EARLY YEARS MATHEMATICS EDUCATION: THE MISSING LINK

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Abstract. In this article, modern standards of early years mathematics education are criticized and a proposal for change is presented. Today's early years mathematics education standards rest on a view of mathematics that became obsolete already at the end of the 19th century while the spirit of children's mathematics is precisely the spirit of modern mathematics. The proposal for change is not a return to the “new mathematics” movement, but something different.

Keywords: early years mathematics education; early years mathematics education standards; National Council of Teachers of Mathematics standards; modern mathematics; child's play

Introduction

With this article, I want to contribute to the changes in the mathematics education of the youngest, which I believe are necessary and for which I hope the time has come. The article gives a serious critique of the early years mathematics (EYM hereafter) education standards and proposes a solution.[1] My goal is to show that today's EYM standards rest on a view of mathematics that became obsolete almost one hundred and fifty years ago, and that the spirit of children's mathematics is precisely the spirit of modern mathematics. The proposal for change is not a return to the “new mathematics” movement, but something different.

My point of view is partly determined by the fact that I entered the world of EYM education as a mathematician who has begun to help my three-and-a-half-year-old granddaughter in her mathematical development. While exploring the world of EYM education, I have experienced EYM standards and usual teaching practice as violence against children. At the same time, I have been pleasantly surprised by the many non-profit organizations and individuals who loosen these standards and teach children in a more appropriate way. However, my conclusion is that the state of EYM education establishment is such that gradual changes will not help, but a revolution is needed that will shake everything up.[2] Maybe my views are not correct. Maybe I will make some people angry or offended. I am ready for all criticism. The only thing I don't want to happen is indifference. I believe that the creators of the standards have an obligation to respond to the critiques presented in this article. I would especially like to get the reaction of people who, unlike me, are full of

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[1] By early years, I primarily mean the years that precede more formal school education, although the criticism and the proposal for change can be extended to later years.

[2] Truly, a revolution is necessary in the entire educational system, and in the entire society, above all in the system of human values which should be developed from an early age. We should all ask ourselves what we are doing with children. The future of our civilization depends on this, as Maria Montessori began to warn humanity almost a hundred years ago. See, for example, (Montessori,1949).

experience in working directly with children. Without these reactions, this article will achieve nothing for children.

Some parts of this article are taken from the author's articles Čulina (2022) and Čulina (2020) with the aim of directly identifying, together with new parts, the missing link in EYM education.

The creators of early childhood education realized long ago, and it has been confirmed by contemporary research, that the first six years of life are the most important period in a person's development, including mathematical development.[3] Recent research has shown that children at this age are much more mathematically capable than previously thought.[4] These results are not surprising: early childhood is a period of exceptional creativity and imagination[5] (if appropriate conditions are ensured for the child), and creativity and imagination are the key elements of mathematical activities. On the other hand, most adults are not only poorly educated in mathematics, but also have a negative attitude towards their mathematical abilities and an aversion to mathematics. The conclusion is inevitable: such an attitude arises in the first phase of mathematics education. What is the problem?

Mathematics education shares the problems of education as a whole, and these problems are substantial. Centres of power (states, big capitals, religious institutions, …) determine the educational systems. Given their interests, they determine how to teach and what to teach.[6] But what is the problem specific to mathematics, which leads to a particularly negative attitude towards mathematics in adults, and which is created in the early stages of mathematics education?

Educators long ago, starting with Fröbel and Montessori (to mention only those who are closest to my heart) established what education in the early years should be like, including EYM education.[7] In short, the child's learning must be a part of the child's world – a part of her daily activities, a part of her play, incorporated into children's stories that she enjoys listening to. Child's learning must have motivation, meaning and value in the child's world, not in the world of adults (from the outside). In developing her abilities, the child must have the freedom and not the pressure to achieve the pre-established learning outcomes. It is up to us that with a lot of love help and guide the child in developing her potential, respecting her world and her individuality – which activities and at what stage of her growth attract her – and providing an appropriate environment for such development. The biggest problem with the education of the youngest is that the existing educational systems are generally contrary to this methodology – in theory, with their uniformity and evaluation system, and in practice

[3] It was this knowledge that motivated Friedrich Fröbel to design and introduce kindergartens into modern society in the first half of the 19th century.

[4] See, for example, English and Mulligan (2013).

[5] See, for example, the chapter on creativity in Bilbao (2020).

[6] In the book Gray (2013), Peter Gray describes the influence of centres of power on education, and in the book Ernest (1991), Paul Ernest describes the influence on mathematics education.

[7] Although its roots can already be found in ancient Greece, this methodology was developed in the 19th century by the founders of modern education: Pestalozzi, Fröbel, Montessori and many others (see, for example Lascarides, Hinitz, 2000). Studying their works, I was personally fascinated by the wealth of educational knowledge they left behind and frustrated by the ignorance of this knowledge in today's wider educational practice.

with challenging working conditions for educators and teachers. The reason is clear: the centres of power do not want free and high-quality education for children. With the help of mediocrities, bureaucrats, and careerists, they marginalize and weaken the individuals and groups who seek different education.[8]

It follows from the above that the teaching of mathematics, as it should be, as well as the one that actually exists, is fundamentally no different from the teaching of other subjects.[9] My conclusion is that the specific problem of the early phase of mathematics education is not how mathematics is taught but what mathematics is taught. And to know what mathematics to teach the youngest, we have to understand well enough what mathematics is. I aim to show in this article that, unlike education in other domains where it is known what the subject of education is, EYM education does not have a clear answer to the question of what mathematics is or has a wrong answer. And that is precisely the missing link in EYM education. If we do not know clearly and correctly enough what mathematics is, how can we properly teach children mathematics? This article gives an answer to the question "What is mathematics?".[10] This answer shows what is wrong with EYM education today and how to fix it.

Even a cursory look at the mathematics education standards for the youngest shows the dominance of numbers and the geometry of figures and bodies. Other mathematical content, if present at all, is subordinated to numbers and geometric figures. Such standards can only come from viewing mathematics as the science of numbers and Euclidean geometry. In mathematics, such a view became obsolete already at the end of the 19th century. Truly, until the middle of the nineteenth century mathematics was described as the science of numbers and (Euclidean) space. The appearance of non-Euclidean geometries which are incompatible with Euclidean geometries but are equally logical in thinking and equally good candidates for the "true" geometry of the world has definitely separated mathematics from the truths about nature. This separation has freed the human mathematical powers, and it has caused the blossom of modern mathematics. With the end of World War II, it became clear that there was a big discrepancy between modern mathematics which proved to be very important for modern society and mathematics taught in school. The "new mathematics" movement of the 1950s and 1960s, which was the most intense in the USA, tried to introduce modern mathematics to school. Unfortunately, this movement failed, not only because of social circumstances but also because the mathematicians who led the movement did it in a one-sided way. They did not respect the developmental psychology of children, and it had the most negative impact on the mathematics education of the youngest. Not only did they fail, but they created a negative attitude towards the introduction of modern mathematics into

[8] Let me mention only those individuals whose works I have read: Friedrich Fröbel, Maria Montessori, Ivan Ilich, Neil Postman, Paul Lockhart, and Peter Gray.

[9] Teaching EYM has one additional complicating factor, which will be discussed below (page 9).

[10] "What is mathematics?" is an open question to which an answer that would be widely accepted has not yet been given. In this article, in my opinion, it has been answered clearly and correctly enough for a correct understanding of EYM Education.

education.[11] Mathematics education of children was once again taken over by educators who, I must say, generally speaking, do not understand modern mathematics, just as the creators of the “new mathematics” did not understand the educators. Mathematics education of the youngest has returned to numbers and geometry, although some traces of modern mathematics can be found in today’s standards. For example, although the mathematics education standards are dominated by numbers and geometry, there are many structural elements in the elaboration of these topics that were highlighted by the creators of the “new mathematics”. Also, additional contents are included: classifying (sets), sorting (equivalence relations), ordering (ordering relations), matching (functions), patterns, chance, change, …. However, they are mostly subordinated to the numbers and geometry of figures. Even if we single out these contents in relation to numbers and geometry, it is all insufficient. The spirit of modern mathematics has been driven out of modern standards for children. This unfortunate history of mathematics education shows that it is necessary for mathematicians and educators to work closely together in designing mathematics education.

Mathematics does not stop at numbers and Euclidean geometry. Mathematics is something much broader, and the numbers and Euclidean geometry are amongst its jewels. The goal of early mathematics education is not to hone mathematical jewels but to acquire the very heart of mathematics that creates such jewels. Today’s real and successful mathematics is precisely modern mathematics, which of course includes numbers and geometry. My goal in this paper is to show that not only modern mathematics is mathematics to be learned, but that modern mathematics is precisely the mathematics of the children’s world. The approach developed in this article is not a return to the “new mathematics” movement, but something different.

The view of the mathematics of children’s world that I will present here is very broad. That width is not dangerous. Looking too narrowly at mathematics carries great danger. Today’s EYM standards carry just such a danger. I will argue that the standards give a narrow and distorted picture of mathematics. Adhering to the standards, we drastically limit the mathematics of the child’s world, hamper the correct mathematical development of a child, and we can turn her away from mathematics. I believe that what many teachers and educators do or want to do is in accordance with the conception presented here. There is a whole host of educators and non-governmental and non-profit associations that “relax” the standards and give a broader view of mathematics. I believe that the view of mathematics presented here can be the missing link: a coherent conceptual basis for their activities. Also, according to the view of mathematics presented in this article, children fortunately develop their mathematical activities through other contents that are widely believed to have nothing to do with mathematics: art, literature, physical culture, socializing through play, organizing their activities and taking care of order and tidiness through, for example, socializing in kindergarten or participation in household chores. The problem is that the child here can show more pronounced mathematical abilities, and the standards will “declare” her as not

[11] Within the mathematical community itself, there was strong criticism of such an approach. A detailed analysis of the “new mathematics” movements in USA can be found in Phillips (2015). An overview of the “new mathematics” movements across the world can be found in de Bock (2023).

good at mathematics because she does not do very well with numbers, probably just because she does not find them interesting at the time.

What is mathematics (for the youngest)?

The view of mathematics I will briefly present here[12] has the same roots as modern mathematics –- in the emergence of non-Euclidean geometries that led to the separation of mathematics from truths about reality. According to this view, mathematics is not a science of the truths of the world, but it is a means of discovering those truths: mathematics is the human activity of building various conceptions whose purpose is to be a tool of our rational cognition and rational activities in general. This purpose significantly influences its design and determines its value. Richard Dedekind summed it up nicely on the example of numbers (Dedekind, 1888, p. vii): "…numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things". Mathematics is a process and result of shaping our intuitions and ideas about reality of our internal world of activities into thoughtful models which enable us to understand and control better the whole reality. For example, we shape our internal activities with quantities into a system of measuring quantities by numbers. Thoughtful modelling of other intuitions about our internal human world of activities, for example, intuitions about symmetry, flatness, closeness, comparison, etc., leads to other mathematical models. By "internal world of activities" I mean the world that consists of activities over which we have strong control, and which we organize and design by our human measure. For example, these include movements in safe space, grouping, arranging, and connecting small objects, spatial constructions and deconstructions with small objects, talking, writing and drawing on paper, shaping and transforming manipulative material, making choices and combining and repeating actions, dynamics of actions and changes in the environment subordinated to us, painting, singing, etc. This does not include activities with objects over which we have no strong control, or our activities are significantly limited by the environment, such as e.g., climbing steep rocks, building a house, etc. It is from these concrete activities that the idea of an idealized mathematical world (model, theory) emerges, the world that expands and supplements the internal world of activities.

Examples of internal activities that I listed above are activities that we can easily recognize in free children's play. These activities form the basis of the world of internal activities of adults. These activities develop as we grow up, but this is primarily the development of their design and conceptualization. Their outstanding presence in the early years indicates their biological basis.[13] They are a unique characteristic of the human species, an essential part of our evolutionarily developed abilities, by which, unlike other species that

[12] A more detailed exposition can be found in Čulina (2022) and a complete exposition in Čulina (2020).

[13] The book Lakoff and Núñez (2000) discusses (page 28): "ordinary cognitive mechanisms as those used for the following ordinary ideas: basic spatial relations, groupings, small quantities, motion, distributions of things in space, changes, bodily orientations, basic manipulations of objects (e.g., rotating and stretching), iterated actions, and so on.".

adapt to the environment, we adapt the environment to ourselves.[14] Of course, cultural evolution and social context play a key role in designing and conceptualizing these activities.[15]

There are many examples of highlighting the importance of internal activities. Dedekind talks about "the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability without which no thinking is possible" (Dedekind, 1888, p. viii). David Hilbert sees the source of his finitist mathematics in "extralogical concrete objects that are intuitively present as immediate experience prior to all thought", and these objects are "the concrete signs themselves, whose shape . . . is immediately clear and recognizable" (Hilbert, 1926, p. 171). Solomon Feferman writes that the source of mathematical conceptions "lies in everyday experience in manifold ways, in the processes of counting, ordering, matching, combining, separating, and locating in space and time" (Feferman, 2014, p.75). Reuben Hersh writes "To have the idea of counting, one needs the experience of handling coins or blocks or pebbles. To have the idea of an angle, one needs the experience of drawing straight lines that cross, on paper or in a sandbox" (Hersh, 1979, p. 46). And so on.

Internal activities are concrete activities with concrete objects: they take place in space and time, and in a given environment – on the table with a pencil and paper, in a ballroom, or on a sandy beach. They are experiential activities, but it is not an experience of the external world, but an experience of our actions in the world available to us and subordinated to us. We experiment not with the objects of the external world but with the possibilities of our actions in the world adapted to us. In this world, Piaget distinguishes two types of knowledge (Piaget, 1970, p. 15-16). An example of the first type of knowledge is when we pick up two objects and determine that one is heavier. This knowledge arose from our action on objects and its source is in these objects: it is knowledge about these objects – knowledge about the world. Piaget calls such knowledge "physical knowledge". But when we order objects, for example by weight, this order was created by our actions and its source is in our activities and not in the objects. The knowledge that has its source in our activities, for example that there are always the same number of objects no matter how they are ordered, is called by Piaget "logical mathematical knowledge". Mathematical knowledge has its source in the world of internal activities precisely in this way: it springs from these activities themselves and not from the objects of these activities. Feferman writes: "Theoretical mathematics has its source in the recognition that these processes are independent of the materials or objects to which they are applied and that they are potentially endlessly repeatable." (Feferman, 2014, p. 75).

However, we cannot completely separate activities from the objects of the activities: generally speaking, they depend on the objects. For internal activities, it is important that this dependence is weak. Our control over these activities is the dominant force, rather than the influence over them by the objects. It is not possible to draw a clear line where the world of

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[14] "Man is a singular creature. He has a set of gifts which make him unique among the animals: so that, unlike them, he is not a figure in the landscape – he is a shaper of the landscape." (Bronowski, 1974, p. 19).

[15] See, for example, Kitcher (1983) for the influence of cultural evolution and Hersh (1997), Ernest (1997) for the influence of human society.

internal activities ends, and the activities become external. Take for exmple constructing and deconstructing objects. When a child does this with Legos, it is surely an internal activity. Although the structure of Lego blocks affects the possibilities of construction, these are activities over which we have a strong control and the possibility of designing them according to our own measure. Constructing and deconstructing stone walls without mortar is certainly not an internal activity: it requires experience working with weights, centres of gravity and forces in contact. But both activities are the source of the same mathematical idea, the idea of analysis and synthesis of what we are researching: let us examine the phenomenon by breaking it into parts, studying those parts and synthesizing the knowledge thus acquired into knowledge about that phenomenon. This idea is a mathematical idea because it has its source in our approach to the world, not in the world itself. Although it is present in both mentioned activities, in the external activity it is burdened with the physical content, while in the internal activity it takes a clear and separate form. Because of the freedom we have in the world subordinated to us, I believe that we can always internalize external activities, represent them with internal activities in which the mathematical idea will come to full expression. To conclude, although it is about concrete activities in a real environment, due to the subordination of that environment to our activities and due to the strong control over these activities and the strong possibilities of their shaping, we can talk about these activities as our internal world of activities, and even about a certain kind of a priority of that world.

In case of children, as in adults, mathematics as a tool of our rational cognition and rational activities in general emerges from internal activities, through their design and conceptualization. In his book (Mac Lane, 1986), Saunders Mac Lane describes this process on a multitude of examples. The table 1.1 on page 35 of the book shows a whole list of examples of activities from which certain ideas are born, and from these ideas mathematical concepts and models arise.

Table 1.1

Activity	Idea	Formulation
Collecting	Collection	Set (of elements)
Counting	Next	Sucessor; order Ordinal number
Comparing	Enumeration	Bijection Cardinal number
Computing	Combination (of nos)	Rules for addition Rules for multiplication Abelian group
Rearranging	Permutation	Bijection Permutation group
Timing	Before and after	Linear order
Observing	Symmetry	Transformation group
Building, shaping	Figure; symmetry	Collection of points
Measuring	Distance; extent	Metric space
Moving	Change	Rigid motion Transformation group Rate of change
Estimating	Approximation	Continuity Limit
	Nearby	Topological space
Selecting	Part	Subset Boolean algebra
Arguing	Proof	Logical connectives
Choosing	Chance	Probability (favorable/total)
Successive actions	Followed by	Composition Transformation group

For example, movements contribute to the idea of change, whose formulation contributes to the concepts of rigid motion, transformation group and rate of change; estimating contributes to the ideas of approximation and closeness, and their formulation contributes to the concepts of continuity, limit, and topological space.

Thus*, the basis of children's mathematics is the children's world of internal activities. Children's world of internal activities manifests itself most expressively and develops best in children's play: it is the key element of the play*.[16] Developing internal activities through play, the child develops intuition (immediate awareness), creativity and imagination, ideas are born to her, and she designs and conceptualizes her activities, freely or with our unobtrusive support. Often the purpose of children's play is to understand the outside world (let us play with dolls, let's play cooking, etc.). When such a purpose is added to the play, then in the world of children, as well as in the world of adults, we have a mathematical model of a phenomenon. Whenever a child organizes her internal activities into a means of understanding a phenomenon, she has made a mathematical model of that phenomenon, whether it is a drawing of an elephant or a doctor-patient game. Children's stories themselves can be understood as mathematical models of certain phenomena. The Witch, for example, represents evil, Hansel and Gretel goodness, which, aided by wisdom, defeats evil and forgives the deceived (their father) but not the incorrigibly evil (the Witch and their stepmother). Here art and mathematics are almost indistinguishable. The lesson is clear: *the more play there is, the more math there is in the children's world.* In addition to play, children develop mathematical skills whenever they try to organize their daily lives with the help of adults: arrange their toys and clothes, plan what they will do, etc. Thus, *children's mathematics consists of the world of children's internal activities that they eventually purposefully organize, design, and conceptualise in order to understand and control the outside world and organize their overall activities in it.*

The spirit of children's mathematics is precisely the spirit of modern mathematics: the creation of various ideas that are not truths about the world (an old view of mathematics) but ultimately serve as a tool for understanding and controlling the world. Children's mathematics is modern mathematics in a more concrete sense. Basic elements of modern mathematics are sets, relations and functions. They correspond to basic children's activities of collecting objects, connecting objects and actions on objects. Just as sets, relations and functions are "Lego blocks" in modern mathematics from which mathematical structures are built, so these basic activities are "Lego blocks" from which children design various ideas: they play games according to various rules, they imitate various situations they observe around them, etc. Natural numbers are only one such structure, distinguished by its success in controlling reality: a set of objects with a distinguished element (the number 1) and a function that assigns to each number the next new number – a structure that enables the process of counting (which is an additional designed activity) by which we control collections. In the children's world, natural numbers are spoken words that have their own natural time order in speaking: one, two, three, ….

[16] And the play is a key element of a child development. I think no one understood this better than Friedrich Fröbel: "Play, then, is the highest expression of human development in childhood, for it alone is the free expression of what is in the child's soul." (Fröbel, 1912, p. 50)

This answer to the question "Which mathematics to teach preschool children?" is completely in harmony with the methodological answer to the question "How to teach children mathematics?", which is described above. It naturally extends the approach of Fröbel, Montessori and others to the field of mathematics. Briefly, a child's mathematics is a part of the child's world of internal activities and is not outside of it. We help the child develop mathematical abilities by developing them in the context of her world and not outside of it. It is up to us to create an environment in which the child can spontaneously develop her activities and to unobtrusively teach the child in those mathematical activities in which she currently shows interest. For example, if she shows interest in folding and cutting paper, let us show her how to make an airplane by folding it, or how to make paper snowflakes by cutting the folded paper. We must just be careful here. If the child refuses or is not interested in what we are showing her, let's not force her. Maybe tomorrow or in a month she will ask: "And how do you make a paper airplane?". We need to support and teach a child exactly the mathematics that we recognize is currently developing or can develop in the child's world.

Creating an appropriate environment is not only creating a physical environment: appropriate space and time (room, children's parks, beach, forest, etc.), toys, manipulative material ("cubes", paper, crayons, clay, sand, leaves, etc.), socializing with other children, etc.; but also creating an appropriate motivational context: a story, a problem situation, a family task, etc., in which the child will naturally show interest in certain activities.[17]

Here I will mention one factor that makes EYM education more burdensome compared to other subjects. I believe that I can best explain this factor by comparing education from two sister fields of human activity: mathematics and art. Both are original products of our internal activities distinguished by their creativity and imagination.[18] During the child's growth, they are separated according to their purpose: whether emotions or intellect are more pronounced in shaping internal activities.[19] Unfortunately, "thanks" to the educational systems, art and math are also later separated by children's attitude towards them. As a child grows up, generally speaking, art and math take on opposite values: art remains attractive, and math becomes repulsive. The main reason for this is that in educational systems the attitude towards art is free and optional, while the attitude towards math is burdened by the repression of "educational achievements". If EYM were taught in an optional way like art, children would later love math as well as art. Instead, EYM education is given a "fateful" importance that burdens both educators and children.

This burden with "educational achievements" puts a strain on EYM education theory and practice. Generally speaking, books in the field are on the one hand full of valuable material,

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[17] For example, Darko Ban, my variant of Peter Pan, helped me a lot. Darko Ban is an astronaut who flies around space in a rocket, and when he has a problem, he flies to Nina's (my granddaughter) house at night, leaves the problem on her table with a request that she help him solve it. Nina solved these problems with great will. In the end, she grew to love solving problems so much that Darko Ban did not need to ask her to handle the problems anymore.

[18] Just as the world of internal activities is a unique characteristic of the human species, so are the subjects that arise from it: art and mathematics.

[19] "May not music be described as the mathematics of the sense, mathematics as music of the reason?" (James Joseph Sylvester).

and on the other hand they incorporate that material into a whole elaborate theory of how to teach the material, a theory that is flawed because it rests on flawed assumptions. Instead of having valuable material that will inspire us in helping children in their mathematical development according to the simple methodology described above, which requires only a feeling for children and creativity, we have a "science" that burdens us with its detailed administrative precision of how children will develop their mathematics "brick by brick" along developed "learning trajectories" which "have three parts: (a) a specific mathematical goal, (b) a path along which children develop to reach that goal, and (c) a set of instructional activities that help children move along that path." (Clements and Sarama, 2014, p. iv). The whole theory is based on questionable concepts and wrong assumptions. A child does not develop her mathematics by adding "brick of knowledge to brick of knowledge" along her "learning trajectory" because there are no such things: these are just false metaphors.[20] EYM should be freed from "specific mathematical goals", children's mathematical development is not "paths (composed of levels) along which children develop to reach the goals", and our helping children in mathematical development is not "a set of instructional activities that help children move along that path". Rather than the modern educational theory of EYM being an extension of the simple methodology well established by Fröbel, Montessori, and others, which is briefly described above, it is an extension of the centres of power that, aided by a pseudo-scientific establishment that claims exclusive rights to EYM education, control and cripple children's mathematical development.

I believe that the understanding of mathematics described above, though general, together with the well-known teaching methodology described above, give a clear conception of the place of mathematics in the children's world and our role in helping children develop their mathematical abilities. Having a clear conception of mathematics is one of the key assumptions to assist parents, educators, and teachers to successfully help the youngest in their mathematical development. I believe that the view of mathematics presented here is the missing link of EYM.

A critique of mathematics education standards and proposals for change

Criticism of the existing standards and proposals for change presented here is incomplete and presented at a conceptual level. The proposals for change only provide frameworks that need to be supplemented with the designed concrete activities. All these activities can be found in various books or on the Internet. The aim of the view of mathematics presented here is to place these activities in the right context. A handbook for parents and educators is in preparation in which I will put into practice these proposals (Čulina 2023).

For definiteness, I chose the NCTM standards (NCTM, 2000), a very clear and precise document with a lot of valuable but, in my opinion, limited and improperly balanced contents, published by the National Council of Teachers of Mathematics – the leading organization of mathematics teachers in the USA and Canada. I will primarily refer to

[20] "Learning is much more similar to biological growth than to manufacture, where component parts are first produced, then fitted together." (Servais and Varga, 1971).

Chapter 4: Standards for Grades Pre-K-2. As far as I know, nothing substantial would have changed in further considerations had I taken some other standards for reference.[21]

For EYM education I advocate, the most important thing is to enable the child to develop mathematical activities through free play and with our possible help – by creating a suitable space in the house, taking the child to the park or into nature, organizing activities with other children, acquiring suitable manipulative material, and creating an appropriate motivational context. Some even consider that it is best at that age to just let the child play (Gray, 2013). It is certainly much better than putting pressure on the child, but I still think it is good to unobtrusively assist the child develop through play. It helps to get books that offer various thinking activities: finding a way through a maze, finding figures that fulfil a certain property, connecting figures, continuing a sequence, looking for differences in drawings, etc. Of course, care should be taken to create a context in which the child will be motivated for such activities.

In the NCTM standards, those activities are not considered mathematics but rather an environment in which mathematical elements should be inserted. NCTM standards consider only isolated mathematical elements as mathematics for children. In this way, the orientation and awareness that children's mathematics encompasses much more than these isolated elements is lost. Much more attention should be paid to the free child's play, stories, and organization of the child's daily life as part of her mathematics. The lack of recognition of these activities in math standards does not necessarily prevent the correct mathematical development of the child as these activities are naturally present in the development and upbringing of the child. However, the lack of recognition can lead the environment, including the child herself, to believe that she is not inclined to mathematics, even though she is.

As for the elements that are more mathematical (in the sense that they empower children for more effective control of reality), they, of course, include natural numbers to control quantities and geometry to control spatial activities. However, the NCTM standards neither cover all the essential mathematical elements nor properly distribute attention to those elements they cover. My main criticism is that in preschool and primary school education the numbers are too prominent and too elaborate, and that other mathematical activities are unnecessarily subordinate to them and thus distorted, while in geometry too much importance is given to figures and bodies that reflect the world of adults rather than the world of children. Reading the NCTM standards we can easily be convinced of this dominance of numbers and geometric figures. The introductory chapter mentions the following about the role of numbers in the mathematics education of children (page 32): "All the mathematics proposed for prekindergarten through grade 12 is strongly grounded in number. The principles that govern equation solving in algebra are the same as the structural properties of systems of numbers. In geometry and measurement, attributes are described with numbers. The entire area of data analysis involves making sense of numbers. Through problem solving, students can explore and solidify their understandings of number. Young children's earliest mathematical reasoning is likely to be about number situations, and their first mathematical representations will probably be of numbers.". Especially for the youngest age, the following is written (page

[21] Some standards seem to me almost like instructions for torturing children.

79): “The concepts and skills related to number and operations are a major emphasis of mathematics instruction in prekindergarten through grade 2.”. The introductory part on geometry begins with the following text (page 41): “Through the study of geometry, students will learn about geometric shapes and structures and how to analyse their characteristics and relationships.”. Especially for the youngest age, the following is written (page 97): “Pre-K–2 geometry begins with describing and naming shapes.”. Such a narrow view of mathematics is simply dangerous. As I wrote in the introduction, it stems from an outdated view of mathematics as the science of numbers and Euclidean geometry.

NCTM’s division of mathematics education into standards also reflects the outdated view of mathematics (page 29): “The Content Standards – Number and Operations, Algebra, Geometry, Measurement, and Data Analysis and Probability – explicitly describe the content that students should learn. The Process Standards – Problem Solving, Reasoning and Proof, Communication, Connections, and Representation – highlight ways of acquiring and using content knowledge.”. I will limit myself to the analysis of these standards for EYM. With the **Bold** font, I will simultaneously indicate a different division that I think corresponds to modern mathematics and is natural for children’s mathematics.

I will start with the standard *Algebra*. In modern mathematics, algebra is the part of mathematics that deals with algebraic structures. Algebraic structure is, roughly speaking, composed of some sets and a few operations (functions) on these sets. Classic examples are various sets of numbers with arithmetic operations on them. The characteristic of algebra is that it has a very efficient symbolic language for examining algebraic structures and connecting them. At the EYM level, the algebra mentioned in NCTM has almost nothing to do with the modern understanding of algebra. The overlap is in observing the properties of operations and in the use of symbols. Observing the properties of operations is a fairly advanced activity, so it should not be a very prominent activity for pre-schoolers. Writing and reading symbols or words should not be present in mathematical content at this level because children are not fluent in these: writing and reading add unnecessary burden and bring additional abstraction that destroys the simplicity of a basic mathematical content. *We must not take written content lightly into mathematical activities of children*.

The only content that NTCM mentions in the *Algebra for Pre-K-2* section (pages 90 - 94), which is suitable for pre-schoolers, and which does not actually belong to algebra, is an impoverished introduction to the basic elements of modern mathematics: **sets, relations, and functions**. Due to their importance, these elements should be singled out as a separate entity in EYM because they are the basic blocks from which modern mathematics is built, and in the children’s world they relate to basic children’s activities: collecting objects, connecting objects and actions on objects. That is why the creators of the “new mathematics” believed that these elements must be at the very basis of mathematics education. So, they thought that the teaching of the youngest should start with these elements – which proved unsuccessful. The reason is simple to me: these concepts are foreign to the children’s world. The concept of set derives from the grouping and classification of objects. However, while it is natural for children to work with concrete objects, it is not natural for them to work with abstract sets of objects, or relations and functions between objects. My granddaughter Nina will talk about objects on the table and not the set of objects on the table. She will say that Ezra and Nina are cousins, but she will certainly not say that they are in a relationship of “being a cousin”. She will say that Anja is Ezra’s mother but not that Anja is a value of the “mom of” function applied to Ezra. *Instead of telling children about sets, relations, and functions, we need to teach children to perceive and construct concrete sets, relations, and functions. Children should use sets, relations and functions and not mention them.* It is important to keep in mind this use-mention distinction in helping children to develop these concepts. This is what of these concepts, in my opinion, should be taught at this age.

The children’s world is full of concrete examples of sets, relations, and functions. Children learn sets by grouping and classifying objects, relations by connecting objects, functions through concrete actions over objects. All these activities are included in the NCTM standards. But that is not enough. Such important concepts require much more attention and the development of the wider range of educational activities. The “new mathematics” movement has given us a wealth of material from the field that we can, taught by history, easily transform into a correct methodological approach. For example, why stop at a comparison relation (smaller – bigger, lighter – heavier, etc.) or an equivalence relation (same height, same shape, same colour, etc.) that are usually associated with some future acquiring of measurements, and not use other relations that are naturally present in the children's world: *x* is the mother of *y*, *x* is a friend of *y*, *x* is a pet of *y*, etc.? These relations are familiar to children and children are happy to explore them. On the example of the rock-paper-scissors game, children can learn the relation “to be stronger” which is not an ordering relation. Using the example of the relation “object x is on top of object y”, children can understand that an ordering relation does not have to be linear. Children naturally classify objects by some similarity (being of the same colour, being of the same shape, etc.), and thus they naturally become familiar with the equivalence relations that are not related to counting and measuring. By replacing the places of objects in a relation, children can be naturally motivated to examine the properties of relations. For example, in the relation *x is a parent of y*, we cannot replace the places of *x* and *y* (if *x* is a parent of *y*, then *y* is not a parent of *x*, so the relation is asymmetric) nor instead *y* we can put *x* (*x* is not the parent of *x*, so the relation is irreflexive). Why not use graphs to represent other types of relations? Graphs allow children to visually analyse the entire menagerie of relations from their world. Such a presentation of relations is very striking. In (Servais and Varga,1971) Willy Servais writes (page 97): “Arrow graphs are used to represent binary relations by sets of arrows …The finished graph, being formed of arrows, preserves the memory of the dynamic operation involved in drawing it. …They are really perceptual drawings fulfilling an abstract purpose. Coloured graphs have made a powerful contribution to the elementary understanding of relational notions …”. For instance, we can paste or draw the human characters on paper and connect them with arrows: blue for “to be a mom of”, red for “to be a dad of”. In this graph, children can explore family relationships; for example, find all a person’s grandparents, or all her siblings. etc. Just as we can expand the mathematical content associated with relations, we can also expand the mathematical content associated with sets and functions. For example, we can introduce operations with sets by collecting objects in piles or putting a lasso around them (if they are on paper, by drawing a closed line around them), or by connecting language conditions using connectives “not”, “and” and “or”. With this last method, children simultaneously acquire the correct logic of the language, as demonstrated by Zoltán Pál Dienes in a lesson in logic (Servais and Varga, 1971). The NCTM standards describe various activities with functions (matching, patterns, geometric transformations, symmetries, etc.), but why not add functions that are constantly present in the children's world, such as "mom of" and "dad of", which can be combined in interesting ways for children, for example using graphs as described above, or "turn left" and "go two steps forward", which can be combined in spatial movement.

While on the one hand the NCTM standards are very silent about the sets, relations and functions of which the children's world is full, on the other hand they almost worship patterns – as if it were "the philosopher's stone" of EYM: "Patterns are a way for young students to recognize order and to organize their world."[22] (page 91). If we look, for example, at a pattern that is in the form of a sequence, it is only a recursively described sequence: based on the sequence built until then, we construct the next member of the sequence using a recursive rule. The same applies to other types of patterns. Interesting, but without any great significance.[23]

The next level, which is not present in the *Algebra* section, would be to design activities with simple mathematical **structures**. Various games and puzzles with moves and rules are simple and interesting examples of such structures for children.

Numbers are the oldest and still the most important mathematics. However, in the mathematics education standards, numbers are too much imposed on the children's world and as such overshadow other mathematical content – they can even turn children away from mathematics due to their more pronounced formal aspect. That is why numbers should be treated more carefully than is currently the case. The numbers should grow slowly in the children's world, perhaps over years. Considering that they can be "boring", we should take good care that they develop naturally in the children's world, and not impose them as learning outcomes that the child must achieve. With such a "fateful" approach and demands to achieve the planned outcomes, we only frustrate the child and distance her from mathematics. Reducing math to numbers and forcing children's achievement with numbers is the main reason we have so many people who do not like math and think they are incapable of math.

My suggestion for pre-schoolers is as follows. By comparing sets by establishing a 1–1 connection between their elements, children turn their intuition of quantities into a precise mathematical model of comparing sets (it is better not to mention sets) – where there are more, where fewer, and where there are equal objects. The next step is to introduce natural numbers and a counting process that establishes a 1–1 connection of a set with the initial

[22] This sounds like a quote from the Bible to me.

[23] To confirm the correctness of "worship of patterns", the authors often refer to Keith Devlin's book *Mathematics: The Science of Patterns* (Devlin, 1996). It is a popular book in which Devlin never explained what patterns are and what it means that mathematics is the science of patterns. The use of that word in the book does not have a specific meaning, but rather an inspirational and metaphorical, I would even add a mystical meaning. It is not clear to me how such a vague word could appear in the mathematics education standards as a very deep mathematical concept.

segment of the set of numbers. Thus, children learn to represent quantities with numbers. At this level, *natural numbers for children are nothing but spoken words that have a time order in speaking*.[24] In the Croatian language we have a time sequence of spoken words: "jedan, dva, tri, …". When children in Croatia learn English, they easily replace Croatian numbers with isomorphic English numbers: a new time sequence of spoken words: "one, two, three, …". It is often imposed on children that numbers are (represented by) written signs (numerals). In my opinion, such an approach is inappropriate for several reasons. First of all, numerals do not have a natural order unlike spoken words which have the natural time order, which is crucial for the counting process. Furthermore, they are symbols and as such introduce at this level unnecessary abstraction into the counting process. In addition, they require a certain child's reading and writing skills, which, as I pointed out above, is a complex process that unnecessarily burdens the mathematical content. By counting, children can easily compare sets of objects by comparing the associated numbers: which numbers occur first and which later in the number sequence. Addition and subtraction of small numbers at this level can be done by adding and subtracting sets of objects that they represent, but not directly by operating with numbers. Direct operations with numbers (apart from the operation of taking the next number) not only require that children know how to write and read numbers, but they are of a formal nature which in my opinion is not part of the children's world at that age.[25]

Geometry arises from the child's movement, navigation, and constructions and deconstructions in space. This includes distinguishing directions and rotations, along with the "amount" of movement in a direction or in rotation. With the development of these activities, the child establishes control in space. Children develop this best on playgrounds, in sports activities, by orienting themselves in nature and making various constructions with "cubes" (wooden bodies, Legos, magnetic figures, …), clay, paper, etc. These activities are described in the NCTM standards, but I think they are far more important than learning geometric shapes which the NCTM standards give priority to. Of course, the figures are present in the surrounding area. But it is a space designed by adults. When we transfer these figures into children's space, we must be aware that these figures do not have the same importance in the children's world as in the adult world. My limited experience has shown that in the children's world, circles, triangles, rectangles, …, are not as prominent as they are represented in the NCTM standards. I cannot imagine a motivation in the children's world that would lead to identifying and analysing the properties of isolated geometric figures. My thesis is that children simply use figures as building blocks for various spatial designs and primarily through these activities they are motivated to analyse their shape and dimensions. Children's space is primarily a space of their movements, navigation in space and constructions in space, and the development of these abilities should be emphasized in their geometric upbringing. In

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[24] This is in line with Kant's well-known claim in the *Prolegomena* that arithmetic "forms its concepts of numbers through successive addition of units in time".

[25] In EYM education literature about natural numbers I came across terms that on the one hand are given too much importance and on the other hand are unclear, such as: "subitising numbers" and that "numbers are made up (composed) of smaller numbers".

developing these abilities today, physical education helps them far more than mathematics education standards.

What is still important about geometry at this level, and which in my opinion is not adequately represented in the NCTM standards, is that geometry provides great opportunities for visual representation of problems by which a child can create mathematical models of various situations. Ordinary drawing of an elephant, for example, is the creation of a mathematical model of an elephant. Here one can follow how the child creates an ever-better model of an elephant over time, even varying the model depending on what interests her in the elephant. We can draw a strong analogy of these children's models with the mathematical models used by adults. These children's models are the first steps in modelling increasingly complex situations. Not to mention that in this way children develop a sense of space and control of lines and shapes in space, especially if they model not on paper but with some material in space. A step forward is sketching the space in which a child lives, from a sketch of the room to a map of the entire area in which she moves, as well as sketching her movement in that space using straight or curved arrows. Making spatial maps as well as using ready-made maps and solving various problems with the help of maps is very important for the development of the child's mathematical abilities and should be given more importance and more attention in mathematics education. For example, it develops perspective taking and scaling. This is very well recognized in The National Geographic Network of Alliances for Geographic Education (National Geographic, 2022).

Here, there are still places for activities with which children gain experience about topological aspects of space (dressing, knots, transformations in clay, stretching rubber, mazes, networks, etc.) or about projective geometry (for example, making shadows).

It is very important to help children develop all aspects of geometry because in modern mathematics, the introduction of geometric insight into the mathematical modelling of various phenomena is just as important as the formulation of efficient algebra for solving problems related to these phenomena. For example, older mathematical physics textbooks were mostly collections of algebraic, analytic, and numerical procedures for solving differential equations, while more modern textbooks are dominated by geometric structures. Even in the very titles of such textbooks, the word *geometry* appears.[26]

Standards *Measure and Measurement*, and *Data Analysis and Probability,* describe only two of the multitude of **other mathematical activities**. They emphasize two very important ideas: the idea of measurement and the idea of chance (the very concept of probability is too complex for this age). However, many other mathematical ideas can be singled out equally: already mentioned simpler mathematical structures (they can be developed through games that do not have to be competitive games but also cooperative games), graphs (to represent spatial networks, relations, states and changes, …), recursion (basic elements plus construction rules), topology (dressing, knots, transformations in clay, stretching rubber, …), chance in general context (games with an element of chance), change (dynamics of movement and activities), choice (games with elements of choice), etc.

[26] Such are, for example, well-known textbooks Frankel: *The Geometry of Physics*, Schutz: *Geometrical Methods of Mathematical Physics*, or Nakahara: *Geometry, Topology and Physics*.

The above-mentioned mathematical elements appear in specific mathematical activities. I will call them **direct mathematical elements**, to distinguish them from the mathematical elements that appear in all mathematical activities, which I will call **background mathematical elements**. Some of the background math elements have already been mentioned above: these are **sets, relations, and functions** that appear in the children's world as primary mathematical elements through concrete examples. However, children should be encouraged to look for sets, relations, and functions in all activities, so these elements also appear as background math elements. Background mathematical elements also include **abstraction**, **representation**, **procedural activities** (algorithms), **logic,** and **language**. But in my opinion, language is the most important, so I will dwell on it, especially since it includes both abstraction and logic. Representation has already been mentioned in the context of geometric representation of problems.

This division is different from the NCTM division into "The Process Standards – Problem Solving, Reasoning and Proof, Communication, Connections, and Representation", which, according to the NCTM, "highlight ways of acquiring and using content knowledge". Background mathematical elements are as important as direct mathematical elements for a child's mathematical development. They are partly represented in NCTM The Process Standards, but they are not recognized there as full-blooded mathematical elements but as a means of achieving mathematical knowledge.

Language elements are concrete means from our world of internal activities by which we control reality. Thus, the language is a very powerful mathematics. By choosing words in a situation, we do an abstraction, extracting from that situation what interests us and abstracting the rest. It is an essential mechanism that helps us deal with the complexity of the world. Furthermore, we use words to control and structure the aspect of the situation that interests us. Through noun expressions we control objects, through predicate expressions we control, and I would say we refine and create concepts. Thus, *language itself is an important type of mathematics that should be developed at the preschool age as well. Like us, a child manages to control and understand reality through language. That is why we help her substantially in her mathematical development whenever we read her stories, when we listen to her when she talks, when we encourage her communication with other children and adults.* Of course, this attention to language development should also be nurtured in all the child's mathematical activities. I would like to mention that at that age, language is the tool of the child's activities and not the subject of her activities. By helping a child to develop language in a given mathematical activity, as well as in any other activity, we help her to learn abstraction and to clarify the concepts or meanings of words. In short, by refining the language, the child refines her mathematics. Furthermore, by using language, the child opens the way to the idealized mathematical world that arises from her activities, thus expanding her mathematics. This step is not a problem for the child. Just as she uses language to specify the story of Snow White and the Seven Dwarfs, she uses language to specify the world of "all numbers". The NCTM standards do not recognize language as a very powerful mathematics and as a means of building idealized mathematical worlds, but they do recognize the importance of language as a means of clarifying and communicating mathematical activities.

The view that language as a means of rational cognition is a part of mathematics is not a common view (Čulina, 2021). However, regardless of whether you consider some language aspects to be a part of mathematics or not, a child's language development should be given much attention.

No matter how we look at **logic**, logic always manifests as the logic of a language. Thus, by acquiring a language, children also acquire logic. I have already mentioned the use of connectives in classifying objects using complex conditions. My experience with Nina showed me that children learn the meaning of negation[27] and of conditional[28] very quickly, but somewhat slower the meaning of conjunction and disjunction. Children also understand the meaning of quantifiers.[29] Logical inference is not foreign to them, especially when it works in their favour.[30] Furthermore, if there is inconsistency in the story, a child immediately registers it.[31] And it is well-known in logic that consistency is the equivalent of logical inference. Although the NCTM standards emphasize reasoning as a separate process in mathematical activities, the standards limit it to the process of establishing mathematical claims, and even in such a limited context, the view of children's reasoning is very limited. What is written in the NCTM standards on page 122 – "Two important elements of reasoning for students in the early grades are pattern-recognition and classification skill." –- may be appropriate for chickens but certainly not for children who are full of imagination. The NCTM standards do not recognize children's thinking as separate mathematics that develops through all children's activities, especially through stories and fantasies, and not only in mathematical activities, nor do they recognize the overall richness of children's thinking. On the contrary*, it is very important to encourage children to retell or invent stories and events themselves, to discuss stories and events with each other or with us, to look for reasons for certain behaviours or events, and to infer other information from available information.*

Procedural thinking (*how to achieve something*) *is more appropriate to the dynamics of the children's world than declarative thinking* (*what is and what is not*). However, these procedures should be meaningful and expressed in spoken and pictorial language. The refinement of procedures should be gradual with the awareness that in this way freedom is lost but efficiency is gained. Finally, we adults do not really like detailed instructions, but only general instructions that leave us a lot of space for our own creation. In my limited experience, this is even more present in children. The transition to formal procedures, such as algorithms with numbers, is a demanding transition, because formal procedures involve writing and they lose content, so they should not be rushed. The NCTM standards deal almost only with formal procedures with numbers. As formal procedures are not appropriate for pre-schoolers, the procedural thinking of the youngest is not present in the NCTM standards. This important mathematical component of the child development can be developed very

[27] "I'm not going to kindergarten!"

[28] Me: "How can I help you stop your knee hurting?", Nina: "If I watch cartoons, it will stop me aching knee."

[29] "All the toys are here.", "Someone is in the basement."

[30] Grandma: "Santa Claus only brings gifts to good children.", Nina: "Then Ezra will not get a gift.", Grandma: "Why?", Nina: "Because he was not good: he hit me." –- there are connectives and quantifiers in this deduction.

[31] Me: "What's your doll's name?", Nina: "Aurora", Me: "Didn't you tell me yesterday that her name was Julia?", Nina: "Yes, but she's constantly changing her name."

efficiently through nursery rhymes, songs, spatial movement instructions, cooking recipes, etc. For example, with the help of "The Enormous Turnip" folktale, children learn the concept of iteration in problem solving (programming loops) and with the help of "Pošla koka na Pazar" (English translation: "When Hen Was on Her Way to Fair") South Slavic folktale[32], children learn the concept of reductive problem solving (subroutine calls in programming). The development of the procedural component in children is also important due to the increasing importance of software in modern society. If we leave out technology, programming is, from a conceptual point of view, a part of mathematics. Praiseworthy is the emergence of simple programming languages and environments, such as Scratch (Scratch Foundation 2022), in which children can easily and vividly create characters, program their behaviour, and compose stories. All this is an important part of mathematics for the youngest to which adequate attention should be paid.

An essential component of mathematics is that it has a purpose: to be a tool of our rational cognition and rational activities in general. This is true for both adults and children. Only, the purpose of children's mathematical activities must be incorporated into their world. Just as all human civilization has developed mathematics as a means of solving various big and small problems, and just as individuals are developing it, in the same way children in their children's world need to build their mathematics by solving problems from their world. As in the world of adults, *this purpose in the world of children gives mathematical activities integrity – a natural framework for their development*. This component, which is usually called "**problem solving**", must be kept in mind when helping a child to develop mathematical skills. This can be solving problems arising from the organization of the child's daily activities (placing goods in drawers), arising from play (how to assemble a crane from Lego bricks) or integrated into the world of a story (e.g., the story of the wolf, goat, and cabbage). Counting on its own can be fun, but it only gets real meaning when counting controls whether all the bears are present at the morning review of stuffed animals.

Conclusion

A specific problem of EYM education is an insufficiently clear or wrong assumption about the nature of mathematics. Modern educational standards are based on the assumption that mathematics is the science of numbers and Euclidean geometry. This assumption has been obsoleted more than a century since the advent of modern mathematics. In EYM standards, numbers are too prominent and too elaborate, and other mathematical activities are unnecessarily subordinate to them and thus distorted, while in geometry too much importance is given to geometric figures and bodies that reflect the world of adults rather than the world of children. The standards give a narrow and distorted picture of mathematics. And looking too narrowly at mathematics carries great danger. Adhering to the standards, we drastically limit the mathematics of the child's world, hamper the correct mathematical development of a child, and we can turn her away from mathematics.

[32] I only know of the English translation in the book by Stanić-Rašin (2018).

Children’s mathematics is much broader. Children’s mathematics consists of the world of children’s internal activities that they eventually purposefully organize, design, and conceptualise in order to understand and control the outside world and organize their overall activities in it. The spirit of children’s mathematics is precisely the spirit of enormously successful modern mathematics: the creation of various ideas that are not the truths about the world (an old view of mathematics) but ultimately serve as a tool for understanding and controlling the world. Given that the child's internal activities are a key part of her play, the child develops mathematics best through play.

This understanding of children’s mathematics is in complete harmony with the methodological approach developed long ago by Fröbel, Montessori and many others, according to which the child’s learning must be a part of the child’s world. Child’s learning must have motivation, meaning and value in the child’s world, and not from the outside, in the world of adults. In developing her abilities, the child must have the freedom and not the pressure to achieve pre-established learning outcomes. It is up to us that with a lot of love help and guide the child in developing her potential, respecting her world and her individuality – which activities and at what stage of her growth attract her –- and providing an appropriate environment for such development.

Briefly, a child’s mathematics is part of a child’s world of internal activities and is not outside of it. We help the child develop mathematical abilities by developing them in the context of her world and not outside of it.

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