Constructive mathematics with the knowledge predicate $\mathcal{K}$ satisfied by every currently known theorem

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Abstract

$\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set $\mathcal{K}$ is time-dependent, publicly available, and contains theorems both from formal and constructive mathematics. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. Mathematical statements with known constructive proofs exist in $\mathcal{K}$ separately and form the set $\mathcal{K}_c \subseteq \mathcal{K}$. We assume that mathematical sets are atemporal entities. They exist formally in $\textrm{ZFC}$ theory although their properties can be time-dependent (when they depend on $\mathcal{K}$) or informal. Algorithms always terminate. We explain the distinction between algorithms whose existence is provable in $\textrm{ZFC}$ and constructively defined algorithms which are currently known. By using this distinction, we obtain non-trivially true statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that belong to constructive and informal mathematics and refer to the current mathematical knowledge on $\mathcal{X}$.

Key words and phrases: constructive mathematics, constructively defined algorithms, current mathematical knowledge, informal mathematics, known algorithms, time-dependent notion of truth in constructive mathematics.

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1 Introduction

This article in an extended and changed version of [19].

$\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set $\mathcal{K}$ is time-dependent, publicly available, and contains theorems both from formal and constructive mathematics. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. Mathematical statements with known constructive proofs exist in $\mathcal{K}$ separately and form the set $\mathcal{K}_c \subseteq \mathcal{K}$. 
We assume that mathematical sets are atemporal entities. They exist formally in ZFC theory although their properties can be time-dependent (when they depend on \(K\)) or informal. The true statement "\(K\) is non-empty" is outside \(K\) forever because \(K\) is not a formal set.

Paul Cohen proved in 1963 that the equality \(2^\aleph_0 = \aleph_1\) is independent of ZFC, see [3]. Before 1963, the statement "There is a constructively defined integer \(n \geq 1\) such that \(2^\aleph_0 = \aleph_n\)" was outside \(K\). Since 1963, this statement is outside \(K\) forever. The true statement "There exists a set \(X \subseteq \{1, \ldots, 49\}\) such that \(\text{card}(X) = 6\) and \(X\) never occurred as the winning six numbers in the Polish Lotto lottery" refers to the current non-mathematical knowledge and is outside \(K\) forever.

Algorithms always terminate. Semi-algorithms may not terminate. There is the distinction between existing algorithms (i.e. algorithms whose existence is provable in ZFC) and known algorithms (i.e. algorithms whose definition is constructive and currently known), see [2], [14, p. 9]. By using this distinction, we obtain non-trivially true statements on decidable sets \(X \subseteq \mathbb{N}\) that belong to constructive and informal mathematics and refer to the current mathematical knowledge on \(X\). For every such statement \(\Phi\), Observations 1 and 2 hold.

**Observation 1.** The truth of \(\Phi\) concerning the current mathematical knowledge is implied by a true statement of the form:

\[
\begin{align*}
&(\text{the conjunction of } i \text{ conditions of the form } \alpha \in K) \land \\
&(\text{the conjunction of } j \text{ conditions of the form } \beta \notin K) \land \\
&(\text{the conjunction of } k \text{ conditions of the form } \gamma \in K_c) \land \\
&(\text{the conjunction of } l \text{ conditions of the form } \delta \notin K_c),
\end{align*}
\]

where \(i, j, k, l \in \mathbb{N}\) and \(\alpha, \beta, \gamma, \delta\) are mathematical statements.

**Observation 2.** The proof of \(\Phi\) uses mathematical theorems. For example, the proof of Statement 6 uses the following implication: if

\[
X = \{n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains more than } 29.5 + \frac{11}{3n + 1} \cdot \sin(n) \text{ primes of the form } k! + 1\}
\]

then \(\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, 501893]\).

**Observation 3.** Church's thesis is based on the fact that the currently known computable functions are recursive, where the notion of a computable function is informal.

In Observation 4 the predicate \(K\) trivially increases constructive mathematics.

**Observation 4.** The largest known prime number has the form \(2^n - 1\).
2 Time-dependent notion of truth in constructive mathematics

Below is the English summary of [18] available at the internet address of [18].

The basic philosophical idea of intuitionism is that mathematical entities exist only as mental constructions and that the notion of truth of a proposition should be equated with its verification or the existence of proof. However different intuitionists explained the existence of a proof in fundamentally different ways. There seem to be two main alternatives: the actual and potential existence of a proof. The second proposal is also understood in two alternative ways: as knowledge of a method of construction of a proof or as knowledge-independent and tenseless existence of a proof. This paper is a presentation and analysis of these alternatives.

In constructive mathematics ([13]) and the traditional Brouwerian intuitionism ([10, p. 135]), the truth of a mathematical statement means that we know a constructive proof. Therefore, the truth of a mathematical statement depends on time, where the statement is formally stated in the classical mathematics without the predicate $K$.

In this article, mathematical statements on decidable sets $X \subseteq \mathbb{N}$ refer to time because they refer to the current mathematical knowledge on $X$. They cannot be formally stated in the classical mathematics without the predicate $K$ and their logical values may change in time.

In Martin-Löf’s terminology ([9, p. 142]), every currently known theorem is actually true whereas every theorem (known or unknown) is potentially true. Actual truth is knowledge dependent and tensed. Potential truth is knowledge independent and tenseless.

3 Basic definitions and examples

Definition 1 applies to sets $X \subseteq \mathbb{N}$ whose infiniteness is false or unproven.

**Definition 1.** We say that a non-negative integer $k$ is a known element of $X$, if $k \in X$ and we know an algebraic expression that defines $k$ and consists of the following signs: 1 (one), + (addition), − (subtraction), · (multiplication), $^*$ (exponentiation with exponent in $\mathbb{N}$), ! (factorial of a non-negative integer), ( (left parenthesis), ) (right parenthesis).

The set of known elements of $X$ is finite and time-dependent, so cannot be defined in the formal language of classical mathematics. Let $t$ denote the largest twin prime that is smaller than $(((((((9!!))))!!!))))!$. The number $t$ is an unknown element of the set of twin primes.

**Definition 2.** Conditions (1) - (5) concern sets $X \subseteq \mathbb{N}$.

1. A known algorithm with no input returns an integer $n$ satisfying $\text{card}(X) < \omega$.
2. A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in X$.
3. For every known algorithm $A$ with no input, the statement "$A$ returns the logical value of the statement $\text{card}(X) = \omega$" is outside $K$.
4. There are many elements of $X$ and it is conjectured, though so far unproven, that $X$ is infinite.
5. $X$ is naturally defined. The infiniteness of $X$ is false or unproven. $X$ has the simplest definition among known sets $Y \subseteq \mathbb{N}$ with the same set of known elements.
Condition (3) implies that no known proof shows the finiteness/infiniteness of $\mathcal{X}$. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions (1)-(4) and is widely known in number theory or naturally defined, where this term has only informal meaning.

Let $\lfloor \cdot \rfloor$ denote the integer part function.

**Example 1.** The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } \lfloor \frac{(((9!)!)!)!]!\pi} \rfloor \text{ is odd} \\ \emptyset, & \text{otherwise} \end{cases}$$

does not satisfy Condition (3) because we know an algorithm with no input that computes $\lfloor \frac{(((9!)!)!)!]!\pi} \rfloor$. The set of known elements of $\mathcal{X}$ is empty. Hence, Condition (5) fails for $\mathcal{X}$.

**Example 2.** ([2], [44, p. 9]). The function

$$\mathbb{N} \ni n \rightarrow \begin{cases} 1, & \text{if the decimal expansion of } \pi \text{ contains } n \text{ consecutive zeros} \\ 0, & \text{otherwise} \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ or there exists $k \in \mathbb{N}$ such that

$$h = ((0, \ldots, k) \times \{1\}) \cup (\{k + 1, k + 2, k + 3, \ldots\} \times \{0\})$$

No known algorithm computes the function $h$.

**Example 3.** The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if the continuum hypothesis holds} \\ \emptyset, & \text{otherwise} \end{cases}$$

is decidable. This $\mathcal{X}$ satisfies Conditions (1) and (3) and does not satisfy Conditions (2), (4), and (5). These facts will hold forever.

4 A consequence of the physical limits of computation

**Statement 1.** No set $\mathcal{X} \subseteq \mathbb{N}$ will satisfy Conditions (1)-(4) forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in 1 second or less.

**Proof.** The proof goes by contradiction. We fix an integer $n$ that satisfies Condition (1). Since Conditions (1)-(3) will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n + 1 \notin \mathcal{X}, \ n + 2 \notin \mathcal{X}, \ n + 3 \notin \mathcal{X}, \ldots$$ (T)
The sentences from the sequence $(T)$ and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m) \cap X = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $X$ is finite, contrary to the conjecture in Condition (4).

The physical limits of computation (8) disprove the assumption of Statement 1.

5 Statements which refer to Conditions (1)-(5)

Edmund Landau’s conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form $n^2 + 1$ is infinite, see [16], [17], [22].

Statement 2. Condition (1) remains unproven for $X = \mathcal{P}_{n^2+1}$.

Proof. For every set $X \subseteq \mathbb{N}$, there exists an algorithm $\text{Alg}(X)$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \text{card}(X) \in \{0, \omega\} \\ \max(X), & \text{otherwise} \end{cases}$$

This $n$ satisfies the implication in Condition (1), but the algorithm $\text{Alg}(\mathcal{P}_{n^2+1})$ is unknown because its definition is ineffective.

Statement 3. The statement

$$\exists n \in \mathbb{N} \ (\text{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, n + 3])$$

remains unproven in ZFC and classical logic without the law of excluded middle.

Statement 4. The set

$$X = \{k \in \mathbb{N} : \text{card}(\mathcal{P}_{n^2+1} \cap [-1, k]) > 10^{10^{10}} \} \cup \{k \in \mathcal{P}_{n^2+1} : \text{card}(\mathcal{P}_{n^2+1} \cap [-1, k]) \leq 10^{10^{10}} \}$$

satisfies Conditions (2)-(4). Condition (1) fails for $X$, $\text{card}(X) < \omega \Rightarrow \text{card}(X) \leq 10^{10^{10}}$. 

\[5\]
Proof. Since \( \text{card}(\mathcal{P}_{n+1} \cap [2, 10^{28}]) = 219989423892 \) (17) and the inequality \( \text{card}(\mathcal{P}_{n+1}) \geq 10^{10^{10}} \) remains unproven, Conditions (3) and (4) hold.

For a non-negative integer \( n \), let \( \theta(n) \) denote the largest integer divisor of \( 10^{10^{10}} \) smaller than \( n \). Let \( \kappa : \mathbb{N} \to \mathbb{N} \) be defined by setting \( \kappa(n) \) to be the exponent of 2 in the prime factorization of \( n + 1 \).

Statement 5. ([20, p. 250]). The set
\[
\mathcal{X} = \{ n \in \mathbb{N} : (\theta(n) + \kappa(n))^2 + 1 \text{ is prime} \}
\]
satisfies Conditions (1)-(5) except the requirement that \( \mathcal{X} \) is naturally defined. Condition (1) holds with \( n = 10^{10^{10}} \).

Let \( \mathcal{P}_{n+1} \) denote the set of primes of the form \( n! + 1 \).

Conjecture 1. ([1, p. 443], [5]). The set \( \mathcal{P}_{n+1} \) is infinite.

For a non-negative integer \( n \), let \( \rho(n) \) denote \( 29.5 + \frac{11!}{3n+1} \cdot \sin(n) \).

Statement 6. The set
\[
\mathcal{X} = \{ n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains more than } \rho(n) \text{ elements of } \mathcal{P}_{n+1} \}
\]
satisfies Conditions (1)-(5) except the requirement that \( \mathcal{X} \) is naturally defined. 501893 \( \in \mathcal{X} \). Condition (1) holds with \( n = 501893 \). \( \text{card}(\mathcal{X} \cap [0, 501893]) = 159827 \). \( \mathcal{X} \cap [501894, \infty) = \{ n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains at least } 30 \text{ elements of } \mathcal{P}_{n+1} \} \).

Proof. For every integer \( n \geq 11! \), 30 is the smallest integer greater than \( \rho(n) \). By this, if \( n \in \mathcal{X} \cap [11!, \infty) \), then \( n + 1, n + 2, n + 3, \ldots \in \mathcal{X} \). Hence, Condition (1) holds with \( n = 11! - 1 \). Since the inequality \( \text{card}(\mathcal{P}_{n+1}) \geq 30 \) remains unproven, Condition (3) holds. The interval \([-1, 11! - 1]\) contains exactly three primes of the form \( k! + 1 : 1! + 1, 2! + 1, 3! + 1 \). For every integer \( n > 503000 \), the inequality \( \rho(n) > 3 \) holds. Therefore, the execution of the following MuPAD code

```muPAD
m:=0:
for n from 0.0 to 503000.0 do
if n<1!+1 then r:=0 end_if:
if n>=1!+1 and n<2!+1 then r:=1 end_if:
if n>=2!+1 and n<3!+1 then r:=2 end_if:
if n>=3!+1 then r:=3 end_if:
if r>29.5+(11!/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of \( \mathcal{X} \). The output ends with the line \([501893.0, 159827] \), which proves Condition (1) with \( n = 501893 \) and Condition (4) with \( \text{card}(\mathcal{X}) \geq 159827 \). □
T. Nagell proved in [11] (cf. [15, p. 104]) that the equation \( x^2 - 17 = y^3 \) has exactly 16 integer solutions, namely \((\pm 3, -2), (\pm 4, -1), (\pm 5, 2), (\pm 9, 4), (\pm 23, 8), (\pm 282, 43), (\pm 375, 52), (\pm 378661, 5234)\). The set
\[
\bigcup \left\{ (x + 8)^8 \right\} 
\begin{align*}
(x^2 - y^3 - 17) \cdot (y^2 - x^3 - 17) &= 0
\end{align*}
\]
has exactly 23 elements. Among them, there are 14 integers from the interval \([1, 2199894223892]\). Let \( \mathcal{W} \) denote the set
\[
\bigcup \left\{ k \in \mathbb{N} : k \text{ is the } (x + 8)^{8\text{th}} \text{ element of } \mathcal{P}_{n^2 + 1} \right\}
\begin{align*}
(x^2 - y^3 - 17) \cdot (y^2 - x^3 - 17) &= 0
\end{align*}
\]
From [17], it is known that \( \text{card}(\mathcal{P}_{n^2 + 1} \cap [2, 10^{28}]) = 2199894223892 \). Hence, \( \text{card}(\mathcal{W} \cap [2, 10^{28}]) = 14 \) and 14 elements of \( \mathcal{W} \) can be practically computed. The inequality \( \text{card}(\mathcal{P}(n^2 + 1)) \geq (378661 + 8)^8 \) remains unproven. The last two sentences and Statement 6 imply the following corollary.

**Corollary 1.** If we add \( \mathcal{W} \) to \( \mathcal{X} \), then the following statements hold:
- **Condition (1) fails for \( \mathcal{X} \),**
- the above lower bound is currently the best known,
- \( \text{card}(\mathcal{X}) < \omega \implies \text{card}(\mathcal{X}) \leq 159827 + 23, \)
- the above upper bound is currently the best known,
- \( \mathcal{X} \) satisfies Conditions (2)-(5) except the requirement that \( \mathcal{X} \) is naturally defined.

**Corollary 2.** Since the inequality \( \text{card}(\mathcal{P}(n^2 + 1)) > 9999 \) remains unproven and \( 10^{953} < 9999 < 10^{954} \), analagous statements hold when we add to \( \mathcal{X} \) the set
\[
\bigcup \left\{ k \in \mathbb{N} : k - 501894 \text{ is the } \left\lfloor \frac{9999}{10^i} \right\rfloor + 1 \text{ th element of } \mathcal{P}_{n^2 + 1} \right\}
\]
which has at most 955 elements.

For a non-negative integer \( i \), let \( d(i) \) denote the smallest prime divisor of \( 31 + \frac{10^6}{i+1} \).

**Statement 7.** The set
\[
\mathcal{X} = \bigcup \left\{ k^i : k \text{ is the } d(i) \text{ th element of } \mathcal{P}_{n^2 + 1} \right\}
\]
satisfies Conditions (2)-(5) except the requirement that \( \mathcal{X} \) is naturally defined. **Condition (1) fails for \( \mathcal{X} \).** \( \text{card}(\mathcal{X}) \geq 946732. \) If \( \text{card}(\mathcal{P}(n^2 + 1)) \leq 28 \), then \( \text{card}(\mathcal{X}) = 946732. \) If \( 29 \leq \text{card}(\mathcal{P}(n^2 + 1)) \leq 30 \), then \( \text{card}(\mathcal{X}) = 946745. \) If \( \text{card}(\mathcal{P}(n^2 + 1)) \geq 31 \), then \( \text{card}(\mathcal{X}) = \omega. \)
Proof. The inequality \( \text{card}(\mathcal{P}_{n+1}) \geq 23 \) holds, see [4]. The inequality \( \text{card}(\mathcal{P}_{n+1}) \geq 29 \) remains unproven. The execution of the following MuPAD code

\[
[m,n]:=[0,0]:
\text{for } i \text{ from } 0 \text{ to } 10^6-1 \text{ do}
A:=\text{numlib::primedivisors}(%\text{floor}(31+(10^6/(i+1)))):
\text{if } A[1] \leq 23 \text{ then } m:=m+1 \text{ end_if:}
\text{if } A[1] \leq 29 \text{ then } n:=n+1 \text{ end_if:}
\text{end_for:}
\text{print}([m,n]):
\]

displays \([946732,946745]\). The last claim of Statement 7 holds because \( d(\left[31 + \frac{10^6}{i+1}\right]) = 31 \) for every integer \( i \geq 10^6 \). Condition (1) fails for \( \mathcal{X} \) because we cannot rule out the possibility that \( 29 \leq \text{card}(\mathcal{P}_{n+1}) \leq 30 \). \( \square \)

6 Satisfiable conjunctions which consist of Conditions (1)-(5) and their negations

Open Problem 1. Is there a set \( \mathcal{X} \subseteq \mathbb{N} \) which satisfies Conditions (1)-(5)?

Open Problem 1 asks about the existence of a year \( t \geq 2024 \) in which the conjunction

\[
(\text{Condition 1}) \land (\text{Condition 2}) \land (\text{Condition 3}) \land (\text{Condition 4}) \land (\text{Condition 5})
\]

will hold for some \( \mathcal{X} \subseteq \mathbb{N} \). For every year \( t \geq 2024 \) and for every \( i \in \{1,2,3\} \), a positive solution to Open Problem i in the year \( t \) may change in the future. Currently, the answers to Open Problems 1-5 are negative.

The set \( \mathcal{X} = \mathcal{P}_{n+1} \) satisfies the conjunction

\[
\neg(\text{Condition 1}) \land (\text{Condition 2}) \land (\text{Condition 3}) \land (\text{Condition 4}) \land (\text{Condition 5})
\]

The set \( \mathcal{X} = \{0, \ldots, 10^6\} \cup \mathcal{P}_{n+1} \) satisfies the conjunction

\[
\neg(\text{Condition 1}) \land (\text{Condition 2}) \land (\text{Condition 3}) \land (\text{Condition 4}) \land \neg(\text{Condition 5})
\]

Let \( f(1) = 10^6 \), and let \( f(n+1) = f(n)^f(n) \) for every positive integer \( n \). The set

\[
\mathcal{X} = \begin{cases} 
\mathbb{N}, & \text{if } 2^2f(9^9) + 1 \text{ is composite} \\
\{0, \ldots, 10^6\}, & \text{otherwise}
\end{cases}
\]

satisfies the conjunction

\[
(\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land \neg(\text{Condition 5})
\]

Open Problem 2. Is there a set \( \mathcal{X} \subseteq \mathbb{N} \) that satisfies the conjunction

\[
(\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land (\text{Condition 5})?
\]
The numbers $2^{2^k} + 1$ are prime for $k \in \{0, 1, 2, 3, 4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k} + 1$, see [7, p. 158] and [12, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [7, p. 159] and [12, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geq 5$, see [6, p. 23].

The set

$$X = \begin{cases} \mathbb{N}, & \text{if } 2^{2^{(9^9)}} + 1 \text{ is composite} \\ \{0, \ldots, 10^6\} \cup \{n \in \mathbb{N} : n \text{ is the sixth prime number of the form } 2^{2^k} + 1\}, & \text{otherwise} \end{cases}$$

satisfies the conjunction

$$\neg(\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land \neg(\text{Condition 5})$$

**Open Problem 3.** Is there a set $X \subseteq \mathbb{N}$ that satisfies the conjunction

$$\neg(\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land (\text{Condition 5})?$$

It is possible, although very doubtful, that at some future day, the set $X = \mathcal{P}_{n^2+1}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

Table 1 shows satisfiable conjunctions of the form

$$\neg(\text{Condition 1}) \land (\text{Condition 2}) \land \neg(\text{Condition 3}) \land (\text{Condition 4}) \land \neg(\text{Condition 5})$$

where $\#$ denotes the negation $\neg$ or the absence of any symbol.

<table>
<thead>
<tr>
<th>(Cond. 1) \land (Cond. 5)</th>
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<th>(Cond. 2) \land (Cond. 3) \land (Cond. 4)</th>
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<tr>
<td>Open Problem 1</td>
<td>Open Problem 2</td>
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<tr>
<th>(Cond. 1) \land (Cond. 5)</th>
<th>$X = {n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains more than } 29.5 + \frac{111}{3n+1} \cdot \sin(n) \text{ primes of the form } k! + 1}$</th>
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<tr>
<td>$\neg(\text{Condition 1}) \land (\text{Condition 5})$</td>
<td>$X = {\mathbb{N}, \text{ if } 2^{2^{(9^9)}} + 1 \text{ is composite} }$</td>
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Open Problem 3

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<th>(Cond. 1) \land (Cond. 5)</th>
<th>$X = {0, \ldots, 10^6} \cup \mathcal{P}_{n^2+1}$</th>
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Table 1 Five satisfiable conjunctions
7 Statements which refer to Conditions (1a)-(5a) and (6)-(11)

Definition 3. Conditions (1a)-(5a) concern sets $X \subseteq \mathbb{N}$.

(1a) A known algorithm with no input returns a positive integer $n$ satisfying $\text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n].$

(2a) A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in X$.

(3a) For every known algorithm $A$ with no input, the statement $A$ returns the logical value of the statement $\text{card}(X) < \omega$ is outside $\mathcal{K}$.

(4a) There are many elements of $X$ and it is conjectured, though so far unproven, that $X$ is finite.

(5a) $X$ is naturally defined. The finiteness of $X$ is false or unproven. $X$ has the simplest definition among known sets $Y \subseteq \mathbb{N}$ with the same set of known elements.

Statement 8. The set

$$X = \left\{ n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains more than } 6.5 + \frac{10^6}{3n+1} \cdot \sin(n) \text{ squares of the form } k! + 1 \right\}$$

satisfies Conditions (1a)-(5a) except the requirement that $X$ is naturally defined. 95151 $\in X$. Condition (1a) holds with $n = 95151$. $\text{card}(X \cap [0,95151]) = 30311$. $X \cap [95152, \infty) = \{n \in \mathbb{N} : \text{the interval } [-1, n] \text{ contains at least 7 squares of the form } k! + 1\}$.

Proof. For every integer $n > 10^6$, 7 is the smallest integer greater than $6.5 + \frac{10^6}{3n+1} \cdot \sin(n)$. By this, if $n \in X \cap (10^6, \infty)$, then $n + 1, n + 2, n + 3, \ldots \in X$. Hence, Condition (1a) holds with $n = 10^6$. It is conjectured that $k! + 1$ is a square only for $k \in \{4, 5, 7\}$, see [21] p. 297. Hence, the inequality $\text{card}(\{k \in \mathbb{N} \setminus \{0\} : k! + 1 \text{ is a square}\}) > 3$ remains unproven. Since 3 < 7, Condition (3a) holds. The interval $[-1,10^6]$ contains exactly three squares of the form $k! + 1$: $4! + 1, 5! + 1, 7! + 1$. Therefore, the execution of the following MuPAD code

```
m:=0:
for n from 0.0 to 1000000.0 do
if n<25 then r:=0 end_if:
if n>=25 and n<121 then r:=1 end_if:
if n>=121 and n<5041 then r:=2 end_if:
if n>=5041 then r:=3 end_if:
if r>6.5+(1000000/(3*n+1))*sin(n) then
  m:=m+1:
  print([n,m])
end_if:
end_for:
```

displays the all known elements of $X$. The output ends with the line $[95151.0, 30311]$, which proves Condition (1a) with $n = 95151$ and Condition (4a) with $\text{card}(X) \geq 30311$. \hfill \Box
Statement 9. The set
\[ X = \{ k \in \mathbb{N} : \text{card}([-1, k] \cap \mathbb{P}_{n+1}) < 10^{10000} \} \]
satisfies the conjunction
\[ \neg (\text{Condition 1a}) \land (\text{Condition 2a}) \land (\text{Condition 3a}) \land (\text{Condition 4a}) \land (\text{Condition 5a}) \]

Statement 10. There exists a naturally defined set \( C \subseteq \mathbb{N} \) which satisfies the following Conditions (6)-(11).

(6) A known and simple algorithm for every \( k \in \mathbb{N} \) decides whether or not \( k \in C \).
(7) For every known algorithm \( A \) with no input, the statement "\( A \) returns the logical value of the statement \( \text{card}(C) = \omega \)" is outside \( K \).
(8) For every known algorithm \( A \) with no input, the statement "\( A \) returns the logical value of the statement \( \text{card}(\mathbb{N} \setminus C) = \omega \)" is outside \( K \).
(9) It is conjectured, though so far unproven, that \( C \) is infinite.
(10) For every known algorithm \( A \) with no input, the statement "\( A \) returns an integer \( n \) satisfying \( \text{card}(C) < \omega \Rightarrow C \subseteq (-\infty, n] \)" is outside \( K \).
(11) For every known algorithm \( A \) with no input, the statement "\( A \) returns an integer \( m \) satisfying \( \text{card}(\mathbb{N} \setminus C) < \omega \Rightarrow \mathbb{N} \setminus C \subseteq (-\infty, m] \)" is outside \( K \).

Proof. Conditions (6)-(11) hold for \( C = \{ k \in \mathbb{N} : 2^{2^k} + 1 \text{ is composite} \} \). It follows from the following three observations. It is an open problem whether or not there are infinitely many composite numbers of the form \( 2^{2^k} + 1 \), see [7, p. 159] and [12, p. 74]. It is an open problem whether or not there are infinitely many prime numbers of the form \( 2^{2^k} + 1 \), see [7, p. 158] and [12, p. 74]. Most mathematicians believe that \( 2^{2^k} + 1 \) is composite for every integer \( k \geq 5 \), see [6, p. 23].

8 Subsets of \( \mathbb{N} \) and their threshold numbers

Definition 4. We say that an integer \( n \) is a threshold number of a set \( X \subseteq \mathbb{N} \), if \( \text{card}(X) < \omega \Rightarrow X \subseteq (-\infty, n] \).

If a set \( X \subseteq \mathbb{N} \) is empty or infinite, then any integer \( n \) is a threshold number of \( X \). If a set \( X \subseteq \mathbb{N} \) is non-empty and finite, then the all threshold numbers of \( X \) form the set \([\text{max}(X), \infty) \cap \mathbb{N} \).

Definition 5. We say that a non-negative integer \( n \) is a weak threshold number of a set \( X \subseteq \mathbb{N} \), if \( \text{card}(X) < \omega \Rightarrow \text{card}(X) \leq n \).

If a set \( X \subseteq \mathbb{N} \) is infinite, then any non-negative integer \( n \) is a weak threshold number of \( X \). If a set \( X \subseteq \mathbb{N} \) is finite, then the all weak threshold numbers of \( X \) form the set \([\text{card}(X), \infty) \cap \mathbb{N} \).

Let \( X = \{ k \in \mathbb{N} : \text{any proof in ZFC of length } k \text{ or less does not show that } 0 = 1 \} \).

Lemma 1. If \( n \in \mathbb{N} \) and \( \text{card}(X) \leq n \), then \( X \subseteq (-\infty, n-1] \).
Theorem 1. For every explicitly given \( n \in \mathbb{Z} \), if ZFC proves that \( n \) is a threshold number of \( X \), then ZFC is inconsistent. For every explicitly given \( n \in \mathbb{N} \), if ZFC proves that \( n \) is a weak threshold number of \( X \), then ZFC is inconsistent.

Proof. If follows from Lemma 1 and the second Gödel incompleteness theorem. \( \square \)

Open Problem 4. Is there a known (weak) threshold number of \( P_{n^2+1} \)?

Open Problem 5. Is there a known (weak) threshold number of \( P_{n!+1} \)?

References

[1] C. K. Caldwell and Y. Gallot, On the primality of \( n! \pm 1 \) and \( 2 \times 3 \times 5 \times \cdots \times p \pm 1 \), Math. Comp. 71 (2002), no. 237, 441–448, [http://doi.org/10.1090/S0025-5718-01-01315-1](http://doi.org/10.1090/S0025-5718-01-01315-1).


[5] Is \( n!+1 \) often a prime?, [http://math.stackexchange.com/questions/853085/is-n-1-often-a-prime](http://math.stackexchange.com/questions/853085/is-n-1-often-a-prime).


