Constructive mathematics with the knowledge predicate $\mathcal{K}$ satisfied by every currently known theorem

Apoloniusz Tyszka

Abstract

$\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set $\mathcal{K}$ is time-dependent, publicly available, and contains theorems both from formal and constructive mathematics. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. Mathematical statements with known constructive proofs exist in $\mathcal{K}$ separately and form the set $\mathcal{K}_C \subseteq \mathcal{K}$. We assume that mathematical sets are atemporal entities. They exist formally in ZFC theory although their properties can be time-dependent (when they depend on $\mathcal{K}$) or informal. Algorithms always terminate. We explain the distinction between algorithms whose existence is provable in ZFC and constructively defined algorithms which are currently known. By using this distinction, we obtain non-trivially true statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that belong to constructive and informal mathematics and refer to the current mathematical knowledge on $\mathcal{X}$.

Key words and phrases: constructive mathematics, constructively defined algorithms, current mathematical knowledge, informal mathematics, known algorithms, time-dependent notion of truth in constructive mathematics.

2020 Mathematics Subject Classification: 03F65.

1 Introduction

This article in an extended and changed version of [24].

$\mathcal{K}$ denotes both the knowledge predicate satisfied by every currently known theorem and the finite set of all currently known theorems. The set $\mathcal{K}$ is time-dependent, publicly available, and contains theorems both from formal and constructive mathematics. Any theorem of any mathematician from past or present forever belongs to $\mathcal{K}$. Mathematical statements with known constructive proofs exist in $\mathcal{K}$ separately and form the set $\mathcal{K}_C \subseteq \mathcal{K}$. We assume that mathematical sets are atemporal entities. They exist formally in ZFC theory although their properties can be time-dependent (when they depend on $\mathcal{K}$) or informal. The true statement "*$\mathcal{K}$ is non-empty*" is outside $\mathcal{K}$ forever because $\mathcal{K}$ is not a formal set.

Paul Cohen proved in 1963 that the equality $2^{\aleph_0} = \aleph_1$ is independent of ZFC, see [3]. Before 1963, the statement "*There is a constructively defined integer $n \geqslant 1$ such that $2^{\aleph_0} = \aleph_n$*" was outside $\mathcal{K}$. Since 1963, this statement is outside $\mathcal{K}$ forever. The true statement "*There exists a set $\mathcal{X} \subseteq \{1, \ldots, 49\}$ such that $\mathrm{card}(\mathcal{X}) = 6$ and $\mathcal{X}$ never occurred as the winning six numbers in the Polish Lotto lottery*" refers to the current non-mathematical knowledge and is outside $\mathcal{K}$ forever.

Algorithms always terminate. Semi-algorithms may not terminate. There is the distinction between *existing algorithms* (i.e. algorithms whose existence is provable in ZFC) and *known algorithms* (i.e. algorithms whose definition is constructive and currently known), see [2], [17, p. 9]. By using this distinction, we obtain non-trivially true statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ that belong to constructive and informal mathematics and refer to the current mathematical knowledge on $\mathcal{X}$. For every such statement Φ, Observations 1–4 hold.

Observation 1. *The truth of Φ concerning the current mathematical knowledge is implied by a true statement of the form:*

(the conjunction of i conditions of the form $\alpha \in \mathcal{K}$) $\wedge$ (the conjunction of j conditions of the form $\beta \notin \mathcal{K}$) $\wedge$

(the conjunction of k conditions of the form $\gamma \in \mathcal{K}_C$) $\wedge$ (the conjunction of l conditions of the form $\delta \notin \mathcal{K}_C$),

where $i, j, k, l \in \mathbb{N}$ and $\alpha, \beta, \gamma, \delta$ are mathematical statements. In particular, the truth of Statement 6 concerning the current mathematical knowledge guarantees that the set

$$\mathcal{X} = \{n \in \mathbb{N} : \ the\ interval\ [-1, n]\ contains\ more\ than\ 29.5 + \frac{11!}{3n+1} \cdot \sin(n)\ primes\ of\ the\ form\ k! + 1\}$$

satisfies the following conjunction:

$$(\text{the statement "}the\ algorithm\ \mathcal{A}_1\ returns\ the\ logical\ value\ of\ the\ statement\ \mathrm{card}(\mathcal{X}) = \omega\text{" is outside } \mathcal{K}) \wedge \ldots \wedge$$

$$(\text{the statement "}the\ algorithm\ \mathcal{A}_p\ returns\ the\ logical\ value\ of\ the\ statement\ card(\mathcal{X}) = \omega\text{" is outside } \mathcal{K}),$$

where $\mathcal{A}_1, \ldots, \mathcal{A}_p$ are the all known algorithms with no input. Statement 6 will be false when someone proves that there is a constructively defined algorithm $\mathcal{A}$ with no input that returns the logical value of the statement $\mathrm{card}(\mathcal{X}) = \omega$.

Observation 2. *The proof of Φ uses mathematical theorems. For example, the proof of Statement 6 uses the fact that the set $\mathcal{X}$ from Observation 1 satisfies $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, 501893]$.*

Observation 3. *The proof of Φ does not use the assumption that we can list the all elements of $\mathcal{K}$.*

Observation 4. *The proof of Φ does not use the assumption that we can list the all elements of $\mathcal{K}_c$.*

Observation 5 is known from the beginning of computability theory and shows that the predicate $\mathcal{K}$ increases intuitive mathematics.

Observation 5. *Church's thesis is based on the fact that the all currently known intuitively computable functions are computable.*

In Observation 6, the predicate $\mathcal{K}$ trivially increases constructive mathematics.

Observation 6. *The largest known prime number has the form $2^n - 1$.*

2 Time-dependent notion of truth in constructive mathematics

Below is the English summary of [22] available at the internet address of [22].

The basic philosophical idea of intuitionism is that mathematical entities exist only as mental constructions and that the notion of truth of a proposition should be equated with its verification or the existence of proof. However different intuitionists explained the existence of a proof in fundamentally different ways. There seem to be two main alternatives: the actual and potential existence of a proof. The second proposal is also understood in two alternative ways: as knowledge of a method of construction of a proof or as knowledge-independent and tenseless existence of a proof. This paper is a presentation and analysis of these alternatives.

In constructive mathematics ([16]) and the traditional Brouwerian intuitionism ([11, p. 135]), the truth of a mathematical statement means that we know a constructive proof. Therefore, the truth of a mathematical statement depends on time, where the statement is formally stated in the classical mathematics without the predicate $\mathcal{K}$.

In this article, mathematical statements on decidable sets $\mathcal{X} \subseteq \mathbb{N}$ refer to time because they refer to the current mathematical knowledge on $\mathcal{X}$. They cannot be formally stated in the classical mathematics without the predicate $\mathcal{K}$ and their logical values may change in time.

In Martin-Löf's terminology ([10, p. 142]), every currently known theorem is actually true whereas every theorem (known or unknown) is potentially true. Actual truth is knowledge dependent and tensed. Potential truth is knowledge independent and tenseless.

3 Basic definitions and examples

Definition 1 applies to sets $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness is false or unproven.

Definition 1. *We say that a non-negative integer k is a known element of $\mathcal{X}$, if $k \in \mathcal{X}$ and we know an algebraic expression that defines k and consists of the following signs: 1 (one), $+$ (addition), $-$ (subtraction), $\cdot$ (multiplication), $\hat{}$ (exponentiation with exponent in $\mathbb{N}$), ! (factorial of a non-negative integer), ((left parenthesis),) (right parenthesis).*

The set of known elements of $\mathcal{X}$ is finite and time-dependent, so cannot be defined in the formal language of classical mathematics. Let t denote the largest twin prime that is smaller than (((((((((9!)!)!)!)!)!)!)!)!. The number t is an unknown element of the set of twin primes.

Definition 2. *Conditions* `(1)-(5)` *concern sets $\mathcal{X} \subseteq \mathbb{N}$.*
`(1)` *A known algorithm with no input returns an integer n satisfying $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.*
`(2)` *A known algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{X}$.*
`(3)` *For every known algorithm $\mathcal{A}$ with no input, the statement* "$\mathcal{A}$ returns the logical value of the statement $\mathrm{card}(\mathcal{X}) = \omega$" *is outside $\mathcal{K}$.*
`(4)` *There are many elements of $\mathcal{X}$ and it is conjectured, though so far unproven, that $\mathcal{X}$ is infinite.*
`(5)` *$\mathcal{X}$ is naturally defined. The infiniteness of $\mathcal{X}$ is false or unproven. $\mathcal{X}$ has the simplest definition among known sets $\mathcal{Y} \subseteq \mathbb{N}$ with the same set of known elements.*

Condition `(3)` implies that no known proof shows the finiteness/infiniteness of $\mathcal{X}$. No known set $\mathcal{X} \subseteq \mathbb{N}$ satisfies Conditions `(1)-(4)` and is widely known in number theory or naturally defined, where this term has only informal meaning.

Let $[\cdot]$ denote the integer part function.

Example 1. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \text{if } [\frac{(((((((9!)!)!)!)!)!)!)!}{\pi}] \textit{ is odd} \\ \emptyset, & \textit{otherwise} \end{cases}$$

does not satisfy Condition `(3)` *because we know an algorithm with no input that computes* $[\frac{(((((((9!)!)!)!)!)!)!)!}{\pi}]$*. The set of known elements of* $\mathcal{X}$ *is empty. Hence, Condition* `(5)` *fails for* $\mathcal{X}$*.*

Example 2. *([2], [17, p. 9]). The function*

$$\mathbb{N} \ni n \xrightarrow{h} \begin{cases} 1, & \textit{if the decimal expansion of } \pi \textit{ contains } n \textit{ consecutive zeros} \\ 0, & \textit{otherwise} \end{cases}$$

is computable because $h = \mathbb{N} \times \{1\}$ *or there exists* $k \in \mathbb{N}$ *such that*

$$h = (\{0, \ldots, k\} \times \{1\}) \cup (\{k+1,\ k+2,\ k+3,\ \ldots\} \times \{0\})$$

No known algorithm computes the function h*.*

Example 3. *The set*

$$\mathcal{X} = \begin{cases} \mathbb{N}, & \textit{if the continuum hypothesis holds} \\ \emptyset, & \textit{otherwise} \end{cases}$$

is decidable. This $\mathcal{X}$ *satisfies Conditions* `(1)` *and* `(3)` *and does not satisfy Conditions* `(2)`*,* `(4)`*, and* `(5)`*. These facts will hold forever.*

4 A consequence of the physical limits of computation

Statement 1. *No set* $\mathcal{X} \subseteq \mathbb{N}$ *will satisfy Conditions* `(1)-(4)` *forever, if for every algorithm with no input, at some future day, a computer will be able to execute this algorithm in* 1 *second or less.*

Proof. The proof goes by contradiction. We fix an integer n that satisfies Condition `(1)`. Since Conditions `(1)-(3)` will hold forever, the semi-algorithm in Figure 1 never terminates and sequentially prints the following sentences:

$$n+1 \notin \mathcal{X},\ n+2 \notin \mathcal{X},\ n+3 \notin \mathcal{X},\ \ldots \qquad \texttt{(T)}$$

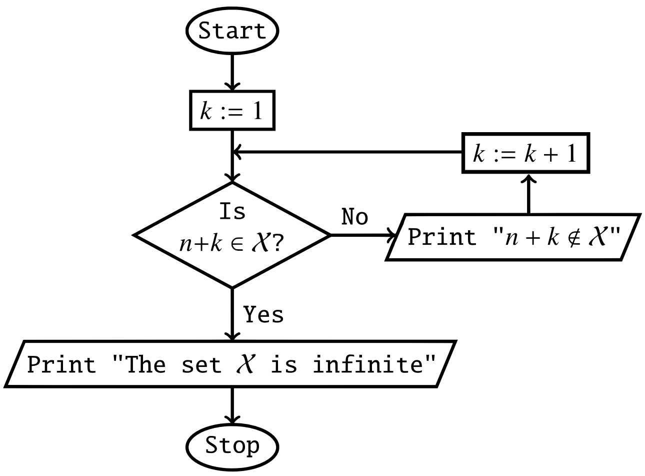

Figure 1 Semi-algorithm that terminates if and only if $\mathcal{X}$ is infinite

The sentences from the sequence `(T)` and our assumption imply that for every integer $m > n$ computed by a known algorithm, at some future day, a computer will be able to confirm in 1 second or less that $(n, m] \cap \mathcal{X} = \emptyset$. Thus, at some future day, numerical evidence will support the conjecture that the set $\mathcal{X}$ is finite, contrary to the conjecture in Condition `(4)`. □

The physical limits of computation ([9]) disprove the assumption of Statement 1.

5 Statements which refer to Conditions `(1)-(5)`

Edmund Landau's conjecture states that the set $\mathcal{P}_{n^2+1}$ of primes of the form n^2+1 is infinite, see [20], [21], [27].

Statement 2. *Condition* `(1)` *remains unproven for* $\mathcal{X} = \mathcal{P}_{n^2+1}$.

Proof. For every set $\mathcal{X} \subseteq \mathbb{N}$, there exists an algorithm $\texttt{Alg}(\mathcal{X})$ with no input that returns

$$n = \begin{cases} 0, & \text{if } \mathrm{card}(\mathcal{X}) \in \{0, \omega\} \\ \max(\mathcal{X}), & \text{otherwise} \end{cases}$$

This n satisfies the implication in Condition `(1)`, but the algorithm $\texttt{Alg}(\mathcal{P}_{n^2+1})$ is unknown because its definition is ineffective. □

Statement 3. *The statement* $\exists k \in \mathbb{N}\ (\mathrm{card}(\mathcal{P}_{n^2+1}) < \omega \Rightarrow \mathcal{P}_{n^2+1} \subseteq [2, k+3])$ *remains unproven in* ZFC *and classical logic without the law of excluded middle.*

Statement 4. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} :\ \mathrm{card}(\mathcal{P}_{n^2+1} \cap [-1,k]) > 10^{10^{10}}\} \cup \{k \in \mathcal{P}_{n^2+1} :\ \mathrm{card}(\mathcal{P}_{n^2+1} \cap [-1,k]) \leqslant 10^{10^{10}}\}$$

satisfies Conditions `(2)-(4)`*. Condition* `(1)` *fails for* $\mathcal{X}$*,* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathrm{card}(\mathcal{X}) \leqslant 10^{10^{10}}$.

Proof. Since $\mathrm{card}(\mathcal{P}_{n^2+1} \cap [2, 10^{28})) = 2199894223892$ ([21]) and the inequality $\mathrm{card}(P_{n^2+1}) \geqslant 10^{10^{10}}$ remains unproven, Conditions `(3)` and `(4)` hold. □

For a non-negative integer n, let $\theta(n)$ denote the largest integer divisor of $10^{10^{10}}$ smaller than n. Let $\kappa : \mathbb{N} \to \mathbb{N}$ be defined by setting $\kappa(n)$ to be the exponent of 2 in the prime factorization of $n+1$.

Statement 5. *([25, p. 250]). The set* $\mathcal{X} = \{n \in \mathbb{N} : (\theta(n) + \kappa(n))^2 + 1 \ is\ prime\}$ *satisfies Conditions* `(1)-(5)` *except the requirement that* $\mathcal{X}$ *is naturally defined. Condition* `(1)` *holds with* $n = 10^{10^{10}}$.

Let $\mathcal{P}_{n!+1}$ denote the set of primes of the form $n!+1$.

Conjecture 1. *([1, p. 443], [6]). The set* $\mathcal{P}_{n!+1}$ *is infinite.*

For a non-negative integer n, let $\rho(n)$ denote $29.5 + \frac{11!}{3n+1} \cdot \sin(n)$.

Statement 6. *The set*

$$\mathcal{X} = \{n \in \mathbb{N} : the\ interval\ [-1,n]\ contains\ more\ than\ \rho(n)\ elements\ of\ \mathcal{P}_{n!+1}\}$$

satisfies Conditions `(1)-(5)` *except the requirement that* $\mathcal{X}$ *is naturally defined.* $501893 \in \mathcal{X}$*. Condition* `(1)` *holds with* $n = 501893$*.* $\mathrm{card}(\mathcal{X} \cap [0, 501893]) = 159827$*.* $\mathcal{X} \cap [501894, \infty) = \{n \in \mathbb{N} : the\ interval\ [-1,n]\ contains\ at\ least\ 30\ elements\ of\ \mathcal{P}_{n!+1}\}$.

Proof. For every integer $n \geqslant 11!$, 30 is the smallest integer greater than $\rho(n)$. By this, if $n \in \mathcal{X} \cap [11!, \infty)$, then $n+1,\ n+2,\ n+3,\ \ldots \in \mathcal{X}$. Hence, Condition `(1)` holds with $n = 11! - 1$. Since the inequality $\mathrm{card}(\mathcal{P}_{n!+1}) \geqslant 30$ remains unproven, Condition `(3)` holds. The interval $[-1, 11!-1]$ contains exactly three primes of the form $k!+1$: $1!+1$, $2!+1$, $3!+1$. For every integer $n > 503000$, the inequality $\rho(n) > 3$ holds. Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 503000.0 do
if n<1!+1 then r:=0 end_if:
if n>=1!+1 and n<2!+1 then r:=1 end_if:
if n>=2!+1 and n<3!+1 then r:=2 end_if:
if n>=3!+1 then r:=3 end_if:
if r>29.5+(11!/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of $\mathcal{X}$. The output ends with the line $[501893.0,\ 159827]$, which proves Condition `(1)` with $n = 501893$ and Condition `(4)` with $\mathrm{card}(\mathcal{X}) \geqslant 159827$. □

T. Nagell proved in [14] (cf. [19, p. 104]) that the equation $x^2 - 17 = y^3$ has exactly 16 integer solutions, namely $(\pm 3, -2)$, $(\pm 4, -1)$, $(\pm 5, 2)$, $(\pm 9, 4)$, $(\pm 23, 8)$, $(\pm 282, 43)$, $(\pm 375, 52)$, $(\pm 378661, 5234)$. The set

$$\bigcup_{\substack{(x,y) \in \mathbb{Z} \times \mathbb{Z} \\ (x^2-y^3-17) \cdot (y^2-x^3-17)=0}} \{(x+8)^8\}$$

has exactly 23 elements. Among them, there are 14 integers from the interval $[1, 2199894223892]$. Let $\mathcal{W}$ denote the set

$$\bigcup_{\substack{(x,y) \in \mathbb{Z} \times \mathbb{Z} \\ (x^2-y^3-17) \cdot (y^2-x^3-17)=0}} \{k \in \mathbb{N}: \ k \ is \ the \ (x+8)^8 - th \ element \ of \ \mathcal{P}_{n^2+1}\}$$

From [21], it is known that $\mathrm{card}(\mathcal{P}_{n^2+1} \cap [2, 10^{28})) = 2199894223892$. Hence, $\mathrm{card}(\mathcal{W} \cap [2, 10^{28})) = 14$ and 14 elements of $\mathcal{W}$ can be practically computed. The inequality $\mathrm{card}(\mathcal{P}_{n^2+1}) \geqslant (378661+8)^8$ remains unproven. The last two sentences and Statement 6 imply the following corollary.

Corollary 1. *If we add $\mathcal{W}$ to $\mathcal{X}$, then the following statements hold:*

Condition `(1)` *fails for $\mathcal{X}$,*

$159827 + 14 \leqslant \mathrm{card}(\mathcal{X})$,

the above lower bound is currently the best known,

$\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathrm{card}(\mathcal{X}) \leqslant 159827 + 23$,

the above upper bound is currently the best known,

$\mathcal{X}$ satisfies Conditions `(2)-(5)` *except the requirement that $\mathcal{X}$ is naturally defined.*

Corollary 2. *Since the inequality $\mathrm{card}(\mathcal{P}_{n^2+1}) > 9^{9^{9}}$ remains unproven and $10^{953} < 9^{9^{9}} < 10^{954}$, analogical statements hold when we add to $\mathcal{X}$ the set*

$$\bigcup_{i \in \mathbb{N}} \left\{k \in \mathbb{N}: \ k - 501894 \ is \ the \ \left(\left[\frac{9^{9^{9}}}{10^i}\right] + 1\right) - th \ element \ of \ \mathcal{P}_{n^2+1}\right\}$$

which has at most 955 elements.

For a non-negative integer i, let $d(i)$ denote the smallest prime divisor of $\left[31 + \frac{10^6}{i+1}\right]$.

Statement 7. *The set*

$$\mathcal{X} = \bigcup_{i \in \mathbb{N}} \left\{k^i: \ k \ is \ the \ d(i) - th \ element \ of \ \mathcal{P}_{n!+1}\right\}$$

satisfies Conditions `(2)-(5)` *except the requirement that $\mathcal{X}$ is naturally defined. Condition* `(1)` *fails for $\mathcal{X}$. $\mathrm{card}(\mathcal{X}) \geqslant 946732$. If $\mathrm{card}(\mathcal{P}_{n!+1}) \leqslant 28$, then $\mathrm{card}(\mathcal{X}) = 946732$. If $29 \leqslant \mathrm{card}(\mathcal{P}_{n!+1}) \leqslant 30$, then $\mathrm{card}(\mathcal{X}) = 946745$. If $\mathrm{card}(\mathcal{P}_{n!+1}) \geqslant 31$, then $\mathrm{card}(\mathcal{X}) = \omega$.*

Proof. The inequality $\mathrm{card}(\mathcal{P}_{n!+1}) \geqslant 23$ holds, see [5]. The inequality $\mathrm{card}(\mathcal{P}_{n!+1}) \geqslant 29$ remains unproven. The execution of the following *MuPAD* code

```
[m,n]:=[0,0]:
for i from 0 to 10^6-1 do
A:=numlib::primedivisors(floor(31+(10^6/(i+1)))):
if A[1]<=23 then m:=m+1 end_if:
if A[1]<=29 then n:=n+1 end_if:
end_for:
print([m,n]):
```

displays $[946732, 946745]$. The last claim of Statement 7 holds because $d\left(\left[31 + \frac{10^6}{i+1}\right]\right) = 31$ for every integer $i \geqslant 10^6$. Condition `(1)` fails for $\mathcal{X}$ because we cannot rule out the possibility that $29 \leqslant \mathrm{card}(\mathcal{P}_{n!+1}) \leqslant 30$. $\square$

6 Satisfiable conjunctions which consist of Conditions `(1)-(5)` and their negations

Open Problem 1. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ which satisfies Conditions* `(1)-(5)`*?*

Open Problem 1 asks about the existence of a year $t \geqslant 2024$ in which the conjunction

$$(Condition\ 1) \land (Condition\ 2) \land (Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)$$

will hold for some $\mathcal{X} \subseteq \mathbb{N}$. For every year $t \geqslant 2024$ and for every $i \in \{1, 2, 3\}$, a positive solution to Open Problem i in the year t may change in the future. Currently, the answers to Open Problems 1–5 are negative.

The set $\mathcal{X} = \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(Condition\ 1) \land (Condition\ 2) \land (Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)$$

The set $\mathcal{X} = \{0, \ldots, 10^6\} \cup \mathcal{P}_{n^2+1}$ satisfies the conjunction

$$\neg(Condition\ 1) \land (Condition\ 2) \land (Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

Let $f(1) = 10^6$, and let $f(n+1) = f(n)^{f(n)}$ for every positive integer n. The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \ if\ 2^{2^{f(9^9)}} + 1\ is\ composite \\ \{0, \ldots, 10^6\}, \ otherwise \end{cases}$$

satisfies the conjunction

$$(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

Open Problem 2. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)?$$

The numbers $2^{2^k} + 1$ are prime for $k \in \{0, 1, 2, 3, 4\}$. It is open whether or not there are infinitely many primes of the form $2^{2^k} + 1$, see [8, p. 158] and [15, p. 74]. It is open whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [8, p. 159] and [15, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geqslant 5$, see [7, p. 23]. The set

$$\mathcal{X} = \begin{cases} \mathbb{N}, \ if\ 2^{2^{f(9^9)}} + 1\ is\ composite \\ \{0, \ldots, 10^6\} \cup \\ \{n \in \mathbb{N} : n\ is\ the\ sixth\ prime\ number\ of\ the\ form\ 2^{2^k} + 1\}, \ otherwise \end{cases}$$

satisfies the conjunction

$$\neg(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land \neg(Condition\ 5)$$

Open Problem 3. *Is there a set $\mathcal{X} \subseteq \mathbb{N}$ that satisfies the conjunction*

$$\neg(Condition\ 1) \land (Condition\ 2) \land \neg(Condition\ 3) \land (Condition\ 4) \land (Condition\ 5)?$$

It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \mathcal{P}_{n^2+1}$ will solve Open Problem 2. The same is true for Open Problem 3. It is possible, although very doubtful, that at some future day, the set $\mathcal{X} = \{k \in \mathbb{N} : 2^{2^k} + 1\ is\ composite\}$ will solve Open Problem 1. The same is true for Open Problems 2 and 3.

Table 1 shows satisfiable conjunctions of the form

$$\#(\text{Condition } 1) \land (\text{Condition } 2) \land \#(\text{Condition } 3) \land (\text{Condition } 4) \land \#(\text{Condition } 5)$$

where $\#$ denotes the negation $\neg$ or the absence of any symbol.

	(Cond. 2) $\wedge$ (Cond. 3) $\wedge$ (Cond. 4)	(Cond. 2) $\wedge$ $\neg$(Cond. 3) $\wedge$ (Cond. 4)
(Cond. 1) $\wedge$ (Cond. 5)	Open Problem 1	Open Problem 2
(Cond. 1) $\wedge$ $\neg$(Cond. 5)	$\mathcal{X} = \{n \in \mathbb{N} :$ *the interval* $[-1, n]$ *contains more than* $29.5 + \frac{11!}{3n+1} \cdot \sin(n)$ *primes of the form* $k! + 1\}$	$\mathcal{X} = \begin{cases} \mathbb{N}, \ \textit{if } 2^{2^{f(9^9)}} + 1 \textit{ is composite} \\ \{0, \ldots, 10^6\}, \ \textit{otherwise} \end{cases}$
$\neg$(Cond. 1) $\wedge$ (Cond. 5)	$\mathcal{X} = \mathcal{P}_{n^2+1}$	Open Problem 3
$\neg$(Cond. 1) $\wedge$ $\neg$(Cond. 5)	$\mathcal{X} = \{0, \ldots, 10^6\} \cup \mathcal{P}_{n^2+1}$	$\mathcal{X} = \begin{cases} \mathbb{N}, \ \textit{if } 2^{2^{f(9^9)}} + 1 \textit{ is composite} \\ \{0, \ldots, 10^6\} \cup \{n \in \mathbb{N} : n \textit{ is} \\ \textit{the sixth prime number of} \\ \textit{the form } 2^{2^k} + 1\}, \ \textit{otherwise} \end{cases}$

Table 1 Five satisfiable conjunctions

7 Statements which refer to Conditions (1a)-(5a) and (6)-(11)

Definition 3. *Conditions* (1a)-(5a) *concern sets* $\mathcal{X} \subseteq \mathbb{N}$.
(1a) *A known algorithm with no input returns a positive integer* n *satisfying* $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.
(2a) *A known algorithm for every* $k \in \mathbb{N}$ *decides whether or not* $k \in \mathcal{X}$.
(3a) *For every known algorithm* $\mathcal{A}$ *with no input, the statement* "$\mathcal{A}$ returns the logical value of the statement $\mathrm{card}(\mathcal{X}) < \omega$" *is outside* $\mathcal{K}$.
(4a) *There are many elements of* $\mathcal{X}$ *and it is conjectured, though so far unproven, that* $\mathcal{X}$ *is finite.*
(5a) $\mathcal{X}$ *is naturally defined. The finiteness of* $\mathcal{X}$ *is false or unproven.* $\mathcal{X}$ *has the simplest definition among known sets* $\mathcal{Y} \subseteq \mathbb{N}$ *with the same set of known elements.*

Statement 8. *The set*

$$\mathcal{X} = \left\{n \in \mathbb{N} : \ \textit{the interval } [-1, n] \textit{ contains more than } 6.5 + \frac{10^6}{3n+1} \cdot \sin(n) \textit{ squares of the form } k! + 1\right\}$$

satisfies Conditions (1a)-(5a) *except the requirement that* $\mathcal{X}$ *is naturally defined.* $95151 \in \mathcal{X}$. *Condition* (1a) *holds with* $n = 95151$. $\mathrm{card}(\mathcal{X} \cap [0, 95151]) = 30311$. $\mathcal{X} \cap [95152, \infty) = \{n \in \mathbb{N} : \textit{the interval } [-1, n] \textit{ contains at least } 7 \textit{ squares of the form } k! + 1\}$.

Proof. For every integer $n > 10^6$, 7 is the smallest integer greater than $6.5 + \frac{10^6}{3n+1} \cdot \sin(n)$. By this, if $n \in \mathcal{X} \cap (10^6, \infty)$, then $n+1, \ n+2, \ n+3, \ \ldots \in \mathcal{X}$. Hence, Condition (1a) holds with $n = 10^6$. It is conjectured that $k! + 1$ is a square only for $k \in \{4, 5, 7\}$, see [26, p. 297]. Hence, the inequality $\mathrm{card}(\{k \in \mathbb{N} \setminus \{0\} : k! + 1 \textit{ is a square}\}) > 3$ remains unproven. Since $3 < 7$, Condition (3a) holds. The interval $[-1, 10^6]$ contains exactly three squares of the form $k! + 1$: $4! + 1, \ 5! + 1, \ 7! + 1$. Therefore, the execution of the following *MuPAD* code

```
m:=0:
for n from 0.0 to 1000000.0 do
if n<25 then r:=0 end_if:
if n>=25 and n<121 then r:=1 end_if:
if n>=121 and n<5041 then r:=2 end_if:
if n>=5041 then r:=3 end_if:
if r>6.5+(1000000/(3*n+1))*sin(n) then
m:=m+1:
print([n,m]):
end_if:
end_for:
```

displays the all known elements of $\mathcal{X}$. The output ends with the line $[95151.0, \ 30311]$, which proves Condition (1a) with $n = 95151$ and Condition (4a) with $\mathrm{card}(\mathcal{X}) \geqslant 30311$. $\square$

Statement 9. *The set*

$$\mathcal{X} = \{k \in \mathbb{N} : \mathrm{card}([-1,k] \cap \mathcal{P}_{n^2+1}) < 10^{10000}\}$$

satisfies the conjunction

$$\neg(Condition~\mathtt{1a}) \wedge (Condition~\mathtt{2a}) \wedge (Condition~\mathtt{3a}) \wedge (Condition~\mathtt{4a}) \wedge (Condition~\mathtt{5a})$$

Statement 10. *There exists a naturally defined set $\mathcal{C} \subseteq \mathbb{N}$ which satisfies the following Conditions* `(6)-(11)`.
`(6)` *A known and simple algorithm for every $k \in \mathbb{N}$ decides whether or not $k \in \mathcal{C}$.*
`(7)` *For every known algorithm $\mathcal{A}$ with no input, the statement* "$\mathcal{A}$ returns the logical value of the statement $\mathrm{card}(\mathcal{C}) = \omega$" *is outside $\mathcal{K}$.*
`(8)` *For every known algorithm $\mathcal{A}$ with no input, the statement* "$\mathcal{A}$ returns the logical value of the statement $\mathrm{card}(\mathbb{N} \setminus \mathcal{C}) = \omega$" *is outside $\mathcal{K}$.*
`(9)` *It is conjectured, though so far unproven, that $\mathcal{C}$ is infinite.*
`(10)` *For every known algorithm $\mathcal{A}$ with no input, the statement* "$\mathcal{A}$ returns an integer n satisfying $\mathrm{card}(\mathcal{C}) < \omega \Rightarrow \mathcal{C} \subseteq (-\infty, n]$" *is outside $\mathcal{K}$.*
`(11)` *For every known algorithm $\mathcal{A}$ with no input, the statement* "$\mathcal{A}$ returns an integer m satisfying $\mathrm{card}(\mathbb{N} \setminus \mathcal{C}) < \omega \Rightarrow \mathbb{N} \setminus \mathcal{C} \subseteq (-\infty, m]$" *is outside $\mathcal{K}$.*

Proof. Conditions `(6)-(11)` hold for $\mathcal{C} = \{k \in \mathbb{N} :~ 2^{2^k} + 1 ~is~composite\}$. It follows from the following three observations. It is an open problem whether or not there are infinitely many composite numbers of the form $2^{2^k} + 1$, see [8, p. 159] and [15, p. 74]. It is an open problem whether or not there are infinitely many prime numbers of the form $2^{2^k} + 1$, see [8, p. 158] and [15, p. 74]. Most mathematicians believe that $2^{2^k} + 1$ is composite for every integer $k \geqslant 5$, see [7, p. 23]. □

8 Subsets of $\mathbb{N}$ and their threshold numbers

Definition 4. *We say that an integer n is a threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathcal{X} \subseteq (-\infty, n]$.*

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer n is a threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of $\mathcal{X}$ form the set $[\max(\mathcal{X}), \infty) \cap \mathbb{N}$.

Definition 5. *We say that a non-negative integer n is a weak threshold number of a set $\mathcal{X} \subseteq \mathbb{N}$, if $\mathrm{card}(\mathcal{X}) < \omega \Rightarrow \mathrm{card}(\mathcal{X}) \leqslant n$.*

If a set $\mathcal{X} \subseteq \mathbb{N}$ is infinite, then any non-negative integer n is a weak threshold number of $\mathcal{X}$. If a set $\mathcal{X} \subseteq \mathbb{N}$ is finite, then the all weak threshold numbers of $\mathcal{X}$ form the set $[\mathrm{card}(\mathcal{X}), \infty) \cap \mathbb{N}$.

Let $\mathcal{X} = \{k \in \mathbb{N} :~ any~proof~in~ZFC~of~length~k~or~less~does~not~show~that~0 = 1\}$.

Lemma 1. *If $n \in \mathbb{N}$ and $\mathrm{card}(\mathcal{X}) \leqslant n$, then $\mathcal{X} \subseteq (-\infty, n-1]$.*

Theorem 1. *For every explicitly given $n \in \mathbb{Z}$, if ZFC proves that n is a threshold number of $\mathcal{X}$, then ZFC is inconsistent. For every explicitly given $n \in \mathbb{N}$, if ZFC proves that n is a weak threshold number of $\mathcal{X}$, then ZFC is inconsistent.*

Proof. If follows from Lemma 1 and the second Gödel incompleteness theorem. □

Open Problem 4. *Is there a known (weak) threshold number of $\mathcal{P}_{n^2+1}$?*

Open Problem 5. *Is there a known (weak) threshold number of $\mathcal{P}_{n!+1}$?*

9 Formal mathematics with the knowledge predicate $\mathcal{K}$ satisfied by every currently known theorem

Theorem 2. *([18, p. 118]). There exists a limit-computable function $\beta_1 : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\delta_1 : \mathbb{N} \to \mathbb{N}$.*

Conjecture 2. *([4, pp. 341–342], [12, p. 42], [13]). Every listable set $\mathcal{X} \subseteq \mathbb{N}^k$ $(k \in \mathbb{N} \setminus \{0\})$ has a single-fold Diophantine representation.*

Statement 11 is non-trivially true, contains the predicate $\mathcal{K}$, and will be false when someone proves Conjecture 2. Since the function β_1 in Theorem 2 is not computable, Statement 11 does not follow from Theorem 2.

Statement 11. *There is a limit-computable function $\beta : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. This is proved in [23]. The term *"dominated"* in the title of [23] means *"eventually dominated"*. Let

$$E_n = \{x_k = 1,\ x_i + x_j = x_k,\ x_i \cdot x_j = x_k :\ i,j,k \in \{0,\ldots,n\}\}$$

For every $n \in \mathbb{N}$, $\beta(n)$ equals the smallest $b \in \mathbb{N}$ such that if system of equations $\mathcal{S} \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0,\ldots,b\}^{n+1}$. We shortly prove that the function β is computable in the limit. For $n, w \in \mathbb{N}$, let $\alpha(n,w)$ denote the smallest $b \in \mathbb{N}$ such that if system of equations $\mathcal{S} \subseteq E_n$ has a unique solution in $\{0,\ldots,w\}^{n+1}$, then this solution belongs to $\{0,\ldots,b\}^{n+1}$. A known algorithm computes the function $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. For every $n \in \mathbb{N}$, there exists $u \in \mathbb{N}$ such that $\beta(n) = \alpha(n,u+1) = \alpha(n,u+2) = \alpha(n,u+3) = \ldots$ $\square$

References

[1] C. K. Caldwell and Y. Gallot, *On the primality of $n! \pm 1$ and $2 \times 3 \times 5 \times \cdots \times p \pm 1$,* Math. Comp. 71 (2002), no. 237, 441–448, *http://doi.org/10.1090/S0025-5718-01-01315-1*.

[2] J. Case and M. Ralston, *Beyond Rogers' non-constructively computable function,* in: *The nature of computation,* Lecture Notes in Comput. Sci., 7921, 45–54, Springer, Heidelberg, 2013, *http://link.springer.com/chapter/10.1007/978-3-642-39053-1_6*.

[3] *Continuum hypothesis, http://en.wikipedia.org/wiki/Continuum_hypothesis*.

[4] M. Davis, Yu. Matiyasevich, J. Robinson, *Hilbert's tenth problem, Diophantine equations: positive aspects of a negative solution;* in: Mathematical developments arising from Hilbert problems (ed. F. E. Browder), Proc. Sympos. Pure Math., vol. 28, Part 2, Amer. Math. Soc., Providence, RI, 1976, 323–378, *http://dx.doi.org/10.1090/pspum/028.2*; reprinted in: The collected works of Julia Robinson (ed. S. Feferman), Amer. Math. Soc., Providence, RI, 1996, 269–324.

[5] *Factorial prime, http://en.wikipedia.org/wiki/Factorial_prime*.

[6] *Is $n! + 1$ often a prime?, http://math.stackexchange.com/questions/853085/is-n-1-often-a-prime*.

[7] J.-M. De Koninck and F. Luca, *Analytic number theory: Exploring the anatomy of integers,* American Mathematical Society, Providence, RI, 2012.

[8] M. Křížek, F. Luca, L. Somer, *17 lectures on Fermat numbers: from number theory to geometry,* Springer, New York, 2001, *http://doi.org/10.1007/978-0-387-21850-2*.

[9] S. Lloyd, *Ultimate physical limits to computation,* Nature 406 (2000), 1047–1054, *http://doi.org/10.1038/35023282*.

[10] P. Martin-Löf, *A path from logic to metaphysics,* in: G. Corsi and G. Sambin (eds.), *Atti del Congresso "Nuovi problemi della logica e della filosofia della scienza",* Viareggio, 8–13 gennaio 1990, vol. II, CLUEB, Bologna, 1991, 141–149, *http://constable.blog/wp-content/uploads/A-path-from-logic-to-metaphysics-1991.pdf*.

[11] E. Martino, *Intuitionistic proof versus classical truth: The role of Brouwer's creative subject in intuitionistic mathematics,* Springer, 2018, *http://link.springer.com/book/10.1007/978-3-319-74357-8*.

[12] Yu. Matiyasevich. *Hilbert's tenth problem: what was done and what is to be done,* in: Proceedings of the Workshop on Hilbert's tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999), Contemp. Math. 270, Amer. Math. Soc., Providence, RI, 2000, 1–47, *http://dx.doi.org/10.1090/conm/270*.

[13] Yu. Matiyasevich, *Towards finite-fold Diophantine representations,* J. Math. Sci. (N. Y.) vol. 171, no. 6, 2010, 745–752, *http://dx.doi.org/10.1007%2Fs10958-010-0179-4*.

[14] T. Nagell, *Einige Gleichungen von der Form $ay^2 + by + c = dx^3$,* Norske Vid. Akad. Skrifter, Oslo I (1930), no. 7.

[15] P. Ribenboim, *The little book of bigger primes,* 2nd ed., Springer, New York, 2004.

[16] F. Richman, *Interview with a constructive mathematician,* Modern Logic 6 (1996), no. 3, 247–271, *http://projecteuclid.org/journals/modern-logic/volume-6/issue-3/Interview-with-a-constructive-mathematician/rml/1204835729.full*.

[17] H. Rogers, Jr., *Theory of recursive functions and effective computability*, 2nd ed., MIT Press, Cambridge, MA, 1987.

[18] J. S. Royer and J. Case, *Subrecursive Programming Systems: Complexity and Succinctness,* Birkhäuser, Boston, 1994.

[19] W. Sierpiński, *Elementary theory of numbers,* 2nd ed. (ed. A. Schinzel), PWN – Polish Scientific Publishers and North-Holland, Warsaw-Amsterdam, 1987.

[20] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences, A002496, Primes of the form* $n^2 + 1$, *http://oeis.org/A002496*.

[21] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences, A083844, Number of primes of the form* $x^2 + 1 < 10^n$, *http://oeis.org/A083844*.

[22] Z. Tworak, *O pojęciu prawdy w intuicjonizmie matematycznym (in Polish) (On the notion of truth in mathematical intuitionism),* Filozofia Nauki (Philosophy of Science) 18 (2010), no. 4, 49–76, *http://www.ceeol.com/search/article-detail?id=269240*.

[23] A. Tyszka, *All functions* $g : \mathbb{N} \to \mathbb{N}$ *which have a single-fold Diophantine representation are dominated by a limit-computable function* $f : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ *which is implemented in MuPAD and whose computability is an open problem*, in: Computation, cryptography, and network security (eds. N. J. Daras, M. Th. Rassias), Springer, Cham, 2015, 577–590, *http://dx.doi.org/10.1007/978-3-319-18275-9_24*.

[24] A. Tyszka, *In constructive and informal mathematics, in contradistinction to any empirical science, the predicate of the current knowledge in the subject is necessary,* Asian Research Journal of Mathematics 19 (2023), no. 12, 69–79, *http://doi.org/10.9734/arjom/2023/v19i12773*.

[25] A. Tyszka, *Statements and open problems on decidable sets* $\mathcal{X} \subseteq \mathbb{N}$ *that contain informal notions and refer to the current knowledge on* $\mathcal{X}$, Creat. Math. Inform. 32 (2023), no. 2, 247–253, *http://semnul.com/creative-mathematics/wp-content/uploads/2023/07/creative_2023_32_2_247_253.pdf*.

[26] E. W. Weisstein, *CRC Concise Encyclopedia of Mathematics,* 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2002.

[27] Wolfram MathWorld, *Landau's Problems, http://mathworld.wolfram.com/LandausProblems.html*.

Apoloniusz Tyszka
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail: *rttyszka@cyf-kr.edu.pl*