# PROVABILITY LOGICS FOR RELATIVE INTERPRETABILITY

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## 0. INTRODUCTION

In this paper the system **IL** for relative interpretability described in Visser (1988) is studied.[1] In **IL** formulae $A \rhd B$ (read: A *interprets* B) are added to the provability logic **L**. The intended interpretation of a formula $A \rhd B$ in an (arithmetical) theory T is: $T + B$ is relatively interpretable in $T + A$. The system has been shown to be sound with respect to such arithmetical interpretations (Švejdar 1983, Montagna 1984, Visser 1986, 1988P).

As axioms for **IL** we take the usual axioms $\Box A \rightarrow \Box \Box A$ and $\Box(\Box A \rightarrow A) \rightarrow \Box A$ (*Löb's Axiom*) for the provability logic **L** and its rules, modus ponens and necessitation, plus the axioms:

(1) $\Box(A \rightarrow B) \rightarrow (A \rhd B)$
(2) $(A \rhd B) \wedge (B \rhd C) \rightarrow (A \rhd C)$
(3) $(A \rhd C) \wedge (B \rhd C) \rightarrow (A \vee B \rhd C)$
(4) $(A \rhd B) \rightarrow (\Diamond A \rightarrow \Diamond B)$
(5) $\Diamond A \rhd A$

With respect to priority of parentheses $\rhd$ is treated as $\rightarrow$.

Furthermore, we will consider the following extensions of **IL**:

**ILM** = **IL** + **M**, where **M** is the axiom $(A \rhd B) \rightarrow (A \wedge \Box C \rhd B \wedge \Box C)$
**ILP** = **IL** + **P**, where **P** is the axiom $(A \rhd B) \rightarrow \Box(A \rhd B)$[2]

We will write $\vdash_{\mathbf{IL}}$ for derivability in **IL**, similarly for the other systems, but sometimes we may leave the subscript off.

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[1] We want to thank Albert Visser who inspired these investigations by asking us to try and find a useful semantics for the system **ILM**. We also thank Rineke Verbrugge for a number of corrections.
[2] The scheme **M** is named after Franco Montagna who showed its soundness with respect to **PA**, even in the more general case when $\Box C$ is replaced by a $\Sigma$-formula. The background of the names of the schemes **P** and **W** is semantic and will be explained in the next section.

The object of the whole study, undertaken together with Smoryński and Visser is to obtain for the standard formal systems an analogon of Solovay's theorem: which are the interpretability logics corresponding to **PA**, **GB** etc? Solovay's Theorem shows that the provability logics of all these systems are the same. However, their interpretability logics are not. Smoryński and Visser have shown that the interpretability logic of **GB** and other finitely axiomatizable systems is **ILP**. It is conjectured that **ILM** is the logic of **PA** and other essentially reflexive systems. A third system

$$\mathbf{ILW} = \mathbf{IL} + \mathbf{W}, \text{ where } \mathbf{W} \text{ is the axiom } (A \triangleright B) \rightarrow (A \triangleright B \wedge \Box \neg A)$$

is weaker than both other logics, and is conjectured to embody the principles common to all "reasonable" arithmetics. For more details one should consult Visser's paper in this volume.

In this paper we restrict ourselves to purely modal properties of the systems in question. In section 1 the semantics for the different logics is described. In section 2 the fixed point theorem of **L** is extended to **IL**. In the remaining sections modal completeness theorems are proved for the systems **IL**, **ILP** and **ILM**. The logics also turn out to have the finite model property, so decidability is a consequence. We are still working on a completeness proof for **ILW**.

## 1. SEMANTICS

It is a well-known fact that the modal logic **L** is complete with respect to the **L**-*frames* $<W, R>$, which consist of a set of worlds W together with a transitive conversely well-founded relation R.

**1.1 Definition**. If $<W, R>$ is a partially ordered set and $w \in W$, then $W[w] = \{w' \in W \mid w \, R \, w'\}$.

**1.2 Definition**. An *IL-frame* is a **L**-frame $<W, R>$ with an additional relation $S_w$, for each $w \in W$, which has the following properties:
(i)   $S_w$ is a relation on $W[w]$,
(ii)  $S_w$ is reflexive and transitive,
(iii) if $w', w'' \in W[w]$ and $w' \, R \, w''$, then $w' \, S_w \, w''$,
We will often write S for $\{S_w \mid w \in W\}$.

**1.3 Definition.** An *IL-model* is given by an **IL**-frame $< W, R, S >$ combined with a

forcing relation with the clauses:

$$u \Vdash \Box A \iff \forall v(u\,R\,v \Rightarrow v \Vdash A)$$

$$u \Vdash A \rhd B \iff \forall v(u\,R\,v \text{ and } v \Vdash A \Rightarrow \exists w(v\,S_u\,w \text{ and } w \Vdash B)).$$

**1.4 Definition**.

    (a) We write $\mathbf{F} \vDash A$ iff $\mathbf{F} = \,< W, R, S >$, and $w \Vdash A$ for every $\Vdash$ on $\mathbf{F}$ and $w \in W$.

    (b) If $K$ is a class of frames, we write $K \vDash A$ iff $\mathbf{F} \vDash A$ for each $\mathbf{F} \in K$.

    (c) $K_{\mathbf{W}}$ is the class of frames satisfying

        (iv) for any $w$, the converse of $R \circ S_w$ is wellfounded

    (d) $K_{\mathbf{M}}$ is the class of frames satisfying

        (iv') if $u\,S_w\,v\,R\,z$, then $u\,R\,z$

    (e) $K_{\mathbf{P}}$ is the class of frames satisfying

        (iv") if $u\,S_w\,v$, then $u\,S_{w'}\,v$ for any $w'$ such that $w\,R\,w'$, $w'\,R\,u$.

The next lemma states that the schemes **W** and **P** characterize the classes of frames $K_{\mathbf{W}}$ and $K_{\mathbf{P}}$ respectively. Their names refer to the character of these classes: in $K_{\mathbf{P}}$ the relation $S_w$ is persistent over R.

**1.5 Lemma** (*Soundness*).

    (a) For each A, if $\vdash_{\mathbf{IL}} A$, then $\mathbf{F} \vDash A$.

    (b) For $\mathbf{S} = \mathbf{W}, \mathbf{M}, \mathbf{P}$, respectively, $\mathbf{F} \vDash \mathbf{ILS} \iff \mathbf{F} \in K_{\mathbf{S}}$ (**ILS** *characterizes* $K_{\mathbf{S}}$ ).

    (c) For $\mathbf{S} = \mathbf{W}, \mathbf{M}, \mathbf{P}$, respectively, if $\vdash_{\mathbf{ILS}} A$, then $K_{\mathbf{S}} \vDash A$.

**Proof**. Straightforward.                 ⊠

In Sections 3 and following completeness will be proved for the three systems **IL**, **ILP** and **ILM**. Actually, **ILP** will be proved complete with respect to the more restricted class of frames in which $S_w$ and $S_{w'}$ are identical on the intersection of their domains. We will keep writing **ILS** if we want leave open which system we are aiming at.

**1.6 Example**. For each of the systems above,

$$\nvdash \neg((p \rhd \neg p) \wedge (\neg p \rhd p)).$$

**Proof**. The following is a countermodel:
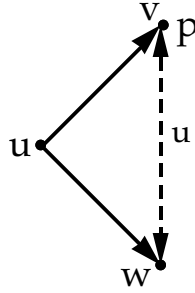
In the above picture only the "extra" arrows for $S_u$ are indicated. Note that in an arithmetical interpretation such a formula would be what is called an *Orey-sentence* (see e.g. Visser 1986). Note also that one could make this model into one in which $S_u$ is antisymmetric; however, the procedure would make the model infinite.    ⊠

In the case of provability logic validity on trees is equivalent to validity on **L**-frames. In the case of interpretability logic this is not generally the case.

**1.7 Proposition**. The formula $\Box(p\to\neg q\wedge\Box\neg q)\wedge(p\rhd q)\to(p\rhd q\wedge\Box\bot)$ is valid on all **ILM**-models on trees, but $K_{ILM}\not\vdash\Box(p\to\neg q\wedge\Box\neg q)\wedge(p\rhd q)\to(p\rhd q\wedge\Box\bot)$ and hence $\not\vdash_{ILM}\Box(p\to\neg q\wedge\Box\neg q)\wedge(p\rhd q)\to(p\rhd q\wedge\Box\neg\bot)$.
**Proof**. Left to the reader.                                                         ⊠

Of course the usual procedure for "stretching out" a partially ordered model into a tree works in this case. The point is that property (iv') will get lost: it will no longer generally hold that, if $w'S_w w'' R u$, then $w' R u$; the only thing one can say of $u$ then is that it will have a forcing relation identical to that of some successor of $w'$, and hence the resulting model will no longer be an **ILM**-model in our sense. For **IL**, **ILW** and **ILP**, on the other hand, one can restrict oneself to tree models.

## 2. FIXED POINTS

From the fact that **IL** is an extension of **L** it is obvious that to prove the existence of explicit fixed points in **IL** it is actually sufficient to find a fixed point for $A(p)\rhd B(p)$, i.e. to find a formula C such that $\vdash_{IL} C\leftrightarrow(A(C)\rhd B(C))$. For, after that we can proceed as in the standard proof for **L** (see Smoryński 1985). One might conjecture that $C=A(T)\rhd B(T)$ would do the trick, and in fact that formula does work for **ILM** (as the reader may check). However, for **IL** a more complicated formula is necessary: $C=A(T)\rhd B(\Box\neg A(T))$.  (The even more complicated, but

more symmetric formula $A(\Box\neg A(T)) \rhd B(\Box\neg A(T))$ is equivalent to C and therefore works too.) We will give a semantic proof. (The results of this section were reached in cooperation with Visser; see Visser 1988P for a syntactic proof). Of course, the present proof does need the completeness of **IL** proved in section 3.

**2.1 Lemma**.

$w \Vdash A(T) \rhd B(\Box\neg A(T)) \Leftrightarrow w \Vdash A(A(T) \rhd B(\Box\neg A(T))) \rhd B(A(T) \rhd B(\Box\neg A(T)))$.

**Proof**. We first establish some simple general facts, for arbitrary w. If we give them without comment their proof is trivial. We write $u \Vdash_{max} A$ iff $u \Vdash A$ and $\forall v (u R v \Rightarrow v \nVdash A)$, and we write $w \underline{R} u$ for $w R u$ or $w = u$.

(1)  $w \Vdash D \rhd E \Leftrightarrow \forall u (w R u \wedge u \Vdash_{max} D \Rightarrow \exists v (u S_w v \wedge v \Vdash E))$;

(2)  if $w \Vdash \Box D$ and $w R u$, then $u \Vdash \Box D$;

(3)  if $w \Vdash_{max} D$, then $w \Vdash \Box\neg D$;

(4)  if $w \Vdash \Box\neg D$, then, if $w \underline{R} u$, then $u \Vdash D \rhd E$;

(5)  if $w \Vdash_{max} D$, then, if $w \underline{R} u$, then $u \Vdash D \rhd E$;

(6)  if $w \Vdash_{max} D$, then $w \Vdash_{max} A(T) \Leftrightarrow w \Vdash_{max} A(D \rhd E)$;

       by (5), as $w \Vdash$ can only be depend on $u \Vdash$ for u for which $w \underline{R} u$, since e.g., $w R v R v' S_v u$ implies $w R u$ by Def.1.2.(i);

(7)  if $w \Vdash_{max} A(T)$, then $w \Vdash_{max} A(A(T) \rhd E)$, by (6);

(8)  if $w \Vdash_{max} A(A(T) \rhd E)$, then $w \Vdash A(T)$;

    for assume $w \Vdash_{max} A(A(T) \rhd E)$, then, by (7), $w R u \Rightarrow u \Vdash \neg A(T)$, hence, for all u with $w \underline{R} u$, $u \Vdash A(T) \rhd E$. As in (6), $w \Vdash A(T)$ follows from $w \Vdash A(A(T) \rhd E)$;

(9)  if $w \Vdash \Box\neg E$, then, for all u with $w \underline{R} u$, $u \Vdash \Box\neg D \Leftrightarrow u \Vdash D \rhd E$;

(10)  if $w \Vdash_{max} E$, then, for all u with $w \underline{R} u$, $u \Vdash B(\Box\neg D) \Leftrightarrow u \Vdash B(D \rhd E)$;

(11)  if $w \Vdash_{max} E$, then, for all u with $w \underline{R} u$,

    $u \Vdash_{max} B(\Box\neg D) \Leftrightarrow u \Vdash_{max} B(D \rhd E)$;

(12)  if $w \Vdash_{max} B(\Box\neg D)$, then $w \Vdash_{max} B(D \rhd B(\Box\neg D))$;

(13)  if $w \Vdash_{max} B(D \rhd B(\Box\neg D))$, then $w \Vdash_{max} B(\Box\neg D)$;

    for assume $w \Vdash_{max} B(D \rhd B(\Box\neg D))$, then, by (12), $w \Vdash \Box\neg B(\Box\neg D)$. So, by (9), for all u with $w \underline{R} u$, $u \Vdash D \rhd B(\Box\neg D) \Leftrightarrow u \Vdash \Box\neg D$; so, $w \Vdash_{max} B(\Box\neg D)$.

Now we establish the main claim:

$\Rightarrow$: Let $w \Vdash A(T) \rhd B(\Box\neg A(T))$. Assume $w R u$ and $u \Vdash_{max} A(A(T) \rhd B(\Box\neg A(T)))$. By (8), $u \Vdash A(T)$. So, for some v with $u S_w v$, $v \Vdash B(\Box\neg A(T))$. We may just as well assume $v \Vdash_{max} B(\Box\neg A(T))$, as $u S_w v R v'$ implies $u S_w v'$ by def. 1.2 (iii). By (12) this implies $v \Vdash B(A(T) \rhd B(\Box\neg A(T)))$.

$\Leftarrow$: Let $w \Vdash A(A(T) \triangleright B(\square \neg A(T))) \triangleright B(A(T) \triangleright B(\square \neg A(T)))$. Assume $w\,R\,u$, $u \Vdash_{max} A(T)$. By (7), $u \Vdash A(A(T) \triangleright B(\square \neg A(T)))$. So, for some $v$ with $u\,S_w\,v$, $v \Vdash B(A(T) \triangleright B(\square \neg A(T)))$. Again we may assume that $v \Vdash_{max} B(A(T) \triangleright B(\square \neg A(T)))$, and (13) gives us $v \Vdash B(\square \neg A(T))$. $\boxtimes$

For completeness' sake we formulate the explicit fixed point theorem which follows from lemma 2.1 by the remarks above.

**2.2 Theorem**. For each **IL**-formula $A(p, q_1, \ldots, q_n)$ in which p occurs only modalized (i.e. all occurrences of p are under some $\square$ or $\triangleright$) there is a provably unique **IL**-formula $B(q_1, \ldots, q_n)$ such that $\vdash_{\textbf{IL}} A(B(q_1, \ldots, q_n), q_1, \ldots, q_n) \leftrightarrow B(q_1, \ldots, q_n)$.

# 3. MODAL COMPLETENESS: PRELIMINARIES

The usual method in modal logic for obtaining completeness proofs is to construct directly or indirectly the necessary countermodels by taking maximal consistent sets of the logic under consideration as the worlds of the model (without necessarily one consistent set standing for only one world) and providing this set of worlds with an appropriate relation R. This method cannot be applied here, since the logic is not compact: some infinite synactically consistent sets of formulae are semantically incoherent. The solution is to restrict the maximal consistent sets to subsets of some finite set of formulae. Such a so-called adequate set has to be rich enough to handle the truth definition, and hence has to be closed under the forming of subformulae and single negations. Furthermore, for each particular logic, additional requirements on the adequate set will be needed to be able to apply the axioms.

**3.1 Definition**. An *adequate* set of formulae is a set $\Phi$ which fulfills the following conditions:
(i)   $\Phi$ is closed under the taking of subformulae
(ii)  if $B \in \Phi$, and B is no negation, then $\neg B \in \Phi$
(iii) $\bot \triangleright \bot \in \Phi$
(iv)  if $B \triangleright C \in \Phi$, then also $\lozenge B, \lozenge C \in \Phi$
(v)   if B as well as C is an antecedent or a consequent of some $\triangleright$-formula in $\Phi$, then $B \triangleright C \in \Phi$.

Obviously, each finite set $\Gamma$ of formulae is contained in a finite adequate set $\Phi$.

**3.2 Definition.** Let $\Gamma$ and $\Delta$ be two maximal **ILS**-consistent subsets of some finite adequate $\Phi$. Then

$\qquad \Gamma \prec \Delta \iff$ for each $\square A \in \Gamma$, $\square A$, $A \in \Delta$, and for some $\square A \notin \Gamma$, $\square A \in \Delta$.

Whenever $\Gamma \prec \Delta$, we say that $\Delta$ is a *successor* of $\Gamma$.

**3.3 Lemma**. Let $\Gamma_0$ be a maximal **ILS**-consistent subset of some finite adequate $\Phi$, and let $W_{\Gamma_0}$ be the smallest set such that

(i)  $\Gamma_0 \in W$

(ii) if $\Delta \in W$ and $\Delta'$ is a maximal **ILS**-consistent subset of $\Phi$ such that $\Delta \prec \Delta'$, then

$\qquad \Delta' \in W$.

Then

(i)  $\prec$ is transitive and irreflexive on $W_{\Gamma_0}$

(ii)  For each $\Gamma \in W_{\Gamma_0}$, $\square A \in \Gamma \iff A \in \Delta$ for every $\Delta$ such that $\Gamma \prec \Delta$.

**Proof**. As in the case of **L** (i) is trivial, and so is $\Rightarrow$ of (ii). For $\Leftarrow$ of (ii) one needs Löb's axiom.                                                                                    ⊠

One might think that this means that, in essence, the completeness problem for **ILS** reduces to defining relations $\prec_\Delta$ on $W_{\Gamma_0}$ such that

$\qquad$ (i) $\prec$ has all the properties of the relation S in $K_S$

$\qquad$ (ii) For each $\Gamma$ in $W_{\Gamma_0}$, $B \rhd C \in \Gamma$ iff, for every $\Delta$ such that $\Gamma \prec \Delta$ and $B \in \Delta$, there

$\qquad\qquad$ is some $\Delta'$ with $\Delta \prec_\Gamma \Delta'$ and $C \in \Delta'$.

The situation is not as simple as that. Before we continue with the completeness proofs, we will give an example to make this clear.

**3.4 Example**. It will be obvious that $\nvdash_{\textbf{ILS}} (p \rhd q \vee r) \to (p \rhd q) \vee (p \rhd r)$. Now, take $\Gamma_0$ to be a maximal **ILS**-consistent set in $\Phi$ that contains $p \rhd q \vee r$, $\neg(p \rhd q)$, and $\neg(p \rhd r)$, as well as the formulae $\square \square \bot$, $\square(p \vee q \vee r)$, $\square \neg(p \wedge q)$, $\square \neg(q \wedge r)$, and $\square \neg(p \wedge r)$. It is then clear that the resulting $W_{\Gamma_0}$ will look as follows:
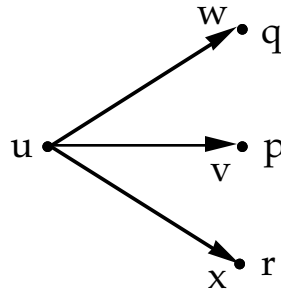


It will also be clear that no $S_u$ can be defined on this model in such a way that

$u \Vdash p \rhd q \vee r$, $\neg(p \rhd q)$, $\neg(p \rhd r)$. By doubling $W[u]$ however an appropriate model can be obtained (the arrows give the additional $S_u$-relations not given by R):
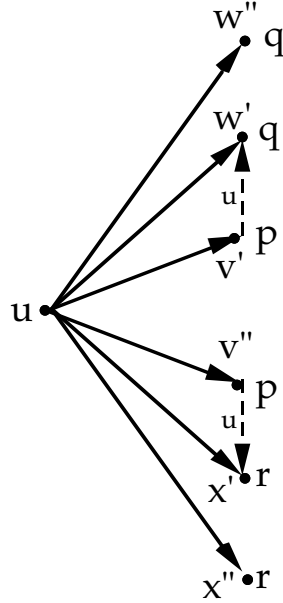


Our strategy in the next section is a generalization of this idea: we will multiply the maximal **ILS**-consistent sets by indexing them with finite sequences of formulae. We write $\tau \subseteq \tau'$ iff the finite sequence $\tau$ is a (not necessarily proper) initial segment of the finite sequence $\tau'$; we write $*$ for concatenation, and, if $w = \langle \Gamma, \tau \rangle$, we write $(w)_0$ for $\Gamma$ and $(w)_1$ for $\tau$.

   Using these pairs we set aside, for each world $w$ and each appropriate formula C, a specific set of the successors of $w$ indexed by C (the so-called critical C-successors of $w$) to provide the counterexamples to the formulae $B \rhd C$ that must be falsified in $w$. We will restrict the relation $S_w$ so that it does not "leave" this set of C-critical successors. Speaking intuitively, the C-critical successors of $w$ will be the ones that contain no formula A that "asks for" C (where A is an antecedent and C the consequent of a $\rhd$-formula in $w$). The next two lemmas show that this whole idea is feasible. The first one says that indeed a counterexample can be found, when needed: for each $\neg(B \rhd C)$ in $w$ a C-critical successor with B in it can be found. The second one says that we will need to have $S_w$ lead from C-critical successors of $w$ to C-critical successors of $w$: if $A \rhd D$ is a member of $w$, and A is a member of a C-critical successor of $w$, then yet another C-critical successor of $w$ with D in it can be found.

**3.5 Definition**. Let $\Gamma$ and $\Delta$ be maximal **ILS**-consistent subsets of some given adequate $\Phi$. Then $\Delta$ is a C-*critical* successor of $\Gamma$ iff

   (i)   $\Gamma \prec \Delta$

(ii)  $\neg A, \square \neg A \in \Delta$ for each A such that $A \triangleright C \in \Gamma$.

Note that successors of C-critical successors of $\Gamma$ are C-critical successors of $\Gamma$.

**3.6 Lemma**. Suppose $\Gamma$ is maximal **ILS**-consistent in $\Phi$ and $\neg(B \triangleright C) \in \Gamma$; then there exists a C-critical successor $\Delta$ of $\Gamma$, maximal **ILS**-consistent in $\Phi$, such that $B \in \Delta$.
**Proof**. Take $\Delta$ to be a maximal **ILS**-consistent extension of

$$\{D, \square D \mid \square D \in \Gamma\} \cup \{\neg A, \square \neg A \mid A \triangleright C \in \Gamma\} \cup \{B, \square \neg B\}$$

Note first that the adequacy of $\Phi$ insures that all the formulae of $\Delta$ are indeed available. Secondly, note that if such a $\Delta$ exists, it is indeed a C-critical successor of $\Gamma$: the fact that

$$\{D, \square D \mid \square D \in \Gamma\} \cup \{\square \neg B\} \subseteq \Delta$$

makes it a successor of $\Gamma$, and the fact that

$$\{\neg A, \square \neg A \mid A \triangleright C \in \Gamma\} \subseteq \Delta$$

makes it C-critical.
Now, if no such $\Delta$ exists, then there are $A_1, \ldots, A_m$ and $D_1, \ldots, D_k$ with

$$D_1, \ldots, D_k, \square D_1, \ldots, \square D_k, \neg A_1, \ldots, \neg A_m, \square \neg A_1, \ldots, \square \neg A_m, B, \square \neg B \vdash \bot.$$

Or, equivalently:

$$D_1, \ldots, D_k, \square D_1, \ldots, \square D_k, \neg(A_1 \vee \ldots \vee A_m), \square \neg(A_1 \vee \ldots \vee A_m), B, \square \neg B \vdash \bot$$

This would mean that:

$$D_1, \ldots, D_k, \square D_1, \ldots, \square D_k, B, \square \neg B \vdash A_1 \vee \ldots \vee A_m \vee \Diamond(A_1 \vee \ldots \vee A_m).$$

In other words:

$$D_1, \ldots, D_k, \square D_1, \ldots, \square D_k \vdash B \wedge \square \neg B \to A_1 \vee \ldots \vee A_m \vee \Diamond(A_1 \vee \ldots \vee A_m).$$

Since **IL** contains **L**:

$$\Box D_1, \ldots, \Box D_k \vdash \Box(B \wedge \Box \neg B \rightarrow A_1 \vee \ldots \vee A_m \vee \Diamond(A_1 \vee \ldots \vee A_m))$$

By axiom (1):

$$\Box D_1, \ldots, \Box D_k \vdash B \wedge \Box \neg B \triangleright A_1 \vee \ldots \vee A_m \vee \Diamond(A_1 \vee \ldots \vee A_m)$$

In view of particularly the axioms (5) and (3) we have that

$$\vdash A_1 \vee \ldots \vee A_m \vee \Diamond(A_1 \vee \ldots \vee A_m) \triangleright A_1 \vee \ldots \vee A_m.$$

So, by axiom (2):

$$\Box D_1, \ldots, \Box D_k \vdash B \wedge \Box \neg B \triangleright A_1 \vee \ldots \vee A_m$$

Given that $A_1 \triangleright C, \ldots, A_m \triangleright C \in \Gamma$, we also have $\Gamma \vdash A_1 \vee \ldots \vee A_m \triangleright C$ (apply axiom (3)), and so by axiom (2):

$$\Gamma \vdash B \wedge \Box \neg B \triangleright C$$

Now, it is not difficult to see that

$$\vdash B \triangleright B \wedge \Box \neg B$$

(To that purpose, note first that $\vdash (B \wedge \Box \neg B) \vee \Diamond(B \wedge \Box \neg B) \triangleright B \wedge \Box \neg B$. Secondly, since **ILS** contains **L**, $\vdash \Box(B \rightarrow (B \wedge \Box \neg B) \vee \Diamond(B \wedge \Box \neg B))$. So by axiom (1), it is provable in **ILS** that $B \triangleright (B \wedge \Box \neg B) \vee \Diamond(B \wedge \Box \neg B)$. Combining these two facts we find $\vdash B \triangleright B \wedge \Box \neg B$.)

Finally, by applying axiom (2) once more, it follows from $\Gamma \vdash B \wedge \Box \neg B \triangleright C$ and $\vdash B \triangleright B \wedge \Box \neg B$ that

$$\Gamma \vdash B \triangleright C$$

This contradicts the consistency of $\Gamma$. ⊠

**3.7 Lemma**. Suppose $B \triangleright C \in \Gamma$ and let $\Delta$ be an E-critical successor of $\Gamma$ with $B \in \Delta$. Then there is an E-critical successor $\Delta'$ of $\Gamma$ with $C \in \Delta'$.
**Proof**. Suppose there is not such a $\Delta'$. Then there would be
$\Box D_1, \ldots, \Box D_n \in \Gamma$, and $F_1 \triangleright E, \ldots, F_k \triangleright E \in \Gamma$ such that

$$D_1, \ldots, D_n, \Box D_1, \ldots, \Box D_n, \neg F_1, \ldots, \neg F_k, \Box \neg F_1, \ldots, \Box \neg F_k, C \vdash \bot$$

and, therefore,

$$D_1, \ldots, D_n, \Box D_1, \ldots, \Box D_n \vdash C \rightarrow F_1 \vee \ldots \vee F_k \vee \Diamond(F_1 \vee \ldots \vee F_k)$$

which as before implies:

$$\Box D_1, \ldots, \Box D_n \vdash C \rhd F_1 \vee \ldots \vee F_k.$$

By axiom (2), $B \rhd C \in \Gamma$ implies that $\Gamma \vdash B \rhd F_1 \vee \ldots \vee F_k$ and, by axiom (3), $\Gamma \vdash B \rhd E$. Given the adequacy conditions, this can be strengthened to $B \rhd E \in \Gamma$. Since $\Delta$ is an E-critical successor of $\Gamma$, this implies $\neg B \in \Delta$, and we have arrived at a contradiction, since it is assumed that $B \in \Delta$. $\boxtimes$


## 4. THE MODAL COMPLETENESS OF **IL**

In this section we just have to carefully adjoin sequences to the maximal **IL**-consistent sets and see that the intuitive ideas of the previous section can be set to work properly.

**4.1 Theorem** (*Completeness and decidability of* **IL**). If $\nvdash_{\mathbf{IL}} A$, then there is a finite **IL**-model **K** such that **K** $\nvDash A$.
**Proof**. Take some finite adequate set $\Phi$ containing $\neg A$. Let $\Gamma$ be a maximally consistent subset of $\Phi$ containing $\neg A$.
Now, set $W_\Gamma$ to be the smallest set of pairs $<\Delta, \tau>$, where $\tau$ is a finite sequence of formulae from $\Phi$, that fulfills the following requirements:

  (i)  $<\Gamma, <>> \in W_\Gamma$
  (ii)  If $<\Delta, \tau> \in W_\Gamma$, then $<\Delta', \tau> \in W_\Gamma$ for every successor $\Delta'$ of $\Delta$
  (iii) If $<\Delta, \tau> \in W_\Gamma$, then $<\Delta', \tau * <C>> \in W_\Gamma$ for every C-critical
        successor $\Delta'$ of $\Delta$.

$W_\Gamma$ is finite. (For every $\Delta$, the number of successors of $\Delta$ is finite. Moreover, if $\Delta \prec \Delta'$, the number of successors of $\Delta'$ is smaller than the number of successors of $\Delta$.)

*Observation*: If $<\Delta, \tau> \in W_\Gamma$ and E occurs in $\tau$, then $\neg E \in \Delta$.

*Proof*: Show with induction on the construction of $W_\Gamma$ that if $<\Delta, \tau> \in W_\Gamma$ and E occurs in $\tau$ then $\neg E, \Box \neg E \in \Delta$.

Define R on $W_\Gamma$ as follows:

$$w \, R \, w' \text{ iff } (w)_0 \prec (w')_0 \text{ and } (w)_1 \subseteq (w')_1 \, .$$

It is easy to check that R has all the properties required.

Finally, let $u \, S_w \, v$ apply if (I) and (II) hold:

(I)   $u, v \in W_\Gamma[w]$

(II)   $(w)_1 = (u)_1 \subseteq (v)_1$ , or $(u)_1 = (w)_1 * <C> * \tau$ and $(v)_1 = (w)_1 * <C> * \sigma$ for some $C, \sigma$ and $\tau$.

We leave it to the reader to check that under this definition $S_w$ will have the required properties:

We are now ready to define

$$w \Vdash p \text{ iff } p \in (w)_0$$

and prove that

for each $A \in \Phi$, $w \Vdash A$ iff $A \in (w)_0$ .

Given (ii) it is immediately clear that the model treats $\Box$-formulae properly.[3] So, the only interesting case to look at in the inductive proof is the one that A is $B \triangleright C$, i.e. we have to show that

$$B \triangleright C \in (w)_0 \iff \forall u \, (w \, R \, u \wedge B \in (u)_0 \Rightarrow \exists v \, (u \, S_w \, v \wedge C \in (v)_0)):$$

$\Leftarrow$: Suppose $B \triangleright C \notin (w)_0$. Then $\neg (B \triangleright C) \in (w)_0$. We must show that $\exists u \, (w \, R \, u \wedge B \in (u)_0 \wedge \forall v \, (u \, S_w \, v \to \neg C \in (v)_0))$. Let $\Delta$ be as in lemma 3.6 with $(w)_0$ as $\Gamma$, and take u to be $<\Delta, (w)_1 * <C>>$. Consider any v such that $u \, S_w \, v$. Then C occurs

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[3] An alternative, perhaps more elegant, set up would be to do without (ii). Then one has to use the equivalence of $\Box A$ with $\neg A \triangleright \bot$ and to adapt the definition of adequate set. Now this equivalence is used in the treatment of $\triangleright$. (See the end of the proof which can be deleted in the alternative set up.)

in $(v)_1$. By the observation above, $\neg C \in (v)_0$.

$\Rightarrow$: Suppose $B \rhd C \in (w)_0$. Consider any $u$ such that $wRu$ and $B \in (u)_0$.
Let us first assume that $(u)_1 = (w)_1 * <E> * \tau$. In that case we can apply lemma 3.7 for
$\Gamma = (w)_0$ and $\Delta = (u)_0$ to obtain an E-critical successor $\Delta'$ of $\Gamma$ with $C \in \Delta'$. It suffices
now to take $v = <\Delta', (w)_1 * <E>>$. It is clear that $v$ fulfills all requirements to make
$u\, S_w\, v$.
If $(u)_1 = (w)_1$, then all we know is that $(w)_0 \prec (u)_0$. Note, however, that every suc-
cessor of $\Gamma$ is a $\perp$-critical successor of $\Gamma$. (By axiom (4), $\vdash F \rhd \perp \to \Box \neg F$; hence if
$F \rhd \perp \in \Gamma$, then $\Box \neg F \in \Gamma$, and therefore $\neg F, \Box \neg F \in \Delta$ for every $\Delta$ such that $\Gamma \prec \Delta$.) So
we can apply lemma 3.7 for $\Gamma = (w)_0$, $\Delta = (u)_0$, and $E = \perp$, in order to obtain a ($\perp$-
critical) successor $\Delta'$ of $\Gamma$ with $C \in \Delta'$. Take $v = <\Delta', (w)_1>$. $\boxtimes$

## 5. THE MODAL COMPLETENESS OF ILP

**5.1 Definition**. A set $\Phi$ of formulae is **ILP**-*adequate* iff
  (i)   $\Phi$ is adequate in the sense of definition 3.1
  (ii)  if $B \rhd C \in \Phi$, then also $\Box (B \rhd C) \in \Phi$.

Clearly, each finite set $\Gamma$ of formulae is contained in a finite **ILP**-adequate set $\Phi$.

**5.2 Theorem** (*Completeness and decidability of* **ILP**). If $\nvdash_{ILP} A$, then there is a finite
**ILP**-model **K** such that $\mathbf{K} \nvDash A$.
**Proof**. Take some finite adequate set $\Phi$ containing $\neg A$. Let $\Gamma$ be a maximally
consistent subset of $\Phi$ containing $\neg A$.
In constructing the model, we multiply the maximal **ILP**-consistent sets similarly
as with **IL** while at the same time transforming the model into a tree in the
standard manner. The purpose of making the model into a tree is insuring that a
unique immediate predecessor exists for each world. A world in the model will be
a sequence of pairs $<<\Gamma_0, \tau_0 >, ..., <\Gamma_{n-1}, \tau_{n-1}>, <\Gamma_n, \tau_n >>$.
More precisely, $W_\Gamma$ is built up according to the following clauses:

  (i)   $<<\Gamma, <>>> \in W_\Gamma$
  (ii)  If $<<\Gamma_0, \tau_0 >, ..., <\Gamma_n, \tau_n >> \in W_\Gamma$, and $\Delta$ is a successor of $\Gamma$ then also
        $<<\Gamma_0, \tau_0 >, ..., <\Gamma_n, \tau_n >, <\Delta, \tau_n >> \in W_\Gamma$;
  (iii) If $<< \Gamma_0, \tau_0 >, ..., < \Gamma_n, \tau_n >> \in W_\Gamma$ and $\Delta$ is a C-critical successor of $\Gamma$,
        then also $<<\Gamma_0, \tau_0 >, ..., <\Gamma_n, \tau_n >, <\Delta, \tau_n * <C>>> \in W_\Gamma$

If $w = <<\Gamma_0, \tau_0>, \ldots, <\Gamma_n, \tau_n>> \in W_\Gamma$, we write $\Delta_w = \Gamma_n$ and $\tau_w = \tau_n$.

We next define R on $W_\Gamma$ as follows: $w\,R\,w'$ iff $w$ is a proper initial segment of $w'$. Thus, R is transitive and irreflexive. More importantly, every world different from $<<\Gamma, <>>>$ has precisely one immediate R-predecessor.

Note that that the model will treat $\square$ properly.

We are now ready to define $u\,S_w\,v$ as applying if (I) and (II) hold:

(I)    $w\,R\,u$, and for every $w'$, if $w'\,R\,u$ then $w'\,R\,v$

(II)   $\tau_u \subseteq \tau_v$

It is easy to check that under this definition $S_w$ will have the required properties. Next we define

$$w \Vdash p \text{ iff } p \in \Delta_w,$$

and prove that

$$\text{for each } A \in \Phi, w \Vdash A \text{ iff } A \in \Delta_w.$$

Again, the only interesting case to look at in the inductive proof is the one that A is $B \triangleright C$, i.e. we have to show that

$$B \triangleright C \in \Delta_w \iff \forall u\,(w\,R\,u \wedge B \in \Delta_u \Rightarrow \exists v\,(u\,S_w\,v \wedge C \in \Delta_v)).$$

$\Leftarrow$: Suppose $B \triangleright C \notin \Delta_w$. Then $\neg(B \triangleright C) \in \Delta_w$. We must show that
$\exists u\,(w\,R\,u \wedge B \in \Delta_u \wedge \forall v\,(u\,S_w\,v \rightarrow \neg C \in \Delta_v))$.
Assume $w = <<\Gamma_0, \tau_0>, \ldots, <\Gamma_n, \tau_n>>$. Let $\Delta$ be as in lemma 3.6 with $\Gamma_n$ as $\Gamma$. Take u to be $<<\Gamma_0, \tau_0>, \ldots, <\Gamma_n, \tau_n>, <\Delta', \tau_n * <C>>>$ with the $\Delta'$ given by that lemma. Consider any v such that $u\,S_w\,v$. Then C occurs in $\tau_v$. As in the previous case, it is easy to see that this means that $\neg C \in \Delta_v$.

$\Rightarrow$: Suppose $B \triangleright C \in \Delta_w$ and $w\,R\,u$ with $B \in \Delta_u$. Let $w'$ the(!) immediate predecessor of u. Note that axiom **P** and the **ILP**-adequacy of $\Phi$ insure that $B \triangleright C \in \Delta_{w'}$.

Let us first assume that $\tau_u = \tau_{w'} * <E>$. In that case we can apply lemma 3.7 with $\Gamma = \Delta_{w'}$ and $\Delta = \Delta_u$ to obtain an E- critical successor $\Delta'$ of $\Gamma$ with $C \in \Delta'$. It suffices now to take $v = w' * <\Delta', \tau_u>$. It is clear that v fulfills all requirements to make $u\,S_w\,v$.

If, on the other hand, $\tau_u = \tau_{w'}$, then all we know is that $\Delta_{w'} \prec \Delta_u$. Recall

however that every successor is a ⊥-critical successor. So, here too, we apply lemma 3.7 for $\Gamma = \Delta_{w'}$, $\Delta = \Delta_u$, and $E = \bot$, in order to obtain a (⊥-critical) successor $\Delta'$ of $\Gamma$ with $C \in \Delta''$. Take $v = w' * <\Delta'', \tau_u>$.　　　　　　　⊠

**5.3 Corollary** (to the proof of theorem 5.2). **ILP** is complete with respect to the frames in which, if $w \, R \, w'$, then $S_{w'} = S_w \upharpoonright W[w']$.
**Proof**. It is clear from the proof that, in the model constructed $u \, S_w \, v$ iff $u \, S_{w'} \, v$ for the immediate predecessor $w'$ of $w$.　　　　　　　⊠

The corollary means that we can take the S-relation in **ILP** to be a rigid relation, essentially independent of w.

# 6. THE MODAL COMPLETENESS OF ILM

The completeness proof for **ILM** is rather more complicated than the ones for the completeness of **IL** and **ILP**. The first problem arises from the fact that to be able to apply the characteristic axiom $(A \triangleright B) \rightarrow (A \wedge \Box C \triangleright B \wedge \Box C)$ we are forced to add the consequent of this formula to the adequate set, whenever we have the antecedent.

**6.1 Definition**. An **ILM**-*adequate* set of formulae is a set $\Phi$ which fulfills the conditions:

  (i)   $\Phi$ is closed under the taking of subformulae
  (ii)  if B and $C \in \Phi$, then for each Boolean combination D of B and C there is a formula **ILM**-equivalent to D in $\Phi$
  (iii) $\bot \triangleright \bot \in \Phi$
  (iv)  if $B \triangleright C \in \Phi$, then also formulae **ILM**-equivalent to $\Diamond B$, $\Diamond C$ in $\Phi$
  (v)   if both B and C are antecedent or consequent of some $\triangleright$-formula in $\Phi$, then $B \triangleright C \in \Phi$
  (vi)  if $B \triangleright C$, $\Box D \in \Phi$, then there is in $\Phi$ a formula **ILM**-equivalent to $B \wedge \Box D \triangleright C \wedge \Box D$.

With this definition it is, of course, not at all obvious that each finite set is contained in a finite adequate one. The problem in keeping things finite is that with $B \wedge \Box D \triangleright C \wedge \Box D$ also $\Diamond (B \wedge \Box D)$ and $\Diamond (C \wedge \Box D)$ will have to be an element of $\Phi$ and these will via clause (vi) generate new formulae in the adequate set, e.g. $B \wedge \Box D \wedge \Box \neg (B \wedge \Box D) \triangleright C \wedge \Box D \wedge \Box \neg (B \wedge \Box D)$. What we have to show is that this does not lead to an infinite regress: after a while the process starts delivering formulae

equivalent to ones which have occurred previously. A little thought will convince the reader that the next lemma shows just that.

**6.2 Lemma**. Starting with a finite set of formulae $\Diamond B_1, \ldots, \Diamond B_n$, and and closing off under the operation of taking $\Diamond(B_i \land \Box \neg B_j)$ (adding each new $\Diamond$-formula to the stock) leads to a finite set of **L**-equivalence classes of formulae.

**Proof**. By induction on n. In the case that there is only one formula $\Diamond B$ the process stops immediately, because $\Diamond(B \land \Box \neg B)$ is **L**-equivalent to $\Diamond B$.

Assume the validity of the lemma for n starting formulae and apply the closing off procedure to $\Diamond B_1, \ldots, \Diamond B_{n+1}$. The formulae obtained will be of the forms $\Diamond(B_i \land \Box \neg D_1 \land \ldots \land \Box \neg D_k)$ ($1 \leq i \leq n+1$). For each of these classes we have to show that they contain only a finite number of equivalence classes. Without loss of generality we restrict ourselves to the case that i=1.

By the induction hypothesis there can be only finitely many formulae $\Diamond(B_1 \land \Box \neg D_1 \land \ldots \land \Box \neg D_k)$ in which the formula $B_1$ has not been used in the construction of $D_1, \ldots, D_k$. Now consider a formula $\Diamond(B_1 \land \Box \neg D_1 \land \ldots \land \Box \neg D_k)$ in which $B_1$ has been used. This formula is **L**-equivalent to $\Diamond(B_1 \land \Box \neg B_1 \land \Box \neg D_1 \land \ldots \land \Box \neg D_k)$. We now use the fact that

$$\vdash_{\mathbf{L}} \Box \neg B_1 \to \Box(B_1 \leftrightarrow \bot) \text{ and } \vdash_{\mathbf{L}} \Box \neg B_1 \to \Box \ldots \Box(B_1 \leftrightarrow \bot).$$

From this it easily follows that $B_1$ can, in each of the $D_1, \ldots, D_k$ occurring in $\Diamond(B_1 \land \Box \neg B_1 \land \Box \neg D_1 \land \ldots \land \Box \neg D_k)$, be **L**-equivalently replaced by $\bot$, since $\Box \neg B_1$ occurs in that formula. Now, each of the $D_i$ is built up in such a manner that $B_1$ occurs only in the context $\Box \neg (B_1 \land \ldots)$. This means that after replacing $B_1$ by $\bot$ we get a tautology, which can be left out altogether. We end up with a formula $\Diamond(B_1 \land \Box \neg E_1 \land \ldots \land \Box \neg E_m)$ in which each of the $E_i$ has been constructed according to procedure from $B_2, \ldots, B_{n+1}$. We already concluded that there can be only finitely many such formulae. ⊠

**6.3 Theorem** (*Completeness and decidability of* **ILM**) If $\nvdash_{\mathbf{ILM}} A$, then there is a finite **ILM**-model **K** such that $\mathbf{K} \nvDash A$.

**Proof**. Take some finite **ILM**-adequate set $\Phi$ containing $\neg A$. Let $\Gamma$ be a maximal **ILM**-consistent subset of $\Phi$ containing $\neg A$. Unfortunately, we need more worlds than present in the $W_\Gamma$ used in the proofs for **IL** and **ILP**.

This time we set $W_\Gamma$ to be the collection of all pairs $\langle \Delta, \tau \rangle$, with

(i)   $\Gamma \prec \Delta$ or $\Gamma = \Delta$

(ii)  $\tau$ is a finite sequence of formulae from $\Phi$, the length of which does not exceed the the depth[4] of $\Gamma$ minus the depth of $\Delta$. (So, $\Gamma$ is only paired off with the empty sequence.)

Clearly, $W_\Gamma$ is finite. Note that the sequence $\tau$ in a pair $<\Delta, \tau>$ provides no longer sufficient information on the "C-critical" status of $\Delta$.

Define R on $W_\Gamma$ as follows:

$$w \, R \, w' \text{ iff } (w)_0 \prec (w')_0 \text{ and } (w)_1 \subseteq (w')_1 .$$

It is easy to check that R has all the properties required.

We say that u is a C-critical R-successor of w if $(u)_0$ is a C-critical successor of $(w)_0$ and $(u)_1 = (w)_1 * <C> * \tau$ .

Let $u \, S_w \, v$ apply if (I)–(IV) hold:

(I)   $u, v \in W_\Gamma[w]$

(II)   $(u)_1 \subseteq (v)_1$

(III) for each A such that $\Box A \in (u)_0$ also $\Box A \in (v)_0$

(IV)  if u is a C-critical R-successor of w, then v is a C-critical R-successor of w.

Let us check right away that under this definition $S_w$ will have the required properties:

(i)   that $u, v \in W[w]$ if $u \, S_w \, v$, is instantaneous.
(ii)  reflexivity and transitivity of $S_w$ are easy to check.
(iii) if $u, v \in W[w]$ and $u \, R \, v$, then (I), (II) and (III) are immediate.

As for (IV) it suffices to recall that successors of C-critical successors are C-critical.
(iv) Suppose $w' \, S_w \, w'' \, R \, u$. We must show that $w' \, R \, u$. That $(w')_1 \subseteq (u)_1$ is immediate. That $(w')_0 \prec (u)_0$ follows from $(w'')_0 \prec (u)_0$ combined with (III) for $w', w''$.

We are now ready to define $w \Vdash p$ iff $p \in (w)_0$ and prove that in that case $w \Vdash A$ iff $A \in (w)_0$, holds for each $A \in \Phi$. Again, we restrict ourselves to the case that A is $B \rhd C$, i.e. we have to show that

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[4] $\Gamma$ has *depth* n if the maximal length of a complete chain $\Gamma = \Gamma_0 \prec \ldots \prec \Gamma_m$ is n+1.

$B \triangleright C \in (w)_0 \Leftrightarrow \forall u (wRu \wedge B \in (u)_0 \Rightarrow \exists v(u S_w v \wedge C \in (v)_0))$:

$\Leftarrow$: Suppose $B \triangleright C \notin (w)_0$. Then $\neg(B \triangleright C) \in (w)_0$. We must show that

$$\exists u (w R u \wedge B \in (u)_0 \wedge \forall v (u S_w v \rightarrow \neg C \in (v)_0)).$$

Let $\Delta$ be as in lemma 3.6 with $(w)_0$ as $\Gamma$, and take u to be $<\Delta, (w)_1 * <C>>$. Consider any v such that $u S_w v$. Since u is a C-critical R-successor of w, v will be one too. Therefore, $\neg C \in (v)_0$.

$\Rightarrow$: Suppose $B \triangleright C \in (w)_0$ and let u be such that $w R u$ and $B \in (u)_0$. Let $\{\Box D_1, \ldots, \Box D_n\} = \{\Box D \mid \Box D \in (u)_0\}$. Note that axiom **M** and the adequacy of $\Phi$ insure that $(w)_0$ contains a formula equivalent to

$$B \wedge \Box D_1 \wedge \ldots \wedge \Box D_n \triangleright C \wedge \Box D_1 \wedge \ldots \wedge \Box D_n.$$

Let us first assume that u is an E-critical R-successor of w. Then, for some $\tau$, $(u)_1 = (w)_1 * <E> * \tau$. In that case we can apply lemma 3.7 with $\Gamma = (w)_0$, $\Delta = (u)_0$ and a formula equivalent to $B \wedge \Box D_1 \wedge \ldots \wedge \Box D_n \triangleright C \wedge \Box D_1 \wedge \ldots \wedge \Box D_n$, rather than $B \triangleright C$ itself, as input. In so doing, we obtain an E-critical successor $\Delta'$ of $\Gamma$ with (i) $C \in \Delta'$ and (ii) $\Box D \in \Delta'$ for each D such that $\Box D \in \Delta$. It suffices now to take $v = <\Delta', (u)_1>$. Given that each $\Box$-formula in $\Delta$ is also an element of $\Delta'$, the depth of $\Delta'$ cannot be larger than the depth of $\Delta$. Therefore $v \in W_\Gamma$. It is clear that v fulfills all requirements to make $u S_w v$.

If, on the other hand u is not an E-critical R-successor of w, then all we know is that $(w)_0 \prec (u)_0$. Recall once more that every successor of $\Gamma$ is a $\bot$-critical successor of $\Delta$. So, an application of lemma 3.7 with $\Gamma = (w)_0$, $\Delta = (u)_0$, $E = \bot$, and $B \wedge \Box D_1 \wedge \ldots \wedge \Box D_n \triangleright C \wedge \Box D_1 \wedge \ldots \wedge \Box D_n$ as input, yields a ($\bot$-critical) successor $\Delta'$ of $\Gamma$ with $C \in \Delta'$ and $\Box D \in \Delta'$ for each D such that $\Box D \in \Delta$. Take $v = <\Delta', (u)_1>$. ⊠

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