empty or infinite, then any $m \in [-1, \infty) \cap \mathbb{Z}$ is a threshold number of X. If a set $X \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of X form the set $\{\max(X), \max(X) + 1, \max(X) + 2, \ldots\}$.

Definition 2. *We say that a non-negative integer m is a weak threshold number of a set* $X \subseteq \mathbb{N}$, *if X is infinite if and only if* $\mathrm{card}(X) > m$.

Proposition 1. *For every* $X \subseteq \mathbb{N}$, *if an integer* $m \in [-1, \infty)$ *is a threshold number of X, then* $m + 1$ *is a weak threshold number of X.*

Proof. For every $X \subseteq \mathbb{N}$, if $m \in [-1, \infty) \cap \mathbb{Z}$ and $\mathrm{card}(X) > m + 1$, then $X \cap [m + 1, \infty) \neq \emptyset$. □

It is conjectured that the set of prime numbers of the form $n^2 + 1$ is infinite, see [15, pp. 37–38]. It is conjectured that the set of prime numbers of the form $n! + 1$ is infinite, see [2, p. 443]. A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that the set of twin primes is infinite, see [15, p. 39]. It is conjectured that the set of composite numbers of the form $2^{2^n} + 1$ is infinite, see [11, p. 23] and [12, pp. 158–159]. A prime p is said to be a Sophie Germain prime if both p and $2p + 1$ are prime, see [21]. It is conjectured that the set of Sophie Germain primes is infinite, see [17, p. 330]. For each of these sets, we do not know any weak threshold number.

Open Problem 1. *Define a set* $X \subseteq \mathbb{N}$ *that satisfies the following conditions:*
(a1) *a known and simple algorithm for every* $n \in \mathbb{N}$ *decides whether or not* $n \in X$,
(b1) *a known and simple algorithm returns an integer n such that X is infinite if and only if* $\mathrm{card}(X) > n$,
(c1) *new elements of X are still discovered,*
(d1) *it is conjectured that X is infinite although we do not know any algorithm deciding the infiniteness of X,*
(e1) *some simple formula* $\phi(x)$ *of Peano arithmetic satisfies* $\{n \in \mathbb{N} : \phi(n)\} = X$ *and* $\phi(n)$ *has the same intuitive meaning for every* $n \in \mathbb{N}$.

The following statement: *for every non-negative integer n there exist*

$$prime\ numbers\ p\ and\ q\ such\ that\ p + 2 = q\ and\ p \in \left[10^n, 10^{n + 1} \right] \tag{T}$$

is a Π_1 statement which strengthens the twin prime conjecture, see [3, p. 43]. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger Π_1 statements, see [1]. The

statement (T) is equivalent to the non-halting of a Turing machine. If a set $X \subseteq \mathbb{N}$ is computable and we know a threshold number of X, then the infiniteness of X is equivalent to the halting of a Turing machine.

The height of a rational number $\frac{p}{q}$ is denoted by $H\left(\frac{p}{q}\right)$ and equals $\max(|p|, |q|)$ provided $\frac{p}{q}$ is written in lowest terms. The height of a rational tuple $(x_1, \ldots, x_n)$ is denoted by $H(x_1, \ldots, x_n)$ and equals $\max(H(x_1), \ldots, H(x_n))$.

Proposition 2. *The equation $x^5 - x = y^2 - y$ has only finitely many rational solutions, see [14, p. 212]. The known rational solutions are $(x, y) = (-1, 0)$, $(-1, 1)$, $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, -5)$, $(2, 6)$, $(3, -15)$, $(3, 16)$, $(30, -4929)$, $(30, 4930)$, $\left(\frac{1}{4}, \frac{15}{32}\right)$, $\left(\frac{1}{4}, \frac{17}{32}\right)$, $\left(-\frac{15}{16}, -\frac{185}{1024}\right)$, $\left(-\frac{15}{16}, \frac{1209}{1024}\right)$, and the existence of other solutions is an open question, see [18, pp. 223–224].*

Proposition 3. *The set $\mathcal{T} = \{n \in \mathbb{N} : \text{the equation } x^5 - x = y^2 - y \text{ has a rational solution of height } n\}$ is finite. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{T}$. We do not know any algorithm which returns a threshold number of $\mathcal{T}$.*

Open Problem 2. *Define a set $X \subseteq \mathbb{N}$ that satisfies the following conditions:*
(a2) *a known and simple algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in X$,*
(b2) *a known and simple algorithm returns an integer n such that X is infinite if and only if X contains an element greater than n,*
(c2) *new elements of X are still discovered,*
(d2) *it is conjectured that X is infinite although we do not know any algorithm deciding the infiniteness of X,*
(e2) *some simple formula $\phi(x)$ of Peano arithmetic satisfies $\{n \in \mathbb{N} : \phi(n)\} = X$ and $\phi(n)$ has the same intuitive meaning for every $n \in \mathbb{N}$.*

Let $\mathcal{P}_{\text{twin}}$ denote the set of twin primes, and let $\mathcal{M}$ denote the set of multiples of twin primes greater than $9^{9^{9^{9^9}}}$.

Proposition 4. *The set $X = \left(\left[2, 9^{9^{9^{9^9}}}\right] \cap \mathcal{P}_{\text{twin}}\right) \cup \mathcal{M}$ satisfies conditions* (a2)-(d2).

Proof. The largest known twin prime is much smaller than $9^{9^{9^{9^9}}}$. $\qquad \square$

Let
$$
\mathcal{H} = \begin{cases} \mathbb{N}, \text{ if } \sin\left(9^{9^{9^{9^9}}}\right) < 0 \\[2mm] \mathbb{N} \cap \left[0, \ \sin\left(9^{9^{9^{9^9}}}\right) \cdot 9^{9^{9^{9^9}}}\right] \text{ otherwise} \end{cases}
$$
We do not know whether or not the set $\mathcal{H}$ is finite.

Proposition 5. *The number $9^{9^{9^{9^9}}}$ is a threshold number of $\mathcal{H}$. We know an algorithm which decides the equality $\mathcal{H} = \mathbb{N}$. If $\mathcal{H} \neq \mathbb{N}$, then the set $\mathcal{H}$ consists of all integers from 0 to a non-negative integer which can be computed by a known algorithm. We know an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{H}$.*

Let
$$
\mathcal{K} = \begin{cases} \{n\}, \text{ if } (n \in \mathbb{N}) \wedge \left(2^{\aleph_0} = \aleph_{n+1}\right) \\[2mm] \{0\}, \text{ if } 2^{\aleph_0} \geqslant \aleph_\omega \end{cases}
$$

3

Theorem 1. *ZFC proves that* $\text{card}(\mathcal{K}) = 1$. *If ZFC is consistent, then for every* $n \in \mathbb{N}$ *the sentences* "n `is a threshold number of` $\mathcal{K}$" *and* "n `is not a threshold number of` $\mathcal{K}$" *are not provable in ZFC. If ZFC is consistent, then for every* $n \in \mathbb{N}$ *the sentences* "$n \in \mathcal{K}$" *and* "$n \notin \mathcal{K}$" *are not provable in ZFC.*

Proof. It suffices to observe that $2^{\aleph_0}$ can attain every value from the set $\{\aleph_1, \aleph_2, \aleph_3, \ldots\}$, see [7] and [10, p. 232]. □

3 A Diophantine equation whose non-solvability expresses the consistency of *ZFC*

Gödel's second incompleteness theorem and the Davis-Putnam-Robinson-Matiyasevich theorem imply the following theorem.

Theorem 2. *([5, p. 35]). There exists a polynomial* $D(x_1, \ldots, x_m)$ *with integer coefficients such that if ZFC is arithmetically consistent, then the sentences* "`The equation` $D(x_1, \ldots, x_m) = 0$ `is solvable in non-negative integers`" *and* "`The equation` $D(x_1, \ldots, x_m) = 0$ `is not solvable in non-negative integers`" *are not provable in ZFC.*

Remark 1. *([4], [9, p. 53]). The polynomial* $D(x_1, \ldots, x_m)$ *is very complicated.*

Let $\mathcal{Y}$ denote the set of all non-negative integers k such that the equation $D(x_1, \ldots, x_m) = 0$ has no solutions in $\{0, \ldots, k\}^m$. Since the set $\{0, \ldots, k\}^m$ is finite, there exists an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{Y}$. Theorem 2 implies the next theorem.

Theorem 3. *For every* $n \in \mathbb{N}$, *ZFC proves that* $n \in \mathcal{Y}$. *If ZFC is arithmetically consistent, then the sentences* "$\mathcal{Y}$ `is finite`" *and* "$\mathcal{Y}$ `is infinite`" *are not provable in ZFC. If ZFC is arithmetically consistent, then for every* $n \in \mathbb{N}$ *the sentences* "n `is a threshold number of` $\mathcal{Y}$" *and* "n `is not a threshold number of` $\mathcal{Y}$" *are not provable in ZFC.*

Let $\mathcal{E}$ denote the set of all non-negative integers k such that the equation $D(x_1, \ldots, x_m) = 0$ has a solution in $\{0, \ldots, k\}^m$. Since the set $\{0, \ldots, k\}^m$ is finite, there exists an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{E}$. Theorem 2 implies the next theorem.

Theorem 4. *The set* $\mathcal{E}$ *is empty or infinite. In both cases, every non-negative integer n is a threshold number of* $\mathcal{E}$. *If ZFC is arithmetically consistent, then the sentences* "$\mathcal{E}$ `is empty`", "$\mathcal{E}$ `is not empty`", "$\mathcal{E}$ `is finite`", *and* "$\mathcal{E}$ `is infinite`" *are not provable in ZFC.*

Let $\mathcal{V}$ denote the set

$$\Big\{ k \in \mathbb{N} : \big(\text{the polynomial } D(x_1, \ldots, x_m) \text{ has no solutions in } \{0, \ldots, k\}^m\big) \wedge$$

$$\big(\text{the polynomial } D(x_1, \ldots, x_m) \text{ has a solution in } \{0, \ldots, k+1\}^m\big) \Big\}.$$

Since the sets $\{0, \ldots, k\}^m$ and $\{0, \ldots, k+1\}^m$ are finite, there exists an algorithm which for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{V}$. According to Remark 1, at present we do not know a simple computer program that realizes such an algorithm. Theorem 2 implies the next theorem.

Theorem 5. (6) *ZFC proves that* $\text{card}(\mathcal{V}) \in \{0, 1\}$. (7) *For every* $n \in \mathbb{N}$, *ZFC proves that* $n \notin \mathcal{V}$. (8) *ZFC does not prove the emptiness of* $\mathcal{V}$, *if ZFC is arithmetically consistent.* (9) *For every* $n \in \mathbb{N}$, *the sentence* "n `is a threshold number of` $\mathcal{V}$" *is not provable in ZFC, if ZFC is arithmetically consistent.* (10) *For every* $n \in \mathbb{N}$, *the sentence* "n `is not a threshold number of` $\mathcal{V}$" *is not provable in ZFC, if ZFC is arithmetically consistent.*

Open Problem 3. *Define a simple algorithm* `A` *such that* `A` *returns 0 or 1 on every input* $k \in \mathbb{N}$ *and the set*

$$\mathcal{V} = \{k \in \mathbb{N} : \text{ the program A returns 1 on input } k\}$$

satisfies conditions (6)-(10).

4 Hypothetical statements $\Psi_3, \ldots, \Psi_{16}$

For an integer $n \geqslant 3$, let $\mathcal{U}_n$ denote the following system of equations:

$$
\begin{cases}
\forall i \in \{1, \ldots, n-1\} \setminus \{2\} \; x_i! &= x_{i+1} \\
x_1 \cdot x_1 &= x_3 \\
x_2 \cdot x_2 &= x_3
\end{cases}
$$

The diagram in Figure 1 illustrates the construction of the system $\mathcal{U}_n$.
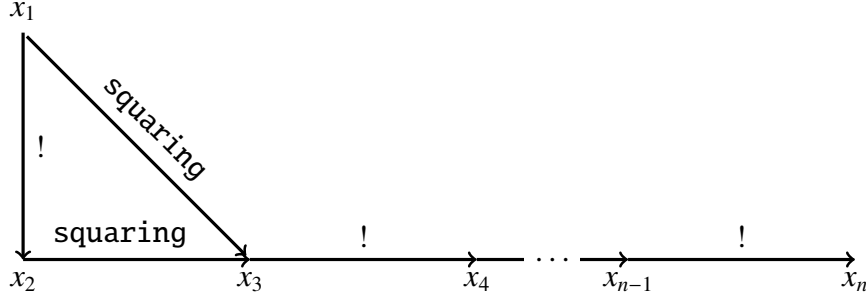

Fig. 1 Construction of the system $\mathcal{U}_n$

Let $g(3) = 4$, and let $g(n+1) = g(n)!$ for every integer $n \geqslant 3$.

Lemma 5. *For every integer $n \geqslant 3$, the system $\mathcal{U}_n$ has exactly two solutions in positive integers, namely* $(1, \ldots, 1)$ *and* $\left(2, 2, g(3), \ldots, g(n)\right)$.

Let

$$
B_n = \left\{ x_i! = x_k \; : \; (i, k \in \{1, \ldots, n\}) \wedge (i \neq k) \right\} \cup \left\{ x_i \cdot x_j = x_k \; : \; i, j, k \in \{1, \ldots, n\} \right\}
$$

For an integer $n \geqslant 3$, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant g(n)$.* The statement Ψ_n says that for subsystems of B_n the largest known solution is indeed the largest possible.

Hypothesis 1. *The statements $\Psi_3, \ldots, \Psi_{16}$ are true.*

Lemma 6. *Every statement Ψ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system B_n has a finite number of subsystems. $\qquad \square$

Lemma 7. *For every statement Ψ_n, the bound $g(n)$ cannot be decreased.*

Proof. It follows from Lemma 5 because $\mathcal{U}_n \subseteq B_n$. $\qquad \square$

Remark 2. *By Lemma 2 and algebraic lemmas in [19, p. 110], the statement $\forall n \in \mathbb{N} \setminus \{0, 1, 2\} \; \Psi_n$ implies that there is an algorithm which takes as input a factorial Diophantine equation, and returns an integer such that this integer is greater than the solutions in positive integers, if these solutions form a finite set. This conclusion is unbelievable because a computable upper bound on non-negative integer solutions does not exist for exponential Diophantine equations with a finite number of solutions, see [13, p. 300]. Therefore, the statement $\forall n \in \mathbb{N} \setminus \{0, 1, 2\} \; \Psi_n$ seems to be false.*

5 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let $\mathcal{A}$ denote the following system of equations:

$$\begin{cases} x_1! &=& x_2 \\ x_2! &=& x_3 \\ x_5! &=& x_6 \\ x_4 \cdot x_4 &=& x_5 \\ x_3 \cdot x_5 &=& x_6 \end{cases}$$

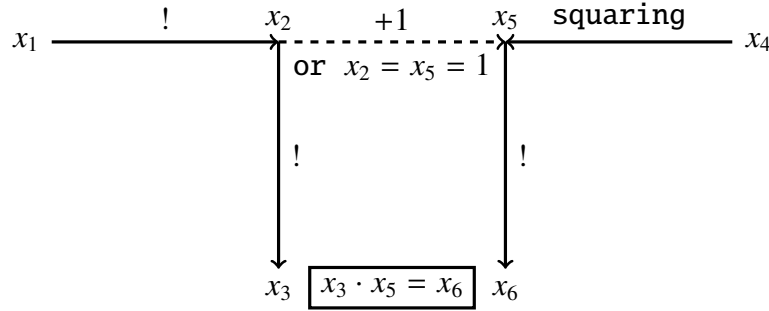Lemma 2 and the diagram in Figure 2 explain the construction of the system $\mathcal{A}$.

Fig. 2 Construction of the system $\mathcal{A}$

Lemma 8. *For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system $\mathcal{A}$ is solvable in positive integers x_2, x_3, x_5, x_6 if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:*

$$\begin{array}{rcl} x_2 &=& x_1! \\ x_3 &=& (x_1!)! \\ x_5 &=& x_1! + 1 \\ x_6 &=& (x_1! + 1)! \end{array}$$

Proof. It follows from Lemma 2. □

It is conjectured that $x! + 1$ is a perfect square only for $x \in \{4, 5, 7\}$, see [20, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [16].

Theorem 6. *If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement Ψ_6 guarantees that each such solution (x_1, x_4) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.*

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 8, the system $\mathcal{A}$ is solvable in positive integers x_2, x_3, x_5, x_6. Since $\mathcal{A} \subseteq B_6$, the statement Ψ_6 implies that $x_6 = (x_1! + 1)! \leqslant g(6) = g(5)!$. Hence, $x_1! + 1 \leqslant g(5) = g(4)!$. Consequently, $x_1 < g(4) = 24$. If $x_1 \in \{1, \ldots, 23\}$, then $x_1! + 1$ is a perfect square only for $x_1 \in \{4, 5, 7\}$. □

6 Are there infinitely many prime numbers of the form $n^2 + 1$?

Edmund Landau's conjecture states that there are infinitely many primes of the form $n^2 + 1$, see [15, pp. 37–38]. Let $\mathcal{B}$ denote the following system of equations:

$$\begin{cases} x_2! &=& x_3 \\ x_3! &=& x_4 \\ x_5! &=& x_6 \\ x_8! &=& x_9 \end{cases} \qquad \begin{array}{rcl} x_1 \cdot x_1 &=& x_2 \\ x_3 \cdot x_5 &=& x_6 \\ x_4 \cdot x_8 &=& x_9 \\ x_5 \cdot x_7 &=& x_8 \end{array}$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system $\mathcal{B}$.
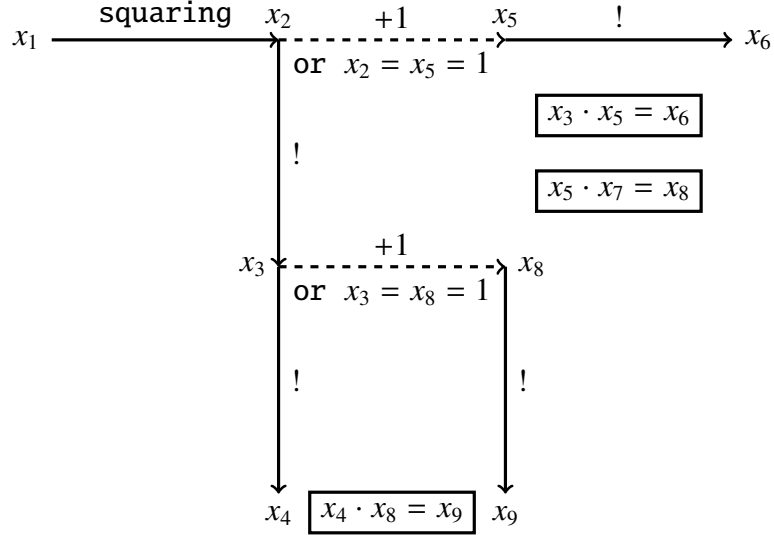
Fig. 3 Construction of the system $\mathcal{B}$

Lemma 9. *For every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ is prime. In this case, the integers $x_2, \ldots, x_9$ are uniquely determined by the following equalities:*

$$
\begin{aligned}
x_2 &= x_1^2 \\
x_3 &= (x_1^2)! \\
x_4 &= ((x_1^2)!)! \\
x_5 &= x_1^2 + 1 \\
x_6 &= (x_1^2 + 1)!
\end{aligned}
\qquad
\begin{aligned}
x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
x_8 &= (x_1^2)! + 1 \\
x_9 &= ((x_1^2)! + 1)!
\end{aligned}
$$

Proof. By Lemma 2, for every integer $x_1 \geqslant 2$, the system $\mathcal{B}$ is solvable in positive integers $x_2, \ldots, x_9$ if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 9 follows from Lemma 4. □

Lemma 10. *There are only finitely many tuples $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system $\mathcal{B}$ and satisfy $x_1 = 1$.*

Proof. If a tuple $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system $\mathcal{B}$ and $x_1 = 1$, then $x_1, \ldots, x_9 \leqslant 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leqslant 2$. □

Theorem 7. *The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geqslant 2$ such that $x_1^2 + 1$ is prime and greater than $g(7)$, then there are infinitely many primes of the form $n^2 + 1$.*

Proof. Suppose that the antecedent holds. By Lemma 9, there exists a unique tuple $(x_2, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple $(x_1, x_2, \ldots, x_9)$ solves the system $\mathcal{B}$. Since $x_1^2 + 1 > g(7)$, we obtain that $x_1^2 \geqslant g(7)$. Hence, $(x_1^2)! \geqslant g(7)! = g(8)$. Consequently,

$$
x_9 = ((x_1^2)! + 1)! \geqslant (g(8) + 1)! > g(8)! = g(9)
$$

Since $\mathcal{B} \subseteq B_9$, the statement Ψ_9 and the inequality $x_9 > g(9)$ imply that the system $\mathcal{B}$ has infinitely many solutions $(x_1, \ldots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 9 and 10, there are infinitely many primes of the form $n^2 + 1$. □

Corollary 2. *Let $\mathcal{X}_9$ denote the set of primes of the form $n^2 + 1$. The statement Ψ_9 implies that we know an algorithm such that it returns a threshold number of $\mathcal{X}_9$, and this number equals $\max(\mathcal{X}_9)$, if $\mathcal{X}_9$ is finite. Assuming the statement Ψ_9, a single query to an oracle for the halting problem decides the infiniteness of $\mathcal{X}_9$. Assuming the statement Ψ_9, the infiniteness of $\mathcal{X}_9$ is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{X}_9 \cap [1, g(7)])$. □

7 Are there infinitely many prime numbers of the form $n! + 1$?

It is conjectured that there are infinitely many primes of the form $n! + 1$, see [2, p. 443].

Theorem 8. *The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geqslant g(6)$ such that $x_1! + 1$ is prime, then there are infinitely many primes of the form $n! + 1$.*

Proof. We leave the analogous proof to the reader. □

8 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [15, p. 39]. Let C denote the following system of equations:

$$
\begin{cases}
x_1! = x_2 \\
x_2! = x_3 \\
x_4! = x_5 \\
x_6! = x_7 \\
x_7! = x_8 \\
x_9! = x_{10} \\
x_{12}! = x_{13} \\
x_{15}! = x_{16}
\end{cases}
\qquad
\begin{aligned}
x_2 \cdot x_4 &= x_5 \\
x_5 \cdot x_6 &= x_7 \\
x_7 \cdot x_9 &= x_{10} \\
x_4 \cdot x_{11} &= x_{12} \\
x_3 \cdot x_{12} &= x_{13} \\
x_9 \cdot x_{14} &= x_{15} \\
x_8 \cdot x_{15} &= x_{16}
\end{aligned}
$$

Lemma 2 and the diagram in Figure 4 explain the construction of the system C.
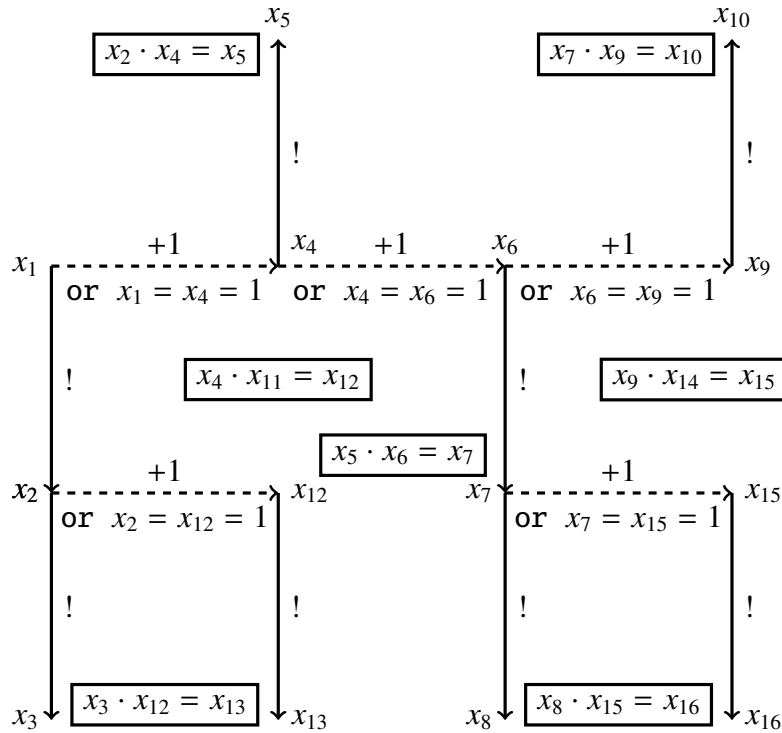

Fig. 4 Construction of the system C

Lemma 11. *For every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ if and only if x_4 and x_9 are prime and $x_4 + 2 = x_9$. In this case, the integers $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$ are uniquely determined by the following equalities:*

$$
\begin{aligned}
x_1 &= x_4 - 1 \\
x_2 &= (x_4 - 1)! \\
x_3 &= ((x_4 - 1)!)! \\
x_5 &= x_4! \\
x_6 &= x_9 - 1 \\
x_7 &= (x_9 - 1)! \\
x_8 &= ((x_9 - 1)!)! \\
x_{10} &= x_9!
\end{aligned}
\qquad
\begin{aligned}
x_{11} &= \frac{(x_4 - 1)! + 1}{x_4} \\
x_{12} &= (x_4 - 1)! + 1 \\
x_{13} &= ((x_4 - 1)! + 1)! \\
x_{14} &= \frac{(x_9 - 1)! + 1}{x_9} \\
x_{15} &= (x_9 - 1)! + 1 \\
x_{16} &= ((x_9 - 1)! + 1)!
\end{aligned}
$$

Proof. By Lemma 2, for every $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$, the system C is solvable in positive integers x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} if and only if

$$
\left(x_4 + 2 = x_9\right) \wedge \left(x_4 | (x_4 - 1)! + 1\right) \wedge \left(x_9 | (x_9 - 1)! + 1\right)
$$

Hence, the claim of Lemma 11 follows from Lemma 4. $\qquad\square$

Lemma 12. *There are only finitely many tuples $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ which solve the system C and satisfy $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$.*

Proof. If a tuple $(x_1, \ldots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$ solves the system C and $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$, then $x_1, \ldots, x_{16} \leqslant 7!$. Indeed, for example, if $x_4 = 2$ then $x_6 = x_4 + 1 = 3$. Hence, $x_7 = x_6! = 6$. Therefore, $x_{15} = x_7 + 1 = 7$. Consequently, $x_{16} = x_{15}! = 7!$. $\qquad\square$

Theorem 9. *The statement Ψ_{16} proves the following implication: if there exists a twin prime greater than $g(14)$, then there are infinitely many twin primes.*

Proof. Suppose that the antecedent holds. Then, there exist prime numbers x_4 and x_9 such that $x_9 = x_4 + 2 > g(14)$. Hence, $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$. By Lemma 11, there exists a unique tuple

$$
(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{14}
$$

such that the tuple $(x_1, \ldots, x_{16})$ solves the system C. Since $x_9 > g(14)$, we obtain that $x_9 - 1 \geqslant g(14)$. Therefore, $(x_9 - 1)! \geqslant g(14)! = g(15)$. Hence, $(x_9 - 1)! + 1 > g(15)$. Consequently,

$$
x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)
$$

Since $C \subseteq B_{16}$, the statement Ψ_{16} and the inequality $x_{16} > g(16)$ imply that the system C has infinitely many solutions in positive integers $x_1, \ldots, x_{16}$. According to Lemmas 11 and 12, there are infinitely many twin primes. $\qquad\square$

Corollary 3. *(cf. [6]). Let $\mathcal{X}_{16}$ denote the set of twin primes. The statement Ψ_{16} implies that we know an algorithm such that it returns a threshold number of $\mathcal{X}_{16}$, and this number equals $\max(\mathcal{X}_{16})$, if $\mathcal{X}_{16}$ is finite. Assuming the statement Ψ_{16}, a single query to an oracle for the halting problem decides the infiniteness of $\mathcal{X}_{16}$. Assuming the statement Ψ_{16}, the infiniteness of $\mathcal{X}_{16}$ is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{X}_{16} \cap [1, g(14)])$. $\qquad\square$

9 Are there infinitely many composite Fermat numbers?

Integers of the form $2^{2^n} + 1$ are called Fermat numbers. Primes of the form $2^{2^n} + 1$ are called Fermat primes, as Fermat conjectured that every integer of the form $2^{2^n} + 1$ is prime, see [12, p. 1]. Fermat correctly remarked that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$, and $2^{2^4} + 1 = 65537$ are all prime, see [12, p. 1].

Open Problem 4. *([12, p. 159]). Are there infinitely many composite numbers of the form $2^{2^n} + 1$?*

Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geqslant 5$, see [11, p. 23]. Let

$$H_n = \left\{ x_i \cdot x_j = x_k : \ i, j, k \in \{1, \ldots, n\} \right\} \cup \left\{ 2^{2^{x_i}} = x_k : \ i, k \in \{1, \ldots, n\} \right\}$$

Let $h(1) = 1$, and let $h(n + 1) = 2^{2^{h(n)}}$ for every positive integer n.

Lemma 13. *The following subsystem of H_n*

$$\begin{cases} x_1 \cdot x_1 &=\ x_1 \\ \forall i \in \{1, \ldots, n - 1\}\ 2^{2^{x_i}} &=\ x_{i+1} \end{cases}$$

has exactly one solution $(x_1, \ldots, x_n) \in (\mathbb{N} \setminus \{0\})^n$, namely $(h(1), \ldots, h(n))$.

For a positive integer n, let ξ_n denote the following statement: *if a system of equations $S \subseteq H_n$ has only finitely many solutions in positive integers $x_1, \ldots, x_n$, then each such solution $(x_1, \ldots, x_n)$ satisfies $x_1, \ldots, x_n \leqslant h(n)$.* The statement ξ_n says that for subsystems of H_n the largest known solution is indeed the largest possible.

Hypothesis 2. *The statements $\xi_1, \ldots, \xi_{13}$ are true.*

Lemma 14. *Every statement ξ_n is true with an unknown integer bound that depends on n.*

Proof. For every positive integer n, the system H_n has a finite number of subsystems. $\qquad \square$

Theorem 10. *The statement ξ_{13} proves the following implication: if $z \in \mathbb{N} \setminus \{0\}$ and $2^{2^z} + 1$ is composite and greater than $h(12)$, then $2^{2^z} + 1$ is composite for infinitely many positive integers z.*

Proof. Let us consider the equation

$$(x + 1)(y + 1) = 2^{2^z} + 1 \tag{E}$$

in positive integers. By Lemma 3, we can transform the equation (E) into an equivalent system of equations $\mathcal{G}$ which has 13 variables (x, y, z, and 10 other variables) and which consists of equations of the forms $\alpha \cdot \beta = \gamma$ and $2^{2^{\alpha}} = \gamma$, see the diagram in Figure 5.
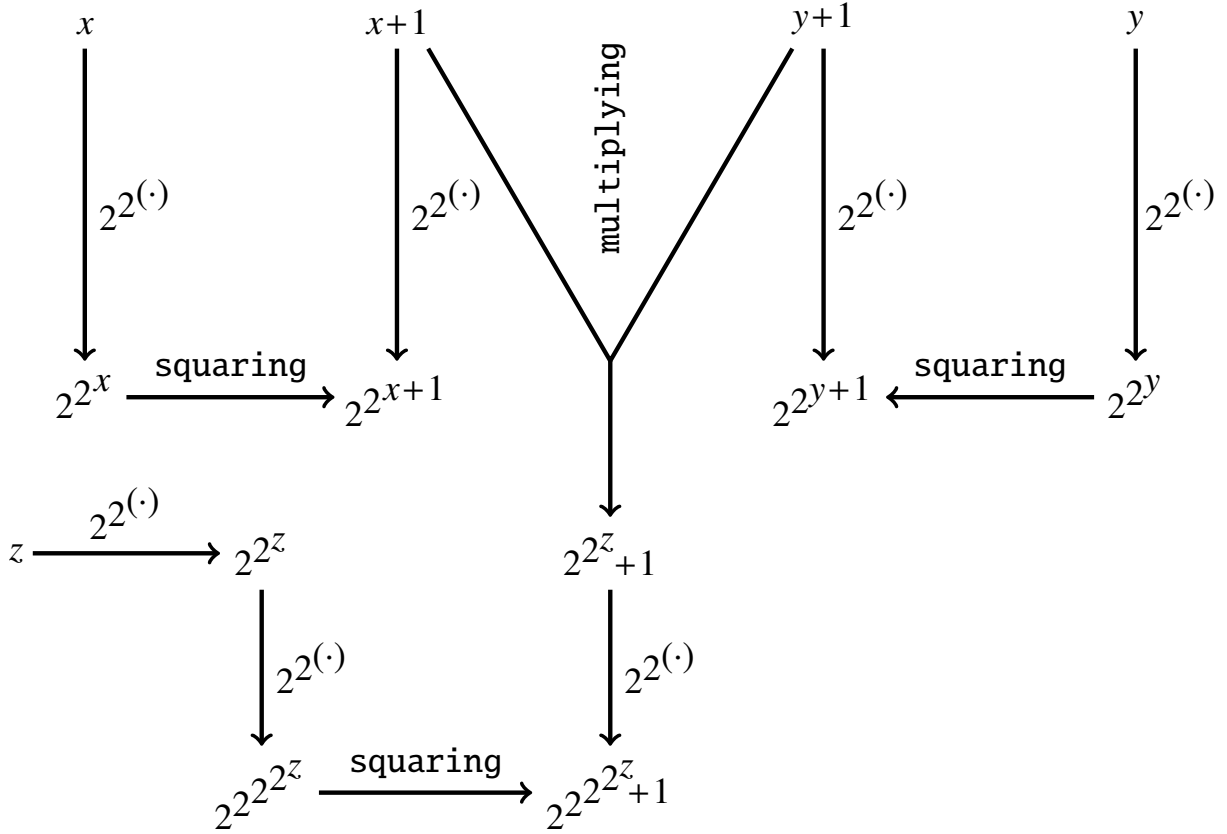
Fig. 5 Construction of the system $\mathcal{G}$

Since $2^{2^z} + 1 > h(12)$, we obtain that $2^{2^{2^{2^z+1}}} > h(13)$. By this, the statement ξ_{13} implies that the system $\mathcal{G}$ has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers. $\qquad\square$

Corollary 4. *Let $\mathcal{W}_{13}$ denote the set of composite Fermat numbers. The statement ξ_{13} implies that we know an algorithm such that it returns a threshold number of $\mathcal{W}_{13}$, and this number equals $\max(\mathcal{W}_{13})$, if $\mathcal{W}_{13}$ is finite. Assuming the statement ξ_{13}, a single query to an oracle for the halting problem decides the infiniteness of $\mathcal{W}_{13}$. Assuming the statement ξ_{13}, the infiniteness of $\mathcal{W}_{13}$ is decidable in the limit.*

Proof. We consider an algorithm which computes $\max(\mathcal{W}_{13} \cap [1, h(12)])$. $\qquad\square$

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