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Choice functions and hard choices☆

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ABSTRACT

A hard choice is a situation in which an agent is unable to make a justifiable choice from a given menu of alternatives. Our objective is to present a systematic treatment of the axiomatic structure of such situations. To do so, we draw on and contribute to the study of choice functions that can be indecisive, i.e., that may fail to select a non-empty set for some menus. In this more general framework, we present new characterizations of two well-known choice rules, the maximally dominant choice rule and the top-cycle choice rule. Together with existing results, this yields an understanding of the circumstances in which hard choices arise.

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1. Introduction

We are concerned with the phenomena of *hard choices*, that is, situations in which an agent is unable to make a justified choice from a given menu. As Ruth Chang – one of the most prominent contemporary scholars of hard choices – writes, "In the most general terms, hard choices are ones in which reasons 'run out': they fail, in some sense, to determine what you should do" (Chang (2012); p. 107). Our objective here is to present a systematic analysis of the axiomatic structure of hard choices with the help of the theory of rational choice.

Conventionally, a choice function f specifies, for any choice situation or menu A, a choice set $f(A)$ which is a *non-empty* subset of A. The non-emptiness of the choice set – or *decisiveness* of the choice function, using Richter's (1971) terminology – is standardly built into the definition of a choice function. We follow Gerasimou (2016, 2018) and others and abandon the assumption that a choice set will always be non-empty.[1] In behavioural terms, Gerasimou interpretes the possibility of an empty choice set as a refusal or deferral on the side of a rational decision maker. In a later moment, some mechanism may lead to the selection of an option after all, but it is either not the agent herself who makes that selection or, if she does, she does not do it in the full agentive sense.[2]

Note that there have also been proposals to model the absence of justifiable options with the help of decisive functions. Eliaz and Ok (2006), for instance, provide a characterization of a decisive choice function whose range coincides with the maximal set.[3] We do not follow this route. Instead, we examine choice functions in general (decisive and indecisive ones) and define a hard choice as a situation to which a given choice function assigns the empty set.

There are several reasons for the possible existence of hard choices.[4] We focus on two in particular. The first refers to cases in which any choice is unacceptable. In the philosophical literature such a case is often illustrated with the story of Sophie who, in

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[1] Earlier studies of indecisive functions include Smith (1979), Hurwicz (1986), Aizerman and Aleskerov (1995), Clark (1995) and Gaertner and Xu (2004). For a textbook treatment that motivates the concept, see Kreps (1990, p. 28–29), and chapter 2 of Aleskerov et al. (2007).

[2] Viewing a choice function as the first step of a two-step procedure is a well-known view in the theory of rational choice. Richter (1971, p. 31), for instance, uses it in his explanation of choice sets that contain multiple elements. Such sets are, in the standard account, viewed as revealing indifference of the agent between her most preferred outcomes. Richter points out that the decision process in these cases can be seen as involving two stages. The agent chooses first and then some random device is used to select a single element from the agent's choice set.

[3] See also, Schwartz (1976), Moulin (1985) and Bandyopadhyay and Sengupta (1993).

[4] Gerasimou (2018) is the relevant source in the literature in economics. He also discusses cases in which an agent cannot make a decision because of the large number of available options. Our analysis does not cover situations of choice overload such as this. More generally, we ignore hard choices that follow from the epistemic limits faced by a deliberating agent (see e.g. Sepielli, 2009, 2014).

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Styron's novel, was forced to choose between her two children. Yet hard choices of this first type need not always be of such a radical or horrendous nature. They can occur in any situation in which an agent has to make a choice between unacceptable or unattractive alternatives. A second possibility is that the alternatives pose incommensurable values against one another (Raz, 1986, 1997). The choice between two very successful, yet very different, career paths may serve as an example of such a hard choice (see Raz, 1986, p. 332). If we cannot compare the underlying values that are involved, we are not able to make a justifiable choice in favour of either option.

Our analysis of indecisive choice functions is motivated by these two types of hard choices.[5] Of course, analysing hard choices in terms of indecisive choice functions does not mean that the classic results of rational choice theory are irrelevant for our understanding of hard choices. On the contrary, they form an important starting point since they show us when hard choices do *not* emerge. If a decisive choice function induces ('reveals') a weak ordering over the set of alternatives, then an optimal choice is defined for every menu of alternatives. Assuming that the availability of an optimal choice entails that a rational agent can make a justified choice, the standard theory teaches us that a rational agent can only face a hard choice if optimality is not ensured.

A failure of optimality can occur if the preference relation violates completeness, that is, when an agent cannot compare all alternatives. Of course, it can also result from a violation of transitivity. In what follows, we shall examine conditions under which the violation of transitivity as such does *not* constitute a hard choice. Thus, given the conditions that we stipulate, a hard choice is always caused by the incompleteness of one's preferences. To see why, it is worth mentioning that we follow Schwartz (1972) and Deb (1977) and use the transitive closure of the base relation as the basis of choice. The idea is that the turn to the transitive closure can be seen as a step in the process of deliberation that a rational agent can make when confronted with a seemingly difficult choice.[6] If, after such deliberation, the agent still faces a deadlock, then it 'really' is a hard choice.[7]

The significance of our analysis is twofold. First, much of the non-formal discussion of hard choices has been concerned with choices restricted to the class of binary situations. A hard choice is then defined as a choice between a pair x and y of alternatives that we are unable to rank *vis-a-vis* each other. But there are many non-binary situations wherein even if we cannot rank two given alternatives, there may be a third alternative that is clearly superior and therefore optimal. We thus cannot straightforwardly apply the account of a hard choice in the binary case to the general setting. Our results, however, show how it can be so generalized. Second, insofar as hard choices have been analysed in non-binary situations, the focus has often been with the question of *how, if at all, an agent should rationally respond to, or deal with, a hard choice.*[8] In contrast, we are concerned here with providing a formal description of a hard choice under different rationality conditions. In doing so, we make a contribution to the existing literature on choice functions.

The structure of the paper is as follows. After setting up the formal framework in Section 2, we study – in Section 3 – the nature of hard choices in situations where the preference relation induced by a choice function (the so-called base relation) is transitive. In Section 4, we turn to studying situations in which the base relation may fail to be transitive.

2. The framework

Let X be a finite set of all alternatives that has at least three elements and $\mathcal{P}$ the power set of X, i.e., the set of all subsets of X. A choice function f is a function that assigns to each non-empty $A \in \mathcal{P}$ an element of $\mathcal{P}$, where $f(A)$ is non-empty for all singleton sets.[9] As stated above, a decisive choice function is one that always yields a non-empty choice set, whereas an indecisive choice function will yield empty choice sets for some menus.

In the setting of choice functions, there is a natural definition of a *hard choice*.

Definition 1 (*Hard Choice*)**.** Given a choice function f, an $A \in \mathcal{P}$ is a hard choice if, and only if, $f(A) = \emptyset$.

Let R (with P and I as its asymmetric and symmetric parts, respectively) denote the base relation induced by f: for all $x, y \in X$, we have xRy if, and only if, $x \in f(\{x, y\})$. Let $\bar{R}(A)$ be the transitive closure of R in A, i.e., $x\bar{R}(A)y$ if there is a subset $\{x = x_1, \ldots, x_k = y\}$ of A such that $x_1Rx_2 \ldots Rx_k$. Next, for any set A and any binary relation R', $B(A, R')$ denotes the set of optimal elements of A defined with respect to R'. That is, an element x is optimal in A with respect to R' if, and only if, for all y in A, $xR'y$. $M(A, R')$ denotes the set of maximal elements of A defined with respect to R'. That is, an element x is a maximal element of A defined with respect to R' if, and only if, there is no y in A such that $yP'x$.

Next, we introduce three classic consistency conditions attributable to Amartya Sen, to wit: the properties α, β, and γ (Sen, 1971).

Axiom 1 (α)**.** *For all $A, B \in \mathcal{P}$ and all $x \in A$, if $x \in f(A \cup B)$, then $x \in f(A)$.*

Property α requires that if some element in A is selected in the union of A and B, it is also selected in A.

Axiom 2 (β)**.** *For all $A, B \in \mathcal{P}$ and all $x, y \in f(A)$: $x \in f(A \cup B)$ if, and only if, $y \in f(A \cup B)$.*

Property β stipulates that if x and y are both selected in A, then one of them cannot be selected in the union of A and B, without the other also being selected in the union of A and B.

Axiom 3 (γ)**.** *For all $A, B \in \mathcal{P}$: $f(A) \cap f(B) \subseteq f(A \cup B)$.*

[5] A related argument can be found in Levi (1986, p. 84). He criticizes the assumption of revealed preference theory, i.e. that the preferences of rational agents are complete (which revealed preference theory presupposes if an agent can indeed always make a choice), and proceeds to claim that this assumption is just as arbitrary as – if not more than – the assumption that the preferences of rational agents are incomplete.

[6] Alternatively, an agent may make this deliberation in order to avoid potential hazards of choice, like being money-pumped. We refer to Mandler (2005) for an outcome-rational motivation of the transitivity of psychological preferences.

[7] It could be objected that we then could have assumed transitivity as a feature of agent's rationality all along. In response to this we concede that the agent is *more* rational when they use the transitive closure, but we dispute the claim that a violation of transitivity as such is irrational.

[8] Arguments include satisficing, or making a satisfactory rather than an optimal selection (Schmidtz, 1992), maximization, or selecting an alternative that is not strictly worse than any other alternative that could have been chosen instead (Sen, 1997, 2004), and V-admissibility, or selecting an option that is optimal according to at least one of the relevant considerations at hand (Levi, 1986).

[9] Gerasimou (2018) lists the condition that singleton sets always have a non-empty choice set as a separate axiom, which he calls 'desirability', and he also examines cases in which the condition is not met. We do not follow him in this regard.

If an element is selected in A as well as in B, then, with γ, the element is also in the selection from the union of both A and B. We shall also use several variants of γ. They are, in increasing logical strength:

Axiom 4 ($\gamma^{=}$)**.** *For all $A, B \in \mathcal{P}$: if $f(A) = f(B) \neq \emptyset$, then $f(A \cup B) \neq \emptyset$.*

Axiom 5 (γ^{-})**.** *For all $A, B \in \mathcal{P}$: if $f(A) \cap f(B) \neq \emptyset$, then $f(A \cup B) \neq \emptyset$.*

Axiom 6 (γ^{+})**.** *For all $A, B \in \mathcal{P}$, if $x \in f(A)$ and $y \in A \cap f(B)$, then $x \in f(A \cup B)$.*

The two weaker versions of γ are, as far as we know, new. Property γ^{+} is due to Salant and Rubinstein (2008).

A classic result of rational choice theory is that a transitive relation is the base relation generated by a choice function that satisfies axioms α and β (see Sen, 1971). We maintain that this result does not extend to indecisive choice functions.

Remark 1. Let an indecisive choice function f satisfy axioms α and β. It does not follow, then, that R is transitive

Remark 1 can be easily demonstrated. Consider the indecisive choice function $f(x, y) = \{x\}$, $f(x, z) = \emptyset$, $f(y, z) = \{y\}$ and $f(x, y, z) = \emptyset$. This indecisive choice function satisfies both α, and β, and yet R is not transitive. The first question for indecisive choice functions, then, is this: *what condition(s), in conjunction with α and β, ensure that the underlying base relation is transitive?*

The following axiom, called property ξ, is new to the literature and relevant to answer this question. This axiom expresses a condition which guarantees that a choice can always be made.

Axiom 7 (ξ)**.** *For all $A, B, C \in \mathcal{P}$: if $f(A)$ and $f(B)$ are non-empty and if $f(A) \subseteq f(A \cup B)$ and $f(B) \subseteq f(B \cup C)$, then $f(A \cup B \cup C) \neq \emptyset$.*

Using our definition of hard choices, property ξ demands that for all choice situations A, B, C, if situation A and situation B do not constitute hard choices, and if the situation involving the union of A and B does not constitute a hard choice (because the element(s) selected from A is also an element(s) that is selected in the union of A and B), and further, if the situation involving the union of B and C does not constitute a hard choice (because the element(s) selected from B is also an element(s) that is selected in the union of B and C), then the situation involving the union of A, B and C does not constitute a hard choice.

We can now answer the question that has just been raised.

Result 1. *Let f be a choice function satisfying axioms α, β and ξ. R, then, is a pre-ordering, i.e. a reflexive and transitive relation on X.*

Proof. Reflexivity follows directly by definition of R and by the fact that $f(\{x\}) \neq \emptyset$ for all $x \in X$. To show transitivity, assume xRy and yRz, that is, $x \in f(\{x, y\})$ and $y \in f(\{y, z\})$. By axiom ξ, $f(\{x, y, z\}) \neq \emptyset$. If $z \in f(\{x, y, z\})$, then by axiom α, $z \in f(\{y, z\})$, and by axiom β, $y \in f(\{x, y, z\})$. Similarly, when $y \in f(\{x, y, z\})$, by axioms α and β, $x \in f(\{x, y, z\})$. Hence, in all scenario's $x \in f(\{x, y, z\})$ if $f(\{x, y, z\})$ is non-empty. Then, by axiom α, $x \in f(\{x, z\})$, hence xRz. ■

We conclude this section by noting a well-known result about pre-orderings.

Remark 2 (*Sen, 1970*)**.** A pre-ordering on X is sufficient to establish that a maximal alternative exists for all $A \in \mathcal{P}$.

3. Characterizations with transitivity

We now present our first characterization result. Assuming the transitivity of R, it establishes under what conditions a choice function always selects the set of optimal elements. We refer to these choice functions as *maximally dominant choice rules*, following Gerasimou (2018), who provides the following characterization for them with the use of α and γ^{+}.

Gerasimou (2018, Theorem 1) *A choice function f satisfies axioms α and γ^{+} if, and only if, R is transitive and $f(A) = B(A, R)$ for all $A \in \mathcal{P}$.*

Our result also uses α and brings out the different intuitions which underlie γ^{+}.[10]

Proposition 1. *A choice function f satisfies axioms α, β, ξ and $\gamma^{=}$ if, and only if, R is transitive and $f(A) = B(A, R)$ for all $A \in \mathcal{P}$.*

Proof. $\Rightarrow$ Let f satisfy the four aforementioned axioms. If A contains one element, the proposition is true by definition of f and the reflexivity of R. If $B(A, R) = \emptyset$, we must have $f(A) = \emptyset$: if there were some $x \in f(A)$, we would have $x \in B(A, R)$ by axiom α. Let A therefore contain at least two elements and let $B(A, R)$ be non-empty.

We first prove that $f(B) = B$ for all non-empty subsets B of $B(A, R)$. If B is a singleton set, the result follows trivially by definition of a choice function. For the case in which the cardinality k of B is 2 or larger, we proceed by induction. If $k = 2$, this is directly implied by the definition of an optimal element and by R. Let it be true for any subset of $B(A, R)$ with cardinality $k \geq 2$ and consider a subset B of $B(A, R)$ with $k+1$ elements, say x, y, z are distinct elements of it. The induction hypothesis implies that $f(B - \{z\}) = B - \{z\}$ and thus $f(B - \{z\}) \neq \emptyset$, in particular $x \in f(B - \{z\})$. By α, $x \in f(B - \{y, z\})$. Since $x \in f(B - \{z\})$, any other element of $f(B - \{y, z\})$ is also an element of $f(B - \{z\})$ by β. Hence, $f(B - \{y, z\}) \subseteq f(B - \{z\})$. Since $y = f(\{y\}) \subseteq f(\{y, z\})$, axiom ξ implies $f(B - \{y, z\} \cup \{y\} \cup \{z\}) = f(B) \neq \emptyset$. With axioms α and β we subsequently get $f(B) = B$.

In the second part of the proof we add, one by one, the elements of $A \backslash B(A, R)$ to the elements of $B(A, R)$ and show that in each step the choice set remains $B(A, R)$. Take any $v \in A \backslash B(A, R)$. We first demonstrate that, for any $x \in B(A, R)$, $v \notin f(\{x, v\})$ must hold. For suppose on the contrary that $v \in f(\{x, v\})$ for some $x \in B(A, R)$. Take any $z \in A$. From $v \in f(\{x, v\})$ and $x \in f(\{x, z\})$, it follows by axiom ξ that $f(\{x, v, z\}) \neq \emptyset$. By axiom β, x must be an element of $f(\{x, v, z\})$, but then, also by β, so must v. We then have, by axiom α, $v \in f(\{v, z\})$. Since this is true for any $z \in A$, we have $v \in B(A, R)$, which is a contradiction. If x is the only element in $B(A, R)$, we have $f(\{x, v\}) = f(B(A, R) \cup \{v\}) = \{x\} = B(A, R)$. Assume $B(A, R)$ contains some $y \neq x$. Step 1 demonstrated that $f(B(A, R) - \{y\}) = B(A, R) - \{y\}$ and $f(B(A, R)) = B(A, R)$. Thus, $f(B(A, R) - \{y\}) \subseteq f(B(A, R))$. Since we also have $f(\{y\}) \subseteq f(\{y, v\})$, we have, by axiom ξ, $f((B(A, R) - \{y\}) \cup \{y\} \cup \{v\}) = f(B(A, R) \cup \{v\}) \neq \emptyset$. Then, by axioms α and β, $f(B(A, R) \cup \{v\}) = B(A, R)$.

Now take a $w \in A \backslash (B(A, R) \cup \{v\})$, if any (otherwise we are done). Analogously, we have $f(B(A, R) \cup \{w\}) = B(A, R)$. Since $f(B(A, R) \cup \{v\}) = f(B(A, R) \cup \{w\})$, by axiom $\gamma^{=}$, we have $f(B(A, R) \cup \{v\} \cup \{w\}) \neq \emptyset$. Application of α and β subsequently shows that $f(B(A, R) \cup \{v, w\}) = B(A, R)$. Continuing this way, we eventually arrive at $f(A) = B(A, R)$.

$\Leftarrow$ Axioms α and $\gamma^{=}$ follow directly from $f(A) = B(A, R)$. Consider axiom β. Let $x, y \in f(A)$ for some A and $x \in f(A \cup B)$ for some B. We then have yRx (because $x, y \in f(A)$) and xRz for

[10] In the Appendix we demonstrate that the axioms used in Proposition 1 are independent of each other.

all $z \in A \cup B$ by definition of an optimal element. Hence, by transitivity, yRz for all $z \in A \cup B$, which needed to be shown. Now consider axiom ξ. Let $f(A) \neq \emptyset \neq f(B)$ and assume $f(A) \subseteq f(A \cup B)$ and $f(B) \subseteq f(B \cup C)$. Let $x \in f(A)$. By definition of an optimal element we have xRy for all $y \in B$, and yRz for all $y \in f(B)$ and all $z \in C$. Hence, by transitivity, xRv for all $v \in A \cup B \cup C$, meaning that $x \in f(A \cup B \cup C)$. ■

From Remark 2 and Proposition 1, we state the following corollary which provides a first characterization of hard choices, namely when the binary relation is transitive. In particular, the corollary equates a hard choice with the absence of an optimal element because the binary relation is incomplete.

Corollary 1. *Let f be a choice function that satisfies α, β, ξ, and $\gamma^{=}$. A choice situation A constitutes a hard choice if, and only if, for some $x, y \in M(A, R)$ we have $f(\{x, y\}) = \emptyset$.*

4. Characterizations without transitivity

Thus far, our line of reasoning has presented a characterization of hard choices in light of a transitive binary relation. It is of course well-known that the restriction to transitive binary relations need not be upheld. Therefore we shall now be concerned with analysing choice functions when this restriction is lifted. We start with the result due to Aizerman and Aleskerov (1995) that extends a well-known result of Sen (1971, p. 314) on decisive choice functions to the setting of all choice functions.

Aizerman and Aleskerov (1995, Theorem 2.7): *A choice function f satisfies axioms α and γ if, and only if, for all $A \in \mathcal{P} : f(A) = B(A, R)$.*

Corollary 2. *Let f be a choice function that satisfies α and γ. A choice situation A constitutes a hard choice if, and only if, $M(A, R) = \emptyset$ or for all $x \in M(A, R)$, there is some $y \in A$ such that $f(\{x, y\}) = \emptyset$.*

The corollary presents a second characterization of hard choices. While it still equates a hard choice with the absence of an optimal element, it drops the transitivity restriction. In particular, it formalizes the intuition that a hard choice involves either a violation of acyclicity, implying that the set of maximal elements is empty, or that the underlying base relation cannot be used to compare a maximal element with all other elements in the menu.[11]

However, one may contest the view that a violation of acyclicity and the consequent lack of a maximal element constitutes a hard choice. The following example, originally presented in Deb (1977), helps illustrate this point.

Example 1. Consider the choice situation $A = \{x, y, z, w\}$ with the following rankings of the relevant pairs in A: xPy, yPz, zPx, xPw, yPw, and zPw.

While the set of optimal elements defined with respect to R is empty in the present example, some have argued that a deliberating agent can make a rationally justified selection here, because the set of optimal elements defined with respect to the transitive closure $\bar{R}(A)$ of R in A is non empty (see, e.g. Schwartz, 1972; Deb, 1977). According to this view, Example 1 does not constitute a hard choice because $B(A, \bar{R}(A)) = \{x, y, z\}$ and one can justifiably select each element of this set. Indeed, *per* this view, a given situation constitutes a hard choice if, and only if, $B(A, \bar{R}(A)) = \emptyset$.

In the context of decisive choice functions, the functions whose range coincides with $B(A, \bar{R}(A))$, so-called *top-cycle choice rules*, have been characterized, among others, by Bordes (1976) and Evren et al. (2019). In what follows, we shall characterize top-cycle choice rules in the extended framework of general choice functions.[12] The following pair of axioms will be relevant.

Axiom 8 (α^{-})**.** *For all $A \in \mathcal{P}$ and all $x, y \in A$, if $x = f(\{x, y\})$, then $y \notin f(A)$ if $x \notin f(A)$.*

Property α^{-}, which is a weakening of α, requires that if x is selected from the binary pair (x, y), then in any situation A where both x and y are present, y is not selected in A if x is not selected in A as well.

Axiom 9 (ρ)**.** *For all $A, B \in \mathcal{P}$, if there are no $x \in A$ and $y \in B$ with $y \in f(\{x, y\})$, then $f(A \cup B) \subseteq A$.*

For decisive choice functions, Property ρ equals the condition of Preference Consistency that was used by Evren et al. (2019) in their characterization of top-cycle choice rules. Its intuition can be explained as follows. We may say that a set A 'is immune from' a set B if an element from B is never chosen in a paired comparison with any element from A. Property ρ then states that the set of chosen elements when A is united with a set B that it is immune from, can only consist of elements that were in A.

Next, we introduce three Lemmas.

Lemma 1. *If a choice function f satisfies α^{-}, β and ξ, then for any R-chain $x_1 R \dots R x_m$ we have $B(\{x_1, \dots, x_m\}, \bar{R}(\{x_1, \dots, x_m\})) \subseteq f(\{x_1, \dots, x_m\})$.*

Proof. Let $x_1 R \dots R x_m$ be an R-chain. Take any k such that $2 < k \leq m$ and assume $f(\{x_1, \dots, x_{k-2}\})$ and $f(\{x_1, \dots, x_{k-1}\})$ are non-empty. Let $x_i \in f(\{x_1, \dots, x_{k-1}\})$. If $i = 1$, we are done. Let $i > 1$. By applying α^{-} or β step by step to x_{i-1}, x_{i-2} *et cetera*, we eventually arrive at $x_1 \in f(\{x_1, \dots, x_{k-1}\})$. By the same reasoning we derive $x_1 \in f(\{x_1, \dots, x_{k-2}\})$. For any $x \in (\{x_1, \dots, x_{k-2}\})$ other than x_1, we get with β and $x_1 \in f(\{x_1, \dots, x_{k-2}\}) \cap f(\{x_1, \dots, x_{k-1}\})$, $x \in f(\{x_1, \dots, x_{k-1}\})$. Hence, $f(\{x_1, \dots, x_{k-2}\}) \subseteq f(\{x_1, \dots, x_{k-1}\})$. Since we also have that $f(\{x_{k-1}\}) \subseteq f(\{x_{k-1}, x_k\})$, we obtain $f(\{x_1, \dots, x_k\}) \neq \emptyset$ by ξ. Thus non-emptiness of $f(\{x_1, \dots, x_{k-2}\})$ and $f(\{x_1, \dots, x_{k-1}\})$ implies non-emptiness of $f(\{x_1, \dots, x_k\})$ $(2 < k \leq m)$. Since $f(\{x_1\}) \neq \emptyset \neq f(\{x_1, x_2\})$ by definition of f and R, repeatedly applying this result yields $f(\{x_1, \dots, x_m\}) \neq \emptyset$.

As above, stepwise application of α^{-} or β to arbitrary $x_i \in f(\{x_1, \dots, x_m\})$ reveals that $x_1 \in f(\{x_1, \dots, x_m\})$. Obviously, $x_1 \in B(\{x_1, \dots, x_m\}, \bar{R}(\{x_1, \dots, x_m\}))$. Now take any other $y \in B(\{x_1, \dots, x_m\}, \bar{R}(\{x_1, \dots, x_m\}))$. There is then by definition of an optimal element an R-chain $yR \dots R x_1$ from y to x_1. Applying α^{-} or β at each step from x_1 to y eventually shows that $y \in f(\{x_1, \dots, x_m\})$. ■

Our next lemma shows that a choice function which satisfies α^{-}, β and ξ will always select the set of $\bar{R}$-optimal elements whenever this set constitutes the choice situation.

Lemma 2. *If a choice function f satisfies α^{-}, β and ξ, then for any set A with non-empty $B(A, \bar{R}(A))$, $f(B(A, \bar{R}(A))) = B(A, \bar{R}(A))$.*

Proof. If $B(A, \bar{R}(A))$ is a singleton set, the result follows trivially. Assume it has $k > 1$ elements. Since $\bar{R}$ is transitive, there are exactly two ways in which $B(A, \bar{R}(A))$ can be non-empty. Either we have xRy for all $x, y \in B(A, \bar{R}(A))$ (indifference 'at the top')

[11] Let a choice function satisfy axioms α and γ. It does not follow, then, that R is acyclic. Consider the choice function $f(x, y) = \{x\}$, $f(x, z) = \{z\}$, $f(y, z) = \{y\}$ and $f(x, y, z) = \emptyset$. This choice function satisfies both α and γ, yet R is cyclic.

[12] Note that for any choice function f, R is well-defined and has a unique transitive closure $\bar{R}$.

or we have a top cycle, that is, there is a chain $x_1R\dots Rx_kRx_1$ with $B(A,\bar{R}(A)) = \{x_1,\dots,x_k\}$ and at least for some i,j, x_iPx_j. In either scenario we can draw a chain for any two optimal elements x,y that starts in x, passes through other elements of $B(A,\bar{R}(A))$, does not pass through elements not in $B(A,\bar{R}(A))$ and ends in y. Since the chains go through elements of $B(A,\bar{R}(A))$ only, and since such chains exist for any $x,y \in B(A,\bar{R}(A))$, we see that $B(B(A,\bar{R}(A)),\bar{R}(B(A,\bar{R}(A)))) = B(A,\bar{R}(A))$. By Lemma 1 we have $B(B(A,\bar{R}(A)),\bar{R}(B(A,\bar{R}(A)))) \subseteq f(B(A,\bar{R}(A)))$. Thus we have $B(A,\bar{R}(A)) \subseteq f(B(A,\bar{R}(A)))$. Since $f(B(A,\bar{R}(A))) \subseteq B(A,\bar{R}(A))$ by definition of a choice function, we see $B(A,\bar{R}(A)) = f(B(A,\bar{R}(A)))$. ■

A third useful lemma is:

Lemma 3. *Let f be a choice function that satisfies α^-, β and ρ. Then (a) For all $A,B \in \mathcal{P}$: if $y \notin f(\{x,y\})$ for all $x \in A$ and $y \in B$, then $f(A\cup B) \subseteq f(A)$, and (b) f satisfies γ^-, if, and only if, it satisfies γ.*

Proof. (a) Let f and A and B be as described. By non-emptiness of $f(\{x\})$ for all x, A and B are disjoint. We have to show that $f(A\cup B) \subseteq f(A)$. Take arbitrary $x \in f(A)$ and $y \in A - f(A)$. If such y does not exist, $A = f(A)$ and the result follows directly from ρ. If $y \in f(\{x,y\})$ we have $y \in f(A)$ by either α^- (if $f(\{x,y\}) = \{y\}$) or β (if $f(\{x,y\}) = \{x,y\}$), which is a contradiction. By assumption, we also have $y \notin f(\{x,y\})$ for all $y \in B$. Hence, $y \notin f(\{x,y\})$ for all $x \in f(A)$ and $y \in [A - f(A)] \cup B$. With ρ we then get $f[f(A) \cup [A - f(A)] \cup B] = f(A \cup B) \subseteq f(A)$.

(b) To prove the non-trivial direction, let $x \in f(A)\cap f(B)$. By γ^-, $f(A\cup B) \neq \emptyset$, say $y \in f(A\cup B)$. If $(A\cup B) = f(A)\cup f(B)$, then $y \in f(A)$ or $y \in f(B)$ and either $x = y$ or $x \in f(A \cup B)$ by β, which we need to show. Next consider $(A\cup B) - [f(A)\cup f(B)] \neq \emptyset$, say z is in it. We cannot have $z \in f\{x,z\}$ for any $x \in f(A)\cup f(B)$: α^- (if $f\{x,z\} = \{z\}$) or β (if $f\{x,z\} = \{x,z\}$) would then entail that $z \in f(A) \cup f(B)$. Hence, it follows from (a) above that $f(A\cup B) \subseteq f(f(A)\cup f(B))$ and, in particular, $y \in f(f(A) \cup f(B))$. This means $y \in f(A) \cup f(B)$ and thus $y \in f(A)$ or $y \in f(B)$. Application of β subsequently shows that $x \in f(A \cup B)$. ■

We now present the main characterization result of this paper.[13]

Proposition 2. *A choice function f satisfies axioms α^-, β, ξ, γ^- and ρ if, and only if, $f(A) = B(A,\bar{R}(A))$ for all $A \in \mathcal{P}$.*

Proof. $\Leftarrow$ Assume $f(A) = B(A,\bar{R}(A))$ for all A. Properties β and γ^- follow directly from the definition of an optimal $\bar{R}$ -element. To prove α^- assume $x = f\{(x,y\})$ for some $x,y \in X$ and $x \notin f(A)$ for some A that contains x,y. There is then some $z \in A$ for which we do not have $x\bar{R}(A)z$. Since xPy, we cannot have $y\bar{R}(A)z$ either. Hence, $y \notin f(A)$. For ξ, let A,B and C be defined as: $f(A)$ and $f(B)$ are non-empty and $f(A) \subseteq f(A \cup B)$ and $f(B) \subseteq f(B \cup C)$. Let $x \in f(A)$. Since $f(A) \subseteq f(A\cup B)$ we have $x\bar{R}(A\cup B)y$ for all $y \in A\cup B$. In particular, $x\bar{R}(A\cup B)y$ for all $y \in f(B)$. Since $f(B) \subseteq f(B\cup C)$ we have for all $y \in f(B)$ and all $z \in C$, $y\bar{R}(B\cup C)z$. By transitivity of $\bar{R}$ we thus have $x\bar{R}(A\cup B\cup C)z$ for all $z \in C$. Hence, $x \in B(A\cup B\cup C, \bar{R}(A\cup B\cup C))$ and thus $f(A \cup B \cup C) \neq \emptyset$. To consider ρ, let A and B be disjoint and assume that there are no $x \in A$ and $y \in B$ with $y \in f(\{x,y\})$. This means that there is no R-chain starting with some $y \in B$ and ending with $x \in A$, for if there were, there must be some x_i, x_{i+1} in the chain with $x_i \in f(x_i, x_{i+1})$, $x_i \in B$ and $x_{i+1} \in A$, which would be a contradiction. Since there cannot be a $y \in (B(A\cup B, \bar{R}(A\cup B))\cap B)$, $f(A \cup B) = B(A \cup B, \bar{R}(A \cup B)) \subseteq f(A) = B(A,\bar{R}(A)) \subseteq A$.

[13] In the Appendix we demonstrate that the axioms used in Proposition 2 are independent of each other.

$\Rightarrow$ Let f satisfy the axioms and take arbitrary A. First, assume $B(A,\bar{R}(A)) = \emptyset$, i.e., there is no $x \in A$ with $x\bar{R}(A)y$ for all $y \in A$. Since $\bar{R}$ is transitive and reflexive, $M(A,\bar{R}(A)) \neq \emptyset$ (see Remark 2). We first show that $M(A,\bar{R}(A))$ is not a singleton set. Take arbitrary $x \in M(A,\bar{R}(A))$. Since $B(A,\bar{R}(A)) = \emptyset$ there is some y such that there is no R -chain from x to y, that is, $x\bar{R}(A)y$ does not hold. Let B_y be the set of all elements z such that $z\bar{P}(A)y$. If $B_y = \emptyset$, then $y \in M(A,\bar{R}(A))$. Assume it is non-empty. By the construction and finiteness of A, there must be some $v \in B_y (v \neq x)$ such that $w\bar{P}(A)v$ for no $w \in A$. But then $v \in M(A,\bar{R}(A))$. Hence, in all scenario's $M(A,\bar{R}(A))$ contains at least one element other than x. Denote $k \geq 2$ the cardinality of $M(A,\bar{R}(A))$.

Next, we show that there are sets $A_1,\dots,A_t$ with $t \leq k$ such that (i) $\cup_{i\in\{1,\dots,t\}}A_i = A$ and where (ii) $\{B(A_1,\bar{R}(A_1)),\dots, B(A_t,\bar{R}(A_t))\}$ is a partition of $M(A,\bar{R}(A))$. We introduce some definitions first. Define $\Phi = \{x \in M(A,\bar{R}(A))|$ for all $y \in M(A,\bar{R}(A))$, $y = x$ if $y\bar{R}(A)x\}$. Let g denote the cardinality of Φ. Take an $x \in \Phi$. Define $A^x = \{y \in A|x\bar{R}(A)y\}$. Then $B(A^x,\bar{R}(A^x)) = \{x\}$. We speak of a $\{x_1,\dots,x_l\}$-top $\bar{R}$ cycle ($l \leq k$) when $\{x_1,\dots,x_l\} \in M(A,\bar{R}(A))$ and $x_1\bar{R}(A)\dots\bar{R}(A)x_l\bar{R}(A)x_1$. Define $A^{\{x_1,\dots,x_l\}} = \{y \in A|x_1\bar{R}(A)y$ or … or $x_l\bar{R}(A)y\}$. Then $B(A^{\{x_1,\dots,x_l\}},\bar{R}(A^{\{x_1,\dots,x_l\}})) = \{x_1,\dots,x_l\}$. Let h be the number of distinct top $\bar{R}$ cycles in A. Note that, if $g = 0$, then $h > 1$ and if $h = 0$, then $g > 1$, otherwise $B(A,\bar{R}(A)) \neq \emptyset$, which is a contradiction. For notational convenience, label the ith top $\bar{R}$ cycle in A as TC_i and denote its cardinality $\#TC_i$. We then have $k = g + \sum_{i=1}^{h}\#TC_i$. Then (i) $A^{x_1}\cup\dots\cup A^{x_g}\cup A^{TC_1}\dots\cup A^{TC_h} = A$ and (ii) $\{B(A^{x_1},\bar{R}(A^{x_1})),\dots, B(A^{x_g},\bar{R}(A^{x_g})), B(A^{TC_1},\bar{R}(A^{TC_1})),\dots, B(A^{TC_h},\bar{R}(A^{TC_h}))\}$ partition $M(A,\bar{R}(A))$. Result (ii) follows by construction but to see (i), assume there is some $A\backslash(A^{x_1}\cup\dots\cup A^{x_g}\cup A^{TC_1}\cup\dots\cup A^{TC_h}) \neq \emptyset$. In particular, let y be an element of it for which it is true that not $x\bar{P}(A)y$ for any other x in $A\backslash(A^{x_1}\cup\dots\cup A^{x_g}\cup A^{TC_1}\cup\dots\cup A^{TC_h}) \neq \emptyset$. Then for all $x \in M(A,\bar{R}(A))$, we would have not $x\bar{R}(A)y$ (otherwise $y \in (A^{x_1}\cup\dots\cup A^{x_g}\cup A^{TC_1}\cup\dots\cup A^{TC_h})$) and not $y\bar{R}(A)x$ (otherwise we have, since not $x\bar{R}(A)y$, $x \notin M(A,\bar{R}(A))$). Since $y \notin M(A,\bar{R}(A))$, there must by the way y was chosen, be some $x \in (A^{x_1}\cup\dots\cup A^{x_g}\cup A^{TC_1}\cup\dots\cup A^{TC_h}) - M(A,\bar{R}(A))$ with $x\bar{P}(A)y$. Let $z \in M(A,\bar{R}(A))$ be an element for which $z\bar{P}(A)x$. Since $x \notin M(A,\bar{R}(A))$ such a z exists. But then we either have $y \in A^z$ or $y \in TC_i$, where z is an element of some top-cycle TC_i and where $z\bar{P}(A)x$.

Now take the following partition $\{B(A^{x_1},\bar{R}(A^{x_1})),\dots, B(A^{x_g},\bar{R}(A^{x_g})), B(A^{TC_1},\bar{R}(A^{TC_1})),\dots, B(A^{TC_h},\bar{R}(A^{TC_h}))\}$ of $M(A,\bar{R}(A))$. For simplicity's sake relabel the elements so that we have $\{A_1,\dots, A_{g+h}\} = \{B(A^{x_1},\bar{R}(A^{x_1})),\dots, B(A^{x_g},\bar{R}(A^{x_g})), B(A^{TC_1},\bar{R}(A^{TC_1})),\dots, B(A^{TC_h},\bar{R}(A^{TC_h}))\}$. Applying Lemma 3a to A_1 and A_2 shows that $f(A_1\cup A_2) \subseteq f(A_1)$ as well as $f(A_1\cup A_2) \subseteq f(A_2)$. Since the sets A_1 and A_2 are disjoint, this can only mean that $f(A_1\cup A_2) = \emptyset$. Now consider the partition $\{A_1\cup A_2, A_3,\dots,A_{h+g}\}$. Applying Lemma 3a now to $A_1\cup A_2$ and A_3 leads, by the same reasoning, to the conclusion that $f(A_1\cup A_2\cup A_3) = \emptyset$. Proceeding in this way we eventually arrive at $f\{A_1\cup\dots\cup A_{h+g}\} = f(M(A,\bar{R}(A))) = \emptyset$. Consider next the sets $M(A,\bar{R}(A))$ and $A - M(A,\bar{R}(A))$. Since there are no $y \in A - M(A,\bar{R}(A))$ and $x \in M(A,\bar{R}(A))$ with $y \in f(\{x,y\})$, Lemma 3a yields $f(M(A,\bar{R}(A)))\cup(A - M(A,\bar{R}(A))) = f(A) \subseteq f(M(A,\bar{R}(A)))$. Since $f(M(A,\bar{R}(A))) = \emptyset$ by the previous step, $f(A) = \emptyset$.

Next, assume $B(A,\bar{R}(A))$ is non-empty. By definition, there is no $x \in A - B(A,\bar{R}(A))$ and $y \in B(A,\bar{R}(A))$ with $x \in f(\{x,y\})$. Hence, by Lemma 3a, $f(A) \subseteq f(B(A,\bar{R}(A)))$ and, by Lemma 2, $f(A) \subseteq B(A,\bar{R}(A))$. By definition of $\bar{R}$ there is, for any $y \in A$, an R-chain from x to y. Let A^x_y denote the set of all elements in the chain from x to y. By Lemma 2, $x \in f(A^x_y)$. By Lemma 3b, f satisfies γ. Repeated application of γ allows us to derive $x \in f(\cup_{y\in A}A^x_y) = f(A)$. Hence, $B(A,\bar{R}(A)) \subseteq f(A)$. Thus $f(A) = B(A,\bar{R}(A))$. ■

Two remarks are worth making. First, γ^+ and ρ together yield a more succinct characterization as a corollary of our main result, when combined with the following observation.[14]

Observation 1. *If a choice function f satisfies γ^+, then it satisfies α^-, β, ξ, and γ^-.*

Corollary 3. *A choice function f satisfies axioms ρ and γ^+ if, and only if, $f(A) = B(A, \bar{R}(A))$ for all $A \in \mathcal{P}$.*

Second, Evren et al. (2019, Theorem 5.1) use the following strengthening of β, first introduced in Bordes (1976), together with ρ in their characterization of the top-cycle choice rule in the context of decisive choice functions.

Axiom 10 (β^+)**.** *For all $A, B \in \mathcal{P}$, if $A \cap f(B) \neq \emptyset$, then $f(A) \subseteq f(B)$.*

Note that given their result, our Proposition 2 and the observation that γ^- and ξ are trivially fulfilled for decisive choice functions, α^-, β^+ and ρ together yield an alternative characterization of decisive top-cycle choice rules.

We conclude with a corollary of Proposition 2 which forms a novel account of what constitutes a hard choice. *Per* this characterization, a hard choice is defined by the incompleteness of the binary relation rather than by the violation of regularity properties such as transitivity or acyclicity.

Corollary 4. *Let f be a choice function f satisfying axioms α^-, β, ξ, γ^- and ρ. A choice situation A constitutes a hard choice if, and only if, for some $x, y \in M(A, \bar{R}(A))$ we have $f(\{x, y\}) = \emptyset$.*

Appendix. Independence of axioms

We demonstrate that the axioms in Propositions 1 and 2 are independent. Let x, y, z be distinct elements in X and let $A = X - \{x, y, z\}$. For the choice functions f_i ($1 \leq i \leq 7$) used below we specify: (a) $f(A^*) = \emptyset$ for all non-singleton subsets A^* of A and (b) $f(A \cup B) = f(B)$ for all non-empty subsets B of $\{x, y, z\}$.

First, consider the axioms used in Proposition 1.

1. Satisfying α, ξ, $\gamma^=$ and violating β: $f_1(x, y) = \{x, y\}$, $f_1(x, z) = \{x, z\}$, $f_1(y, z) = \{y, z\}$ and $f_1(x, y, z) = \{y, z\} = f_1(X) \neq B(X, R) = \{x, y, z\}$.
2. Satisfying β, ξ, $\gamma^=$ and violating α: $f_2(x, y) = \{x\}$, $f_2(x, z) = \{z\}$, $f_2(y, z) = \{y, z\}$ and $f_2(x, y, z) = \{y, z\} = f_2(X) \neq B(X, R) = \{z\}$.
3. Satisfying α, β, $\gamma^=$ and violating ξ: $f_3(x, y) = \{y\}$, $f_3(x, z) = \{x\}$, $f_3(y, z) = \{y, z\}$ and $f_3(x, y, z) = \emptyset = f_3(X) \neq B(X, R) = \{y\}$.
4. Satisfying α, β, ξ, and violating $\gamma^=$: $f_4(x, y) = \{x\}$, $f_4(x, z) = \{x\}$, $f_4(y, z) = \emptyset$ and $f_4(x, y, z) = \emptyset = f_4(X) \neq B(X, R) = \{x\}$.

Next, consider the axioms used in Proposition 2.

1. Satisfying α^-, β, ξ, ρ and violating γ^-: Take f_4 defined above. We have $f_4(X) = \emptyset \neq B(X, \bar{R}(X)) = \{x\}$.
2. Satisfying α^-, β, γ^-, ρ and violating ξ: $f_5(x, y) = \{x\}$, $f_5(y, z) = \{y\}$, $f_5(x, z) = \{z\}$ and $f_5(x, y, z) = \emptyset = f_5(X) \neq B(X, \bar{R}(X)) = \{x, y, z\}$.
3. Satisfying α^-, ξ, γ^-, ρ and violating β: $f_6(x, y) = \{x, y\}$, $f_6(x, z) = \{x, z\}$, $f_6(y, z) = \{z\}$ and $f_6(x, y, z) = \{x, z\} = f_6(X) \neq B(X, \bar{R}(X)) = \{x, y, z\}$.
4. Satisfying β, ξ, γ^-, ρ and violating α^-: Take f_2 above. We have $f_2(X) = \{y, z\} \neq B(X, \bar{R}(X)) = \{x, y, z\}$.
5. Satisfying α^-, β, ξ, γ^- and violating ρ: $f_7(x, y) = \{x, y\}$, $f_7(x, z) = \{x\}$, $f_7(y, z) = \{y\}$ and $f_7(x, y, z) = \{x, y, z\} = f_7(X) \neq B(X, \bar{R}(X)) = \{x, y\}$.

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