

On *ZFC*-formulae  $\varphi(x)$  for which we know a non-negative integer  $n$  such that  $\{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$  if the set  $\{x \in \mathbb{N} : \varphi(x)\}$  is finite

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### Abstract

Let  $\mathcal{P}_{\text{twin}}$  denote the set of twin primes, and let  $\mathcal{M}$  denote the set of multiples of twin primes greater than  $99999$ . The set  $\mathcal{X} = ([2, 99999] \cap \mathcal{P}_{\text{twin}}) \cup \mathcal{M}$  satisfies the following conditions: (1) a known and simple algorithm for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{X}$ , (2) a known and simple algorithm returns an integer  $n$  such that  $\mathcal{X}$  is infinite if and only if  $\mathcal{X}$  contains an element greater than  $n$ , (3) new elements of  $\mathcal{X}$  are still discovered, (4) it is conjectured that  $\mathcal{X}$  is infinite although we do not know any algorithm deciding the infiniteness of  $\mathcal{X}$ . The following problem is open: **define a set  $\mathcal{X} \subseteq \mathbb{N}$  such that  $\mathcal{X}$  satisfies conditions (1)–(4) and a known and simple formula  $\phi(x)$  of Peano arithmetic satisfies  $\{n \in \mathbb{N} : \phi(n)\} = \mathcal{X}$  and  $\phi(n)$  has the same intuitive meaning for every  $n \in \mathbb{N}$**  (5). The problem remains open if condition (2) states that a known and simple algorithm returns an integer  $n$  such that  $\mathcal{X}$  is infinite if and only if  $\text{card}(\mathcal{X}) > n$ . Let  $g(3) = 4$ , and let  $g(n+1) = g(n)!$  for every integer  $n \geq 3$ . For an integer  $n \in \{3, \dots, 16\}$ , let  $\Psi_n$  denote the following statement: if a system of equations  $\mathcal{S} \subseteq \{x_i! = x_k : (i, k \in \{1, \dots, n\}) \wedge (i \neq k)\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq g(n)$ . For every statement  $\Psi_n$ , the bound  $g(n)$  cannot be decreased. The author's guess is that the statements  $\Psi_3, \dots, \Psi_{16}$  are true. The statement  $\Psi_9$  implies that the set of primes of the form  $n^2 + 1$  and the set of primes of the form  $n! + 1$  satisfy conditions (1)–(5). The statement  $\Psi_{16}$  implies that the set of twin primes satisfies conditions (1)–(5).

**Key words and phrases:** finiteness of a set, incompleteness of *ZFC*, infiniteness of a set, prime numbers of the form  $n^2 + 1$ , prime numbers of the form  $n! + 1$ , twin primes.

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## 1 Introduction and basic lemmas

The phrase "we know a non-negative integer  $n$ " in the title means that we know an algorithm which returns  $n$ . The title cannot be formalised in *ZFC* because the phrase "we know a non-negative integer  $n$ " refers to currently known non-negative integers  $n$  with some property. A formally stated title may look like this: On *ZFC*-formulae  $\varphi(x)$  for which there exists a non-negative integer  $n$  such that *ZFC* proves that

$$\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) < \infty \implies \{x \in \mathbb{N} : \varphi(x)\} \subseteq \{x \in \mathbb{N} : x \leq n - 1\}$$

Unfortunately, this formulation admits formulae  $\varphi(x)$  without any known non-negative integer  $n$  such that *ZFC* proves the above implication.

**Lemma 1.** For every non-negative integer  $n$ ,  $\text{card}(\{x \in \mathbb{N} : x \leq n - 1\}) = n$ .

**Corollary 1.** The title altered to "On ZFC-formulae  $\varphi(x)$  for which we know a non-negative integer  $n$  such that  $\text{card}(\{x \in \mathbb{N} : \varphi(x)\}) \leq n$  if the set  $\{x \in \mathbb{N} : \varphi(x)\}$  is finite" involves a weaker assumption on  $\varphi(x)$ .

**Lemma 2.** For every positive integers  $x$  and  $y$ ,  $x! \cdot y = y!$  if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

**Lemma 3.** For every non-negative integers  $b$  and  $c$ ,  $b + 1 = c$  if and only if

$$2^{2^b} \cdot 2^{2^b} = 2^{2^c}$$

**Lemma 4.** (Wilson's theorem, [8, p. 89]). For every positive integer  $x$ ,  $x$  divides  $(x - 1)! + 1$  if and only if  $x = 1$  or  $x$  is prime.

## 2 Subsets of $\mathbb{N}$ and their threshold numbers

**Definition 1.** We say that an integer  $m \in [-1, \infty)$  is a threshold number of a set  $X \subseteq \mathbb{N}$ , if  $X$  is infinite if and only if  $X$  contains an element greater than  $m$ , cf. [22] and [23].

If a set  $X \subseteq \mathbb{N}$  is empty or infinite, then any  $m \in [-1, \infty) \cap \mathbb{Z}$  is a threshold number of  $X$ . If a set  $X \subseteq \mathbb{N}$  is non-empty and finite, then the all threshold numbers of  $X$  form the set  $\{\max(X), \max(X) + 1, \max(X) + 2, \dots\}$ .

**Definition 2.** We say that a non-negative integer  $m$  is a weak threshold number of a set  $X \subseteq \mathbb{N}$ , if  $X$  is infinite if and only if  $\text{card}(X) > m$ .

**Proposition 1.** For every  $X \subseteq \mathbb{N}$ , if an integer  $m \in [-1, \infty)$  is a threshold number of  $X$ , then  $m + 1$  is a weak threshold number of  $X$ .

*Proof.* For every  $X \subseteq \mathbb{N}$ , if  $m \in [-1, \infty) \cap \mathbb{Z}$  and  $\text{card}(X) > m + 1$ , then  $X \cap [m + 1, \infty) \neq \emptyset$ . □

It is conjectured that the set of prime numbers of the form  $n^2 + 1$  is infinite, see [15, pp. 37–38]. It is conjectured that the set of prime numbers of the form  $n! + 1$  is infinite, see [2, p. 443]. A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that the set of twin primes is infinite, see [15, p. 39]. It is conjectured that the set of composite numbers of the form  $2^{2^n} + 1$  is infinite, see [11, p. 23] and [12, pp. 158–159]. A prime  $p$  is said to be a Sophie Germain prime if both  $p$  and  $2p + 1$  are prime, see [21]. It is conjectured that the set of Sophie Germain primes is infinite, see [17, p. 330]. For each of these sets, we do not know any weak threshold number.

**Open Problem 1.** Define a set  $X \subseteq \mathbb{N}$  that satisfies the following conditions:

- (a1) a known and simple algorithm for every  $n \in \mathbb{N}$  decides whether or not  $n \in X$ ,
- (b1) a known and simple algorithm returns an integer  $n$  such that  $X$  is infinite if and only if  $\text{card}(X) > n$ ,
- (c1) new elements of  $X$  are still discovered,
- (d1) it is conjectured that  $X$  is infinite although we do not know any algorithm deciding the infiniteness of  $X$ ,
- (e1) a known and simple formula  $\phi(x)$  of Peano arithmetic satisfies  $\{n \in \mathbb{N} : \phi(n)\} = X$  and  $\phi(n)$  has the same intuitive meaning for every  $n \in \mathbb{N}$ .

The following statement: for every non-negative integer  $n$  there exist

$$\text{prime numbers } p \text{ and } q \text{ such that } p + 2 = q \text{ and } p \in [10^n, 10^n + 1] \tag{T}$$

is a  $\Pi_1$  statement which strengthens the twin prime conjecture, see [3, p. 43]. C. H. Bennett claims that most mathematical conjectures can be settled indirectly by proving stronger  $\Pi_1$  statements, see [1]. The

statement (T) is equivalent to the non-halting of a Turing machine. If a set  $X \subseteq \mathbb{N}$  is computable and we know a threshold number of  $X$ , then the infiniteness of  $X$  is equivalent to the halting of a Turing machine.

The height of a rational number  $\frac{p}{q}$  is denoted by  $H\left(\frac{p}{q}\right)$  and equals  $\max(|p|, |q|)$  provided  $\frac{p}{q}$  is written in lowest terms. The height of a rational tuple  $(x_1, \dots, x_n)$  is denoted by  $H(x_1, \dots, x_n)$  and equals  $\max(H(x_1), \dots, H(x_n))$ .

**Proposition 2.** *The equation  $x^5 - x = y^2 - y$  has only finitely many rational solutions, see [14, p. 212]. The known rational solutions are  $(x, y) = (-1, 0), (-1, 1), (0, 0), (0, 1), (1, 0), (1, 1), (2, -5), (2, 6), (3, -15), (3, 16), (30, -4929), (30, 4930), \left(\frac{1}{4}, \frac{15}{32}\right), \left(\frac{1}{4}, \frac{17}{32}\right), \left(-\frac{15}{16}, -\frac{185}{1024}\right), \left(-\frac{15}{16}, \frac{1209}{1024}\right)$ , and the existence of other solutions is an open question, see [18, pp. 223–224].*

**Proposition 3.** *The set  $\mathcal{T} = \{n \in \mathbb{N} : \text{the equation } x^5 - x = y^2 - y \text{ has a rational solution of height } n\}$  is finite. We know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{T}$ . We do not know any algorithm which returns a threshold number of  $\mathcal{T}$ .*

**Open Problem 2.** *Define a set  $X \subseteq \mathbb{N}$  that satisfies the following conditions:*

- (a2) *a known and simple algorithm for every  $n \in \mathbb{N}$  decides whether or not  $n \in X$ ,*
- (b2) *a known and simple algorithm returns an integer  $n$  such that  $X$  is infinite if and only if  $X$  contains an element greater than  $n$ ,*
- (c2) *new elements of  $X$  are still discovered,*
- (d2) *it is conjectured that  $X$  is infinite although we do not know any algorithm deciding the infiniteness of  $X$ ,*
- (e2) *a known and simple formula  $\phi(x)$  of Peano arithmetic satisfies  $\{n \in \mathbb{N} : \phi(n)\} = X$  and  $\phi(n)$  has the same intuitive meaning for every  $n \in \mathbb{N}$ .*

Let  $\mathcal{P}_{\text{twin}}$  denote the set of twin primes, and let  $\mathcal{M}$  denote the set of multiples of twin primes greater than  $999999$ .

**Proposition 4.** *The set  $X = \left(\left[2, 999999\right] \cap \mathcal{P}_{\text{twin}}\right) \cup \mathcal{M}$  satisfies conditions (a2)–(d2).*

*Proof.* The largest known twin prime is much smaller than  $999999$ . □

Let

$$\mathcal{H} = \begin{cases} \mathbb{N}, & \text{if } \sin\left(999999\right) < 0 \\ \mathbb{N} \cap \left[0, \sin\left(999999\right) \cdot 999999\right) & \text{otherwise} \end{cases}$$

We do not know whether or not the set  $\mathcal{H}$  is finite.

**Proposition 5.** *The number  $999999$  is a threshold number of  $\mathcal{H}$ . We know an algorithm which decides the equality  $\mathcal{H} = \mathbb{N}$ . If  $\mathcal{H} \neq \mathbb{N}$ , then the set  $\mathcal{H}$  consists of all integers from 0 to a non-negative integer which can be computed by a known algorithm. We know an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{H}$ .*

Let

$$\mathcal{K} = \begin{cases} \{n\}, & \text{if } (n \in \mathbb{N}) \wedge (2^{\aleph_0} = \aleph_{n+1}) \\ \{0\}, & \text{if } 2^{\aleph_0} \geq \aleph_{\omega} \end{cases}$$

**Theorem 1.** *ZFC proves that  $\text{card}(\mathcal{K}) = 1$ . If ZFC is consistent, then for every  $n \in \mathbb{N}$  the sentences " $n$  is a threshold number of  $\mathcal{K}$ " and " $n$  is not a threshold number of  $\mathcal{K}$ " are not provable in ZFC. If ZFC is consistent, then for every  $n \in \mathbb{N}$  the sentences " $n \in \mathcal{K}$ " and " $n \notin \mathcal{K}$ " are not provable in ZFC.*

*Proof.* It suffices to observe that  $2^{\aleph_0}$  can attain every value from the set  $\{\aleph_1, \aleph_2, \aleph_3, \dots\}$ , see [7] and [10, p. 232].  $\square$

### 3 A Diophantine equation whose non-solvability expresses the consistency of ZFC

Gödel's second incompleteness theorem and the Davis-Putnam-Robinson-Matiyasevich theorem imply the following theorem.

**Theorem 2.** *([5, p. 35]). There exists a polynomial  $D(x_1, \dots, x_m)$  with integer coefficients such that if ZFC is arithmetically consistent, then the sentences "The equation  $D(x_1, \dots, x_m) = 0$  is solvable in non-negative integers" and "The equation  $D(x_1, \dots, x_m) = 0$  is not solvable in non-negative integers" are not provable in ZFC.*

**Remark 1.** *([4], [9, p. 53]). The polynomial  $D(x_1, \dots, x_m)$  is very complicated.*

Let  $\mathcal{Y}$  denote the set of all non-negative integers  $k$  such that the equation  $D(x_1, \dots, x_m) = 0$  has no solutions in  $\{0, \dots, k\}^m$ . Since the set  $\{0, \dots, k\}^m$  is finite, there exists an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{Y}$ . Theorem 2 implies the next theorem.

**Theorem 3.** *For every  $n \in \mathbb{N}$ , ZFC proves that  $n \in \mathcal{Y}$ . If ZFC is arithmetically consistent, then the sentences " $\mathcal{Y}$  is finite" and " $\mathcal{Y}$  is infinite" are not provable in ZFC. If ZFC is arithmetically consistent, then for every  $n \in \mathbb{N}$  the sentences " $n$  is a threshold number of  $\mathcal{Y}$ " and " $n$  is not a threshold number of  $\mathcal{Y}$ " are not provable in ZFC.*

Let  $\mathcal{E}$  denote the set of all non-negative integers  $k$  such that the equation  $D(x_1, \dots, x_m) = 0$  has a solution in  $\{0, \dots, k\}^m$ . Since the set  $\{0, \dots, k\}^m$  is finite, there exists an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{E}$ . Theorem 2 implies the next theorem.

**Theorem 4.** *The set  $\mathcal{E}$  is empty or infinite. In both cases, every non-negative integer  $n$  is a threshold number of  $\mathcal{E}$ . If ZFC is arithmetically consistent, then the sentences " $\mathcal{E}$  is empty", " $\mathcal{E}$  is not empty", " $\mathcal{E}$  is finite", and " $\mathcal{E}$  is infinite" are not provable in ZFC.*

Let  $\mathcal{V}$  denote the set

$$\left\{ k \in \mathbb{N} : \left( \text{the polynomial } D(x_1, \dots, x_m) \text{ has no solutions in } \{0, \dots, k\}^m \right) \wedge \left( \text{the polynomial } D(x_1, \dots, x_m) \text{ has a solution in } \{0, \dots, k+1\}^m \right) \right\}.$$

Since the sets  $\{0, \dots, k\}^m$  and  $\{0, \dots, k+1\}^m$  are finite, there exists an algorithm which for every  $n \in \mathbb{N}$  decides whether or not  $n \in \mathcal{V}$ . According to Remark 1, at present we do not know a simple computer program that realizes such an algorithm. Theorem 2 implies the next theorem.

**Theorem 5.** (6) *ZFC proves that  $\text{card}(\mathcal{V}) \in \{0, 1\}$ .* (7) *For every  $n \in \mathbb{N}$ , ZFC proves that  $n \notin \mathcal{V}$ .* (8) *ZFC does not prove the emptiness of  $\mathcal{V}$ , if ZFC is arithmetically consistent.* (9) *For every  $n \in \mathbb{N}$ , the sentence " $n$  is a threshold number of  $\mathcal{V}$ " is not provable in ZFC, if ZFC is arithmetically consistent.* (10) *For every  $n \in \mathbb{N}$ , the sentence " $n$  is not a threshold number of  $\mathcal{V}$ " is not provable in ZFC, if ZFC is arithmetically consistent.*

**Open Problem 3.** *Define a simple algorithm A such that A returns 0 or 1 on every input  $k \in \mathbb{N}$  and the set*

$$\mathcal{V} = \{k \in \mathbb{N} : \text{the program A returns 1 on input } k\}$$

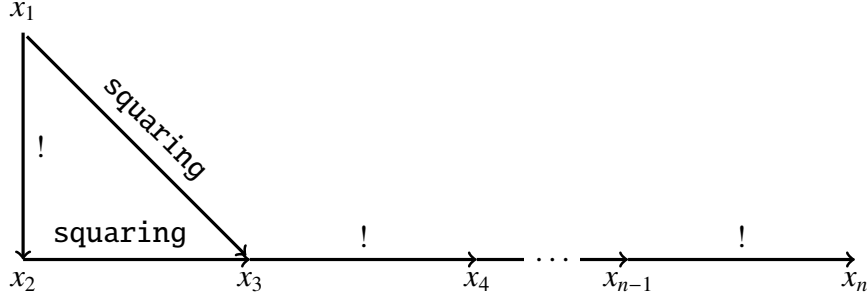
*satisfies conditions (6)–(10).*

## 4 Hypothetical statements $\Psi_3, \dots, \Psi_{16}$

For an integer  $n \geq 3$ , let  $\mathcal{U}_n$  denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \dots, n-1\} \setminus \{2\} & x_i! = x_{i+1} \\ & x_1 \cdot x_1 = x_3 \\ & x_2 \cdot x_2 = x_3 \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system  $\mathcal{U}_n$ .



**Fig. 1** Construction of the system  $\mathcal{U}_n$

Let  $g(3) = 4$ , and let  $g(n+1) = g(n)!$  for every integer  $n \geq 3$ .

**Lemma 5.** *For every integer  $n \geq 3$ , the system  $\mathcal{U}_n$  has exactly two solutions in positive integers, namely  $(1, \dots, 1)$  and  $(2, 2, g(3), \dots, g(n))$ .*

Let

$$B_n = \{x_i! = x_k : (i, k \in \{1, \dots, n\}) \wedge (i \neq k)\} \cup \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$$

For an integer  $n \geq 3$ , let  $\Psi_n$  denote the following statement: *if a system of equations  $\mathcal{S} \subseteq B_n$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq g(n)$ .* The statement  $\Psi_n$  says that for subsystems of  $B_n$  the largest known solution is indeed the largest possible.

**Hypothesis 1.** *The statements  $\Psi_3, \dots, \Psi_{16}$  are true.*

**Lemma 6.** *Every statement  $\Psi_n$  is true with an unknown integer bound that depends on  $n$ .*

*Proof.* For every positive integer  $n$ , the system  $B_n$  has a finite number of subsystems. □

**Lemma 7.** *For every statement  $\Psi_n$ , the bound  $g(n)$  cannot be decreased.*

*Proof.* It follows from Lemma 5 because  $\mathcal{U}_n \subseteq B_n$ . □

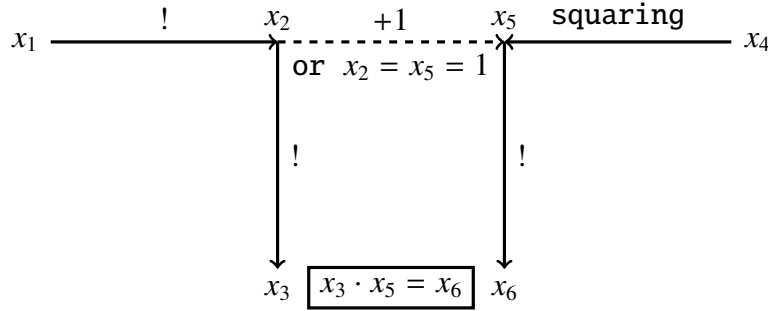
**Remark 2.** *By Lemma 2 and algebraic lemmas in [19, p. 110], the statement  $\forall n \in \mathbb{N} \setminus \{0, 1, 2\} \Psi_n$  implies that there is an algorithm which takes as input a factorial Diophantine equation, and returns an integer such that this integer is greater than the solutions in positive integers, if these solutions form a finite set. This conclusion is unbelievable because a computable upper bound on non-negative integer solutions does not exist for exponential Diophantine equations with a finite number of solutions, see [13, p. 300]. Therefore, the statement  $\forall n \in \mathbb{N} \setminus \{0, 1, 2\} \Psi_n$  seems to be false.*

## 5 The Brocard-Ramanujan equation $x! + 1 = y^2$

Let  $\mathcal{A}$  denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_5! = x_6 \\ x_4 \cdot x_4 = x_5 \\ x_3 \cdot x_5 = x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system  $\mathcal{A}$ .



**Fig. 2** Construction of the system  $\mathcal{A}$

**Lemma 8.** For every  $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$ , the system  $\mathcal{A}$  is solvable in positive integers  $x_2, x_3, x_5, x_6$  if and only if  $x_1! + 1 = x_4^2$ . In this case, the integers  $x_2, x_3, x_5, x_6$  are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

*Proof.* It follows from Lemma 2. □

It is conjectured that  $x! + 1$  is a perfect square only for  $x \in \{4, 5, 7\}$ , see [20, p. 297]. A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation  $x! + 1 = y^2$ , see [16].

**Theorem 6.** If the equation  $x_1! + 1 = x_4^2$  has only finitely many solutions in positive integers, then the statement  $\Psi_6$  guarantees that each such solution  $(x_1, x_4)$  belongs to the set  $\{(4, 5), (5, 11), (7, 71)\}$ .

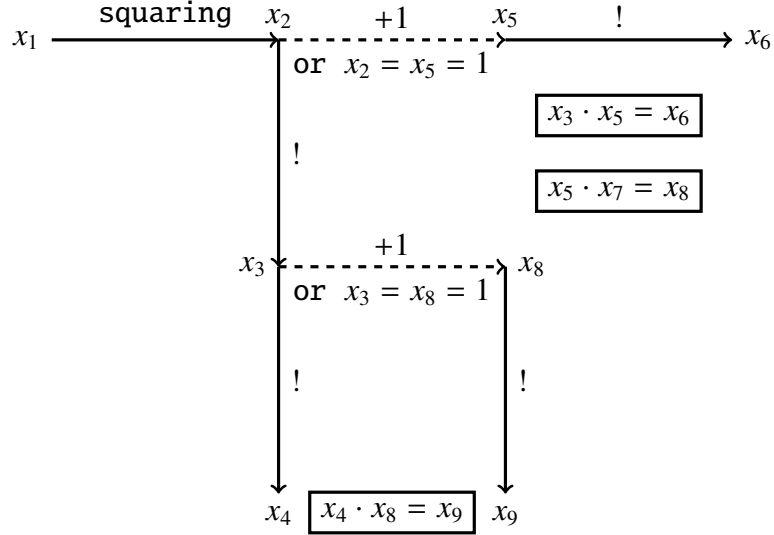
*Proof.* Suppose that the antecedent holds. Let positive integers  $x_1$  and  $x_4$  satisfy  $x_1! + 1 = x_4^2$ . Then,  $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$ . By Lemma 8, the system  $\mathcal{A}$  is solvable in positive integers  $x_2, x_3, x_5, x_6$ . Since  $\mathcal{A} \subseteq \mathcal{B}_6$ , the statement  $\Psi_6$  implies that  $x_6 = (x_1! + 1)! \leq g(6) = g(5)!$ . Hence,  $x_1! + 1 \leq g(5) = g(4)!$ . Consequently,  $x_1 < g(4) = 24$ . If  $x_1 \in \{1, \dots, 23\}$ , then  $x_1! + 1$  is a perfect square only for  $x_1 \in \{4, 5, 7\}$ . □

## 6 Are there infinitely many prime numbers of the form $n^2 + 1$ ?

Edmund Landau's conjecture states that there are infinitely many primes of the form  $n^2 + 1$ , see [15, pp. 37–38]. Let  $\mathcal{B}$  denote the following system of equations:

$$\begin{cases} x_2! = x_3 & x_1 \cdot x_1 = x_2 \\ x_3! = x_4 & x_3 \cdot x_5 = x_6 \\ x_5! = x_6 & x_4 \cdot x_8 = x_9 \\ x_8! = x_9 & x_5 \cdot x_7 = x_8 \end{cases}$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system  $\mathcal{B}$ .



**Fig. 3** Construction of the system  $\mathcal{B}$

**Lemma 9.** For every integer  $x_1 \geq 2$ , the system  $\mathcal{B}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  is prime. In this case, the integers  $x_2, \dots, x_9$  are uniquely determined by the following equalities:

$$\begin{aligned}
 x_2 &= x_1^2 & x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\
 x_3 &= (x_1^2)! & x_8 &= (x_1^2)! + 1 \\
 x_4 &= ((x_1^2)!)! & x_9 &= ((x_1^2)! + 1)! \\
 x_5 &= x_1^2 + 1 & & \\
 x_6 &= (x_1^2 + 1)! & & 
 \end{aligned}$$

*Proof.* By Lemma 2, for every integer  $x_1 \geq 2$ , the system  $\mathcal{B}$  is solvable in positive integers  $x_2, \dots, x_9$  if and only if  $x_1^2 + 1$  divides  $(x_1^2)! + 1$ . Hence, the claim of Lemma 9 follows from Lemma 4.  $\square$

**Lemma 10.** There are only finitely many tuples  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$  which solve the system  $\mathcal{B}$  and satisfy  $x_1 = 1$ .

*Proof.* If a tuple  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$  solves the system  $\mathcal{B}$  and  $x_1 = 1$ , then  $x_1, \dots, x_9 \leq 2$ . Indeed,  $x_1 = 1$  implies that  $x_2 = x_1^2 = 1$ . Hence, for example,  $x_3 = x_2! = 1$ . Therefore,  $x_8 = x_3 + 1 = 2$  or  $x_8 = 1$ . Consequently,  $x_9 = x_8! \leq 2$ .  $\square$

**Theorem 7.** The statement  $\Psi_9$  proves the following implication: if there exists an integer  $x_1 \geq 2$  such that  $x_1^2 + 1$  is prime and greater than  $g(7)$ , then there are infinitely many primes of the form  $n^2 + 1$ .

*Proof.* Suppose that the antecedent holds. By Lemma 9, there exists a unique tuple  $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$  such that the tuple  $(x_1, x_2, \dots, x_9)$  solves the system  $\mathcal{B}$ . Since  $x_1^2 + 1 > g(7)$ , we obtain that  $x_1^2 \geq g(7)$ . Hence,  $(x_1^2)! \geq g(7)! = g(8)$ . Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (g(8) + 1)! > g(8)! = g(9)$$

Since  $\mathcal{B} \subseteq \mathcal{B}_9$ , the statement  $\Psi_9$  and the inequality  $x_9 > g(9)$  imply that the system  $\mathcal{B}$  has infinitely many solutions  $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ . According to Lemmas 9 and 10, there are infinitely many primes of the form  $n^2 + 1$ .  $\square$

**Corollary 2.** Let  $\mathcal{X}_9$  denote the set of primes of the form  $n^2 + 1$ . The statement  $\Psi_9$  implies that we know an algorithm such that it returns a threshold number of  $\mathcal{X}_9$ , and this number equals  $\max(\mathcal{X}_9)$ , if  $\mathcal{X}_9$  is finite. Assuming the statement  $\Psi_9$ , a single query to an oracle for the halting problem decides the infiniteness of  $\mathcal{X}_9$ . Assuming the statement  $\Psi_9$ , the infiniteness of  $\mathcal{X}_9$  is decidable in the limit.

*Proof.* We consider an algorithm which computes  $\max(\mathcal{X}_9 \cap [1, g(7)])$ .  $\square$

## 7 Are there infinitely many prime numbers of the form $n! + 1$ ?

It is conjectured that there are infinitely many primes of the form  $n! + 1$ , see [2, p. 443].

**Theorem 8.** *The statement  $\Psi_9$  proves the following implication: if there exists an integer  $x_1 \geq g(6)$  such that  $x_1! + 1$  is prime, then there are infinitely many primes of the form  $n! + 1$ .*

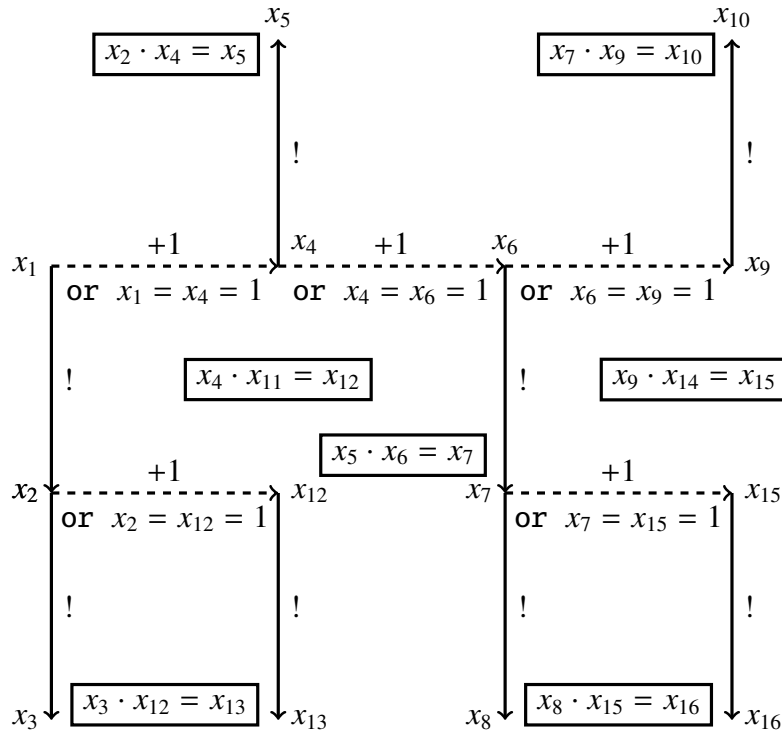
*Proof.* We leave the analogous proof to the reader. □

## 8 The twin prime conjecture

A twin prime is a prime number that differs from another prime number by 2. The twin prime conjecture states that there are infinitely many twin primes, see [15, p. 39]. Let  $C$  denote the following system of equations:

$$\left\{ \begin{array}{ll} x_1! = x_2 & x_2 \cdot x_4 = x_5 \\ x_2! = x_3 & x_5 \cdot x_6 = x_7 \\ x_4! = x_5 & x_7 \cdot x_9 = x_{10} \\ x_6! = x_7 & x_4 \cdot x_{11} = x_{12} \\ x_7! = x_8 & x_3 \cdot x_{12} = x_{13} \\ x_9! = x_{10} & x_9 \cdot x_{14} = x_{15} \\ x_{12}! = x_{13} & x_8 \cdot x_{15} = x_{16} \\ x_{15}! = x_{16} & \end{array} \right.$$

Lemma 2 and the diagram in Figure 4 explain the construction of the system  $C$ .



**Fig. 4** Construction of the system  $C$

**Lemma 11.** *For every  $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$ , the system  $C$  is solvable in positive integers  $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$  if and only if  $x_4$  and  $x_9$  are prime and  $x_4 + 2 = x_9$ . In this case, the integers  $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$  are uniquely determined by the following equalities:*



$$\begin{array}{ll}
x_1 = x_4 - 1 & x_{11} = \frac{(x_4 - 1)! + 1}{x_4} \\
x_2 = (x_4 - 1)! & x_{12} = (x_4 - 1)! + 1 \\
x_3 = ((x_4 - 1)!)! & x_{13} = ((x_4 - 1)! + 1)! \\
x_5 = x_4! & x_{14} = \frac{(x_9 - 1)! + 1}{x_9} \\
x_6 = x_9 - 1 & x_{15} = (x_9 - 1)! + 1 \\
x_7 = (x_9 - 1)! & x_{16} = ((x_9 - 1)! + 1)! \\
x_8 = ((x_9 - 1)!)! & \\
x_{10} = x_9! & 
\end{array}$$

*Proof.* By Lemma 2, for every  $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$ , the system  $C$  is solvable in positive integers  $x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}$  if and only if

$$(x_4 + 2 = x_9) \wedge (x_4 | ((x_4 - 1)! + 1)) \wedge (x_9 | ((x_9 - 1)! + 1))$$

Hence, the claim of Lemma 11 follows from Lemma 4.  $\square$

**Lemma 12.** *There are only finitely many tuples  $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$  which solve the system  $C$  and satisfy  $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$ .*

*Proof.* If a tuple  $(x_1, \dots, x_{16}) \in (\mathbb{N} \setminus \{0\})^{16}$  solves the system  $C$  and  $(x_4 \in \{1, 2\}) \vee (x_9 \in \{1, 2\})$ , then  $x_1, \dots, x_{16} \leq 7!$ . Indeed, for example, if  $x_4 = 2$  then  $x_6 = x_4 + 1 = 3$ . Hence,  $x_7 = x_6! = 6$ . Therefore,  $x_{15} = x_7 + 1 = 7$ . Consequently,  $x_{16} = x_{15}! = 7!$ .  $\square$

**Theorem 9.** *The statement  $\Psi_{16}$  proves the following implication: if there exists a twin prime greater than  $g(14)$ , then there are infinitely many twin primes.*

*Proof.* Suppose that the antecedent holds. Then, there exist prime numbers  $x_4$  and  $x_9$  such that  $x_9 = x_4 + 2 > g(14)$ . Hence,  $x_4, x_9 \in \mathbb{N} \setminus \{0, 1, 2\}$ . By Lemma 11, there exists a unique tuple

$$(x_1, x_2, x_3, x_5, x_6, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}) \in (\mathbb{N} \setminus \{0\})^{14}$$

such that the tuple  $(x_1, \dots, x_{16})$  solves the system  $C$ . Since  $x_9 > g(14)$ , we obtain that  $x_9 - 1 \geq g(14)$ . Therefore,  $(x_9 - 1)! \geq g(14)! = g(15)$ . Hence,  $(x_9 - 1)! + 1 > g(15)$ . Consequently,

$$x_{16} = ((x_9 - 1)! + 1)! > g(15)! = g(16)$$

Since  $C \subseteq B_{16}$ , the statement  $\Psi_{16}$  and the inequality  $x_{16} > g(16)$  imply that the system  $C$  has infinitely many solutions in positive integers  $x_1, \dots, x_{16}$ . According to Lemmas 11 and 12, there are infinitely many twin primes.  $\square$

**Corollary 3.** *(cf. [6]). Let  $\mathcal{X}_{16}$  denote the set of twin primes. The statement  $\Psi_{16}$  implies that we know an algorithm such that it returns a threshold number of  $\mathcal{X}_{16}$ , and this number equals  $\max(\mathcal{X}_{16})$ , if  $\mathcal{X}_{16}$  is finite. Assuming the statement  $\Psi_{16}$ , a single query to an oracle for the halting problem decides the infiniteness of  $\mathcal{X}_{16}$ . Assuming the statement  $\Psi_{16}$ , the infiniteness of  $\mathcal{X}_{16}$  is decidable in the limit.*

*Proof.* We consider an algorithm which computes  $\max(\mathcal{X}_{16} \cap [1, g(14)])$ .  $\square$

## 9 Are there infinitely many composite Fermat numbers?

Integers of the form  $2^{2^n} + 1$  are called Fermat numbers. Primes of the form  $2^{2^n} + 1$  are called Fermat primes, as Fermat conjectured that every integer of the form  $2^{2^n} + 1$  is prime, see [12, p. 1]. Fermat correctly remarked that  $2^{2^0} + 1 = 3$ ,  $2^{2^1} + 1 = 5$ ,  $2^{2^2} + 1 = 17$ ,  $2^{2^3} + 1 = 257$ , and  $2^{2^4} + 1 = 65537$  are all prime, see [12, p. 1].

**Open Problem 4.** ([12, p. 159]). *Are there infinitely many composite numbers of the form  $2^{2^n} + 1$ ?*

Most mathematicians believe that  $2^{2^n} + 1$  is composite for every integer  $n \geq 5$ , see [11, p. 23]. Let

$$H_n = \{x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\} \cup \{2^{2^{x_i}} = x_k : i, k \in \{1, \dots, n\}\}$$

Let  $h(1) = 1$ , and let  $h(n+1) = 2^{2^{h(n)}}$  for every positive integer  $n$ .

**Lemma 13.** *The following subsystem of  $H_n$*

$$\begin{cases} x_1 \cdot x_1 = x_1 \\ \forall i \in \{1, \dots, n-1\} 2^{2^{x_i}} = x_{i+1} \end{cases}$$

*has exactly one solution  $(x_1, \dots, x_n) \in (\mathbb{N} \setminus \{0\})^n$ , namely  $(h(1), \dots, h(n))$ .*

For a positive integer  $n$ , let  $\xi_n$  denote the following statement: *if a system of equations  $S \subseteq H_n$  has only finitely many solutions in positive integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $x_1, \dots, x_n \leq h(n)$ .* The statement  $\xi_n$  says that for subsystems of  $H_n$  the largest known solution is indeed the largest possible.

**Hypothesis 2.** *The statements  $\xi_1, \dots, \xi_{13}$  are true.*

**Lemma 14.** *Every statement  $\xi_n$  is true with an unknown integer bound that depends on  $n$ .*

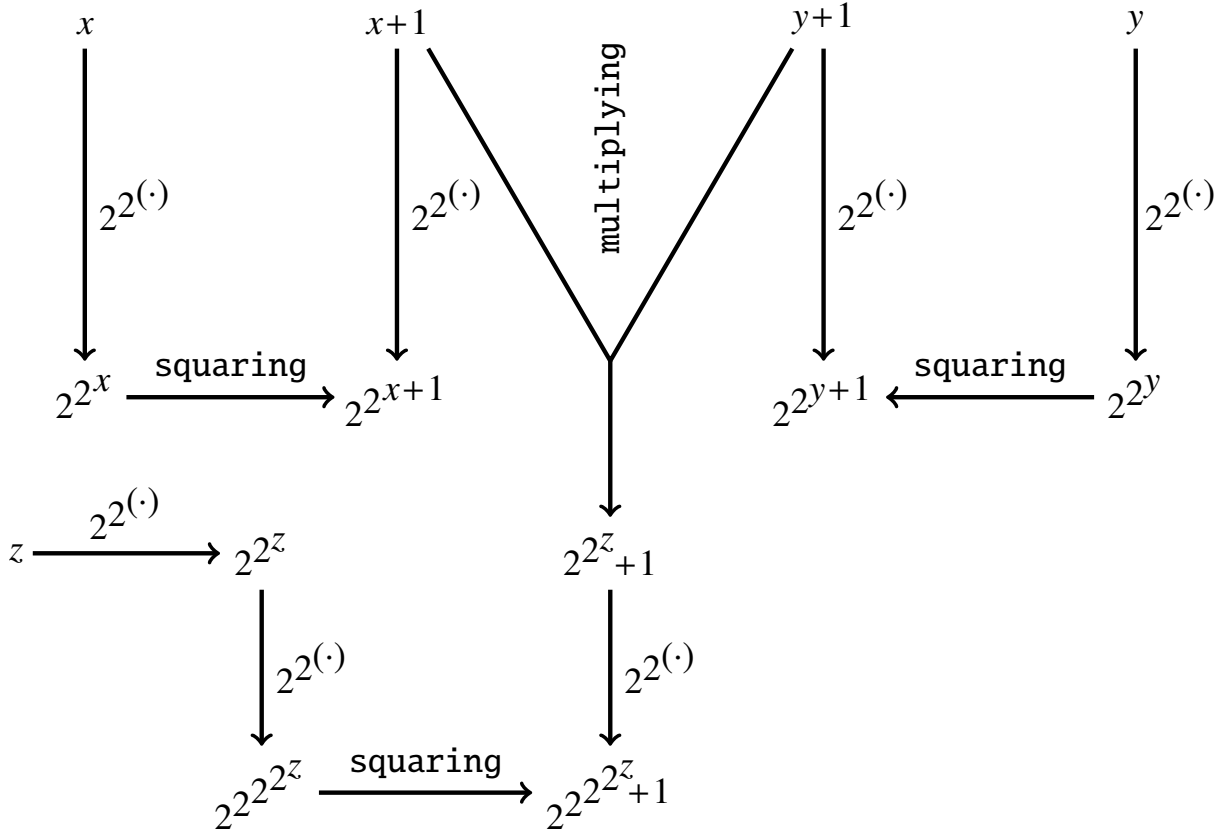
*Proof.* For every positive integer  $n$ , the system  $H_n$  has a finite number of subsystems. □

**Theorem 10.** *The statement  $\xi_{13}$  proves the following implication: if  $z \in \mathbb{N} \setminus \{0\}$  and  $2^{2^z} + 1$  is composite and greater than  $h(12)$ , then  $2^{2^z} + 1$  is composite for infinitely many positive integers  $z$ .*

*Proof.* Let us consider the equation

$$(x+1)(y+1) = 2^{2^z} + 1 \tag{E}$$

in positive integers. By Lemma 3, we can transform the equation (E) into an equivalent system of equations  $\mathcal{G}$  which has 13 variables ( $x, y, z$ , and 10 other variables) and which consists of equations of the forms  $\alpha \cdot \beta = \gamma$  and  $2^{2^\alpha} = \gamma$ , see the diagram in Figure 5.



**Fig. 5** Construction of the system  $\mathcal{G}$

Since  $2^{2^z} + 1 > h(12)$ , we obtain that  $2^{2^{2^z+1}} > h(13)$ . By this, the statement  $\xi_{13}$  implies that the system  $\mathcal{G}$  has infinitely many solutions in positive integers. It means that there are infinitely many composite Fermat numbers.  $\square$

**Corollary 4.** *Let  $\mathcal{W}_{13}$  denote the set of composite Fermat numbers. The statement  $\xi_{13}$  implies that we know an algorithm such that it returns a threshold number of  $\mathcal{W}_{13}$ , and this number equals  $\max(\mathcal{W}_{13})$ , if  $\mathcal{W}_{13}$  is finite. Assuming the statement  $\xi_{13}$ , a single query to an oracle for the halting problem decides the infiniteness of  $\mathcal{W}_{13}$ . Assuming the statement  $\xi_{13}$ , the infiniteness of  $\mathcal{W}_{13}$  is decidable in the limit.*

*Proof.* We consider an algorithm which computes  $\max(\mathcal{W}_{13} \cap [1, h(12)])$ .  $\square$

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