Rejection in Łukasiewicz's and Słupecki's Sense

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Abstract The idea of rejection originated by Aristotle. The notion of rejection was introduced into formal logic by Łukasiewicz. He applied it to complete syntactic characterization of deductive systems using an axiomatic method of rejection of propositions. The paper gives not only genesis, but also development and generalization of the notion of rejection. It also emphasizes the methodological approach to biaspectual axiomatic method of characterization of deductive systems as acceptance (asserted) systems and rejection (refutation) systems, introduced by Łukasiewicz and developed by his student Słupecki, the pioneers of the method, which becomes relevant in modern approaches to logic.

Keywords Aristotle's syllogistic · Genesis of rejection notion · Rejection in Łukasiewicz's sense · Słupecki's solution of decidability problem of Aristotle's syllogistic · Deductive systems · Axiomatic method of rejection · Refutation systems · Słupecki's rejection function · Generalization of rejection notion

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1 Introduction

The European logic as a science arose in fourth century BC in ancient Greece. Aristotle is thought to be the creator of logic. First of all, he is recognized as the creator of the first formal—logic system, deductive system, the so called Aristotle's syllogistic, which together with the theory of immediate reasoning (square of opposition, conversion, obversion, contraposition, inversion) is treated as *traditional logic* in a narrower meaning. It is the *logic of names*.

The idea of rejection of some sentences on the basis of others was originated by Aristotle, who in his systematic investigations regarding syllogistic forms not only proves the proper (true) forms but also rejects the false (invalid, erroneous) ones. Aristotle to reject some false syllogistic forms very often used examples, but he also used another way of rejecting false forms by reducing them to other ones, already rejected (see Łukasiewicz [22, 23]).

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The method of rejecting sentences always functioned in empirical sciences in connection with the procedure of refutation of hypotheses, and we can assume that it was also known to Stoics: because within the five 'unprovable', hypothetical syllogistic rules, their logic of sentences, formulated probably by Chrisippus, the rule of deduction called in Latin *modus tollendo tollens*, or *modus tollens* (If p, then q; not-q, so not-p) appears as the second one.

Although the method of rejection of sentences has always existed in empirical science, in relation to the procedure of refutation of hypothesis, the Aristotle's idea of rejection of some sentences on the basis of the others has never been properly understood by logicians or mathematicians, especially the ones convinced that rejection of closed sentences of the language of deductive system can be always replaced by introducing into such system the negation of such sentences.

The proper understanding of Aristotle's ideas and implementation of the concept of rejection into the formal researches on logical deductive systems (including on Aristotle's syllogistics) we owe to Jan Łukasiewicz—the co-founder of world famous Warsaw Logical School, which functioned in the interwar period (1918–1939). Łukasiewicz and his pupil—Jerzy Słupecki are classified as pioneers of meta-logical studies on the concept of rejection and related to it the notion of saturation (in Słupecki's terminology—*Ł-decidability*) of deductive systems.

The notion of rejection was introduced into formal logic by Łukasiewicz in his work "Logika dwuwartościowa" ("Two-valued logic") [20], in which, apart from the term "assertion" (introduced by Frege), Łukasiewicz introduces also the term "rejection". In adding "rejection" to "assertion" he, as he states himself, followed Brentano, but he did not mention any more about it. The notion of rejected proposition later played an important role in his research on Aristotle's syllogistic [22, 23], as well as in his metalogical studies of some propositional calculi [24, 25].

In that research Łukasiewicz makes use of the idea of rejection originated by Aristotle. Łukasiewicz applied it to complete a syntactical characterization of deductive systems using an axiomatic method of rejection, introduced by him into the formal logic in his paper on Aristotle's syllogistic [22], and then, after the war, in a monograph [23], which followed many years of research on Aristotelian logic, and which included the results presented in the paper prepared before the war.

As was pointed out by Łukasiewicz (see [23, p.67]),

- Aristotle, in his systematic investigations of syllogistic forms, not only proves the true ones but also shows that all the others are false, and must be rejected.

Further on (p.74), Łukasiewicz observes:

- Aristotle rejects invalid forms by exemplification through concrete terms. This procedure is logically correct, but it introduces into the systems terms and propositions not germane to it. There are, however, cases where he applies a more logical procedure, reducing one invalid form to another already rejected. On the basis of this remark, a rule of rejection could be stated corresponding to the rule of detachment by assertion; this can be regarded as the commencement of a new field of logical inquiries and of new problems that have to be solved.

- Modern formal logic, as far as I know—writes Łukasiewicz (see [23, p.71])—does not use 'rejection' as an operation opposed to Frege's 'assertion'. The rules of rejection are not yet known.

As a rule of rejection corresponding to the rule of detachment by assertion, Łukasiewicz adopts [22, 23] the following rule, which was anticipated by Aristotle:

- the rule of rejection by detachment:

  if the implication "If α, then β" is asserted, but its consequent β is rejected, then its antecedent α must be rejected, too.

As a rule of rejection corresponding to the rule of substitution for assertion Łukasiewicz adopts [22, 23] the following rule, which was unknown to Aristotle:

- the rule of rejection by substitution:

  if β is a substitution instance of α, and β is rejected, then α must be rejected, too.

Both rules enable us to reject some syllogistic forms, provided that some other forms have already been rejected.

As we mentioned above, Aristotle used the procedure of rejection of some forms by means of concrete terms, but such a procedure, though correct, introduces into logic terms and propositions that are not germane to it.

To avoid this difficulty, Łukasiewicz rejects some forms axiomatically, which leads him to biaspectual axiomatic characterization of deductive systems analyzed by him [22–25]. The idea of rejection was first used by Łukasiewicz for two-level syntactic description of Aristotle's axiomatic system of syllogistics: as a system with respect to acceptance (the first level) and as a system with respect to rejection (the second level). The sentence rejected from the system he understood as a false sentence, or a sentence which due to some reasons we cannot classify as a thesis of this system. The idea of rejection was used by Łukasiewicz while studying the decidability (saturation) of the system: each sentence, which is not the thesis of a decidable (saturated) system, is rejected.

Reconstruction of concepts of rejection and decidability (saturation) used by Łukasiewicz, was done by his pupil—Jerzy Słupecki [39, 40] (see also Słupecki et al. [45]). Słupecki modified, developed and later generalized the concept of rejected sentence, he also made its certain formalization. He also inspired systematic, formal studies on this and related concepts, and also initiated research on the decidability of many deductive systems.

In this paper[1] I am starting from Słupecki's and mine reconstruction of concepts rejection and decidability—the notions introduced and used by Łukasiewicz (Sect. 2).

Later on (Sect. 3), I am discussing the problem of decidability of Aristotle's syllogistic, set by Łukasiewicz, and I outline its solution given by Słupecki. To follow with, I am describing Słupecki's modification and generalization of the concept of rejection and also presenting the importance of Słupecki's research on syllogistics for contemporary metalogic (Sect. 4). Later on, I am showing the relation of idea of decidability in Łukasiewicz's sense with the common concept of decidability given by Słupecki, and I am discussing the decidability of more important logical systems (Sect. 5). Finally, I

[1] The paper is elaborated on the basis of my works [63–65] and Słupecki et al. [45].

am presenting Słupecki's generalization of the concept of rejection to the function of rejection, and its formalization in the theory of rejected sentences (Sect. 6). The final notes relate to presentation of different than Łukasiewicz's way of two-level formalization of deductive systems (Sect. 7).

2 Reconstruction of Concepts of Rejection and Decidability: The Notions Introduced and Used by Łukasiewicz

The notion of decidability of the deductive system that was used by Łukasiewicz in his research on Aristotle's syllogistic [22, 23] and systems of propositional calculi [24, 25] is based on the notion of a rejected sentence introduced by him.

The main idea of a syntactic biaspectual characterization of deductive systems in Łukasiewicz's sense is compatible with providing both:

- the axioms and inference rules for the given deductive system, which intuitively lead from some true formulas to true ones of this system

and

- the rejected axioms (treated as false formulas of this system) and rejection (refutation) rules of this system, which intuitively lead from some false formulas to false ones of this system.[2]

Łukasiewicz used the terms 'decidable system' and 'consistent system' in the meaning different from the one accepted in logic. Łukasiewicz does not give clear definitions of these terms, but the context points out that he used them in the following meaning:

- The system is *decidable* if every its expression which is not its thesis is rejected on the ground of finite number of axiomatically rejected expressions;
- The system is *consistent* if none of its thesis is rejected.

Łukasiewcz did not use the term 'decidable system' consequently. He also employed interchangeably the terms 'saturated system' or 'categorical system'.

In nomenclature introduced by Słupecki, decidability of a deductive system in Łukasewicz's sense was called Ł-*decidability* and its consistency was called Ł-*consistency*.

The meaning of the term 'decidable system' compatible with the understanding of the notion of a decidable system by Łukasiewicz gives the following definition:

- The deductive system determined by means of the ordered triple:

$$\langle F, A, R \rangle$$

[2]Such a syntactic formalization of some propositional calculi was also, probably independently, introduced by Rudolf Carnap [6, 7]; see Citkin [10].

where F is the set of all well-formed formulas of this system, A is the set of its axioms and R is the set of its primitive inference rules, is Ł-*decidable* if and only if there exist finite sets: the set A^{-1} of rejected axioms (included in F) and the set R^{-1} of primitive rejected rules, such that the following two conditions are satisfied:

$$\text{I. } T \cap T^{-1} = \emptyset \text{ and II. } T \cup T^{-1} = F,$$

where T is the set of all theses of the system, T^{-1} is the set of all rejected formulas (the smallest set including the set A^{-1} and closed with respect to every relation determined by rejected rules of the set R^{-1}).

The conditions I and II we call, respectively, Ł-*consistency* of the deductive system and Ł-*completeness* of the system.

Characterizing the deductive system by means of tuples

$$\langle F, A, R; A^{-1}, R^{-1} \rangle \text{ or } \langle F, T; T^{-1} \rangle,$$

the definition given above can be defined as follows:

The deductive system is Ł-*decidable* if and only if the following conditions are satisfied:

I. the set of all its theses (asserted expressions) determining the system is disjoined with the set of all its rejected formulas,
II. every propositional formula is either asserted or rejected.

3 The Problem of Decidability of Aristotle's Syllogistics, Set by Łukasiewicz, and Its Solution Given by Słupecki

3.1 Łukasiewicz's Biaspectual Formalization of Aristotelian Logic AS

On the first level **AS** is characterized as follows (cf. Łukasiewicz [21]):

The Vocabulary of ***AS***:

- constant symbols of classical logic *CL*, i.e., the connectives of *CL*,
- primitive terms of **AS**: constants a and i which are sentence forming functors of two-term arguments: 'all ... are ... ' and 'some ... are ... ',
- nominal variables: S, P, M, N, ...

Well-Formed Expressions of ***AS***:

- atomic affirmative expressions: formulas of the form of $S\ a\ P$ and $S\ i\ P$, which are read: 'all S are P', 'some S are P', respectively, compound expressions: formulas that are created from the atomic ones using the connectives of *CL*,
- the set F of all well-formed expressions—the smallest set of formulas including the atomic expressions and closed under the connectives of *CL*.

*Defined Terms of **AS***:

- the remaining constants of Aristotelian logic: e and o, i.e. the functors: 'no ... are ...' and 'some ... are not ...' which are defined as follows:

D1. $S\ e\ P \stackrel{\text{df}}{=} {\sim}S\ i\ P$,
D2. $S\ o\ P \stackrel{\text{df}}{=} {\sim}S\ a\ P$.

The negative expressions $S\ e\ P$ and $S\ o\ P$ are read, respectively: 'no S are P' and 'some S are not P'. Apart from the atomic expressions, we include them into the so called *simple expressions*.

*Axioms of **AS***:

$A^{+}1$. $S\ a\ S$,
$A^{+}2$. $S\ i\ S$,
$A^{+}3$. $M\ a\ P\ \wedge\ S\ a\ M \rightarrow S\ a\ P$ (*Barbara*),
$A^{+}4$. $M\ a\ P\ \wedge\ M\ i\ S \rightarrow S\ i\ P$ (*Datisi*).

Axioms $A^{+}1$ and $A^{+}2$ are the two laws of identity; Aristotle did not accept them.

*Primitive Inference Rules for **AS***:

rŁ^: *the rule of definitional replacement* (according to D1, $S\ e\ P$ may be everywhere replaced by ${\sim}S\ i\ P$, and, according to D2, $S\ o\ P$ may be everywhere replaced by ${\sim}S\ a\ P$);
$r^{+}1$: *the rule of detachment (modus ponens)* (if '$\alpha \rightarrow \beta$' and α are asserted expressions of a system, then β is an asserted expression);
$r^{+}2$: *the rule of substitution* (if α is an asserted expression of the system, then any expression produced from α by a valid substitution is also an asserted expression; the valid substitution is to put, for term-variables, other term-variables).

The schemes of the rules $r^{+}1$ and $r^{+}2$ are as follows:

$$r^{+}1:\ \frac{\vdash \alpha \rightarrow \beta \quad \vdash \alpha}{\vdash \beta} \qquad r^{+}2:\ \frac{\vdash \alpha}{\vdash e(\alpha)}$$

The symbol '$\vdash$' is a sign of assertion introduced by Frege, whereas, the expression '$e(\alpha)$' denotes a substitution instance of α.

Characterization of the system **AS** on the second level consists in supplementing it with rejected axioms and rejection rules.

Łukasiewicz formulates the following rejected axioms and rejection rules:

*Rejected Axioms of **AS***:

$A^{-}1$. $P\ a\ M\ \wedge\ S\ a\ M \rightarrow S\ i\ P$,
$A^{-}2$. $P\ e\ M\ \wedge\ S\ e\ M \rightarrow S\ i\ P$.

*Primitive Rejection Rules for **AS***:

$r^{-}1$: the rule of rejection by detachment (reverse *modus ponens*),
$r^{-}2$: the rule of rejection by substitution.

The schemes of the rules are the following:

$$r^{-}1:\quad \frac{\vdash \alpha \to \beta \qquad \dashv \beta}{\dashv \alpha} \qquad\qquad r^{-}2:\quad \frac{\dashv e(\alpha)}{\dashv \alpha}$$

The symbol '$\dashv$' is the sign of rejection.

The characterized system **AS** is determined by the following ordered 5-tuple, which may be called the *basis* of **AS**:

$$\langle F, A^{+}, R^{+}; A^{-}, R^{-}\rangle, \tag{B}$$

where F is the set of all well-formed formulas of this system; A^{+}—the set of its axioms; R^{+}—the set of its primitive inference rules; A^{-}—the set of its rejected axioms, and R^{-}—the set of its primitive rejection rules. The tuples

$$\langle F, A^{+}, R^{+}\rangle \text{ and } \langle F, A^{-}, R^{-}\rangle$$

determine, respectively, the set T^{+} of all *theses* of this system and the set T^{-} of all its *rejected formulas*.

The first tuple may be called the *assertion system* for **AS**, whereas, the second one may be called the *refutation system* for **AS**.

- T^{+} is the set of all well-formed formulas derivable from the set of theses of metalogically formulated CL and axioms of A^{+} by means of inference rules of R^{+}, while
- T^{-} is the set of all well-formed formulas derivable from the rejected axioms A^{-} by means of theses of T^{+} and rejection (refutation) rules of R^{-}. So

$$T^{+} = Cn^{+}(CL \cup A^{+}, R^{+}),$$

and T^{+} is the smallest set including $CL \cup A^{+}$ and closed under the inference rules of R^{+}.

$$T^{-} = Cn^{-}(T^{+} \cup A^{-}, R^{-}), \tag{D(Ł)}$$

and T^{-} is the smallest set including A^{-} and closed under the rejection rules of R^{-}. The set T^{-} is the *set of all rejected expressions of the system in Łukasiewicz's sense*.

To the set T^{+} there also belong all 24 valid syllogistic forms, the laws of logical square and the laws of conversion, and, to the set T^{-} of all rejected formulas, there belong all the remaining 232 invalid forms. However, it turned out that there exists such well-formed expression of **AS**, which is neither a thesis of this system nor a rejected expression of the set T^{-}. Such, for example, is the formula:

$$S\ i\ P \to (\sim S\ a\ P\ \wedge\ P\ a\ S). \tag{Fl}$$

In order to remove this difficulty, we could reject the expression (Fl) axiomatically. However, a question arises whether there exists some other formula of the same kind as (Fl), or, may be, an infinite number of such formulas, which can be called undecidable on the strength of our basis (B). Therefore, we may only claim that the following condition holds:

$$T^{+} \cup T^{-} \subset F.$$

3.2 The Problem of Ł-Decidability of AS

The system **AS** whose basis is (B) analyzed by us, is not *saturated* or *decidable* in the sense that it is both 1° Ł-*consistent* and 2° Ł-*complete*, i.e.

$$1^{\circ} \quad T^{+} \cap T^{-} = \emptyset \quad \text{and} \quad 2^{\circ} \quad T^{+} \cup T^{-} = F.$$

As we know, a system satisfying both conditions was called by Słupecki an Ł-*decidable system* (see Sect. 2).

The problem concerning the finite Ł-decidability of Aristotelian syllogistic was raised by Łukasiewicz in December 1937, during his seminar on Mathematical Logic at the University of Warsaw.

Łukasiewicz presented the problem in the form of the following questions:

Q1. Are the axioms of A^{+} for **AS** together with the inference rules of R^{+} for **AS** sufficient to prove all true expressions of the **AS**?

Q2. Are the rules of rejection of $R^{-} = \{r^{-}1, r^{-}2\}$ for **AS** sufficient to reject all false expressions (every formula of F that is not a thesis of T^{+}), provided that a finite number of them are rejected axiomatically?

3.3 Słupecki's Solution of the Problem of Ł-Decidability of AS

Jerzy Słupecki, who participated in Łukasiewicz's seminar, solved in 1938 the problem providing a basis for which the system of Aristotelian syllogistic is Ł-decidable (see Łukasiewicz [22, 23]). His answer to the question Ql was positive; to the second one, negative.

Słupecki was able to prove that it is not possible to reject all the false expressions of **AS** by means of the rules $r^{-}1$ and $r^{-}2$, provided a finite number of them is rejected axiomatically.

This way Słupecki gave a negative answer to the question Q2:

(i) The system **AS** with the basis (B) is not Ł-complete with any finite set of rejected axioms of A^{-}.

Słupecki extended the system **AS**, adding to it a new rejection rule, called by Łukasiewicz *Słupecki's rule of rejection*. It is denoted by $r^{-}S$ and has the following scheme:

$$r^{-}S\colon \quad \frac{\dashv \alpha \to \gamma \qquad \dashv \beta \to \gamma}{\dashv \alpha \wedge \beta \to \gamma}$$

where α and β denote negative expressions in the form: $S\ e\ P$ or $S\ o\ P$ and γ denotes a simple expression or an implication the consequent of which is a simple expression and the antecedent, a conjunction of such expressions.

The Słupecki's rule says: *If the expression* γ *does not follow from any of two negative expressions then it does not follow from their conjunction*.

Słupecki's rule is closely related to the principle of traditional logic (*ex mere negative nihil sequitur*).

As was noticed by Łukasiewicz, having added Słupecki's rule, it is enough to adopt merely one rejected axiom, namely, $A^{-}1$.

Słupecki demonstrated that:

(ii) System of Aristotle's syllogistic determined by the following base:

$$\langle F, A^{+}, R^{+}; \{A^{-}1\}, R^{-} \cup \{r^{-}S\}\rangle \tag{BS}$$

with the refutation system:

(iii) $$\langle F, \{A^{-}1\}, R^{-} \cup \{r^{-}S\}\rangle,$$

is Ł-decidability system, i.e.

$$1^{\circ} \quad T^{+} \cap T^{-S} = \emptyset \quad \text{and} \quad 2^{\circ} \quad T^{+} \cup T^{-S} = F, \text{ where}$$

$$T^{-S} = Cn^{-}(T^{+} \cup \{A^{-}1\}, R^{-} \cup \{r^{-}S\}), \tag{D(\underline{Ł})}$$

is the *set of all rejected propositions*, i.e. the set of propositions derivable from the axiom $A^{-}1$ by means of the thesis of **AS** and Łukasiewicz's rejection rules and Słupecki's rejection rule.

And the problem of Ł-decidability of **AS** has been solved: any well-formed formula of **AS** is either a thesis or is a rejected formula of **AS**.

It is clear that Słupecki extended the notion of the rejected proposition used by Łukasiewicz, because:

$$T^{-} \subset T^{-S}. \tag{D(Ł) ⊂ D(\underline{Ł})}$$

The results obtained by Słupecki were summarized by Łukasiewicz in his work [22] containing also the text of his paper on Aristotle's syllogistic.

The results of research of both Łukasiewicz and Słupecki were later, after the war, presented in detail in Łukasiewicz's monograph [23]. In both works Łukasiewicz expressed his high opinion of Słupecki's findings, which, in the words of Łukasiewicz [22] were

> organically united with researches of the author [...] the author regards as the most significant discovery made in the field of syllogistic since Aristotle.

3.4 Słupecki's Definition of a Rejected Proposition

Słupecki failed to publish his findings before the war. After the war ends, Słupecki published them in [38] and in a monograph [39]. In the monograph [39], in his proof of Ł-completeness (condition (ii), 2°), he also used a definition of the rejected proposition different than Łukasiewicz D(Ł), and additionally, he modified its extension D(Ł̲), adopted earlier by himself. Instead of Łukasiewicz's definition D(Ł), Słupecki adopts the following equivalent definition:
D(Sł). *A rejected proposition* on the ground of the basis

$$\langle F, A^{+}, R^{+}; A^{-}, \emptyset \rangle, \qquad (B \setminus R^{-})$$

is such an expression for which there exists a rejected axiom of the set A^{-} *which is derivable from it and theses of the set* T^{+} *by means of inference rules of* R^{+}*.*
Denoting a set of all rejected propositions in the sense of the definition D(Sł) by $Cn^{-1}(T^{+} \cup A^{-}, R^{+})$, we obtain the following symbolic notation of it:

$$\alpha \in Cn^{-1}(T^{+} \cup A^{-}, R^{+}) \Leftrightarrow \exists \beta \in A^{-}(\beta \in Cn^{+}(T^{+} \cup \{\alpha\}, R^{+})), \qquad \text{D(Sł)}$$

where Cn^{+} is a consequence operation with respect to the set T^{+} of all theses of the system and its set of rules R^{+}.

The definition of a rejected proposition D(Sł) is closer to Aristotle's idea of refutation of syllogisms by means of reducing them to syllogisms rejected earlier.

We note (see D(Ł)) that:

$$T^{-} = Cn^{-}(T^{+} \cup A^{-}, R^{-}) = Cn^{-1}(T^{+} \cup A^{-}, R^{+}). \qquad \text{D(Ł)} \approx \text{D(Sł)}$$

Thus, the notions of rejected propositions, both the one used by Łukasiewicz and that introduced by Słupecki in the form of the definition D(Sł), are equivalent.

3.5 Słupecki's Definition of an Extended Notion of Rejected Proposition

We will reconstruct Słupecki's definition of an extended notion of the rejected proposition, equivalent to the definition D(Ł̲).

D($\underline{\text{Sł}}$). *A rejected proposition* on the ground of the basis

$$\langle F, A^+, R^+; \{A^-1\}, \{r^-S\}\rangle \tag{BSł}$$

is either a rejected axiom A^-1 *or a proposition rejected with respect to those rejected earlier* in the sense of D(Sł; A^-/X), *or an expression rejected on the basis of those rejected earlier, by means of using Słupecki's rule*.

Let $Cn'(T^+ \cup \{A^-1\}, R^+; \{r^-S\})$ be the set of all rejected propositions in the sense of D($\underline{\text{Sł}}$). We may note that

$$T^{-S} = Cn^-(T^+ \cup \{A^-1\}, R^- \cup \{r^-S\}) = Cn'(T^+ \cup \{A^-1\}, R^+; \{r^-S\}).$$

D($\underline{\text{Ł}}$) $\approx$ D($\underline{\text{Sł}}$)

3.6 Three Different Ways of Understanding the Notion of Rejected Proposition

It is easy to see that the given definitions of rejected propositions provide three different ways of understanding this notion:

$$\begin{array}{lccc}
\mathrm{D(Sł)} \subset \mathrm{D}(\underline{\mathrm{Sł}}). & Cn^{-1}(T^+ \cup A^-, R^+) & \subset & Cn'(T^+ \cup \{A^-1\}, R^+; \{r^-S\}) \\
 & \| & & \| \\
 & T^- & & T^{-S} \\
 & \| & & \| \\
\mathrm{D(Ł)} \subset \mathrm{D}(\underline{\mathrm{Ł}}). & Cn^-(T^+ \cup A^-, R^-) & \subset & Cn^-(T^+ \cup \{A^-1\}, R^- \cup \{r^-S\})
\end{array}$$

The first of them refers to Łukasiewicz's understanding of the rejected proposition (see D(Ł)) and to its strengthening given by Słupecki (see D($\underline{\text{Ł}}$)); the second and the third ones, to Słupecki's understanding of the rejected proposition (see D(Sł) and D($\underline{\text{Sł}}$). At the same time, the second one refers directly to Aristotle's method of rejection of syllogisms by reducing them to previously rejected syllogisms and makes it possible to simplify the procedure of rejection without supplementing the system with the rules of rejection, and the third one (see D($\underline{\text{Sł}}$)) is a combination of both former methods.

4 Notions of Rejection in a Deductive System and Notions of Ł-Decidability

4.1 Three Different Notions of Rejection

We are adapting the previous definitions of rejection for **AS** for any deductive system.

Let S be any deductive system with biaspectual formalization and with the basis

$$\langle F_S, A_S^+, R_S^+; A_S^-, R_S^- \cup R'_S\rangle \tag{B$_S$}$$

determined, respectively, by computable sets: the set F_S of all well-formed formulas, the set A_S^+ of axioms (asserted axioms), the set R_S^+ of inference rules, the set A_S^- of rejected axioms and by the set $R_S^- \cup R'_S$ of rejection rules, with an assumption that the sets R_S^+ and R_S^- are sets of mutual dual rules, while the set R'_S is a set of non-dual rejection rules.

Let T_S^+ be a set of all theses of S. Then, according to Łukasiewicz's conception, on the analogy to D(Ł̲),

the set ${}^1T_S^-$ *of all rejected propositions of* S, with respect to the set T_S^+ and the basis (B_S) (the refutation system $\langle F_S, A_S^-, R_S^- \cup R'_S \rangle$) is defined as follows:

$$ {}^1T_S^- = Cn_S^-(T_S^+ \cup A_S^-, R_S^- \cup R'_S) \qquad \mathrm{D}_S(\underline{\text{Ł}}) $$

And ${}^1T_S^-$ is *a set of all formulas with rejection proofs in Łukasiewicz's sense*, i.e. it is a set of all formulas derivable from rejected axioms of A_S^- by means of theses of T_S^+ and rejection rules of $R_S^- \cup R'_S$; i.e. ${}^1T_S^-$ is the smallest set including A_S^- and closed under the rejection rules of $R_S^- \cup R'_S$.

If $R'_S = \emptyset$, the basis (B_S) of the system S can be replaced by the basis

$$ \langle F_S, A_S^+, R_S^+; A_S^-, \emptyset \rangle \qquad (\mathrm{B}_S \setminus R_S^-) $$

and *the set* ${}^2T_S^-$ *of all rejected propositions of* S with respect to the set T_S^+ and the basis ($\mathrm{B}_S \setminus R_S^-$), on the analogy to D(Sł), is defined as follows:

$$ {}^2T_S^- = Cn_S^{-1}(T_S^+ \cup A_S^-, R_S^+) \qquad \mathrm{D}_S(\text{Sł}) $$

and ${}^2T_S^-$ is *a set of all propositions with rejection proofs in Słupecki's sense*, i.e. it is a set of all such formulas from which, and from theses of T_S^+, and by means of inference rules of R_S^+ a rejected axiom of A_S^- is derivable.

If $R'_S \neq \emptyset$, the basis (B_S) can be replaced by the basis

$$ \langle F_S, A_S^+, R_S^+; A_S^-, R'_S \rangle \qquad (\underline{\mathrm{B}}_S) $$

and *the set* ${}^3T_S^-$ *of all rejected propositions of* S with respect to the set T_S^+ and the basis ($\underline{\mathrm{B}}_S$), on the analogy to D(S̲ł̲), can be defined as follows:

$$ {}^3T_S^- = Cn'_S(T_S^+ \cup A_S^-, R_S^+; R'_S) \qquad \mathrm{D}_S(\underline{\text{Sł}}) $$

and the set ${}^3T_S^-$ is *a set of rejected propositions with rejected proofs in Słupecki-Łukasiewicz's sense*, i.e. it is a set of all such propositions every one of which is either:

1° a rejected axiom of A_S^-, or 2° a proposition rejected with respect to those rejected earlier in Słupecki's sense, or 3° a proposition rejected on the basis of those rejected earlier by means of rejection rules of R'_S.

Let us note that the given above definitions $\mathrm{D}_S(\underline{\text{Ł}})$, $\mathrm{D}_S(\text{Sł})$ and $\mathrm{D}_S(\underline{\text{Sł}})$ are in some extent a simplification and require the precise definitions of the above-mentioned rejection proofs on the basis of any set $X \subseteq F_S$, with respect to the bases (B_S), ($\mathrm{B}_S \setminus R_S^-$) and ($\underline{\mathrm{B}}_S$) (see Słupecki [43], Słupecki and Bryll [44], Wybraniec-Skardowska [63, 64]).

Let us observe that among the set of rejected propositions defined above, the following relationships hold:

a. If $R'_S = \emptyset$ then ${}^2T_S^- = {}^1T_S^-$ b. ${}^2T_S^- \subseteq {}^1T_S^- = {}^3T_S^-$.

4.2 Three Different Notions of Ł-Decidability; Decidability

The three different definitions of the sets of rejected propositions of the system S entail three different definitions of Ł-decidability of this system.

Definition The system S is Ł-*decidable* if and only if, for some $i = 1, 2, 3$, S is ^{i}Ł-*decidable*, i.e. it satisfies the two following conditions:

$$1^\circ \quad T_S^+ \cap {}^iT_S^- = \emptyset, \qquad 2^\circ \quad T_S^+ \cup {}^iT_S^- = F_S.$$

The condition 1° is called ^{i}Ł-*consistence* condition and the condition 2° is called ^{i}Ł-*completeness* condition of the system S.

A question arises: What the relationship between Ł-decidability and decidability in the usual sense is? The answer was given by J. Słupecki.

Słupecki's Theorem [43]: *If the system S is Ł-decidable and any rejection rule, except for the rule of rejection by detachment, is computable, then the system S is decidable in the usual meaning.*[3]

It is easy to show that

Corollary *The system* **AS** *of Aristotle's syllogistic with the bases* (BSŁ) *is decidable.*

In the next section we will present Ł-decidable systems that are also decidable.

5 More Important Findings Concerning Ł-Decidability of Deductive Systems

The axiomatic rejection method introduced by Lukasiewicz to complete, biaspectual characterization of deductive systems (which was effectively used and developed by Słupecki) provided a broad response in literature after the Second World War. Łukasiewicz, already living in Dublin, uses such method in his research on intuitionistic logic [24], as well

[3] Słupecki's theorem is an immediate consequence of the following theorem of the theory recursion, which we quote from Grzegorczyk's book [14, p. 355 in the Eng. ed.]: *If the union of two recursively enumerable disjoint sets T and S is computable set, then the sets T and S are also computable.*

as in a four-valued modal system of propositional calculus [25], built by himself. At the same time, in Poland, further studies, inspired by Słupecki on Ł-decidability of deductive systems and the very notion of rejected proposition were taken up. In studies on Ł-decidability and providing complete refutation systems for logical systems (assertion systems) one of three methods, modeled on those described in the previous section, is usually used. In the next sections we will present a few, more important results, connected with this research.[4]

5.1 Calculi of Names

J. Słupecki's research on Aristotle's syllogistic was continued mainly by B. Iwanuś. Using the Słupecki-Łukasiewicz's methods, Iwanuś managed to prove Ł-decidability of a few systems of calculi of names.

- Iwanuś [17] gave a proof of Ł-decidability of the whole traditional calculus of names, i.e. the system of Aristotle's syllogistic enriched by nominal negation.
- Another interesting, though much later obtained result of Iwanuś's research [18] is a proof of Ł-decidability of the system of Aristotle's syllogistic built by Słupecki [38]. In this system, the two initial Łukasiewicz's axioms (laws of identity) of the system **AS** (those that are absent in Aristotle's logic) are replaced with the following axioms:

$$S\ a\ P \Rightarrow S\ i\ P, \qquad S\ i\ P \Rightarrow P\ i\ S.$$

  In Słupecki's system, unlike in the system **AS**, it is permissible for variables to represent empty names. A complete refutation system for the system given by Iwanuś [18] is based on three rejected axioms and one rejected rule that is germane to the traditional calculus of names.
- Iwanuś also, in [16], gave a proof of Ł-decidability of a certain version of *elementary ontology*, distinguished from the system of *Leśniewski's Ontology* (both terms were invented by Słupecki in [41], who presented a system of calculus of names, based on Leśniewski's findings). In elementary ontology it is possible to interpret both the asserted system **AS** and other asserted syllogistic systems richer than **AS**, with nominal negation. Iwanuś gave a complete refutation system for the version of the system of elementary ontology; Iwanuś's refutation system consist of two independent rejected axioms and one non-Łukasiewicz's specific rejection rule.

[4]G. Bryll (as well as his book [2]) and T. Skura have been very helpful in verifying certain significant facts.

5.2 *Propositional Logic*

As we have already mentioned, Łukasiewicz used to apply the axiomatic method of rejection also to some systems of propositional calculus.

5.2.1 Classical Logic

Łukasiewicz mentioned in [25], that the classical propositional calculus with the inference rules $r^{+}1$ and $r^{+}2$ is Ł-decidable. A complete refutation system for it determines one rejected axiom, namely the sentential variable p, and two Łukasiewicz's rules: $r^{-}1$ and $r^{-}2$.

Let us note that the method of rejection of false formulas (i.e. non-theses) used by Łukasiewicz can be replaced by Słupecki's method, omitting Łukasiewicz's rules, i.e. applying the following principle: *If, from a formula of propositional calculus and the set of theses, it is possible to deduce, according to the rules* $r^{+}1$ *and* $r^{+}2$*, the rejected axiom* p*, then the formula is rejected* (see D(Sł)).

For the classical first-order calculus rejected axioms and rejected rules were formulated by T. Skura in [32]. Skura made use of the 'tableaux' method.

The rejection procedure accompanying the syntactic characterization of systems of propositional logic became a standard among logicians.

We will limit our presentation of results in the scope of the rejection procedure in the systems to a brief mention of only intuitionistic logic and extensions, modal logic and Łukasiewicz's logics.

5.2.2 Intuitionistic Logic and Extensions

In his research on intuitionistic propositional calculus, Łukasiewicz [24] advanced the hypothesis that it is Ł-decidable. Moreover, he supposed that a sole rejected axiom of it is a propositional variable, and that the rejection rules are: $r^{-}1$, $r^{-}2$, and one special rule (Gödel rule) which states that the disjunction $\alpha \vee \beta$ is rejected whenever so are α and β. Thanks to Kreisel-Putnam [19], we know that the rules proposed by Łukasiewicz do not suffice to reject all non-theses of the intuitionistic calculus. It is also known that there is no finite set of rejected axioms that, together with Łukasiewicz's rules, gives a complete refutation of intuitionistic system (cf. Maduch [28]).

Ł-decidability of the intuitionistic propositional logic was achieved by Scott [29] using a countable number of non-structural rejection rules. However, Scott's results, which were presented in *Summaries of Talks* at Cornell University, were inaccessible to Słupecki's circle in behind the Iron Curtain (then) Poland.

Independently of Scott's results, the proof of Ł-decidability of the intuitionistic propositional calculus was provided by Dutkiewicz [11]. In his approach, a complete refutation intuitionistic system is compounded of one axiom and three rejection rules: Łukasiewicz's rules, and a new, original rejection rule, which is, in fact, an infinite countable class of rejection rules of a common scheme. In his proof, Dutkiewicz uses the method of rejection modeled on the one applied by Łukasiewicz, as well as the method of Beth's semantic tableaux.

Another proof of Ł-decidability for the intuitionistic calculus was given by Skura [30, 36], who, while defining a complete intuitionistic refutation system, added to Łukasiewicz's rules a new rule, or rather, a class of structural rules of rejection, the number of which is infinite. Skura in [31], provided a complete refutation system for certain intermediate logics.

5.2.3 Modal Logic

The first research into a complete syntactic characterization of a modal propositional calculus was undertaken by Łukasiewicz [25]: he extends the four-valued modal system, built by himself, by two rejected axioms and his rejection rules, obtaining Ł-decidability of this system.

Afterwards, Słupecki initiated research on Ł-decidability of Lewis system S5. In his and Bryll's paper [44], the proof of Ł-decidability was achieved with an assumption of one rejected axiom (the prepositional variable p) and, apart from Łukasiewicz's rules, a class of rejection rules of the common scheme.

Skura [31, 32, 34] gave a simpler proof of Ł-decidability of Lewis system S5, assuming that the language of this system was supplemented with a symbol '$\bot$', i.e. *the constant of falsity*. Skura adopts this constant as the rejected axiom and extends the systems of Łukasiewicz's rules by: 1) a rule stating that: if formula $\Box p$ is rejected, the formula p is rejected, too; and 2) a class of structural rejection rules of the same scheme.

Skura in [31, 35, 36], using the algebraic method, also provided a complete refutation system for the logic S4 and for some of its extension (Grzegorczyk's logic).

A little earlier, Goranko [12, 13] formulated a complete refutation system for some normal modal propositional logics (including S4 and Grzegorczyk's logic) that are characterized by a class of finite trees. His refutation systems for these logics are based on the same rejected axiom (the constant '$\bot$'), Łukasiewicz's rules, and a class of non-structural rejected rules of the same scheme.

The method of constructing refutation systems corresponding to classes of finite models was used by Skura [31] for intermediate logics and, by Skura [33] and Goranko [13], for certain normal modal logics. Skura in [33], showed that the refutation systems can be useful in such cases when a given system of logic cannot be characterized by any class of finite models: there is a decidable modal logic without a finite model property that has a simple refutation system.

Tomasz Skura is regarded as an expert on the methods in refutation systems; his book [37] is devoted to refutation methods in modal propositional logics.

5.2.4 Łukasiewicz's Many-Valued Logics

Researches into Ł-decidability of Łukasiewicz's sentential calculus were conducted in the Opole circle of logical research, which, for many years, was led by Jerzy Słupecki. Bryll and Maduch [3], formulated a uniform method of rejection of formulas in an $n+1$-valued implicational, implicative-negative and definitionally complete Łukasiewicz's calculus. In these systems the same formula may be adopted as the sole rejected axiom[5]

$$C(Cp)^n q (Cp)^{n-1} q$$

(for $n = 1$ we get, in particular, the rejected axiom: $CCpqq$ in the classical implicational calculus). The only rejection rules are, here, Łukasiewicz's rules.

A complete refutation system for $\aleph_0$-valued Łukasiewicz's calculus was built by Skura [32] by extension of Łukasiewicz's rejection rules. Research into Ł-decidability of this system has been also conducted by Bryll [2]. His research was continued by R. Sochacki.

Sochacki [50] gave complete refutation systems for all invariant Łukasiewicz's many-valued logics (in which the rule of rejection by substitution was eliminated; see also Sochacki [48], Bryll and Sochacki [4, 5].

Sochacki also built refutation systems for selected many-valued logics: the k-valued logic of Sobociński and some systems of nonsense logic [49, 51].

5.3 The Generalized Method of Natural Deduction

The method of rejection introduced to metalogical investigations by Łukasiewicz in many cases can be replaced with the generalized method of natural deduction. The latter is applicable to all propositional calculi which have finite adequate matrices, as well as to the intuitionistic propositional calculus and the first-order predicate calculus, that is, to almost all logics discussed in this section.

The basis for such systems consists then of only assertion rules and rejection rules; the sets asserted and the rejected axioms are empty sets. The method used in the proofs is similar to Słupecki-Łukasiewicz's method of rejection, though they are apagogic proofs (by *reductio ad absurdum*).

This method refers to the 'tableaux' method. It is presented by Bryll [2], who in his studies refers to results obtained by Hintikka [15], Smullyan [47], Suchoń [54], Surma [55, 56] and Carnielli [8, 9].

6 Rejection Operation

As was noticed by Słupecki, the notion of rejected proposition is so general that it is most convenient to base studies concerning this notion on Tarski's theory of deductive systems, i.e. the axiomatic theory of consequence built by Alfred Tarski [57]. Let us recall that the only primitive notions of this theory are:

[5]It is saved in Łukasiewicz's, so called *Polish notation*.

- the set F of all propositions (well-formed formulas) of an arbitrary but fixed language of a given system and the consequence operation:

$$Cn^{+}: 2^{F} \to 2^{F};$$

- the symbol '$Cn^{+}(X)$' denotes the set of all consequences of the set $X \subseteq F$.

Properties of deductive systems characterized by the bases with refutation systems can be established by means of Tarski's theory of consequence enriched by the following definition of rejection operation:

$$Cn^{-1}: 2^{F} \to 2^{F},$$

determined by the consequence operation Cn^{+} (see D(Sł)): For any $X \subseteq F, \alpha \in F$

$$\alpha \in Cn^{-1}(X) \Leftrightarrow \exists \beta \in X(\beta \in Cn^{+}(\{\alpha\})) \qquad \text{MD(Sł)}$$

According to the definition MD(Sł): a proposition α is a *rejected proposition* on the ground of the set X if and only if at least one of the propositions of X is derivable from (is a consequence of) α; the symbol '$Cn^{-1}(X)$' denotes the set of all propositions rejected on the ground of propositions of X.

The following theorem helps to understand the intuitive sense of the definition MD(Sł):

$$\forall X \subseteq F(X \subseteq Y \Rightarrow Cn^{+}(X) \subseteq Y) \Rightarrow \forall X \subseteq F(X \subseteq Y' \Rightarrow Cn^{-1}(X) \subseteq Y').$$

If Y is the set of all true propositions of F (then the set Y' is the set of all false propositions of F), then, according to the above theorem, we can state that:

If consequence operation always leads from true propositions to true propositions, then a rejected operation always leads from false propositions to false propositions.

So, the defined rejection operation (function) Cn^{-1} is a generalization of the notion of rejection introduced by Łukasiewicz.

The definition MD(Sł) of the function Cn^{-1} was formulated by Słupecki [42]. Słupecki proved that the function satisfies all axioms of Tarski's general theory of deductive systems [57] for the consequence Cn^{+} and that it is additive. Thus, the rejected operation is another consequence operation and is called a *rejection consequence*. It is clear that

Every deductive system with a bi-level formalization (with the assertion system and the refutation system) can be characterized by the basis:

$$\langle F, Cn^{+}, Cn^{-1} \rangle$$

and that the extension of Tarski's theory of the definition MD(Sł) describes every such system.

This theory has been developed in the form of the *theory of rejected propositions* by Wybraniec-Skardowska [62], and later also by Bryll [1]. Their researches have been a continuation of investigations initiated by J. Słupecki and have been conducted under his supervision to be later presented in co-authored papers (see Słupecki et al. [45, 46]).

The theory of rejected propositions contains many significant theorems that are not counterparts of any theorem of Tarski's theory [58]. The following are examples of such theorems about the rejection operation Cn^{-1}:

- $Cn^{-1}(\emptyset) = \emptyset$ – *it is normal*,
- $Cn^{-1}(\bigcup\{X \subseteq G : G \subseteq F\}) = \bigcup\{Cn^{-1}(X) : X \subseteq G \subseteq F\}$ – *it is complete additive*,
- $Cn^{-1}(X) = \bigcup\{Cn^{-1}(\{\alpha\}) : \alpha \in X\}$ – *it is unit operation*,
- $\alpha \in Cn^{-1}(X) \Leftrightarrow \exists\beta \in X(Cn^{-1}(\{\alpha\} \subseteq Cn^{-1}(\{\beta\}))$ – *it is a unit consequence*.

In the theory the following formulation of the rule of rejection by detachment is valid:

- $c\,\alpha\beta \in Cn^{+}(X) \ \wedge \ \beta \in Cn^{-1}(Y \cup X) \Rightarrow \alpha \in Cn^{-1}(Y \cup X)$,

while in the Tarski's theory the formulation of rule of detachment has the form:

- $c\,\alpha\beta \in Cn^{+}(Y) \ \wedge \ \alpha \in Cn^{+}(Y \cup X) \Rightarrow \beta \in Cn^{+}(Y \cup X)$.

The set $Y \subseteq F$ can be understood as a set of axioms A^{+} or the set of theses T^{+} of a given deductive system.

7 Rejection Operation as a Primitive Notion

There is a possibility of the axiomatization of a theory of rejected propositions in a dual and equivalent way, i.e. assuming that a primitive notion is the rejection consequence Cn^{-1}, while Cn^{+} is defined operation.

Such dual theory of rejected propositions was formulated by Wybraniec-Skardowska [62] (see also [66, 67]). It can be understood as the theory describing the deductive systems with the basis

$$\langle F, Cn^{-1}, Cn^{+}\rangle.$$

A theory of rejected propositions was developed also to formalize some problems of methodology of empirical sciences, mainly by Bryll [1] (see also Słupecki et al. [46]).

The theories of rejected propositions have a natural interpretation, which was given by Staszek [53]: the set $Cn^{-1}(X)$ can be understood as a set of all rejection proofs on the ground of a proposition of the set X, in the sense relating to the nature of rejection proofs used by Słupecki in his researches on Aristotle's syllogistic.

In the theory of rejected propositions can be defined the notion of Ł-decidability.

7.1 Rejection Operation as a Finitistic Consequence: Dual Consequences

Rejection function Cn^{-1} can be generalized into a **dual, finitistic consequence** in the usual meaning. The notion of the dual consequence dCn^{+} relating to the consequence

Cn^+ was introduced by Wójcicki [61] by definition:

$$\alpha \in \mathrm{d}Cn^+(X) \Leftrightarrow \exists Y \subseteq X \wedge \mathrm{card}(Y) < \aleph_0(\bigcap\{Cn^+(\{\beta\})\colon \beta \in Y\} \subseteq Cn^+(\{\alpha\})).$$

The dual consequence $\mathrm{d}Cn^+$ is stronger than the rejected consequence Cn^{-1} (i.e. $Cn^{-1} \leq \mathrm{d}Cn^+$), though the former is linked with the latter by a number of interesting relationships (see Spasowski [52]).
Facts: a. $\mathrm{d}Cn^{+1} = Cn^{-1}$ and b. $\mathrm{d}Cn^{-1} = Cn^{+1}$,
where the unit consequence Cn^{+1} is defined as follows:

$$\alpha \in Cn^{+1}(X) \Leftrightarrow \exists \beta \in X(\alpha \in Cn^+(\{\beta\}).$$

Certain studies of generalization of the notion of rejected expression in the form of a function of a consequence of rejection or a dual consequence are discussed by Wybraniec-Skardowska and Waldmajer [68].

The dual consequences Cn^+ and $\mathrm{d}Cn^+$ as well as Cn^{+1} and Cn^{-1} can be used to study both true (asserted), and respectively, false (rejected) contents of a given theory. The fact was noticed by Woleński [60] in his studies relating to Popper's conception of a comparison of scientific theories by their contents.

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