PROBABILITIES OF CONDITIONALS

> 'He says he'll pay me every pfennig if he gets this job as barman at the Lady Windermere ... if, if...' Frl. Schroeder sniffs with intense scorn: 'I dare say! If my grandmother had wheels, she'd be an omnibus!'
>
> Christopher Isherwood, *Goodbye to Berlin*

Both conditionals and probabilities have been the subject of lively philosophical debate. Lately their interaction has been in the limelight, through the disputed thesis that $P(A \to B) = P(B/A)$, *the probability of the conditional is the conditional probability* (of consequent on antecendent). This thesis is tenable for the Stalnaker conditional if nesting of arrows is not allowed; for nested arrows I have weaker results. For ease of reading, I have limited the body of this paper to an exposition of the philosophical disputes, while the technical results are collected in a many-sectioned appendix.[1]

Both for probabilities and conditionals, I shall make a distinction between interpretations and paradigms. An interpretation, in this context, is a full-fledged account of the subject. A paradigm is something that guides our attempts to arrive at an interpretation: it is an idea that explains 'clear' cases, breaks down immediately for more complex cases, but is returned to for inspiration whenever more formal attempts at interpretation run into their own difficulties.

1. Conditionals

The first paradigm that guided the explication of conditionals was the idea of this situation: a person asserts $A \to B$ (read 'if A then B') to signify that the argument with premise A and conclusion B, is valid. The first obvious extrapolation of this idea is the assertion that it is the exact and sole function of a conditional to express the statement that a certain

Harper and Hooker (eds.), Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science, Vol. I, 261–308.

corresponding argument is valid. And this extrapolation then yields the following principles for reasoning with conditionals.

(1) If A and $A \rightarrow B$ are true, then B is true
(Symbolically: $A, A \rightarrow B \Vdash B$) *(Modus Ponens)*

(2) $A \rightarrow B \Vdash (A \,\&\, C) \rightarrow B$ (Weakening)

(3) $A \rightarrow B, B \rightarrow C \Vdash A \rightarrow C$ (Transitivity).

There are many logics of conditionals that incorporate these principles, for the above paradigm led to a number of different (detailed) interpretations of the conditional.

However, these principles can be held only at the cost of ignoring a large class of cases of conditional assertion, which apparently do not fit the paradigm. These cases were discussed in the forties by Goodman and Chisholm, under the heading of *contrary-to-fact* or *counterfactual conditionals.* A typical example is the statement 'If this match were struck (now), it would light', said with reference to a match held up for inspection. Agreement that this statement is true does not allow the inference that if this match were a burnt match, and struck now, it would light. Similarly, it seems that I could truly say of a drinking glass: 'Were this glass dropped now, it would break' but not 'Were this glass dropped, and were the floor covered with foam, the glass would break'. These examples contravene principle 2 above (Weakening), and once you see the trick, you can also provide examples that contravene Transitivity *(though not Modus Ponens)*.

The trick, as everyone saw at once, is no trick at all: many conditional statements in English carry a tacit *ceteris paribus* clause on their antecedent. There are two paradigms for this case: what I shall call the Ramsey paradigm and the Sellars paradigm. The Ramsey paradigm centres on a person with body of information K. This person asserts $A \rightarrow B$ in each of two cases: A is compatible with K, and the argument from K and A to B is valid; A is not compatible with K, but minimal changes in K yield an alternative $K(A)$ compatible with A, such that the argument from $K(A)$ and A to B is valid. Goodman's original arguments seem to establish pretty well that we cannot give anything like a general recipe for finding the correct body $K(A)$, or even for saying what changes in K are more minimal than others. But the logician adhering to the Ramsey paradigm is not worried by this, since he seeks generality: he asks what principles of reasoning remain correct whatever the recipe be.

The second paradigm is due to Wilfrid Sellars, who considers the relevant class of conditionals $A \to B$ to have a restricted syntactic form: A is an 'input' statement and B an 'output' statement. The person asserting the conditional has a general background theory with principles of form

(4) ϕ-ing a thing of kind K in conditions C makes it ψ

The conditional (4) has no *ceteris paribus* clause; it fits the original 'valid argument' paradigm. The tacit *ceteris paribus* clause in conditionals that violate principles of Weakening and Transitivity is a tacit specification of kind K and/or circumstances C for the antecedent of, say,

(5) If X be ϕ-ed, it will (would) ψ.

To a logician this paradigm may seem very limited, because of the restricted syntactic form. But it is very likely that many problems are artificially created in logic through syntactic generality, going beyond the original context of philosophical problems.

There are two main theories of conditionals in which the interpretation is sufficiently detailed to allow a complete characterization of the logic of reasoning with conditionals. These are due to Robert Stalnaker and David Lewis; they are cast in the terminology of the current semantic analysis of modal logic.

Simplifying a bit, Stalnaker's theory is this: every statement is true or false in each possible world (or possible situation, or set-up, if you like); there is for each world α a *nearness ordering* of worlds such that α is nearest α, and if there are any worlds in which A is true, there is a nearest world to α among those in which A is true ('the nearest A-world to α'); $A \to B$ is true in α if and only if: B is true at the nearest A-world to α, or there are no A-worlds.

Lewis accepts the basic approach but denies that there must be a unique nearest A-world to α. Hence he says: $A \to B$ is true in α if and only if B is true at all the nearest A-worlds to α. Besides their more basic agreements, we find therefore that both Stalnaker and Lewis reject Weakening and Transitivity, and accept *Modus Ponens* and a kind of weakened transitivity:

(6) $A \to B, B \to A, A \to C \Vdash B \to C$

But Stalnaker alone, and not Lewis, accepts

(7) $(A \rightarrow B) \vee (A \rightarrow \neg B)$ must always be true
(Symbolically: $\Vdash (A \rightarrow B) \vee (A \rightarrow \neg B)$

where $\vee$ is the sign of inclusive disjunction ('and/or') and $\neg$ the sign of negation ('not').

To the philosophical status of the 'possible world' talk and its intelligibility I shall return below. But first I must discuss a technical question: the question of nestings of arrows, of conditionals whose antecedents or consequents are themselves conditionals.

2. Nested Conditionals

The preceding section does not presuppose that nesting of arrows, as in $(A \rightarrow B) \rightarrow C$ or $A \rightarrow (B \rightarrow C)$, yields syntactically well-formed sentences, or that similar constructions in English are meaningful before rephrasing them. One might hold, for example, that $A \rightarrow (B \rightarrow C)$ is to be understood as equivalent to $(A \,\&\, B) \rightarrow C$, and that when it sounds as if some-one is saying $(A \rightarrow B) \rightarrow C$ he really intends the metalinguistic $(A \rightarrow B) \Vdash C$. More abstruse constructions might be considered totally useless or unintelligible.

Much to my regret, the facts of discourse do not seem to allow of such simplifications. I was convinced of this by Richmond Thomason, with such examples as

(8) If the glass would break if thrown against the wall, then it would break if dropped on the floor.

This cannot be construed as of form

(8a) $(A \rightarrow B) \Vdash (C \rightarrow D)$

which asserts a relation between two statements, but must be accepted as having the form

(8b) $(A \rightarrow B) \rightarrow (C \rightarrow D)$

because it suffers from typical conditional trouble, in that (8) does not imply

(9) If this glass would break if thrown against the wall, and the floor were covered with foam rubber, then it would break if dropped on the floor.

Of course, it is still possible to react that one is not interested in the facts of discourse *per se*, and that such assertions as (8) and (9) are of no interest for more substantive reasons either. Much of what follows in this paper can be read while ignoring nested arrows, or abhorring them; only the last result I shall discuss benefits from restricting our attention to statements less complex than (8).

In any case, both Stalnaker and Lewis consider it a virtue of their account that the truth-conditions for statements involving nested arrows are provided automatically. For example, let β be the nearest A-world to α, and γ the nearest B-world to β; then clearly, on their account, $A \rightarrow (B \rightarrow C)$ is true at α exactly if C is true at γ.

To give our discussion somewhat more precision, I shall now state, in simplified form, the semantic account (essentially) due to Stalnaker.[2] A *model structure* is a couple $M = \langle K, s \rangle$ with K a non-empty set (the *possible worlds*), and s a map such that for each member α of K, and each subset X of K, $s_\alpha(X)$ is a subset of K also; subject to the conditions

(10a) $s_\alpha(X)$ is included in X

(10b) $s_\alpha(X)$ contains at most one member

(10c) $s_\alpha(X) = \{\alpha\}$ if α is in X

(10d) If $s_\alpha(X) \subseteq Y$ and $s_\alpha(Y) \subseteq X$ then $s_\alpha(X) = s_\alpha(Y)$.

The set $s_\alpha(X)$ is the set of nearest worlds in X to α. Lewis denies (10b), but adds other clauses.

An *interpretation* (this technical usage of the term is to be distinguished from my earlier use of it) is a function I which assigns to each sentence A a subset of K (intended meaning: $I(A)$ is the set of worlds in which A is true), subject to the clauses

(11a) $I(\neg A) = K - I(A)$

(11b) $I(A \,\&\, B) = I(A) \cap I(B)$

(11c) $I(A \vee B) = I(A) \cup I(B)$

(11d) $I(A \rightarrow B) = \{\alpha : s_\alpha(I(A)) \subseteq I(B)\}$

We say that A is *valid* ($\Vdash A$) exactly if $I(A)=K$ for each interpretation I on each model structure; and that $A_1,\ldots, A_n$ *semantically imply* $B(A_1,\ldots, A_n \Vdash B)$ exactly if $I(A_1)\cap\ldots\cap I(A_n)\subseteq I(B)$ for each interpretation I on each model structure.

A set of possible worlds is called a *proposition*; to say that sentence A 'expresses' the proposition X means that X is the set of worlds in which A is true. The family $\{I(A): A \text{ is a sentence}\}$ is called an algebra of propositions with Boolean set operations, plus binary operator

$$X\rightarrow Y=\{\alpha: s_\alpha(X)\subseteq Y\}.$$

All this is general for model structures of any conditional logic; in the present case these algebras of propositions may suitably be called *Stalnaker algebras.* The question: what is the logic, i.e. what principles govern valid inferences for this language? is clearly answered if and only if we can give an exact account of the class of Stalnaker algebras. This was done axiomatically by Stalnaker and Thomason; see Appendix Section 1 for a simplified account.

3. A PHILOSOPHICAL DIGRESSION

To the question what principles govern deductive reasoning involving conditionals, Stalnaker and Lewis give exact replies. But the validity of an argument does not depend on whether its premises are true; and indeed, Stalnaker and Lewis have not notably increased our ability to decide whether particular conditionals are true or false.

I opened this paper with a quote from Isherwood; his Frl. Schroeder seems to challenge the whole world to refute even the wildest consistent counterfactual conditional. Possible world discourse may seem to meet the challenge in principle: even if we cannot tell whether the dear lady's grandmother would have been an omnibus if she had been endowed with wheels, there is, as a matter of objective fact, a set of nearest worlds to the actual one in which the grandmother is so endowed. And the question whether in those worlds, she is an omnibus, has an objective answer.

However, one needs to swallow a great deal of metaphysics to take this seriously. Let me say at once that I do not; there are no possible worlds except the actual one, in the literal sense of 'there are'. I see the possible world machinery just as Duhem saw the rope-and-pulley models of the

English physicists: such fictions are useful when giving an account of the surface phenomena – and there is, in reality, nothing below the surface. In our case the phenomena are the inferential relations among statements, attested in the inferential behaviour of those engaged in such discourse. Within the model structure, what is meant to mirror these phenomena is the algebra of propositions. We introduce possible worlds, and relations on them, because that yields an intuitively simple, but formally indirect, way of defining that proposition algebra.

Let me return to the truth-values of conditionals. If there are facts only about this world, and no counterfacts or facts about other possible worlds, sentences involving conditionals cannot be evaluated by asking whether they correspond to the facts. For they are about what is not a fact. So except for ones that cannot be true – like $[A \,\&\, (A \rightarrow \neg A)]$, which just has to be false; and other limiting cases – the truth-value of such sentences seems to be indeterminate. Stalnaker and also Thomason have indicated how this may be accepted without imperiling the logic of inference involving conditionals; and I have elsewhere shown how this yields a perspicuous way to relate Lewis' and Stalnaker's theories to each other.[3]

However, this is a bit of an uncomfortable position, for one would like to say that all sorts of contingent conditionals are true. If this butter were heated to 150 °C, it would melt. Well, these may be the cases in which the context makes very clear the exact content of that tacit *ceteris paribus* clause. But the semantic analysis is in terms of possible worlds, and an alternative semantics based on the ceteris paribus idea yields a logic that is a *proper* part of both Lewis' and Stalnaker's logics, and indeed also of any other logic which we shall encounter in this paper.[4] So we feel the inclination to say that even if the truth-values of some conditionals are indeterminate, there are yet many non-trivial ones that we are entirely prepared to assert.

I must now state my own position: counterfactual conditionals do not have the function of stating facts, and in a strict sense, none deserve to be called true or false. Their function is different, just like the function of sentences in the interrogative or imperative mood is different.

Before I elaborate this position, I may as well mention at once that David Lewis considers such positions (in his discussion of Adams) and rejects them. Lewis writes:[5]

I cannot think of any conclusive objection to the hypothesis that indicative conditionals are non-truth-valued sentences, governed by a special rule of assertability.... I have an inconclusive objection, however: the hypothesis requires too much of a fresh start. It burdens us with too much work still to be done, and wastes too much that has been done already. So far we have nothing but a rule of assertability for conditionals with truth-valued antecedents and consequents. But what about compound sentences that have such conditionals as subsentences?

I assume that Lewis would voice these same admirably conservative sentiments in response to similar hypotheses about any conditional or modal connective. Yet he probably does not mean that it is not a virtue of a philosophical position, if it spurs its adherents on to new ventures, such as the development of a pragmatics-oriented alternative to truth-value semantics. What Lewis intends, presumably, is a challenge: do not claim any advantages for your hypothesis until you have substantiated its feasibility in detail. And Lewis assumes that, since the hypothesis must also eventually yield a characterization of the logical laws of inference, substantiation will require a full-fledged alternative to the usual semantics, in which concepts like *assertability* replace concepts such as truth.

This assumption accompanying the challenge I deny. Clearly, if anyone claims that conditionals, and sentences with conditionals as parts, have a different function from statements of facts, then he must give an account of that function. But the account I shall now give implies that, to ferret out the logic of discourse involving conditionals, it is appropriate exactly to engage in the usual (possible worlds & truth-values) semantics.

Let us return for a moment to the Sellars paradigm.[6] A person asserts that this pat of butter would/will melt if heated. He knows very well that his experience of heat melting butter in the past does not warrant his assertion; he has no such simple faith in straight rule induction as Russell's chicken. But in asserting the conditional he signifies his allegiance to a certain background theory, in which relevant principles of form 4, about butter and heating, hold. He may be more or less vague on the theory and the theory in question will be more or less sophisticated; it might surprise you to find what vague and unsophisticated theories some of your nearest and dearest have about butter. But his allegiance is strong, and he will reiterate the conditional with rising volume and timbre if pressed. For when a man enters a commitment, whether to a scientific theory or an ideology, he assumes *ipso facto* the office of explainer. He undertakes to answer questions *ex cathedra*, *qua* adherent of the theory.

And his use of conditionals has the function of signalling his commitment.

It would be nice to have a formal model for this situation, so that we could calculate what a man's *ex cathedra* answers would be, given his theoretical commitments, plus observations. I admit this, and that I have no such model to give. But I add that I can nevertheless describe informally how such a situation develops, and that in a way which shows the correct approach to the central question: what is the logic of inferences involving conditionals?

Your run of the mill conditional-user has no exact account of how theories plus facts warrant the assertion of conditionals or complex sentences involving them. So how does he keep straight how he is to behave in argument, carried on in such discourse? The situation is similar to that of a physicist unacquainted with the axiomatic basis of his discipline. He is willingly bewitched by a heuristic picture, and this picture guides his inferences. It is also similar to the pictorial way in which we give set-theoretic or algebraic proofs without reference to any axiomatic basis. (In a more serious vein, Milton chose to use the language of an earlier cosmography than that of his own day, when writing *Paradise Lost*: and today too, a scientifically educated person can give expression to his religious commitments through texts or hymns talking of the seven days of creation and four corners of the earth.) If there is an axiomatic basis in existence, the heuristic pictures or pictorial talk has no *theoretical* interest any more. But when there is no axiomatic basis and we try to produce one, we proceed by carefully examining the pictures. Recall for example Helmholtz's pictorial 'axiom of free mobility' and Lie's formalization in terms of geometric transformations.

By what picture, then, is the conditional-user willingly bewitched, when he uses such discourse? If pressed, he will talk about imagining alternative possibilities, and such; he may even give such elegant arguments as Lewis'

> I believe that there are possible worlds other than the one we happen to inhabit. If an argument is wanted, it is this. It is uncontroversially true that things might be otherwise than they are. I believe, and so do you, that things could have been different in countless ways. But what does this mean? Ordinary language permits the paraphrase: there are many ways things could have been besides the way they actually are. On the face of it, this sentence is an existential quantification. It says that there exist many entities of a certain description,

to wit 'ways things could have been'. I believe that things could have been different in countless ways; I believe permissible paraphrases of what I believe; taking the paraphrase at face value, I therefore believe in the existence of entities that might be called 'ways things could have been'. I prefer to call them 'possible worlds'.[7]

The point to which I agree is that logician's possible world machinery is a passable explication of the picture that bewitches users of modal and conditional discourse. Since that picture is what guides inference, the logical catalogue of valid patterns of inferences can only be searched out by exploring this picture, or the logician's explication thereof. Hence the success of the usual form of semantics.

4. Probability

As every knows, there are five Schools or interpretations of probability: the logical, frequency, subjective, propensity, and Kyburg. I do not wish to discuss these interpretations, nor align myself with one, but to discuss paradigms. There seem to me to be two paradigms guiding the interpretation of probability, and their use cuts across the five Schools (though at least among philosophers if not among statisticians, some Schools show a distinct preference for one or other). These paradigms I shall call the *epistemic ticker tape* and the *finite state machine.* Each of these is a conceived situation, or type of situation, in which some probability talk at least appears eminently intelligible.

A philospher adhering to the first paradigm imagines a subject who has both knowledge and degrees of belief. His knowledge comes to him on a ticker tape bearing simple sentences, and his total knowledge at time t is the content of the tape at t. The tape delivers one message per unit time, and each time it does, the subject's knowledge is increased, and he revises his degrees of belief. And the subject asserts 'Prob$(A)=r$' if and only if his degree of belief that A, equals r.

A philosopher adhering to the second paradigm imagines a subject (or nature) feeding inputs into a black box with finitely many states. Call the possible inputs $I_1,\ldots, I_m$ and the possible states $B_1,\ldots, B_n$. The box comes with an instruction sheet giving for each state B_i a matrix $[p(i, j, k)]$ which purports to mean that if the machine is in state B_i and input I_j is applied, then the machine transits to state B_k with probability $p(i, j, k)$ – the *transition probability.* The philosopher likes to refer to this situation as a *chance set-up*, and sees it as his task to explain what the instruction sheet really means.

Conditional probabilities have a place in both paradigms, but the guidance which the paradigms give for defining conditional probability is distressingly less than complete. Consider first the ticker tape. Bayesians established early on that, if to be rational is to be such that no one can make book against you, then the degrees of belief of a rational subject at a given time t must be, mathematically speaking, an assignment of probabilities. Professor Teller's paper in this volume proceeds in adherence to the epistemic ticker tape paradigm, and asks whether the conditional probability $P(A/B)$ of A given B must be defined by

(12) $P(A/B) = P(A \,\&\, B)/P(B)$

which is the usual formula.

The basic constraints on $P(A/B)$, namely that $P(B) = 1$, and that the function $P(A/B)$ is itself a probability assignment, are nowhere near enough to deduce (12). Mathematically at least, there are many alternatives to (12); and David Lewis has defined one for Stalnaker models. However, Teller's paper gives a proof, also due to Lewis, that the Bayesian requirements of rationality also require (12). That is, if the ticker tape subject revised his degrees of belief other than by conditionalizing to his increasing knowledge by formula (12), it would be possible to make book against him. This is a very nice result. Its philosophical significance, however, is somewhat marred, to my mind, by the lack of realism in the rational ticker tape subject as model for the scientific inquirer. The latter will frame theories, suggested but not entailed by his evidence, commit himself to them, and conditionalize his degrees of belief accordingly. Although these commitments are open to revision in the light of new evidence, he will nevertheless make his bets in accordance with these theoretical commitments in the meanwhile. So a real scientific pilgrim's progress violates the rationality conditions on which Lewis' proof is predicated.[8]

It might seem that the finite state machine paradigm is in good shape to guide the definition of conditional probability. For after all, is the transition probability not a conditional probability

(13) $p(i, j, k) = P(B_k \text{ next} \mid B_i \text{ now} \,\&\, I_j \text{ now})$

of the next state given present state and input?

Indeed; and it is possible to reconstruct the $p(i, j, k)$ from other probabilities so that (13) follows from (12) (see Appendix, Section 2). But the moment we start varying the conditions, we run into trouble. How are we to interpret

(14) $P(B_k \text{ next} \mid \text{one of } B_k, \ldots, B_m \text{ next})$

for example? Well, the transition matrix describes tendencies of the machine due to its physical make-up; hence the condition in (14) is to be conceived of as a condition imposed on the physical make-up of the machine. We put a 'damper' on the machine, 'closing off' possible states $B_1, \ldots, B_{k-1}$. Thus the machine has an offspring, the 'dampened' machine, with new transition probability matrices $[p^*(i, j, k)]$. What are these new matrices to be like? Well, that will depend on exactly what the damper does; the only real constraint is that after the damper is put on, P (one of $B_k, \ldots, B_m$ next) = 1. But there must be lots of dampers having that effect, and only one kind of damper will allow us to calculate $p^*(i, j, k)$ using formula (12). This kind of damper can be described (see Teller's proof that (12) must be true if, roughly speaking, $P(A/B)$ is to be functionally determined by $P(A \,\&\, B)$ alone, and not by other factors). But in the present context, this is still only one case, and it is hard to see why it should have such a preferential status.

In both paradigms, there are simple, clear cases, in which formula (12) appears the correct one. If I am a relatively dull subject, not given to much theorizing, like Conrad's Winnie Verloc of the conviction that things don't bear looking into much, then I shall have reason only to revise my degrees of belief by conditionalizing on new knowledge via formula (12). If I consider a finite state machine, and look only at those conditional probabilities listed in the transition probability matrices, again I shall see formula (12) satisfied. Outside these simple realms, all is grey. However, having noted that (12) cannot be considered sacrosanct, I shall now accept the usual extrapolation, and henceforth consider conditional probability to be defined by formula (12) everywhere.

5. THE STALNAKER THESIS

The English statement of a conditional probability sounds exactly like that of the probability of a conditional. What is the probability that I throw a six if I throw an even number, if not the probability that: if I throw an even number, it will be a six? And if we do not allow nesting of conditionals, nor conjoining conditionals with other sentences, '$P(B \rightarrow A)$' is surely no more than a harmlessly rewritten '$P(A/B)$'. But the rewriting may not be so harmless if we regard $B \rightarrow A$ as a full-fledged sentence and make logical claims about the arrow – even if we do not allow nesting of arrows. But even if a thesis is not harmless or trivial, it may be true; and Robert Stalnaker advanced

(15) Stalnaker's Thesis.
$P(A/B) = P(B \rightarrow A)$ whenever $P(B)$ is positive

in conjunction with the usual (formula 12) definition of $P(A/B)$. I shall refer to this briefly as the Thesis. It must be distinguished from Stalnaker's secondary claim that the Thesis holds with $\rightarrow$ being the Stalnaker conditional.[9]

This secondary claim goes well beyond the Thesis, which supposes only that we are speaking of some logically respectable conditional. In the Appendix (Section 2) I shall show that chance set-ups may be regarded as Stalnaker models; but there is then in general no *obvious* way of getting the Thesis to hold, for the probabilities $p(i, j, k)$ of the transition probability matrix. Later on (Section 6) it will appear that there is an unobvious way to do that, involving the fictional introduction of infinitely many further possible states not observationally distinguishable from the ones originally countenanced. In any case, the matter is not obvious even for simple probability statements.

David Lewis has an argument to show that instead, the Thesis is untenable, on pain of triviality. He deduces, starting from the Thesis, that no probability assignment can have more than four distinct values. This demonstration he gave at the Canadian Philosophical Association in June 1972, and it was a veritable bombshell; for months afterward everyone believed that the Thesis was defunct.

But Lewis' demonstration has subtle auxiliary assumptions that go into his formulation of the problem. He introduces his formulation with the following persuasive commentary:[10]

> What needs explaining is the fact that for all speakers at all times ... the assertability of indicative conditionals goes by conditional subjective probability. If we hope to explain this general fact by the hypothesis that the probabilities of conditionals always equal the corre-

sponding conditional probabilities, and if we assume that → means the same for speakers with different beliefs, then the fixed interpretation of → must be such as to guarantee that those equalities hold ho matter what the speaker's subjective probability function may be.

It may not be immediately obvious how this commentary affects the formal reasoning. The key phrase is 'the fixed interpretation of →'. What Lewis is actually demanding is that the model structures be such that any probability assignment (of a large enough class) can be superimposed without violating the Thesis, and without corresponding changes elsewhere in the model structure.

If persuasive English lacks some perspicuity, so does the formalistic jargon in which I have just restated it. Let us imagine the following situation. A certain person, who has a certain amount of information, and a commitment to certain theories, represents the world to himself by means of a model structure with probability measure P on its set of possible worlds. In this way he properly allows for his partial ignorance: there are many ways things might be compatible with his information and theories. His structure also has a nearness relation among worlds, so that he can tell in which worlds $A \rightarrow B$ is true. But he does not know which world is the actual one; however, if X is a set of these possible worlds in the domain of P, then $P(X)$ is the probability, according to him, that the actual world is in X.

Now a revision occurs; this person's information and/or theoretical commitment changes in a certain way. His ideas about what the world is like change; and also the degrees of belief he attaches to the sentences in his language. So he revises his model structure cum probability measure.

And here, Lewis introduces the requirement that it should be possible to make this revision by changing the probability measure alone – and not the constitution of the possible worlds or the nearness relation on them. What inspires this requirement, which is crucial to Lewis' reductio? Would it not seem rather, that our probabilities are inextricably involved in the way we represent the possibilities, and nearness relations among them, to ourselves? In the finite state machine paradigm, the transition probability matrix presumably reflects the machine's physical structure; if our ideas about the one change, will we not revise our modelling of the other?

The inspiration for the requirement must doubtlessly be Lewis' metaphysics, according to which one should always be able to say: let the possible worlds in my model structure be those which there actually are, let the nearness relation on them be the one reflecting their actual and objective similarities. In this scheme, the probability measure is nothing but a device to picture our ignorance. Hence it has nothing to do with the internal constitution of the model structure, which is reality itself. For this reality of possible worlds exists independent of the mind, its evolution flows on in its own even tenor;

The Moving Finger writes; and, having writ,
Moves on: nor all thy Piety nor Wit
Shall lure it back to cancel half a Line,
Nor all thy Tears wash out a Word of it.

How very different it looks to those of us who locate all of reality in the actual world and the representing subject, seeing nothing but manipulable fictions in the possible world menagerie!

So the logical disaster was precipitated not by Stalnaker's Thesis, but by the Thesis coupled with Lewis' metaphysical realism.

Lewis also has another demonstration about the Thesis, in which it is considered independently of formula (12), the usual definition of conditional probability. Assume the Thesis, and assume that $P(-/B)$ is a revision of P such that $P(B/B)=1$, and the revision is 'in some sense minimal'. Then Lewis derives the conclusion that $\rightarrow$ must be the Stalnaker conditional. The same results would follow *a fortiori* with (12) assumed; and this was also proved by William Harper.[11] But in this demonstration too a crucial role is played by the assumption that the probability measure on the model structure may be revised, without violating the Thesis, and without changing any other aspect of the structure. (The specific revisions used are to the zero-one measures P_α giving probability 1 to the set $\{\alpha\}$, where α is any one of the possible worlds.) But this is exactly the assumption found in the preceding demonstration, justified only by metaphysics, and not acceptable to me.

In conclusion then, I see the state of the issue as follows: the Thesis coupled with the usual definition of conditional probability, and the assumption that $\rightarrow$ is a logically respectable conditional, does not reduce to absurdity, and also does not imply that $\rightarrow$ is the Stalnaker conditional. There are two questions I want now to explore: *first*, what is the minimal logic of conditionals such that the Thesis holds nontrivially, and *second*, is it tenable to assert the Thesis for the Stalnaker conditional?

6. A MINIMAL LOGIC

Let us now assume the Thesis, and the usual definition of conditional probability, and the respectability of $\rightarrow$ as a conditional connective. What logical principles must hold then? I shall need to introduce some Auxiliary Assumptions which are meant as explicating the respectability of the arrow. Henceforth capital letters denote propositions, i.e. sets of possible worlds; & is intersection, $\rightarrow$ a certain binary operation, $\vee$ is union and $\neg A$ or $\bar{A}$ denotes $K-A$ where K is the set of all possible worlds.

(16) *Assumption.* If two propositions must always have the same probability, then they are identical.

This is not logically precise: I mean that if $P(\phi(X_1,\ldots, X_n))=P(\psi(X_1,\ldots, X_n))$ for every model structure with probability measure, and all propositions $X_1,\ldots, X_n$ in that structure, then $\phi(X_1,\ldots, X_n)=\psi(X_1,\ldots, X_n)$ for every model structure etc.

(17) *Theorem.* $A\rightarrow A=K$, the set of all possible worlds.

For $P(A\rightarrow A)=P(A/A)=1=P(K)$, using the Thesis and Assumption (16).

(18) *Assumption.* If A is not the empty proposition then $A\rightarrow B$ and $A\rightarrow C$ are disjoint if B and C are disjoint.

(19) *Theorem.* $A\rightarrow(B\vee C)=(A\rightarrow B.\vee.A\rightarrow C)$.

The proof, for the case $P(A)\neq 0$, is as follows.

$$\begin{aligned}P(A\rightarrow.B\vee C)&=P(B\vee C/A)=\\&=P(B\,\&\,C.\vee.B\,\&\,\bar{C}.\vee.\bar{B}\,\&\,C/A)=\\&=P(B\,\&\,C/A)+P(B\,\&\,\bar{C}/A)+P(\bar{B}\,\&\,C/A)=\\&=P(A\rightarrow.B\,\&\,C)+P(A\rightarrow.B\,\&\,\bar{C})+\\&\qquad+P(A\rightarrow.\bar{B}\,\&\,C).\end{aligned}$$

By Assumption (18), the propositions involved are disjoint, so

$$=P[(A\rightarrow.B\,\&\,C)\vee(A\rightarrow.B\,\&\,\bar{C})\vee(A\rightarrow\bar{B}\,\&\,C)]$$

Now by (16), we conclude

(20) $[A \to B \vee C] = [(A \to B \ \& \ C) \vee (A \to B \ \& \ \bar{C}) \vee (A \to \bar{B} \ \& \ C)]$

However, by exactly similar reasoning we get, because $B = (B \ \& \ C) \vee \vee (B \ \& \ \bar{C})$, the following

(21) $A \to B = (A \to B \ \& \ C) \vee (A \to B \ \& \ \bar{C})$

(22) $A \to C = (A \to B \ \& \ C) \vee (A \to \bar{B} \ \& \ C)$

and noting that the first disjunct on the right in (20) can be repeated, we derive Theorem (19) by substitution of equals for equals in (20). Again, the case of $P(A) = 0$, if allowed to be defined at all, is obvious.

The last theorem has a corollary that $(A \to B. \vee .A \to \bar{B}) = K$, which is the peculiar Stalnaker principle first denied by Lewis. To this extent, at least, Stalnaker's Thesis is intrinsically connected with Stalnaker's theory of conditionals. A second corollary is that, if $P(A) \neq 0$, then, because $A \to B. \vee .A \to \bar{B} = K$, $\overline{A \to B} = A \to \bar{B}$. And with this second corollary in hand, we derive that $P(A \to B \ \& \ C) = P(A \to B. \&. \ A \to C)$ in all cases, because of De Morgan's law and Theorem (19). Hence by (16) again,

(23) *Theorem.* $A \to (B \ \& \ C) = A \to B. \ \&. \ A \to C.$

We now have three theorems, but do not know yet that modus ponens holds – as it must, or the name 'conditional' is inappropriate. Indeed, in both Stalnaker and Lewis's systems, one finds the stronger law

(24) *Assumption.* $(A \to B) \ \& \ A = (A \ \& \ B)$

to which I hereby agree.

This is where I stop; let me call a *proposition algebra* (*tout court*, as opposed to 'algebra of propositions in model structure *M*') any field of sets with unit *K* and binary operation → (possibly a partial operation, e.g. not defined for $A \to X$), such that

(I) $(A \to B) \ \& \ (A \to C) = (A \to .B \ \& \ C)$

(II) $(A \to B) \vee (A \to C) = (A \to .B \vee C)$

(III) $A \ \& (A \to B) = (A \ \& \ B)$

(IV) $A \to A = K$

hold where defined. The minimal logic of conditionals suitable for probabilification I shall call *CE*; it is adequately described by saying that in any acceptable model structure with probability, the algebra of propositions must be a proposition algebra in the above sense.

So the Thesis requires logic *CE*; but does the Thesis allow non-trivial probability measures for the model structures countenanced by *CE*? Indeed, as I shall prove in the Appendix, Section 3,

(25) *Theorem.* Any antecedently given probability measure on a countable field of sets can be extended into a model structure with probability, in which Stalnaker's Thesis holds, while the field of sets is extended into a proposition algebra.

which theorem is as essential as the soundness and completeness theorems for *CE*, with respect to the criteria of adequacy for our present problem area.

Principle (III) (Assumption (24)) was introduced somewhat cavalierly perhaps, as being previously agreed to by all contestants. Even if it is only reasonable to approach the problem with a strong an agreement with Stalnaker and Lewis as seems possible, it may be well to look, momentarily, at the status of the Thesis purely *vis-à-vis* this area of agreement.

First, assume the Thesis, and also the nesting-reducing principle

(26) $A \to (A \to B) = (A \to B)$

also common to all contestants. Then we derive

(27) $P(A \to B) = P(A \to (A \to B)) = P(A \to B/A)$

that is, that the conditional is stochastically independent of its antecedent. Conversely, assume principle (III), plus (27), and derive

(28) $$\begin{aligned} P(A \to B) &= P(A \to B/A) = \\ &= P(A \to B.\ \&\ A)/P(A) = \\ &= P(A\ \&\ B)/P(A) = \\ &= P(B/A) \end{aligned}$$

which is the Thesis. So the agreement, by all concerned, about the Thesis is that it is equivalent to the stochastic independence between the conditional and its antecedent. This might provide a new fulcrum for the application of a philosophical critique or defence of the Thesis.

However that may be, in Theorems (17)–(25) we have a complete solution to the problem of the minimal logic *CE* of conditionals required by the Thesis. Also, since nesting of arrows plays, at most, an inessential role in all the demonstrations involved, these results remain if such nesting is disallowed. I turn now to the Stalnaker logic of conditionals.

6. Stalnaker Bernoulli Models

I may as well say at once that I cannot prove an analogue to Theorem (25) for Stalnaker algebras. (See note (12)). From the Appendix, Section 1, it appears that a Stalnaker algebra is a proposition algebra satisfying two further principles:

(V) $(A\to B)\ \&\ (B\to A)\ \&\ (A\to C) .\supset. (B\to C)$

(VI) if A is included in B, then $B\to A$ is included in $A\to A$.

The second of these is about conditionals with impossible antecedents, which is like probabilities, conditionalized on a condition with zero probability, an I shall ignore this (as a mainly technical matter). And (V), I consider debatable, and not established by philosophical argument; but I shall not debate it here.

In this section, I mean to show that one can adhere to the use of Stalnaker models and maintain the Thesis for all conditionals of a few simple kinds. (If no nesting of arrows is allowed, this solves the problem entirely; but I cannot so restrict myself with good conscience.) The kinds of conditionals I shall handle have the forms:

$$A\to B$$

$$A\to(B\to C)$$

$$(A\to B)\to C$$

in which A, B, and C themselves contain no arrows.

Imagine the possible worlds are balls in an urn, and someone selects a ball, replaces it, selects again, and so forth. We say he is carrying out a series of Bernoulli trials. Now it is possible, is it not, that the selects first α, then the nearest world to α, then the nearest to that, and so on. In that case, $A\to B$ is true at α exactly if the first A-ball he selects is a B-ball.

This suggests a way of constructing Stalnaker models, in which probabilities enter very straightforwardly. Indeed, the proofs I shall then give about the Thesis in such models, proceed by quite simple probability calculations. (They were not simple to me until Ian Hacking provided me with a crucial lemma; but they seem simple now.)

I must interject here that, from the point of view of Stalnaker's semantics, these models are of a very special variety. (Hence, although I do not know that the Thesis fails in them for more complicated conditionals, I expect it does.) To explain this limitation, it is easiest to refer to Lewis' reformulation of Stalnaker's semantics. In Lewis' version, a Stalnaker model is a couple $M=\langle K, \leqslant\rangle$ in which, for each α in K, $\leqslant_\alpha$ is a total pre-ordering of K. (For all present purposes, $\leqslant_\alpha$ may be taken to be a discrete linear ordering.) Then the nearest A-world to α is the minimal element by the $\leqslant_\alpha$ ordering, of the set $\{\beta\in K\colon A \text{ is true at } \beta\}$ $(=I(A)$ for the interpretation at issue). Now note that for each world there is another such ordering; because, intuitively, not all the worlds may lie on one straight line. In the models which I construct in this section, however, the following happens: if $\alpha\leqslant_\alpha\beta$ and $\beta\leqslant_\beta\gamma$, then $\alpha\leqslant_\alpha\gamma$. Indeed, think of $\leqslant_\alpha$ as determining a series $\alpha, \beta, \delta, \gamma, \ldots$ and $\leqslant_\beta$ another series, say $\beta, \gamma', \delta', \ldots$. Then in *these* models, but not in general, the series $\beta, \gamma', \delta', \ldots$ is in fact the series $\beta, \gamma, \delta, \ldots$. So we can think of these models as having their worlds arranged on a set of parallel lines very far apart from each other, and on each line, the worlds are arranged like the numbers, $0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$. The reason I have explained this at length is to make clear that the tenability of the Thesis *for more complex sentences* has little to do with what happens in these special models.

Now let me explain how the construction works. We begin with a Stalnaker model $M=\langle K, \leqslant\rangle$, and a field F of propositions in M. These propositions are to be thought of as expressed by zero-degree sentences; sentences in which there are no arrows. Finally, we have a probability P which is defined at least for all of F. If A and B are in F, then $A\rightarrow B$ is a well-defined proposition in M, of course; but at this point we don't know what $P(A\rightarrow B)$ is, nor even whether it is defined.

We now make up the bigger model $M^*=\langle K^*, \leqslant^*\rangle$ where K^* is the set of all maps π of the natural numbers into K. The relation $\leqslant^*_\pi$, on K^*, where π is in K^*, is defined by:

$\rho\leqslant^*_\pi\sigma$ iff there is a number k such that $\rho(k+m)=\sigma(k+m)$ for all natural numbers m.

For obvious reasons, such a map ρ may be referred to as the sequence $\rho(1), \rho(2), \ldots$.

In this much bigger model M^*, the old model M reappears if we identify member α of M with the series $\alpha^* = (\alpha, \beta, \gamma, \ldots)$ which is the domain of $\leqslant_\alpha$ ordered by that relation.

Now we look at the field F^* of subsets of K formed this way: F^* is the family of all sets

$$A_1 \otimes \cdots \otimes A_n \otimes K^*: A_1, \ldots, A_n \text{ in } F$$

where $\otimes$ is the Cartesian product, and this notation is imprecise, but perspicuous for:

$$\{\pi : \pi(i) \in A_i \quad \text{for} \quad i = 1, \ldots, n\}.$$

This construction of F^* is such that if F is a (Borel) field, so is F^*.

What we have so far is what is called a *product* construction, and this has a familiar correlate extension for the probability; I shall use the same symbol 'P' for the new measure as well as the old:

$$P(A_1 \otimes \cdots \otimes A_n \otimes K^*) = P(A_1) \ldots P(A_n).$$

The sets $A \otimes K^*$ in F^* may be called *zero-degree propositions* in M^*. These are the reappearance of the old field F of course, and have the same probabilities as the correlate old propositions: $P(A \otimes K^*) = P(A)$. Henceforth it will be convenient to abbreviate '$A \otimes K^*$' to 'A' when this is not confusing.

In M^* we also have conditional propositions, of course, defined by

$$X \to Y = \{\pi \in K^* : \text{for all } m, \text{ if } m \text{ is the first number such that } \pi_m \in X, \text{ then } \pi_m \in Y\}$$

where π_m is the series that results from π when you cut off the initial segment $\pi(1), \ldots, \pi(m-1)$.

The nice thing shown in the Appendix at this point is that F^* is closed under $\to$: if X and Y are in F^*, so is $X \to Y$. Hence in M^*, all propositions constructible from the zero-degree propositions (by $\to, \cap, \cup, -$), automatically receive a probability. Note also that if A and B are zero degree, then A, and also $A \to B$, is true at α in K iff it is true at α^*.

Now I shall show intuitively that if A and B are zero-degree, then $P(A \to B) = P(B/A)$. Choose at random a series π of worlds in K. This is itself a series formed by making a random selection of a world from K, again and again, ad infinitum. It is just like tossing a die forever. At some point, say at $\pi(m)$ we find a world in which A is true. Since previous selections do not influence this one in the least, it is itself nothing more nor less than a random selection from K. Hence the probability that this first A-world is a B-world, is just the probability that *any* world satisfies B given that it satisfies A. So that probability is $P(B/A)$. Hence the probability that $A \to B$ is true in the actual world (about which nothing is specified a priori *except* that it belongs to K) is just $P(B/A)$. So $P(A \to B) = P(B/A)$.

This proof is made precise in the Appendix, as well as similar, but longer, proofs that $P(A \to .B \to C)$ and $P(A \to B. \to C)$ conform to the Thesis as well. So at least for these simple conditionals, the Thesis is tenable in conjunction with Stalnaker's logic *and* Stalnaker's model theory.

University of Toronto

APPENDIX

For a summary of Stalnaker's semantics, see the end of Section 2 in the body of the text. In this Appendix, I shall use distinct symbols for set-theoretic operations and sentential connectives ($\cap, \cup, -; \&, \vee, \neg$) except in the case of the arrow.

1. STALNAKER ALGEBRAS

David Lewis gave essentially the following simplified axiomatization[1] of Stalnaker's logic C2:

(A1) $\vdash A$ when A is a propositional tautology
(A2) $\vdash A \to A$
(A3) $\vdash (A \to B) \& (B \to A) \& (A \to C). \supset B \to C$
(A4) $\vdash (A \vee B) \to A. \vee (A \vee B) \to B. \vee : (A \vee B) \to C \equiv A \to C. \&. B \to C$
(A5) $\vdash (A \to B) \supset (A \supset B)$
(A6) $\vdash A \to B. \vee. A \to \neg B$
(A7) $\vdash (A \& B) \supset (A \to B)$
(R1) if $\vdash A$ and $\vdash A \supset B$ then $\vdash B$

(R2) if $\vdash (A_1 \& \ldots \& A_n) \supset B$ then $\vdash (C \to A_1) \& \ldots \& (C \to A_n). \supset (C \to B)$

THEOREMS:

(1) $\vdash (A \to B) \& (A \to C). \equiv A \to (B \& C)$

(2) $\vdash (A \to B) \vee (A \to C). \equiv A \to (B \vee C).$

Almost all of this can be proved using (R2); there remains only the following part:

(2′) $\vdash A \to (B \vee C). \supset .(A \to B. \vee .A \to C)$

Suppose the consequent is false; then by (A6) and theorem (1), $A \to (\neg B \& \& \neg C)$ is true. So then if the antecedent is true, by (1) and (R2), $A \to B$ is true after all, contrary to supposition.

Now (A4) can be simplified: suppose that $(A \vee B) \to A$ and $(A \vee B) \to B$ are false; then by (A6) and (T1), $B \vee B \to (\overline{A \vee B})$ and so, via (A2) and (R2), $A \vee B \to C$ for all C. So $A \vee B$ is an impossibility and (A4) reduces to: if $A \vee B$ is an impossibility, so is A (and so is B mutatis mutandis). Hence (A4) can be replaced by:

(A4′) $\vdash (A \vee B) \to \overline{A \vee B}. \supset. A \to \bar{A}$

Also, (A5) and (A7) can together be replaced by

(A5′) $\vdash A \& (A \to B), \equiv (A \& B).$

Accordingly, a Stalnaker algebra of propositions is a field of propositions (sets) with unit K such that the following principles hold (I)–(IV) are same as in Section 6 in body of text):

(I) $(A \to B) \cap (A \to C) = (A \to .B \cap C)$

(II) $(A \to B) \cup (A \to C) = (A \to .B \cup C)$

(III) $A \cap (A \to B) = (A \cap B)$

(IV) $A \to A = K$

(V) $(A \to B) \cap (B \to A) \cap (A \to C) \subseteq (B \to C)$

(VI) If $A \subseteq B$ then $B \to \Lambda \subseteq A \to \Lambda$.

(Here (VI) comes from (A4′).)

THEOREMS

(1) If $A \subseteq B$ then $A \to B = K$.

For if $A \subseteq B$ then $B = A \cup B$; but $A \to A \cup B = A \to A \cup . A \to B = K$.

(2) If $A \subseteq B$ then $C \to A \subseteq C \to B$.

For if $A \subseteq B$ then $B = A \cup B$ so $C \to B = C \to A \cup . C \to B$.

(3) If $A_1 \cap \ldots \cap A_n \subseteq B$ then $C \to A_1 \cap \ldots \cap C \to A_n \subseteq C \to B$.

For $C \to A_1 \cap \ldots \cap C \to A_n = C \to . A_1 \cap \ldots \cap A_n$, and so by (2), $\subseteq C \to B$. This recaptures algebraically the effect of (R2).

2. Chance Set-ups as Stalnaker Models

To produce a simple Stalnaker model, consider a probabilistic finite state machine with input. Let its present state be B_1, its possible states $B_1, \ldots, B_n$, and the possible inputs $I_1, \ldots, I_m$. There is a transition probability matrix $[p_{ij}]$ relevant given that the present state is B_1: the probability that the next state is B_j if one applies input I_i, equals p_{ij}: Since each input leads to some outcome, $\sum_j (p_{ij}) = 1$. In the usual picture, the rows sum to one:

	B_1	B_j	B_n
I_1	p_{11}	p_{1j}	p_{1n}
I_i		p_{ij}	
I_m			p_{mn}

We can also put this as follows:

$$P(B_j \text{ next} \mid B_1 \text{ now} \ \& \ I_i) = p_{ij}$$

Let the set of possible worlds (meaning, possible immediate futures) be the set of couples $\langle I_i, B_j \rangle$. So the actual world is the one in which the present state is B_1, the input I_i, and the next state B_j, and it is represented by the couple $\langle I_i, B_j \rangle$.

With respect to this set of worlds let the proposition I_k expressed by sentence I_k be $\{\langle I_k, B_j \rangle : 1 \leqslant j \leqslant n\}$ and similarly let $B_k = \{\langle I_i, B_k \rangle : 1 \leqslant i \leqslant m\}$. The set of worlds is linearly ordered in some way. The probabilities are as follows: for simplicity, let the input be chosen randomly, hence $P(I_i)$

should be $(1/m)$. So let $P(\{\langle I_i, B_j\rangle\})=(1/m)\, p_{ij}$.

$$P(I_k)=\sum_j (1/m)\, P_{kj}=(1/m)\sum_j p_{kj}=(1/m)$$

$$P(B_k)=\sum_i (1/m)\, p_{ik}=(1/m)\sum_i p_{ik}$$

$$P(I_k \cap B_j)=P(\{\langle I_k, B_j\rangle\})=(1/m)\, p_{kj}$$

$$P(B_j \mid I_k)=P(I_k \cap B_j)/P(I_k)= \\ =[(1/m)\, p_{kj}]/(1/m)= \\ =p_{kj}$$

all as it should be.

Now we consider the nearest ordering. It is linear, and since the set is finite, discrete. So there is for each world $\langle I_i, B_j\rangle$ a nearest I_k world, namely $\langle I_k, B_{kij}\rangle$ where kij is a number between 1 and n inclusive. The restriction on this is that if $i=k$ then $kij=j$. Apart from that, you would think, it could be anything.

$$(I_k \rightarrow B_1)=\{\langle I_i, B_j\rangle : kij=1\}$$

$$P(B_1 \mid I_k)=P(I_k \rightarrow B_1)=\sum_{ij}\{P(\langle I_i, B_j\rangle) : kij=1\}.$$

So

$$P_{k1}=\sum_{ij}(i/m)\, p_{ij}\delta(1, kij)$$

where $\delta(x, y)=1$ if $x=y$ and $=0$ otherwise.

For any two indices a and b, we have therefore the equation

$$(1) \qquad p_{ab}=(1/m)\sum_{ij} p_{ij}\delta(b, aij)$$

But that is very surprising!

There are now two questions: will there always be an ordering such that (1) is true? *and* when (1) is true, what is the ordering like? The answer to the first question is *NO*: in fact, it is difficult to cook up an example in which (1) *is* true. As one counterexample, let the matrix in question be

	B_1	B_2	B_3
I_1	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{7}{10}$
I_2	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{5}{9}$

and consider p_{11} in (1). The equation then becomes $\frac{3}{5} = \frac{1}{5} + ?$ and there just are no other entries summing to $\frac{2}{5}$. (But this problem can presumably be met by throwing in a large number of extra possible worlds, with fictional states, and a larger matrix; see the last section of this Appendix.)

3. The minimal logic CE

I will define model structures with probability; in accordance with my own preferences, the algebra of propositions in the model structure is specified directly. The procedure of having a privileged family of propositions (as opposed to the family of *all* sets of possible worlds) is familiar from quantum logic, but has only recently become current in modal logic (for example, in S. K. Thomason).

3.1. *Frames*

Motivation: in a propositional logic of the usual stripe, the propositions (= sets of worlds) form a field but not a Borel field. Any field can be extended to a Borel field; and a function on the original field would not be a probability measure unless it could be extended to a probability measure on the generated Borel field. *Terminology*: a Borel field of subsets of V is a sigma-ring of such subsets, to which V itself belongs. A triple $\langle V, F, P\rangle$ I shall call a *probability space* exactly if P is a probability measure with Borel field F of subsets of V as its domain.

DEFINITION. A *frame* is a triple $\varphi = \langle V, F, P\rangle$ such that $V \neq \Lambda$ and $F \subseteq$ domain of P is a field and $\varphi' = \langle V, \text{domain of } P, P\rangle$ is a probability space.

Note that φ' is a frame also; I shall say that $\varphi \subseteq \varphi'$ just because their second members are so related, and the other members pairwise identical. A member X of F with $P(X) \neq 0$ will be called a *non-zero* set in φ (to be distinguished from *non-empty*).

DEFINITION. Frame $\varphi = \langle V, F, P\rangle$ is *full* exactly if every non-zero set X in φ is the union of a family X^* such that $P \mid X^*$ is *onto* $[0, P(X)]$.

The vertical bar denotes restriction of a function; a probability space is full exactly if every set of measure r has a subset of measure q, for each positive $q \leqslant r$.

THEOREM. Every frame can be homomorphically extended to a full frame (with measure preserved).

Proof. It is necessary only to consider frames which are probability spaces. If $\varphi = \langle V, F, P \rangle$ is thus, let $\varphi^* = \varphi \otimes [0, 1] = \langle V^*, F^*, P^* \rangle$ where $V^* = V \otimes [0, 1]$ the Cartesian product; the members of F^* are the sets $X \otimes E$ with X in F and E a Borel set on $[0,1]$; $P^*(X \otimes E) = P(X).\ \mu(E)$ where μ is the usual 'length' measure on $[0,1]$ with $\mu([a, b]) = |b-a| = \mu([a, b])$ and similarly for other kinds of intervals. This is a probability space – see any textbook under the heading 'Product measure'. The map $X \to X \otimes [0, 1]$ is a set homomorphism of F into F^* and preserves measure: $P(X) = P(X).\ 1 = P(X)\ \mu([0, 1]) = P^*(X \otimes [0, 1])$.

Finally, this is a full frame. For let r be any fraction between zero and one, and Borel set $E = \bigcup_{i=1}^{\infty} e_i$ where the e_i are disjoint intervals. If e_i has end points a and $a+m$ let re_i be the similar interval with end points a and $a+rm$. Then we argue

$$\begin{aligned} P^*\left(X \otimes \bigcup_{i=1}^{\infty} re_i\right) &= \sum_{i=1}^{\infty} P^*(X \otimes re_i) = \\ &= \sum_{i=1}^{\infty} P(X)\ \mu(re_i) = \\ &= \sum_{i-1}^{\infty} P(X)\ r\mu(e_i) = rP^*(X \otimes E) \end{aligned}$$

This suffices; any Borel set differs by measure zero from a countable union of half-open intervals.

Finally, I will call a frame $\langle V, F, P \rangle$ *finite* or *denumerable* if the field F is so.

3.2. *Proposition Algebras*

Let F be a field of subsets of V. I shall call $\langle V, F, \to \rangle$ a *proposition algebra* if $\to$ is a partial operation with domain $G \otimes F$, $G \subseteq F$ and range $\subseteq F$ such that (where defined)

(I) $(A \to B) \cap (A \to C) = (A \to .B \cap C)$

(II) $(A \to B) \cup (A \to C) = (A \to .B \cup C)$

(III) $A \cap (A \to B) = A \cap B$

(IV) $(A \to A) = V.$

Deducible properties of $\rightarrow$, where defined, are:

(tp0) If $A \subseteq B$ then $(A \rightarrow B) = V$
(tp1) $A \rightarrow A = V$; from (p5)
(tp2) $(A \rightarrow B) = A \rightarrow (A \cap B)$; from (p1) and tp1)
(tp3) $(A \rightarrow B) = (A \rightarrow . B \cap C) \cup (A \rightarrow . B \cap \bar{C})$; from (p2)
(tp4) $(A \rightarrow B) \cup (A \rightarrow \bar{B}) = V$

If $\Rightarrow$ has properties (I)–(IV), and we define $A \rightarrow B$ to be $A \Rightarrow B$ when defined and to be $\bar{A} \cup B$ otherwise, for given sets A, B, then $\rightarrow$ also has properties (I)–(IV). This shows that the stipulation about the domain of $\rightarrow$ is rather innocuous.

The definition of proposition algebra is suggested by the properties of a model structure in which the sentences A receive propositions A and $(A \rightarrow B) = \{x : s(A, x) \in B\}$ where s is a (partial) operation subject to the restrictions

(s1) $s(A, x) \in A$
(s2) if $x \in A$ then $s(A, x) = x$

but no others. I do not think all proposition algebras could be produced this way, since $\cap \{(A \rightarrow B_i) : i = 1, 2, \ldots\}$ might be empty while no member of that set is. But could any proposition algebra be homomorphically extended to one so produced? I don't know.

With reference to (III), define $[A, B] = (A \rightarrow B) - (A \cap B)$. Then

(q1) $[A, B \cap C] = [A, B] \cap [A, C]$
(q2) $[A, B \cup C] = [A, B] \cup [A, C]$
(q3) $[A, B] \subseteq \bar{A}$
(q4) $[A, B] \cup [A, \bar{B}] = \bar{A}$
(q5) if $A \subseteq B$ then $[A, B] = \bar{A}$
(tq1) $[A, A] = \bar{A}$
(tq2) $[A, B] = [A, A \cap B]$
(tq3) $[A, B] = [A, B \cap C] \cup [A, B \cap \bar{C}]$

follow from (I)–(IV); and (q1), (q2), (q4), (q5) together imply (I)–(IV). (Note that (q4) implies (q3).) ...

Hence I shall take (q1), (q2), (q4), (q5) as defining a proposition algebra when convenient.

3.3. *Models*

A model is a combination of frame and proposition algebra:

A *model* is a quadruple $M=\langle V, F, P, \rightarrow\rangle$ such that $\langle V, F, P\rangle$ is a frame and $\langle V, F, \rightarrow\rangle$ is a proposition algebra, and

(m1) $A \rightarrow B$ is defined at least if A and B are in F and A is non-zero

(m2) $P(A\rightarrow B)=P(A\cap B)/P(A)$ if $P(A)\neq 0$ for all A, B in F.

The proviso on (m2) is not operative if $\rightarrow$ is not defined except as required by (m1), but there is no need to be so stringent. I am willing to say even that $P(A\rightarrow B)=P(B/A)$ whenever $A\rightarrow B$ is defined, for I regard conditional probability as an undetermined concept when the antecedent is a zero set; $P(\Lambda/\Lambda)$ and $P(B/\Lambda)$, for example, may be any two numbers you like between 0 and 1 inclusive, or undefined if you like. The mathematical theory of probability certainly requires no choice among these options.

THEOREM. If φ is a denumerable full frame, then there is a model $M=\langle\varphi', \rightarrow\rangle$ such that $\varphi\leqslant\varphi'$.

The proof will be long and I shall begin with intuitive commentary and lemmas. Let $\varphi=\langle V, F, P\rangle$ be a denumerable full frame (recall that this means that F is denumerable). For A and B in F and A non-zero, I shall want to choose a set $[A, B]\subseteq\bar{A}$.

LEMMA 1. $\dfrac{P(A\cap B)}{P(A)}-P(A\cap B)\leqslant 1-P(A)$ if $P(A)\neq 0$

For let $P(A\cap B)=y$ and $P(A)=x=y+m$ then it is necessary to show that $(y/x)-y\leqslant 1-x$ hence that $f(y)=(x^2-xy+y)\leqslant x$. Actually this function increases from x^2 when $y=0$ to x when $y=x$:

$$x^2-xy+y\leqslant x^2-x(y+n)+(y+n)$$
$$\leqslant x^2-xy-xn+y+n$$
$$\text{because } 0\leqslant n-xn \text{ for } 0\leqslant x\leqslant 1.$$

So it appears that for $[A, B]$ we can choose an appropriately large subset of $\bar{A}$.

Now let us go on to $[A, C]$, $[A, D]$, and so on for all the other sets in F. Well, in choosing $[A, B]$ we also chose $[A, \bar{B}]$, namely $\bar{A}-[A, B]$. To choose $[A, C]$ it will suffice to choose $[A, B \cap C]$ and $[A, \bar{B} \cap C]$; see (tq3). So we can choose $[A, B \cap C]$ and $[A, \bar{B} \cap C]$; then $[A, C]$ is determined and we choose $[A, B \cap C \cap D]$, $(A, \bar{B} \cap C \cap D]$, $[A, B \cap \bar{C} \cap D]$, $[A, \bar{B} \cap \bar{C} \cap D]$ and that determines $[A, D]$, and so on. The question is only whether $[A, B \cap C]$, which is a subset of $[A, B]$ is not meant, ever, to have a probability greater than that of $[A, B]$. No:

LEMMA 2. $\dfrac{P(A \cap B \cap C)}{P(A)} - P(A \cap B \cap C) \leqslant \dfrac{P(A \cap B)}{P(A)} - P(A \cap B)$ when $P(A) \neq 0$

For let $P(A \cap B \cap C) = z$; $P(A \cap B) = y = z + m$; $P(A) = x = y + n$. Then

$$
\begin{aligned}
(z/x) - z &\leqslant (y/x) - y \\
&\leqslant (z + m/x) - (z + m) \\
&\leqslant (z/x) + (m/x) - z - m
\end{aligned}
$$

because $0 \leqslant (m/x) - m$ when $0 \leqslant x \leqslant 1$.

So it appears that $[A, B_1 \cap \ldots \cap B_{n+1}]$ can be chosen as an appropriately large subset of $[A, B_1 \cap \ldots \cap B_n]$.

After this long preamble – which, I hope, was *inhaltlich* rather than contentious – the proof should be transparent.

Proof of theorem. Let $\varphi = \langle V, F, P \rangle$ be a denumerable full frame and G the domain of P. Given that φ is full, G must be uncountable (having a cardinality at least as high as the range of P). Let F be enumerated as $A_1, \ldots, A_m, \ldots$

The strategy of the proof is to define a series of fields $F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots$ and to define $\rightarrow$ first for F_0, then for F_1, and so on. Here F_0 must be finite and included in F, F_{m+1} is defined to be the least field containing F_m and every set $A \rightarrow B$ with A a non-zero set and A, B both in F_m, and A_{m+1}. The union $F^* = \bigcup_{i=1}^{\infty} F_i$ is then a field closed under $\rightarrow$.

The process of definition of $\rightarrow$ is exactly the same at each stage, since there are no principles of form 'if $R(A, B)$ then $R(A \rightarrow C, B \rightarrow C)$'; in this I fall short of Stalnaker's logic.

This process of definition consists simply in choosing subsets $[A, B]$ of $\bar{A}$ – this yields also $[A, \bar{B}]$, its relative complement; then $[A, B \cap C]$ of $[A, B]$ and $[A, \bar{B} \cap C]$ of $[A, B]$ – which yields also $[A, B \cap \bar{C}]$, $[A, \bar{B} \cap \bar{C}]$; and so on. In each case, one chooses a subset of appropriate measure, which is possible given the above lemmas and the reflection that the frame is full.

I will add only this *stipulation*: if choosing subset X of set Y in the above construction, then X must be Λ if the appropriate measure is zero, and X must be Y if that measure is $P(Y)$.

The quadruple $\langle V, F^*, P, \rightarrow \rangle$ is now clearly a model, provided only that $\langle V, F^*, \rightarrow \rangle$ is a proposition algebra. It suffices to check (q1), (q2), (q4), (q5). The operation $[A_i, A_j]$ for A_i non-zero is defined to be the union of the sets $[A_i, A_1^* \cap \ldots \cap A_{j-1}^* \cap A_j]$ where A_k^* is either A_k or $\bar{A}_k$. These are all disjoint sets, and subsets of $\bar{A}_i$. The definition guarantees (q1) and (q2) immediately. Also (q4) holds because the union of *all* the sets $[A_i, A_1^* \cap \ldots \cap A_{j-1}^* \cap A_j^*]$ covers $\bar{A}$.

Finally, if $A_i \subseteq A_j$ then

$$\frac{P(A_i \cap A_1^* \cap \ldots \cap A_{j-1}^* \cap \bar{A}_j)}{P(A_i)} = 0$$

so (because of my stipulation above) the sets of which $[A_i, A_j]$ is composed by union already cover $\bar{A}$. This establishes (q5).

3.4. *Logical System*

The logical system which corresponds to the proposition algebras is CE;

(A1) Axiom schemata as for propositional calculus

(A2) $\vdash (A \rightarrow B) \equiv (A \rightarrow . A \,\&\, B)$

(A3) $\vdash A \rightarrow (B \supset C) \supset . (A \rightarrow B) \supset (A \rightarrow C)$

(A4) $\vdash (A \rightarrow B) \supset (A \supset B)$

(A5) $\vdash (A \,\&\, B) \supset (A \rightarrow B)$

(A6) $\vdash (A \rightarrow B) \vee (A \rightarrow \neg B)$

(R1) $A, A \supset B \vdash B$

(R2) If $\vdash A \equiv B$ then $\vdash (C \rightarrow A) \equiv (C \rightarrow B)$

(R3) If $\vdash A \equiv B$ then $\vdash (A \rightarrow C) \equiv (B \rightarrow C)$

(R4) If $\vdash A \supset B$ then $\vdash A \rightarrow B$

Here (A2) and (R2) are redunbant. CE is sound for the described family of models.

Details. Let $M=\langle V, F, P, \rightarrow\rangle$ be a model and v a map of the sentences into F such that $v(A \& B)=v(A)\cap v(B)$, $v(\neg A)=v-v(A)$, and $v(A\rightarrow B)= =v(A)\rightarrow v(B)$ where defined, and $=v(A\supset B)$ otherwise. Then all the theorems of CE receive value V:

(ad A3) $v(A)\rightarrow v(B\supset C).\cap.v(A)\rightarrow v(B)=v(A)\rightarrow.v(B\supset C)\cap \cap v(B)\subseteq v(A)\rightarrow v(C)$ by (I) and (II); if $(A\rightarrow(-))$ is not defined, similarly for the horseshoe.

(ad A4) $v(A)\cap v(A\rightarrow B)\subseteq v(B)$ in either case; see (III)

(ad A5) see (III)

(ad A6) see (tp4)

(ad R2 and R3) if $v(A\equiv B)=V$ then $v(A)=v(B)$

(ad R4) if $v(A\supset B)=V$ then $v(A)\subseteq v(B)$; see (tp0).

Completeness. This is proved by putting together the usual completeness proof for CE and Stalnaker's completeness proof for his probability semantics.[2] Let Σ be the set of all maximal consistent CE theories, $\Sigma(A)=\{\alpha\in\Sigma\colon A\in\alpha\}$ and define $\Sigma(A)\rightarrow\Sigma(B)=\Sigma(A\rightarrow B)$. Finally, for an arbitrary member α of Σ and any subset X of Σ, let $P(X)=1$ if $\alpha\in X$ and $=0$ otherwise. Then $\langle\Sigma, F=\{\Sigma(A)\colon A \text{ a sentencel}\}, P, \rightarrow\rangle$ is a model.

Details. That F is a field follows from (A1). That $\langle\Sigma, F, \rightarrow\rangle$ is a proposition algebra:

(l1) $\vdash A\rightarrow(B \& C).\equiv.(A\rightarrow B) \& (A\rightarrow C)$ via (A3), (R4)

(l2) $\vdash A\rightarrow(B\vee C).\equiv.(A\rightarrow B)\vee(A\rightarrow C)$ via (A3) and (A6)

(l3) $\vdash A \& (A\rightarrow B).\equiv.A \& B$ via (A4) and (A5)

(l4) $\vdash(A\rightarrow B)\vee(A\rightarrow\neg B)$: see (A6)

(l5) if $\vdash A\supset B$ then $\vdash A\rightarrow B$; see (R4).

This yields conditions (I)–(IV). That P is a probability measure on the powerset of Σ is trivial. That F is closed under $\rightarrow$ is also trivial. So only condition (m2) remains. Suppose $P(\Sigma(A))\neq 0$. Then it equals 1 and $A\in\alpha$. But then $A\rightarrow B\in\alpha$ iff B is in α too, via (l3). above or via (A4) and (R5). So then $P(\Sigma(A)\rightarrow\Sigma(B))=1$ iff $P(\Sigma(A)\cap\Sigma(B))=1$; otherwise both are zero. So for X, $Y\in F$, we have $P(X\rightarrow Y)=P(X\cap Y)/P(X)$ when $P(X)\neq 0$.

This ends the proof.

4. Stalnaker Bernoulli Models

In Section 1 above, Stalnaker algebras are equationally defined. They differ from proposition algebras in general by obeying two further principles; and also by having → everywhere defined (not a partial operation, as I have allowed in Section 3 above).

We begin with a probability space $\langle K, F, P\rangle$ as in the preceding section of this Appendix. From this we form the structure $M=\langle K^*, F^*, P\rangle$; the symbol '$P$' is used ambiguously between the original measure and the product measure. The construction of M is the usual product construction (for sequences of stochastically independent events, as in repeated tosses of a die). Hence: K^* is the set of denumerable sequences of members of K (here identified with maps π of the natural numbers into K, but also depicted as $\pi=\langle\pi(1), \pi(2), \pi(3), \ldots\rangle$). F^* is the family of sets *(generating sets)*

$$A_1\otimes\cdots\otimes A_n\otimes K^*=\{\pi\in K^*: \pi(1)\in A, \&\ldots\& \ \pi(n)\in A_n\}$$

formed from sets $A_1,\ldots, A_n$ in F. Finally P is the product measure with domain F^* (which is a Borel field because F is) defined by

$$P(A_1\otimes\cdots\otimes A_n\otimes K)=P(A_1)\ldots P(A_n).$$

If π is in K^*, let π_m be the sequence defined by:

$$\pi_m(k)=\pi(m+k-1)$$

so that $\pi_m=\langle\pi(m), \pi(m+1),\ldots\rangle$. Define the operation → on the powerset of K^* by

$$X\rightarrow Y=\{\pi\in K^*: \text{for all } m, \text{ if } m \text{ is the first natural number such that } \pi_m \text{ is in } X, \text{ then } \pi_m \text{ is in } Y\}$$

(For the case in which there is no X-world 'accessible to' π – i.e. no m such that π_m is in X – Stalnaker introduced an 'absurd world', but the same effect is gotten by the present definition; namely that π is in $X\rightarrow Y$ in that case for any Y.)

Now, F^* is closed under →. For define $X(k)=K^{k-1}\otimes X$ with $X(1)=X$. Note that if X is a generating set, so is $X(k)$; moreover, $\bar{X}(k)=\overline{X(k)}$ and $\bigcup_{i=1}^{\infty}[X_i(k)]=[\bigcup_{i=1}^{\infty} X_i](k)$. So not only for the generating sets, but for all sets generated from them in the Borel way, we find that $X(k)$ is in F^* if

X is. As a special case, define $X(0)=K$. Then we have

$$X \Rightarrow Y = \bigcup_{k\,1}^{\infty} \left[\bigcap_{i=1}^{k-1} \bar{X}(i) \cap (X \cap Y)(k) \right]$$

as the set of worlds π such that for some i, π is in $X(i)$ *and* π is in $(X \to Y)$. So $X \Rightarrow Y$ differs from $X \to Y$ only by leaving out the 'impossible antecedent' case. Therefore

$$X \to Y = (X \Rightarrow Y) \cup \left[\bigcap_{i=1}^{\infty} \bar{X}(i) \right]$$

All the operations thereby used to define $\to$ are such that a Borel field is closed under them; hence F^* is closed under $\to$.

I turn now to probabilities, and shall designate sets $A \otimes K^*$ with A in F as *zero-degree propositions*. Moreover, abbreviate '$A \otimes K$' to 'A' when convenient; the capital letters A, B, C, D are to stand only for zero-degree propositions and members of F (via this symbolic identification).

LEMMA 1. $P(A \to X) = P(A \Rightarrow X)$ when $P(A) \neq 0$. The reason is that $P(\bigcap_{i=1}^{\infty} \bar{A}(i)) = P(\bar{A}) \ldots P(\bar{A}) \ldots$ which equals zero unless $P(\bar{A}) = 1$.

LEMMA 2. (Fraction Lemma)[4] In any probability space $\langle W, G, P \rangle$, if $P(A) \neq 0$ then $\sum_{k=0}^{\infty} P(\bar{A})^k = 1/P(A)$.

For proof consider $s = P(A) \sum_{k=0}^{\infty} P(\bar{A})^k =$

$$\begin{aligned}
&= P(A) + P(\bar{A})\, P(A) + P(\bar{A})^2\, P(A) + \cdots = \\
&= P(A \otimes K^*) + P(\bar{A} \otimes A \otimes K^*) + \cdots = \\
&= P\left(A \otimes K^* \cup \left[\bigcup_{m=1}^{\infty} \bar{A}^m \otimes A \otimes K^* \right] \right)
\end{aligned}$$

because the sets in the third line are disjoint. We note that we can continue these equations as

$$\begin{aligned}
&= P(\{\pi \in K^* \colon \pi(m) \in A \text{ for some } A\}) \\
&= P(K^* - \{\pi \in K^* \colon \pi(m) \in \bar{A} \text{ for all } m\}) \\
&= 1 - P(\bar{A} \otimes \bar{A} \otimes \bar{A} \otimes \ldots) \\
&= 1
\end{aligned}$$

since we are given that $P(A)\neq 1$. But from the fact that $s=1$, the lemma follows.

THEOREM. $P(A\to B)=P(B/A)$ if $P(A)\neq 0$, for all zero degree propositions A and B.

Intuitively: let m be the first number such that $\pi(m)$ is in A. The probability that $\pi(m)$ is in B is then $P(B/A)$, if B too is in F, the consequtive events $\pi(1), \pi(2), \ldots$ being independent. Now m could be 1, or 2, or 3,... the probabilities of these disjoint cases being $P(A)$, $P(\bar{A})\,P(A)$, $P(\bar{A})\,P(A)\,P(A)$, and so forth. Hence we calculate

$$\begin{aligned} P(A\Rightarrow B) &= P(A)\,P(B/A)+P(\bar{A})\,P(A)\,P(B/A)+ \\ &\quad + P(\bar{A})^2\,P(A)\,P(B/A)+\cdots= \\ &= P(A)\sum_{k=0}^{\infty} P(\bar{A})^k\,P(B/A)= \\ &= P(B/A) \end{aligned}$$

by the Fraction Lemma.

LEMMA 3. $P(A\to C.\to B)=P(A\to C.\Rightarrow B)$ if $P(A\to C)\neq 0$. This is a generalization of Lemma 1, the proof being similar and hinging on the point that $P(\bigcap_{m=1}^{\infty}(A\to C)(m))=P(Z)=0$. Now the set Z introduced by definition contains the sequences π such that for each number m, there are one or more numbers $n>m$ such that $\pi(n)\in A$, while the first of these is not in C. So a fortiori, none of them is in C. So a somewhat bigger set than Z is

$$Z_1=\{\pi\colon [\text{for all } m,\ \pi(m)\in\bar{A}\cup\bar{C}]\ \&\ [\text{for some } m,\ \pi(m)\ A]\}$$

and a still bigger set is

$$Z_2=\{\pi\colon \text{for all } m,\ \pi(m)\in\bar{A}\cup\bar{C}\}.$$

Now $P(Z_2)=0$ unless $P(\bar{A}\cup\bar{C})=1$. But if $P(\bar{A}\cup\bar{C})=1$ then $P(A\to\bar{C})=1$, so then $P(A\to C)=0$ – which is ruled out – or else $P(A)=0$. In both cases, however, we see that $P(Z_2)=0$; hence also $P(Z)=0$.

THEOREM. $P(A\to C.\to B)=P(B/A\to C)$ if $P(A\to C)\neq 0$ for zero-degree propositions A, B, C.

By the preceding lemma, we may concentrate on $P(A\to C.\Rightarrow B)$. Let the first number m such that π_m is in $A\to C$ be k; the probability that π_m is in B is entirely independent of the initial sequent $\pi(1),\ldots \pi(m-1)$; hence this is just $P(B/A\to C)$ – more precisely, in this case, it is $P(B\otimes K^*/(A\otimes K^*)\to\to(C\otimes K^*))$, the very probability that *any*, random, sequence in K^* is in B if it is in $A\to C$. The cases are $K=1, 2,\ldots$; and these cases can be described for $k>1$ as:

(a) $\pi(1),\ldots, \pi(k-2)$ are in $\bar{A}\cup\bar{C}$ [if any]
(b) $\pi(k-1)$ is in $A\cap\bar{C}$
(c) $\pi(k)$ is in $A\to C$

while for case $k=1$, we leave out (a) and (b). So we have the calculation (omitting intersection signs where convenient, and so replacing also $\bar{A}\cup\bar{C}$ by $\overline{AC}$):

$$\begin{aligned}
&\text{Let } r=P(B/A\to C)\\
&P(A\to C\Rightarrow B)=P(A\to C)\,r+P(\overline{AC})\,P(A\to C)\,r+\\
&\qquad +P(\overline{AC})\,P(A\bar{C})\,P(A\to C)\,r+\\
&\qquad +P(\overline{AC})^2\,P(A\bar{C})\,P(A\to C)\,r+\cdots=\\
&\qquad =P(A\to C)\,r\left[1+\left(\sum_{k=0}^{\infty}P(\overline{AC})^k\right)P(A\bar{C})\right]=\\
&\qquad =P(A\to C)\,r\left[1+\frac{P(A\bar{C})}{P(AC)}\right]
\end{aligned}$$

by the Fraction Lemma for case $P(AC)\neq 0$; set this

$$=P(A\to C)\,rs$$

but then

$$s=\frac{P(AC)+P(A\bar{C})}{P(AC)}=\frac{P(A)}{P(AC)}=1/P(C/A)=$$
$$=1/P(A\to C)$$

by the preceding theorem. Hence

$$P(A\to C\Rightarrow B)=P(A\to C)\,rs=r=P(B/A\to C).$$

Now in the remaining case, $P(AC)=0$. In that case $P(A\to C)=0$ – which is ruled out – unless $P(A)=0$. But if $P(A)=0$ then $P(A\to C)=1$ and

$P(B/A \to C) = P(B)$. However, in that case also $P(A \to C \Rightarrow B) = P(B)$, because, except for a set of measure zero, any sequence π we pick will be in $A \to C$. More precisely, if $P(X) = 1$, then

$$P(X \Rightarrow Y) = P\left(\bigcup_{k=1}^{\infty} \left[\bigcap_{i=0}^{k-1} \bar{X}(i) \cap (X \cap Y)(k)\right]\right)$$

which is the sum of the terms

$$P(X \cap Y(1)) + P(X(1) \cap (X \cap Y)(2)) + \cdots$$

of which all but the first has probability zero; and $P(X \cap Y(1)) = P(X \cap Y) = P(Y)$ because $P(X) = 1$.

So the theorem holds in all cases. Let me call $A \to C. \to B$ a *left* conditional and $C \to .A \to B$ a *right* conditional. To prove a corresponding result for right conditionals I need one more lemma.

LEMMA 4. (Independence Lemma) $P(\bar{A} \cap C \cap A \to B) = P(\bar{A}C)\, P(A \to B)$ for zero-degree propositions A, B, C.

Consider first $P(A) = 0$. Then the lemma holds for any C, because each side equals zero.

Consider next $P(A) \neq 0$. Abbreviate $\bar{A} \otimes \bar{A} \otimes \ldots$ as $\bar{A}^*$. Then $A \to B = \bar{A}^* \cup (A \Rightarrow B)$, so

$$P(\bar{A}C \cap (A \to B)) = P(\bar{A}C \cap \bar{A}^* \cup \bar{A}C \cap (A \Rightarrow B)) = P(\bar{A}C \cap (A \Rightarrow B))$$

because $P(\bar{A}^*)$ and hence $P(\bar{A}C \cap \bar{A}^*)$, equals zero. Now $\bar{A}C = \bar{A}C(1)$ while $(A \Rightarrow B) = AB(1) \cup \bar{A}(1)\, AB(2) \cup \bar{A}(1)\, \bar{A}(2)\, AB(3) \cup \ldots$. Hence $\bar{A}C \cap (A \Rightarrow B) = \Lambda \cup \bar{A}C(1)\, AB(2) \cup \bar{A}C(1)\, \bar{A}(2)\, AB(3) \cup \ldots$. Hence also

$$\begin{aligned}P(\bar{A}C \cap A \Rightarrow B) &= P(\bar{A}C)\, P(AB) + P(\bar{A}C)\, P(\bar{A})\, P(AB) + \\ &\quad + P(\bar{A}C)\, P(\bar{A})^2\, P(AB) + \cdots = \\ &= P(\bar{A}C)\, P(AB) \sum_{k=0}^{\infty} P(\bar{A})^k = \\ &= P(\bar{A}C)\, P(AB)/P(A) = \\ &= P(\bar{A}C)\, P(B/A) = \\ &= P(\bar{A}C)\, P(A \to B)\end{aligned}$$

as required.

THEOREM. $P(C \rightarrow A \rightarrow B) = P(A \rightarrow B/C)$ if $P(C) \neq 0$, for zero-degree propositions A, B, C.

By Lemma 1, we can concentrate on $C \Rightarrow A \rightarrow B$. Consider first $P(A)=0$. Then $P(A \rightarrow B)=1$ as we have seen; so $P(A \rightarrow B/C)=1$ also. But in addition, $P(C \Rightarrow A \rightarrow B)=1$ because in general, if $P(Y)=1$ then the probability that, for random π, the first m such that π_m is in C, is also in Y, equals 1. So consider henceforth that $P(A) \neq 0$. Now $C \Rightarrow A \rightarrow B$ is the union of the disjoint sets

$$\begin{aligned} X_k &= \bar{C}(1) \ldots \bar{C}(k-1)\,[C \cap (A \rightarrow B)]\,(k) = \\ &= \bar{C}(1) \ldots \bar{C}(k-1)\,\{[CAB(k) \cup C\bar{A}(k)\,AB(k+1) \cup \\ &\quad \cup C\bar{A}(k)\,\bar{A}(k+1)\,AB(k+2) \cup \ldots] \cup C(k)\,\bar{A}^*(k)\}. \end{aligned}$$

Because $P(A) \neq 0$, $\bar{A}^*(k)$ has probability zero, so that term can be ignored. Thus

$$\begin{aligned} P(X_k) &= P(\bar{C})^{k-1}\,[P(CAB) + P(C\bar{A})\,P(AB) + \\ &\quad + P(C\bar{A})\,P(\bar{A})\,P(AB) + P(C\bar{A})\,P(\bar{A})^2\,P(AB) + \cdots] = \\ &= P(\bar{C})^{k-1}\,[P(CAB) + P(C\bar{A})\,P(AB) \sum_{m=0}^{\infty} P(\bar{A})^m] = \\ &= P(\bar{C})^{k-1}\,[P(CAB) + P(C\bar{A})\,P(AB)/P(A)] = \\ &= P(\bar{C})^{k-1}\,[P(CAB) + P(C\bar{A})\,P(A \rightarrow B)] = \\ &= P(\bar{C})^{k-1}\,[P(CAB) + P(C\bar{A} \cap A \rightarrow B)] \end{aligned}$$

(by the Independence Lemma),

$$= P(\bar{C})^{k-1}\,[P(CA \cap A \rightarrow B) + P(C\bar{A} \cap A \rightarrow B)]$$

(because $A \cap (A \rightarrow B) = AB$),

$$= P(\bar{C})^{k-1}\,[P(C \cap A \rightarrow B)].$$

Now we can finish the calculation as follows:

$$\begin{aligned} P(C \Rightarrow A \rightarrow B) &= P\left(\bigcup_{k=1}^{\infty} X_k\right) \\ &= \sum_{k=1}^{\infty} P(\bar{C})^{k-1}\,P(\cap A \rightarrow B) = \\ &= P(C \cap A \rightarrow B)/P(C) = \\ &= P(A \rightarrow B/C) \end{aligned}$$

as required.

The conclusion is that any antecedently given probability assignment to the zero-degree propositions can be reflected in a Stalnaker model, in such a way that the first degree conditionals, and what I have called left and right (second degree) conditionals are in accord with the Stalnaker Thesis.

NOTES

[1] The research for this paper was supported by Canada Council grant S72-0810. I also wish to thank Yvon Gauthier, Ian Hacking, William Harper, David Lewis, Robert Meyer, and Richmond Thomason for helpful comments on earlier drafts of the Appendices. Further debts to Hacking and Lewis are detailed below; it will be clear that my reaction to Lewis' writings colours almost every section.
[2] I give the Stalnaker semantics essentially as simplified by Lewis in his [3].
[3] This refers to comments by Thomason at the end of [11], and my [12, 13]. See also the critical comments by Lewis on this in his [4, 5].
[4] See the Appendix to my [13].
[5] Lewis [6], p. 10.
[6] See also my discussion in [13].
[7] Lewis [4], p. 84
[8] I realize that on some views, the scientist does not make theoretical commitments, and does not believe his theories. So this criticism is based on my views on scientific theories; see [13].
[9] Stalnaker [7].
[10] Lewis [6], p. 5.
[11] Given in an unpublished paper circulated in April 1972.
[12] (*Added January 1974.*) I have just seen a proof by Stalnaker that no such result as Theorem 25 can hold for his C2.

Notes to Appendix

[1] In his 'Completeness and Decidability…'. I have modified the axioms and rules in some obvious ways to ensure continuity with the remainder of my discussion.
[2] In his 'Probability and Conditionals'; this proof is not affected by the apparent vulnerability of Stalnaker's paper to Lewis' triviality results.
[3] *Ibid.*
[4] This is the lemma for which I am indebted to Ian Hacking.

BIBLIOGRAPHY

[1] Chisholm, R., 'The Contrary-to-Fact Conditional', *Mind* **55** (1946), 289–302.
[2] Goodman, N., 'The Problem of Counterfactual Conditionals', *Journal of Philosophy* **44** (1947), 113–128.
[3] Lewis, D., 'Completeness and Decidability of Three Logics of Counterfactual Conditionals', *Theoria* **37** (1971), 74–85.
[4] Lewis, D., *Counterfactuals*, Blackwell, Oxford, 1973.
[5] Lewis, D., 'Counterfactuals and Comparative Possibility', *Journal of Philosophical Logic* **2** (1973), in No. 2–4.

[6] Lewis, D., 'Probabilities of Conditionals and Conditional Probabilities' (ditto'd), Princeton, March 1973.
[7] Stalnaker, R., 'Probability and Conditionality', *Philosophy of Science* **37** (1970), 64–80.
[8] Stalnaker, R., 'A Theory of Conditionals', in N. Rescher (ed.), *Studies in Logical Theory* (APQ Supplementary Volume), Blackwell, Oxford, 1968.
[9] Stalnaker, R. and Thomason, R. H., 'A Semantic Analysis of Conditional Logic', *Theoria* **36** (1970), 23–42.
[10] Teller, P., 'Conditionalization, Observation, and Change of Preference', this volume, p. 205.
[11] Thomason, R. H., 'A Fitch-Style Formulation of Conditional Logic', *Logique et Analyse* **13** (1970), 397–412.
[12] van Fraassen, B. C., 'Hidden Variables in Conditional Logic', *Theoria*, forthcoming.
[13] van Fraassen, B. C., 'Theories and Counterfactuals', forthcoming in a festschrift for W. Sellars (ed. by H-N. Castañeda).

DISCUSSION

Commentator: Giere: I should like you to fill out the motivations to the paper a little more. We have two paradigms for probability, the ticker tape paradigm and the sweepstake paradigm. In both cases it seems that we can say all that needs to be said using regular conditional probabilities. So why should we bother looking for a way of assigning probabilities to conditional statements?

Van Fraassen: Actually they arise in both. If you look upon conditionals the way Ramsey did, then iterated conditionals don't make sense, but there are many practical situations in which there are iterated conditionals which people do assert. The question arises whether we should assign probabilities to these iterated conditionals, and this is a question whether we should extend the probability calculus to these English sentences. In my own view we should extend probabilities to such sentences. For example, I may say "if this glass breaks when I drop it on the floor then these glasses will break if I throw them against that wall". Then we have a claim that is surely not a necessary truth and so one wants to know the probability of its being correct.

Giere: Still it is not clear why we cannot equally well regard the cases you mention as examples of complex systems for which we need never ask about the probability of an iterated conditional but rather simply about probabilities of various outcomes of trials on the entire complex system.

Van Fraassen: Yes there are some philosophers who claim that we get by in science with very little resources in this respect – if they are right, then the answer to the question doesn't carry a lot of practical weight. But the problem of explicating probability discourse in general remains, for philosophy of language and of logic, if not of science.

STALNAKER TO VAN FRAASSEN

January 22, 1974

Dear Bas:

When David Lewis brought in his artillary and fired his bombshell at Stalnaker's assumption, Stalnaker meekly capitulated. But Van Fraassen, chivalrously, has rushed to its defense, opening a second front by uncovering a different assumption which is involved, and diverting the fire to that one in the hope of saving the first. It is time for me to reenter the battle – this time on the other side.

I am not sure whether I am a "metaphysical realist" in the relevant sense, or whether the assumption made by Lewis (and by me) that you criticize is necessarily connected to such realism. But I am convinced by your argument that the assumption was essential to Lewis's argument, that it was essential to my argument using probability semantics to justify C2, and that the assumption is untenable. But I am not tempted to reembrace Stalnaker's assumption, since I can get the same trivialization result from it without "metaphysical realism," as long as I assume that the logic of conditionals is C2.

I list the following six theses for reference. A, B and C are any *propositions.* P is any *probability function.* In my original paper, I used probability functions which allowed non-trivial conditional probability values even when the absolute probability of the condition was zero. This complication is irrelevant to the current discussion, and so I will ignore it. It may be assumed that the conditional probability, $P(B/A)$, is defined only when $P(A) \neq 0$. A *subfunction*, P_A, is a function defined for any probability function P and proposition A such that $P(A) \neq 0$ as follows: $P_A(B) =_{df} P(B/A)$.

(1) If $P(A) \neq 0$, $P(A > B) = P(B/A)$.

(2) Any subfunction is a probability function.

(3) 'Metaphysical Realism': The proposition expressed by a conditional sentence is independent of the probability function defined on it. So, for example, the content of the sentence $B > C$ is the same in the context $P(B > C)$ as it is in the context $P_A(B > C)$.

(4) If $P(A \& C) \neq 0$, $P(A > B/C) = P(B/A \& C)$.

(5) The logic of the conditional corner is C2.
(6) For any probability function, there are at most two disjoint propositions which have non-zero probability.

The aim of my original paper was to derive (5) from (1), and so to provide an independent motivation for C2. My argument assumed (2) more or less explicitly, and (3) implicitly. (2) is not really an assumption, since it follows from elementary probability calculus, and depends on no special assumptions about conditional propositions. Lewis's argument derived (6) from (4). Since (4) follows from the premisses of my argument, this was sufficient to defeat my project. The argument generalizing (1) to (4) goes as follows:

1. $P_C(A \,\&\, B) = P_C(A) \times P_C(B/A)$
2. $\quad = P_C(A) \times P_C(A > B)$
3. $\quad = P(A/C) \times P(A > B/C)$
4. $P_C(A \,\&\, B) = P(A \,\&\, B/C)$
5. $\quad = P(A/C) \times P(B/A \,\&\, C)$
6. So, assuming $P(A/C) \neq 0$, $P(A > B/C) = P(B/A \,\&\, C)$.

How does thesis (3) enter into this argument? Without (3), the move from step 2 to step 3 will involve an equivocation, since $A > B$ need not express the same proposition in both places. So Lewis's argument, which makes no assumptions about the logic of conditionals, does depend on (3). But another argument gets exactly the same conclusion from (1) and (5), without (3), or (2).

Assumptions: (1), (5), and the denial of (6). The argument will show that a contradiction can be derived from these assumptions. By the denial of (6), there are at least three disjoint propositions that are assigned non-zero probability by some probability function. Call them $A \,\&\, B$, $A \,\&\, \bar{B}$, and $\bar{A}$.

I will make use of the following abbreviation:

$$C =_{\mathrm{df}} A \vee (\bar{A} \,\&\, (A > \bar{B})),$$

and the following lemmas which can be proved from the assumptions (1) and (5):

(7) If $P(\bar{X}) \neq 0$, then $P(X > Y/\bar{X}) = P(X > Y)$.
(8) $\bar{C}$ entails $C > \sim(A \,\&\, \bar{B})$.

Finally, since $\bar{A}$ entails $(\bar{A} \,\&\, (A > B)) \vee (\bar{A} \,\&\, (A > \bar{B}))$, and $P(\bar{A}) \neq 0$, it follows that either $P(\bar{A} \& (A > B)) \neq 0$ or else $P(\bar{A} \& (A > \bar{B})) \neq 0$. I will assume the former, but in case it is instead the latter, interchange B and $\bar{B}$ everywhere in the argument, including in the definition of C.

Now, from all these assumptions, it follows that

(9) The following propositions all have non-zero probability: $A \& B$, $A \& \bar{B}$, $\bar{A}$, C, $\bar{C}$.

Now for the inconsistency argument:

1.	$P(C > \sim(A \& \bar{B})/\bar{C}) = 1$	by (8)
2.	$P(\bar{C}) \neq 0$	by (9)
3.	$P(C > \sim(A \& \bar{B})) = 1$	from 1, 2, by (7)
4.	$P(C) \neq 0$	by (9)
5.	$P(\sim(A \& \bar{B})/C) = 1$	from 3, 4, by (1)
6.	$\dfrac{P(C \& \sim(A \& \bar{B}))}{P(C)} = 1$	from 5
7.	$\dfrac{P((A \& B) \vee (\bar{A} \& (A > \bar{B})))}{P((A \& B) \vee (A \& \bar{B}) \vee (\bar{A} \& (A > \bar{B})))} = 1$	from 6
8.	$P(A \& \bar{B}) = 0$	from 7

which contradicts (9).

Note that if we had substituted B for $\bar{B}$ everywhere, the conclusion would have been $P(A \& B) = 0$, which also contradicts (9).

As you say, Stalnaker's logic is one thing, his models are another. The above argument makes no assumptions about models, and so applies to the logic, however interpreted. But if one considers things semantically for a minute, one can see a connection between the above argument and your shielding effect (Notes, III). (My argument was worked out before I went through your Notes III, so the connection was unexpected.)

You define the shielding effect in terms of normal models, but as you point out, the notion is more general. Say that a proposition X *encloses* a proposition Y iff for every possible world $i \in \bar{X}$, $s(X, i) \in \bar{Y}$. Then the following is the generalization of the result you state on page 4 of the Notes: If thesis (1) holds, then there is no pair of propositions X and Y such that $P(X) < 1$, $P(Y) > 0$, and X encloses Y. But, for any probability function defined on any model which assigns non-zero probability to at least three disjoint propositions, there will be a pair of propositions meeting those conditions. The above argument can be taken to show this, since $P(C) < 1$, $P(A \& \bar{B}) > 0$, and C encloses $(A \& \bar{B})$.

So where does this result leave me? Since I want to keep (5), I can avoid disaster only by rejecting (1). But it is not only to avoid disaster that I take this option since I think there are good intuitive arguments against (1). In fact, I am as taken with the distinction between the probability of the conditional and the conditional probability as I once was with their supposed identity.

Where does the result leave you? It does answer, I think, the question you ask on page 297 of your paper. It is not tenable (although it is consistent) to accept the thesis for the C2 conditional. But you don't like C2 that much anyway, so you can just keep (1) and reject (5). But I think this would be a mistake.

Suppose there is a general theoretical proposition T which entails a conditional $A>B$, or perhaps it entails the theoretical claim that $P(A>B)=r$. Since T might be a theory with diverse consequences, the evidence for or against T – and so the evidence which bears on $A>B$, or $P(A>B)=r$ – could be anything. I see no reason to rule out a priori the possibility that A itself be part of the evidence for or against T. If it were, then the conditional might not be stochastically independent of its antecedent (which as you point out the thesis requires). Consider a concrete example: H_1 and H_2 are propositions describing the outcomes of two flips of a coin. Say H_1 says that flip 1 comes up heads, and H_2 says that flip 2 comes up heads. Suppose (as is very plausible) that it is known on the basis of general theoretical considerations that the flips are causally independent of each other, but it is unknown what the bias of the coin is. Suppose it was drawn at random from a bag of coins half of which are biased for heads, half biased to the same degree for tails. On the basis of the causal independence, we know that the truth value of H_2 would have been the same as it in fact was whatever the truthvalue of H_1, and this holds for all possible worlds compatible with the general theoretical assumption of causal independence. Hence, although we may not know the value of $P(H_2)$, we know that $P(H_1>H_2)=P(H_2)$. But H_1 still is *evidence* for H_2, since it is evidence for the proposition that the chosen coin is biased for heads. Thus, in this example, $P(H_2/H_1)>P(H_2)$, and so $P(H_2/H_1)>P(H_1>H_2)$. This result seems intuitively plausible to me, and it does not depend on any assumptions about the logic of conditionals.

I see that (1) is formally compatible with your logic of conditionals, CE, and with any probability distribution defined on all the zero degree sentences. But I am not convinced that any kind of intuitively satisfactory semantics can be given for CE which explains why the thesis should be true. Whatever the truth conditions for conditional propositions (assuming they have *truth* conditions), the effect of the thesis is to impose the following requirement: for any A with probability between 0 and 1 exclusive, and for any B, the ratio of the measure of worlds in which $A > B$ is true to the measure of worlds in which it is false will be the same in the A worlds as it is in the $\bar{A}$ worlds. But why should this be true? One does need some intuitive explanation.

Best.

Yours,

BOB STALNAKER