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A Mathematical Formalization of Qualia Space

--Manuscript Draft--

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Full Title:	A Mathematical Formalization of Qualia Space
Abstract:	Our conscious experiences are qualitative and unitary. The qualitative universals given in particular experiences, i.e. qualia, combine into the seamless unity of our conscious experience. The problematics of quality and cohesion are not unique to consciousness studies. In mathematics, the study of qualities (e.g. shape) resulting from quantitative variations in cohesive spaces led to the axiomatization of cohesion and quality. Using the mathematical definition of quality, herein we model qualia space as a categorical product of qualities. Thus modeled qualia space is a codomain space wherein composite qualities (e.g. shape AND color) of conscious experiences can be valued. As part of characterizing the qualia space, we provide a detailed exemplification of the mathematics of quality and cohesion in terms of the categories of idempotents and reflexive graphs. More specifically, with qualities as commutative triangles formed of cohesion-preserving functors, first we calculate the product of commutative triangles. Next, we explicitly show that the category of idempotents is a quality type. Lastly, as part of showing that the category of reflexive graphs is cohesive, we characterize the adjointness between functors relating cohesive graphs to discrete sets. In conclusion, our category theoretic construction of qualia space is a formalization of the binding of qualitative features (colors and shapes) into the cohesive objects (colored-shapes) of conscious experiences. Compared to the feature-vector accounts of conscious experiences, our product-of-qualities account of consciousness is a substantial theoretical advance.
Keywords:	Binding problem; Category theory; Cohesion; Conscious experience; Quality.
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Author Comments:	Dear Editor, We are herewith submitting our original manuscript entitled "A Mathematical Formalization of Qualia Space" for publication in the Theory and Methodology section of your journal Psychometrika. The qualitative nature of our conscious experiences, though routinely acknowledged, has rarely been explicitly modeled simply because we did not have a mathematical definition of the notion of quality. Recently, as part of the axiomatization of cohesion, F. William Lawvere defined QUALITY. Based on Lawvere's definition of quality, we formalized qualia space as a categorical product of qualities. Our manuscript, to the best of our knowledge, is the first paper to apply the category theory of quality and cohesion to formalize the binding of qualitative qualia into the unity of consciousness. As such, we are confident that our work will inspire further in-depth mathematical investigations of the categorical product of qualities as a model of the qualia space. We hope you will find our manuscript suitable for publication in your esteemed journal Psychometrika. We thank you for your consideration of our manuscript and we look forward to your reply.

	Thanking you, Yours truly, Venkata Rayudu Posina ORCID ID: orcid.org/0000-0002-3040-9224 Consciousness Studies Programme National Institute of Advanced Studies Indian Institute of Science Campus, Bengaluru - 560012, Karnataka, India Tel: +91-80-22185000, Fax: +91-80-22185028 Email: posinavrayudu@gmail.com
Suggested Reviewers:	Francis William Lawvere, PhD Professor Emeritus, University at Buffalo - The State University of New York wlawvere@buffalo.edu Our model of qualia space is based on Professor F. William Lawvere's axiomatic cohesion.
	Richard P Stanley, PhD Professor, Massachusetts Institute of Technology rstan@math.mit.edu Professor Stanley pioneered the mathematical characterization of qualia space.
	Andrée C Ehresmann, PhD Chief-Editor, Cahiers de Topologie et Geometrie Differentielle Categoriques ehres@u-picardie.fr Professor Ehresmann made foundational contributions to category theory and pioneered its applications to consciousness studies.
	Giulio Tononi, PhD Professor, Medical College of Wisconsin gtononi@wisc.edu Professor Tonini recently provided a mathematical characterization of qualia space in terms of his integrated information theory.
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To,

Professor Irini Moustaki

Editor-in-Chief

Psychometrika

Dear Professor Moustaki,

We are herewith submitting our original manuscript entitled "A Mathematical Formalization of Qualia Space" for publication in the **Theory and Methodology** section of your journal *Psychometrika*.

The qualitative nature of our conscious experiences, though routinely acknowledged, has rarely been explicitly modeled simply because we did not have a mathematical definition of the notion of quality. Recently, as part of the axiomatization of cohesion, F. William Lawvere defined QUALITY. Based on Lawvere's definition of quality, we formalized qualia space as a categorical product of qualities. Our manuscript, to the best of our knowledge, is the first paper to apply the category theory of quality and cohesion to formalize the binding of qualitative qualia into the unity of consciousness. As such, we are confident that our work will inspire further in-depth mathematical investigations of the categorical product of qualities as a model of the qualia space.

We hope you will find our manuscript suitable for publication in your esteemed journal *Psychometrika*. We thank you for your consideration of our manuscript and we look forward to your reply.

Thanking you,

Yours truly,

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Abstract

Our conscious experiences are qualitative and unitary. The qualitative universals given in particular experiences, i.e. qualia, combine into the seamless unity of our conscious experience. The problematics of quality and cohesion are not unique to consciousness studies. In mathematics, the study of qualities (e.g. shape) resulting from quantitative variations in cohesive spaces led to the axiomatization of cohesion and quality. Using the mathematical definition of quality, herein we model qualia space as a categorical product of qualities. Thus modeled qualia space is a codomain space wherein composite qualities (e.g. shape AND color) of conscious experiences can be valued. As part of characterizing the qualia space, we provide a detailed exemplification of the mathematics of quality and cohesion in terms of the categories of idempotents and reflexive graphs. More specifically, with qualities as commutative triangles formed of cohesion-preserving functors, first we calculate the product of commutative triangles. Next, we explicitly show that the category of idempotents is a quality type. Lastly, as part of showing that the category of reflexive graphs is cohesive, we characterize the adjointness between functors relating cohesive graphs to discrete sets. In conclusion, our category theoretic construction of qualia space is a formalization of the binding of qualitative features (colors and shapes) into the cohesive objects (colored-shapes) of conscious experiences. Compared to the feature-vector accounts of conscious experiences, our product-of-qualities account of consciousness is a substantial theoretical advance.

I. Introduction

Consciousness continues to elude the reach of science (Albright et al., 2000; Menon, Sinha and Sreekantan, 2014). How are we going to account for the qualitative character of qualia? The problem of qualia (such as the qualitative feel of seeing brilliant orange sunset as distinct from, say, hearing a soothing lullaby) is a problem of relating descriptions to that which is described. Given that conscious experiences are mediated by the brain, the structure of qualia is to be related not only to the structure of physical stimulus spaces but also to that of the spaces of neural processing. Though we have a reasonably good understanding of the quantitative aspects (cf. increasing / decreasing along a stimulus feature dimension, resultant increases / decreases in neural responses, and the attendant changes in conscious experiences), our understanding of the qualitative nature of qualia is rather rudimentary. It is not clear how to formalize the qualitative distinction between sights and sounds that is so palpable in our everyday experience (Clark, 1993; O'Regan, 2011). This shortcoming, however, is not specific to consciousness studies. Scientific study of qualities such as shapes and types encountered in physics and mathematics is also challenging (Lawvere and Rosebrugh, 2003, p. 232). It is only in the past decade, the notion of QUALITY has been mathematically defined (Lawvere, 2007).

Here we begin with an in-depth study of the mathematics of quality so as to develop a mathematical formalism required for the scientific study of consciousness. Our objective, in the present note, is limited to elaborating the mathematics of cohesion and quality in terms of the categories of reflexive graphs and idempotents (Lawvere and Schanuel, 2009, pp. 135-146), so as to make it accessible to theoretical cognitive neuroscientists.

Brain can be thought of as a universal measurement device (Grossberg, 1983), measuring the physical world, and with conscious experiences as values of the neural measurements. Though there is much within a given conscious experience that lends itself to be quantified (e.g. intensity of pain), there is also much that remains beyond the reach of quantities (i.e. the quality of pain as distinct from, say, pleasure). These qualitative universals given in particular conscious experiences are called qualia (Lewis, 1929, p. 121). Within this framework, we need a space: qualia space (Balduzzi and Tononi, 2009; Stanley, 1999), to serve as codomain in which the neural measures can be valued. In the case of quantitative measurements, for example, the real number line serves as a codomain of values. In the case of brains measuring things in the world, we need a space which can serve as a space of values for qualities such as taste and smell. Analogous to the case of quantitative measurements, wherein additional dimensions are introduced (e.g. plane) to deal with more than one quantity (Lawvere and Rosebrugh, 2003, p. 59), we need a product space of qualities that can serve as a codomain space of values for composite qualities. Once we have a qualia space of composite qualities, we can characterize its geometry (figures and their incidences) and algebra (functions and their determinations). This qualia space can then be related to the physical stimulus spaces and the corresponding spaces of neural processing.

To place the theory of qualia we are developing in perspective, currently conscious experiences are modeled, after reducing qualities of conscious experiences into numbers (cf. intensity of pain), as feature lists and, in turn, as points in a vector space (Stanley, 1999). For example, colored shapes such as ‘red square’ are represented as points (red, square) in a two-dimensional space (Color × Shape). However, color and shape are qualities, which are much more structured than mere points (Hardin, 1988). For example, color is an intensive quality, while shape is an extensive quality. Our approach, building on Lawvere’s definition of qualities (Lawvere, 2007), is a direct formalization of the qualities of conscious experiences.

More broadly, category theory has been put forward as the language of consciousness (Struppa et al., 2002). In line with these suggestions, we are working out the mathematics of composite qualities defined as categorical product of qualities. In the present note, with qualities as commutative triangles formed of cohesion-preserving functors, first we calculate the product of commutative triangles. Next, we show that the category of idempotents is a quality type and has central idempotents required of quality types. Lastly, we characterize the adjointness between functors relating reflexive graphs to discrete sets as part of showing that the category of reflexive graphs is cohesive.

II. Composite Qualities

Quality is that which remains upon identifying all quantitative variations (Lawvere, 1992). A quality is a cohesion-preserving functor q: $\boldsymbol{C} \rightarrow \boldsymbol{T}$ on a cohesive category $\boldsymbol{C}$ and valued in a quality type $\boldsymbol{T}$ (Lawvere, 2007). Cohesion of a category $\boldsymbol{C}$ is relative to a base category $\boldsymbol{S}$ of discrete sets and is characterized by an adjoint string:

components ($c_!$) –| **discrete** (c^*) –| **points** (c_*) –| **codiscrete** ($c^!$)

(‘–|’ denotes ‘is left adjoint to’) of four functors:

components $c_!$: $\boldsymbol{C} \rightarrow \boldsymbol{S}$

discrete c^*: $\boldsymbol{S} \rightarrow \boldsymbol{C}$

points c_*: $\boldsymbol{C} \rightarrow \boldsymbol{S}$

codiscrete $c^!$: $\boldsymbol{S} \rightarrow \boldsymbol{C}$

(see Fig. 1A). The **components** functor maps each cohesive object (in ***C***) to its set of components (in ***S***), while the **points** functor maps each cohesive object to its set of points. The functors **discrete** and **codiscrete** map sets (in ***S***) to corresponding discrete and codiscrete objects (in ***C***). The category of reflexive graphs is an example of a cohesive category (serving as domain of qualities). In the section 'Cohesive Category', we show that the **discrete** functor is left adjoint to **points** functor as part of showing that the category of reflexive graphs is cohesive.

Quality type ***T***, the codomain category of quality, is also relative to the base category ***S*** of sets and is characterized by an adjoint string:

$$\textbf{components}\ (t_!) \dashv \textbf{discrete}\ (t^*) \dashv \textbf{points}\ (t_*) \dashv \textbf{codiscrete}\ (t^!)$$

of four functors collapsed to two functors (Johnstone, 1996):

$$\textbf{components}\ t_! = \textbf{points}\ t_*: \boldsymbol{T} \to \boldsymbol{S}$$

$$\textbf{codiscrete}\ t^! = \textbf{discrete}\ t^*: \boldsymbol{S} \to \boldsymbol{T}$$

as a result of which there is exactly one point in every component of each object of a quality type ***T***. The category of idempotents is an example of quality type. In the section 'Quality Type', we show that the category of idempotents is a quality type and that it has central idempotents required of quality types.

In an effort to systematically characterize the geometry and algebra of qualities, we define a category ***Q*** of qualities. Objects of the category of qualities are cohesion-preserving functors q: ***C*** –> ***T*** satisfying $t \cdot q = c$, where c: ***C*** –> ***S*** and t: ***T*** –> ***S*** are Set-labeled categories and '·' denotes composition. Qualities, in other words, are commutative triangles with functors as edges and categories as vertices (Fig. 1B). Within this formalism, qualities are broadly classified as extensive and intensive. An extensive quality *e* is a components-preserving functor *e*: ***C*** –> ***T*** satisfying $t_! \cdot e = c_!$, as a result of which the number of components of an extensive quality of a cohesive object is same as the number of components of the cohesive object. An intensive quality, on the other hand, is a points-preserving functor *i*: ***C*** –> ***T*** satisfying $t_* \cdot i = c_*$, as a result of which the number of points of an intensive quality of a cohesive object is same as the number of points of the cohesive object. Both extensive and intensive qualities are thus commutative triangles of functors on cohesive categories and valued in quality types. Morphisms of qualities are commutativity-preserving transformations of one triangle into another.

Composite qualities (cf. colored-shapes) are defined as categorical products of qualities. With color *c* and shape *s* as two qualities (two objects in the category ***Q***), composite quality colored-shape is an object:

$$p_C: c \times s \to c,\ p_S: c \times s \to s$$

in the category $\boldsymbol{Q}_{CS}$ of pairs of maps to the two factors *c* and *s* (Lawvere and Schanuel, 2009, pp. 255-256). The natural correspondence:

$$\frac{w \to c,\ w \to s}{w \to c \times s}$$

between pairs of *w*-shaped figures in the two factors and *w*-shaped figures in the product object can be used to calculate composite qualities. In order to calculate products of qualities using this method, we need to know the basic shapes of the category ***Q*** of qualities. Basic shapes of a category are those objects of the category in terms of which every object of the category can be completely determined. In the category of sets, there is one basic shape, which is the terminal set **1** = {•} consisting of exactly one element. Put differently, every set is completely determined by its elements (Lawvere and Schanuel, 2009, p. 245). The basic shapes of structured categories, unlike that of sets with zero structure, are more structured than mere element, and many categories have more than one basic shape. In the category of graphs, for example, there are two basic shapes: dot and arrow (Lawvere and Schanuel, 2009, p. 250). Since the objects of the category of qualities are commutative triangles of specific functors between chosen categories, the basic shapes of the category of qualities are going to be rather intricate. Once we identify the basic shapes of the category of qualities, we need to enumerate all pairs of figures (of each one of these basic shapes) in the two factors (qualities). These pairs of figures (of specific basic shapes) correspond to figures in the product object. Once we have all the basic-shaped figures in the product object, we need to determine the incidence relations between figures and the projection maps to factors to obtain composite qualities. As a preliminary step towards characterizing composite qualities, we calculated the product of commutative triangles. With vertices, edges, and triangles as basic shapes, the product of two generic triangles consists of a triangulated surface with nine vertices, twenty seven edges, and thirty seven commutative triangles (Fig. 2). In subsequent work, we plan to interpret the vertices and edges as categories and functors, respectively, so as to characterize composite qualities completely.

III. Quality Type

In this section we show that the functor **discrete**: ***S*** –> ***F*** (from the category ***S*** of sets to the category ***F*** of idempotents) has a right adjoint **points**: ***F*** –> ***S***, which is also left adjoint to the **discrete** functor, and hence makes the category ***F*** of idempotents a quality type over the category ***S*** of sets (Lawvere, 2007). First, we show that the functor **discrete** is left adjoint to the **points** functor, and then show that the functor **points** is [also] left adjoint to the **discrete** functor. To show that the **discrete** functor is left adjoint to the **points** functor, we have to show that there is a natural transformation *n*: **discrete · points** –> 1_F from the composite functor **discrete · points**: ***F*** –> ***S*** –> ***F*** to the identity functor 1_F: ***F*** –> ***F*** (Lawvere and Schanuel, 2009, pp. 372-377).

The functor **points**: ***F*** –> ***S*** assigns to each object in the category ***F*** of idempotents i.e. to each idempotent *e*: X –> X, *e* · *e* = *e* its set of fixed-points Y in the category ***S*** of sets. The set Y of fixed-points can be obtained by splitting the idempotent *e*: X –> X into its retract-section pair X – *r* –> Y – *s* –> X satisfying *s* · *r* = *e* (Lawvere and Schanuel, 2009, p. 102, 117) i.e. **points** (*e*: X –> Y –> X) = Y. The functor **points** assigns to each morphism <*f*, *f*>: *e* –> *e'* of idempotents, i.e. to each commutative diagram satisfying *e'* · *f* = *f* · *e* (Fig. 3A), a function *g*: Y –> Y' (from the set Y of fixed-points of the idempotent *e*: X –> Y –> X to the set Y' of fixed-points of the idempotent *e'*: X' –> Y' –> X') satisfying: *g* · *r* = *r'* · *f* and *s'* · *g* = *f* · *s* (where X' – *r'* –> Y' – *s'* –> X' is the splitting of the idempotent *e'*: X' –> X').

Next, the functor **discrete**: ***S*** –> ***F*** assigns to each set A (in ***S***) its identity function 1_A: A –> A (an idempotent in ***F***) and to each function *v*: A –> B a commutative square satisfying $1_B \cdot v = v \cdot 1_A$ (Fig. 3B), which is a morphism of idempotents in ***F***.

In order to have a natural transformation *n*: **discrete · points** –> 1_F, we need, for each idempotent *e*: X –> X, *e* · *e* = *e* (in ***F***) a map n_e: **discrete · points** (*e*) –> 1_F (*e*) in ***F***. Since **discrete · points** (*e*: X –> Y –> X) = **discrete** (Y) = 1_Y and 1_F (*e*: X –> Y –> X) = *e*, we need a map *h* from 1_Y: Y –> Y to *e*: X –> X satisfying $e \cdot h = h$ (Fig. 3C). Since Y is the set of fixed-points of the idempotent *e*: X –> X obtained by splitting *e*, i.e. X – *e* –> X = X – *r* –> Y – *s* –> X, we take *h* = *s*: Y –> X and find that $e \cdot h = s \cdot r \cdot s = s \cdot 1_Y = s = h$ since $r \cdot s = 1_Y$ (Lawvere and Schanuel, 2009, pp. 108-113). So, we can take sections *s*: Y –> X of the splitting $e = s \cdot r$ as components n_e: 1_Y –> *e* of the natural transformation *n*: **discrete** · **points** –> $1_{\boldsymbol{F}}$. Next, for each morphism (in the category ***F*** of idempotents) $<f, f>$: *e* –> *e'* (from *e*: X –> X to *e'*: X' –> X'), we need a commutative square in ***F*** satisfying $n_{e'}$ · **discrete** · **points** ($<f, f>$) = $1_{\boldsymbol{F}}$ ($<f, f>$) · n_e (Fig. 3D). Since **points** ($<f, f>$: *e* –> *e'*) = **points** (*e*) –> **points** (*e'*) = *g*: Y –> Y' (satisfying $g \cdot r = r' \cdot f$ and $s' \cdot g = f \cdot s$, where $s \cdot r = e$ and $s' \cdot r' = e'$), **discrete** · **points** ($<f, f>$) = **discrete** (*g*: Y –> Y') = $<g, g>$: 1_Y –> $1_{Y'}$, and $1_{\boldsymbol{F}}$ ($<f, f>$: *e* –> *e'*) = $<f, f>$: *e* –> *e'*, we find that we need a commutative diagram satisfying: $f \cdot n_e = n_{e'} \cdot g$ (Fig. 3E). With sections as components we have the required commutative diagram (Fig. 3F) satisfying $f \cdot s = s' \cdot g$, and in turn a natural transformation *n*: **discrete** · **points** –> $1_{\boldsymbol{F}}$**,** which in turn tells that the functor **discrete**: ***S*** –> ***F*** is left adjoint to the functor **points**: ***F*** –> ***S***.

Next, in order to show that the functor **points**: ***F*** –> ***S*** is left adjoint to the functor **discrete**: ***S*** –> ***F***, we need a natural transformation *u*: $1_{\boldsymbol{F}}$ –> **discrete** · **points** with components u_e: $1_{\boldsymbol{F}}$ (*e*: X –> X) –> **discrete** · **points** (*e*: X –> X) satisfying: $u_{e'} \cdot <f, f> = <g, g> \cdot u_e$ (Fig. 4A). Taking the retract *r*: X –> Y of the splitting X – *r* –> Y – *s* –> X of an idempotent X – *e* –> X as the component corresponding to the idempotent i.e. with $u_e = r$: X –> Y we find that $g \cdot r = r' \cdot f$ (Fig. 4B). So, we do have a natural transformation from the identity functor $1_{\boldsymbol{F}}$: ***F*** –> ***F*** to the composite functor **discrete** · **points**: ***F*** –> ***F***.

Given both natural transformations **discrete** · **points** –> 1_F and 1_F –> **discrete** · **points**, we say that the functor **points**: ***F*** –> ***S*** is both right and left adjoint of the functor **discrete**: ***S*** –> ***F***. Thus the **discrete** functor from the category ***S*** of sets to the category ***F*** of idempotents, with **points** functor as its right and left adjoint, makes the category ***F*** of idempotents a quality type over the category ***S*** of sets (Lawvere, 2007).

Next, we show that the category of idempotents has a central idempotent. A central idempotent is a natural endomorphism (of an identity functor) all of whose components are idempotents (Lawvere, 2004). Given a category ***E***, the identity functor 1_E: ***E*** –> ***E*** maps every object, morphism in ***E*** to the same object, morphism, respectively, in the same ***E*** i.e. 1_E (A) = A and 1_E (*f*: A –> B) = *f*: A –> B. A central idempotent is a natural transformation $\theta = 1_E$ –> 1_E assigning to each object A, a map $\theta_A = 1_E$ (A) –> 1_E (A) and to each morphism *f*: A –> B a commutative diagram satisfying $1_E (f) \cdot \theta_A = \theta_B \cdot 1_E (f)$ (Fig. 5A), and with each component an idempotent: $\theta_A \cdot \theta_A = \theta_A$ and $\theta_B \cdot \theta_B = \theta_B$. Since 1_E (A) = A, 1_E (B) = B, and 1_E (*f*: A –> B) = *f*: A –> B, we need commutative squares satisfying $f \cdot \theta_A = \theta_B \cdot f$, $\theta_A \cdot \theta_A = \theta_A$, and $\theta_B \cdot \theta_B = \theta_B$ (Fig. 5B).

Let ***E*** denote the category of idempotents. A morphism *f* from one idempotent (A, α: A –> A; α · α = α) to another idempotent (B, β: B –> B; β · β = β) is a function *f*: A –> B satisfying $f \cdot \alpha = \beta \cdot f$. The identity functor 1_E: ***E*** –> ***E*** maps each object, morphism to itself i.e. 1_E (A, α) = (A, α), 1_E (B, β) = (B, β), and 1_E (*f*: (A, α) –> (B, β)) = *f*: (A, α) –> (B, β). A central idempotent is a natural transformation $\theta = 1_E$ –> 1_E assigning to each object (A, α) a map θ_A: (A, α) –> (A, α), and to each morphism *f*: (A, α) –> (B, β) a commutative diagram satisfying $f \cdot \theta_A = \theta_B \cdot f$, $\theta_A \cdot \theta_A = \theta_A$, and $\theta_B \cdot \theta_B = \theta_B$ (Fig. 5C). Since α: (A, α) –> (A, α) is a morphism in the category ***E*** of idempotents i.e. satisfies α · α = α · α and α · α = α (Lawvere and Schanuel, 2009, p. 179), we take $\theta_A = \alpha$: A –> A and $\theta_B = \beta$: B –> B. With these components, we obtain a commutative diagram satisfying the required $f \cdot \alpha = \beta \cdot f$ (Fig. 5D), since f: (A, α) –> (B, β) is a morphism in the category ***E*** of idempotents i.e. satisfies $f \cdot \alpha = \beta \cdot f$. We also find that the components α: (A, α) –> (A, α) and β: (B, β) –> (B, β) of the natural transformation $\theta = 1_E$ –> 1_E satisfy $\alpha \cdot \alpha = \alpha$ and $\beta \cdot \beta = \beta$ since α: A –> A and β: B –> B are objects of the category ***E*** of idempotents. Thus the natural transformation $\theta = 1_E$ –> 1_E (of the identity functor 1_E: ***E*** –> ***E*** of the category ***E*** of idempotents), each of whose components θ_A: (A, α) –> (A, α) is the corresponding structural map i.e. $\theta_A = \alpha$: A –> A satisfying $\alpha \cdot \alpha = \alpha$, is a central idempotent.

In the next section, as part of characterizing reflexive graphs as a cohesive category (which is the domain of qualities), we show that the functor **discrete**: ***S*** –> ***R*** (from the category ***S*** of sets to the category ***R*** of reflexive graphs) is left adjoint to the functor **points**: ***R*** –> ***S***.

IV. Cohesive Category

The qualities in our product-of-qualities formalization of qualia space are morphisms on cohesive categories (Lawvere, 2007; Lawvere and Menni, 2015). Since calculation of products requires knowledge of basic shapes of the category, we need to be explicit about cohesive categories. Cohesive categories are characterized in terms of adjoint functors to and from the category of discrete sets, relative to which cohesion is measured. In this section, as part of showing that the category of reflexive graphs is a cohesive category, we verify that the functor **discrete**: ***S*** –> ***R*** from the category ***S*** of sets to the category ***R*** of reflexive graphs is left adjoint to the functor **points**: ***R*** –> ***S*** (Lawvere and Schanuel, 2009, pp. 372-377).

A reflexive graph X consists of two component sets: a set X_A of arrows and a set X_D of dots, and three structural maps s_x: X_A –> X_D, t_x: X_A –> X_D, and i_x: X_D –> X_A (assigning to each arrow in X_A its source, target dot in X_D and to each dot in X_D its preferred loop in X_A) satisfying $s_x \cdot i_x = 1_{XD}$ and $t_x \cdot i_x = 1_{XD}$ (Lawvere and Schanuel, 2009, pp. 145-146). A graph morphism *f*: X –> Y (from a graph X to a graph Y) is a pair of set maps f_A: X_A –> Y_A, f_D: X_D –> Y_D satisfying: $s_y \cdot f_A = f_D \cdot s_x$, $t_y \cdot f_A = f_D \cdot t_x$, and $i_y \cdot f_D = f_A \cdot i_x$ (where s_y: Y_A –> Y_D, t_y: Y_A –> Y_D, and i_y: Y_D –> Y_A satisfying $s_y \cdot i_y = 1_{YD}$ and $t_y \cdot i_y = 1_{YD}$ are the structural maps corresponding to the codomain graph Y).

The functor **points**: ***R*** –> ***S*** maps each graph X = (s_x: X_A –> X_D, t_x: X_A –> X_D, i_x: X_D –> X_A) in the domain category ***R*** to its set of dots X_D in the codomain category ***S*** i.e. **points** (X) = X_D, and maps each graph morphism *f*: X –> Y = <f_A: X_A –> Y_A, f_D: X_D –> Y_D> in ***R*** to its dot component f_D: X_D –> Y_D in ***S*** i.e. **points** (*f*: X –> Y) = f_D: X_D –> Y_D.

The functor **discrete**: ***S*** –> ***R*** maps each set P in ***S*** to a graph (also denoted) P in ***R*** with the set P as both of its component sets (set P_A of arrows and set P_D of dots) i.e. $P_A = P_D = P$, and with the identity function on P i.e. 1_P: P –> P as all three structural maps i.e. $s_p = t_p = i_p = 1_P$: P –> P satisfying $s_p \cdot i_p = 1_P$ and $t_p \cdot i_p = 1_P$. Thus **discrete** (P) = (1_P: P –> P, 1_P: P –> P, 1_P: P –> P). A function *z*: P –> Q in ***S*** is mapped by the **discrete** functor to a graph morphism in ***R*** (also denoted) *z*: P –> Q, which has the function *z* as both (arrow and dot) component functions i.e. *z* = <z_A, z_D> = <*z*, *z*> satisfying the three required commutative conditions: $1_Q \cdot z = z \cdot 1_P$, $1_Q \cdot z = z \cdot 1_P$, and $1_Q \cdot z = z \cdot 1_P$ preserving the source, target, and identity structure of reflexive graphs. Thus **discrete** (*z*: P –> Q) = <z: P –> Q, *z*: P –> Q>.

The composite functor **discrete** · **points** = ***R*** –> ***S*** –> ***R*** maps each graph $X = (s_x: X_A \text{–>} X_D, t_x: X_A \text{–>} X_D, i_x: X_D \text{–>} X_A)$ to a discrete graph $X_D = (1_{XD}: X_D \text{–>} X_D, 1_{XD}: X_D \text{–>} X_D, 1_{XD}: X_D \text{–>} X_D)$ consisting of preferred loops only on dots, and maps each graph morphism f: X –> Y i.e. each $f = \langle f_A: X_A \text{–>} Y_A, f_D: X_D \text{–>} Y_D \rangle$ (satisfying: $s_y \cdot f_A = f_D \cdot s_x$, $t_y \cdot f_A = f_D \cdot t_x$, and $i_y \cdot f_D = f_A \cdot i_x$) to the graph morphism $f_D = \langle f_D: X_D \text{–>} Y_D, f_D: X_D \text{–>} Y_D \rangle$ satisfying: $1_{YD} \cdot f_D = f_D \cdot 1_{XD}$ (Fig. 6).

To show that the functor **discrete**: ***S*** –> ***R*** is left adjoint to the functor **points**: ***R*** –> ***S*** we have to show that there is a natural transformation n: **discrete** · **points** –> $1_{\boldsymbol{R}}$ from the composite functor **discrete** · **points** to the identity functor $1_{\boldsymbol{R}}$ on the category ***R*** of reflexive graphs. In other words, we have to show that for each graph $X = (s_x: X_A \text{–>} X_D, t_x: X_A \text{–>} X_D, i_x: X_D \text{–>} X_A)$ there is a graph morphism n_X: **discrete** · **points** (X) –> $1_{\boldsymbol{R}}$ (X) i.e. a graph morphism n_X: **discrete** · **points** $(s_x: X_A \text{–>} X_D, t_x: X_A \text{–>} X_D, i_x: X_D \text{–>} X_A)$ –> $1_{\boldsymbol{R}}$ $(s_x: X_A \text{–>} X_D, t_x: X_A \text{–>} X_D, i_x: X_D \text{–>} X_A)$ which is

$n_X: (1_{XD}: X_D \text{–>} X_D, 1_{XD}: X_D \text{–>} X_D, 1_{XD}: X_D \text{–>} X_D) \text{–>} (s_x: X_A \text{–>} X_D, t_x: X_A \text{–>} X_D, i_x: X_D \text{–>} X_A)$,

which, in turn, is a pair of maps $n_X = \langle n_{XA}: X_D \text{–>} X_A, n_{XD}: X_D \text{–>} X_D \rangle$ preserving the source, target, and identity (preferred loop) structure of reflexive graphs. With $n_{XA} = i_X: X_D \text{–>} X_A$ as the arrow component and with $n_{XD} = 1_{XD}: X_D \text{–>} X_D$ as the dot component, we have a graph morphism preserving the source, target, and identity structure of reflexive graphs i.e. satisfying the three required commutativity conditions: $s_X \cdot i_X = 1_{XD} \cdot 1_{XD}$ (since $s_X \cdot i_X = 1_{XD}$), $t_X \cdot i_X = 1_{XD} \cdot 1_{XD}$ (since $t_X \cdot i_X = 1_{XD}$), and $i_X \cdot 1_{XD} = i_X \cdot 1_{XD}$. Finally, we have to show that for each graph morphism f: X –> Y (in ***R***) the commutativity condition:

$1_{\boldsymbol{R}}$ (f: X –> Y) · $n_X = n_Y$ · **discrete** · **points** (f: X –> Y) is satisfied. In other words, we have to show that

$1_{\boldsymbol{R}}$ ($<f_A$: X_A –> Y_A, f_D: X_D –> Y_D>) · n_X = n_Y · **discrete** · **points** (<f_A: X_A –> Y_A, f_D: X_D –> Y_D>). Since $1_{\boldsymbol{R}}$ (<f_A: X_A –> Y_A, f_D: X_D –> Y_D>) = <f_A: X_A –> Y_A, f_D: X_D –> Y_D> and **discrete** · **points** (<f_A: X_A –> Y_A, f_D: X_D –> Y_D>) = <f_D: X_D –> Y_D, f_D: X_D –> Y_D>, and with n_X = <n_{XA}: X_D –> X_A, n_{XD}: X_D –> X_D> = <i_X: X_D –> X_A, 1_{XD}: X_D –> X_D> and n_Y = <n_{YA}: Y_D –> Y_A, n_{YD}: Y_D –> Y_D> = <i_Y: Y_D –> Y_A, 1_{YD}: Y_D –> Y_D>, we have to show that $f_A \cdot i_X = i_Y \cdot f_D$, which is already given in the identity preserving graph morphism *f* from X to Y. Hence, we have a natural transformation *n*: **discrete** · **points** –> $1_{\boldsymbol{R}}$ (Fig. 7), and, in turn, the **discrete** functor from sets ***S*** to reflexive graphs ***R*** is left adjoint to the **points** functor from reflexive graphs to sets.

There are two additional axioms that a category has to satisfy in order to be cohesive. They are product-preserving **components** functor and connected truth value object (see axioms 1 and 2 in Lawvere, 2005). The **components** functor (from the category of reflexive graphs to the category of sets, and assigning to each reflexive graph its set of components) preserves products (**components** (A × B) = **components** (A) × **components** (B), where A, B are reflexive graphs). Also, the truth value object **Ω** of the category of reflexive graphs is connected i.e. one component (**components** (Ω) = 1). Thus the category of reflexive graphs is a cohesive category.

V. Conclusion

The problem of conscious experience is particularly challenging in view of the seeming incongruity of the qualities of qualia on one hand, and the seamless cohesion of conscious experience on the other hand. We need a mathematical framework that can capture not only qualitative qualia, but also the combination of these qualities into the unity of our perceptual experiences (Roskies, 1999). There is, within mathematics, an analogous research program directed towards objectification of the unity of mathematics. These foundational investigations seeking to reunite analysis, algebra, combinatorics, geometry, and logic, all arising from the study of qualities resulting from the variation of quantities within cohesive spaces (Lawvere, 2014), led to axiomatization of the hitherto vague notions of cohesion and quality (Lawvere, 2007; Lawvere and Menni, 2015). It is this category theoretic study of cohesion and quality that we are applying to the problem of combining qualities into cohesive consciousness.

We defined composite qualities of our conscious experience as categorical products of qualities. The product-of-qualities account of consciousness serves as an abstract theoretical framework to conceptualize how qualities such as color and shape are combined into the colored-shapes of our visual experience. Our product-of-qualities formalization can be thought of as a refinement of the basic idea of feature conjunctions, wherein the percept of 'red square' is associated with the activation of 'red' neuron and 'square' neuron. The theoretical refinement is essentially in taking into account the structure of qualities (intensive colors *vs.* extensive shapes) and of their composition into the cohesiveness of consciousness. More specifically, with the objective of conceptualizing the binding of extensive shapes and intensive colors into the colored shapes of our visual experience, we focused on the particular case of the product of extensive and intensive qualities. Extensive and intensive qualities are functors, from the domain category (reflexive graphs) of one functor (defining cohesion) to the domain category (idempotents) of another functor (defining quality type). Cohesion and quality type are defined relatively i.e. relative to the category of sets, which is the common codomain of these two functors. Hence qualities (both extensive and intensive) are commutative triangles. With extensive and intensive qualities as commutative triangles, composite qualities are products of commutative triangles. In the present note, as part of characterizing composite qualities, we calculated the product of commutative triangles. We also showed that the category of idempotents is a quality type. As part of showing that the category of reflexive graphs is cohesive, we characterized the adjointness between functors relating reflexive graphs to sets. In our subsequent work, we plan to interpret the vertices and edges of commutative triangles, whose products we calculated, as categories and functors, respectively, so as to calculate products of qualities. Furthermore, we plan to thoroughly characterize the geometry and algebra of composite qualities, which can serve as a basis for precise reasoning and definitive calculations about qualia and consciousness. We also plan to provide, in our subsequent work, an in-depth comparison of Ehresmann's formalization of the binding problem in terms of colimits (Ehresmann and Vanbremeersch, 2007) with our product-of-qualities model of qualia space.

In closing, the main problem with cognitive neuroscience is a lack of good theories to guide neuroscientific investigations (Stevens, 2000). Moreover, the need for an explicit mathematical framework, analogous to the calculus of physics, which can facilitate the advancement of the science of consciousness has long been recognized (Lawvere, 1994, 1999). Our mathematical characterization of qualia space as a categorical product of qualities provides rudiments of the mathematical framework needed for the development of the science of consciousness.

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Figure Legends

Figure 1: Qualities as commutative triangles. (**A**) Quality is defined as a functor *q* on a cohesive category ***C*** and valued in a quality type ***T***. Cohesion of a category ***C*** is relative to a base category ***S*** of sets and is characterized by four functors: **components** $c_!$: ***C*** –> ***S***, **discrete** c^*: ***S*** –> ***C***, **points** c_*: ***C*** –> ***S***, and **codiscrete** $c^!$: ***S*** –> ***C***; with **components** ($c_!$) left adjoint to **discrete** (c^*) left adjoint to **points** (c_*) left adjoint to **codiscrete** ($c^!$). Quality type ***T***, the codomain category of quality, is also relative to the base category ***S*** of sets and is characterized by four functors satisfying: **components** $t_!$ = **points** t_*: ***T*** –> ***S*** and **codiscrete** $t^!$ = **discrete** t^*: ***S*** –> ***T***. (**B**) Quality *q*: ***C*** –> ***T*** is a cohesion-preserving functor satisfying $t \cdot q = c$, where '·' denotes composition. With the functors *c* and *t* as **points** functors c_*: ***C*** –> ***S*** and t_*: ***T*** –> ***S***, respectively, a points-preserving functor *i*: ***C*** –> ***T*** satisfying $t_* \cdot i = c_*$ is an intensive quality. With *c* and *t* as **components** functors $c_!$: ***C*** –> ***S*** and $t_!$: ***T*** –> ***S***, respectively, a components-preserving functor *e*: ***C*** –> ***T*** satisfying $t_! \cdot e = c_!$ is an extensive quality.

Figure 2: Product of commutative triangles. Consider a commutative triangle G with three objects A, B, and C, three maps *f*, *g*, and *h*, and the commutativity equation $gf = h$. The commutative triangle can be modeled as a set V = {A, B, C} of vertices, a set $E = \{1_A, f, 1_B, g, 1_C, h\}$ of edges, and a set $T = \{1_A1_A=1_A, f1_A=f, 1_Bf=f, 1_B1_B=1_B, g1_B=g, 1_Cg=g, 1_C1_C=1_C, h1_A=h, 1_Ch=h, gf = h\}$ of

triangles. The set E of edges includes vertices as identities (e.g. 1_A), while the set T of triangles includes one identity commutative triangle (e.g. $1_A 1_A = 1_A$) for each vertex and two identity commutative triangles (e.g. $f1_A=f$, $1_B f=f$) for each edge. The product G × G of two commutative triangles consists of nine vertices (V × V = {AA, AB, AC, BA, BB, BC, CA, CB, CC}), thirty six edges (E × E), and hundred triangles (T × T). Of the thirty six edges of the product, nine are identities (corresponding to the nine vertices in V × V), and the remaining twenty seven non-identity edges are displayed above. These twenty seven edges form thirty seven non-identity commutative triangles. Of the hundred commutative triangles in T × T, nine are identities corresponding to the nine vertices and fifty four are identities corresponding to twenty seven edges (two identity commutative triangles for each non-identity edge), with thirty seven non-identity triangles remaining.

Figure 3: The functor discrete: *S* –> *F* is left adjoint to the functor points: *F* –> *S*. (**A**) The functor **points**: ***F*** –> ***S*** assigns to each morphism i.e. to each commutative square (depicted) satisfying *e’* · *f* = *f* · *e* (in the category ***F*** of idempotents) a function *g*: Y –> Y’ (from the set Y of fixed-points of the idempotent *e*: X – *r* –> Y – *s* –> X to the set Y’ of fixed-points of the idempotent *e’*: X’ – *r’* –> Y’ – *s’* –> X’) satisfying: *g* · *r* = *r’* · *f* and *s’* · *g* = *f* · *s*. (**B**) The functor **discrete**: ***S*** –> ***F*** assigns to each function *v*: A –> B a commutative square satisfying $1_B \cdot v = v \cdot 1_A$, which is a morphism of idempotents in ***F***. (**C**) A map *h*: Y –> X making the (depicted) diagram commute i.e. satisfying *e* · *h* = *h* is required as component n_e: 1_Y –> *e* of the natural transformation *n*: **discrete** · **points** –> $1_{\boldsymbol{F}}$. Taking the section *s*: Y –> X of the splitting *e* = *s* · *r* as *h*: Y –> X, we find that it satisfies the required *e* · *h* = *h*. (**D**) For each morphism (in the category ***F*** of idempotents) <*f*, *f*>: *e* –> *e’* (from *e*: X –> X to *e’*: X’ –> X’) we need the

displayed square to commute (in ***F***) i.e. satisfy: $n_{e'}$ · **discrete** · **points** $(\langle f, f \rangle) = 1_{\boldsymbol{F}} (\langle f, f \rangle) \cdot n_e$. (**E**) With **points** $(\langle f, f \rangle) = g$ and **discrete** · **points** $(\langle f, f \rangle) =$ **discrete** $(g) = \langle g, g \rangle$, and $1_{\boldsymbol{F}} (\langle f, f \rangle) = \langle f, f \rangle$, we need the square displayed to commute i.e. satisfy: $f \cdot n_e = n_{e'} \cdot g$. (**F**) With sections *s*: Y –> X as components n_e: 1_Y –> *e* of the natural transformation *n*: **discrete** · **points** –> $1_{\boldsymbol{F}}$, we have the required commutative diagram satisfying $f \cdot s = s' \cdot g$.

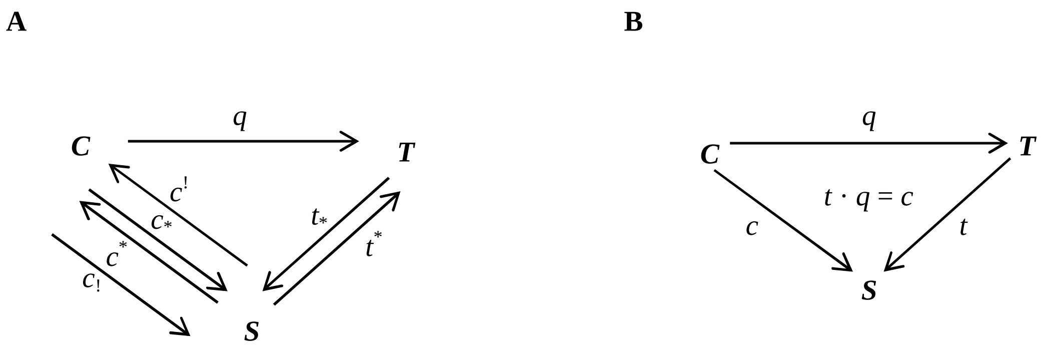
Figure 4: The functor points: *F* –> *S* is left adjoint to the functor discrete: *S* –> *F*. (**A**) A natural transformation *u*: $1_{\boldsymbol{F}}$ –> **discrete** · **points** with components u_e: $1_{\boldsymbol{F}}$ (*e*: X –> X) –> **discrete** · **points** (*e*: X –> X) satisfying: $u_{e'} \cdot \langle f, f \rangle = \langle g, g \rangle \cdot u_e$, i.e. making the displayed square commutative, makes the functor **points**: ***F*** –> ***S*** left adjoint to the functor **discrete**: ***S*** –> ***F***. (**B**) With retracts *r*: X –> Y as components u_e: *e* –> 1_Y, i.e. with $u_e = r$: X –> Y, we obtain the required commutative square satisfying $g \cdot r = r' \cdot f$.

Figure 5: Central idempotent. (**A**) A central idempotent is a natural transformation θ assigning to each morphism *f*: A –> B a commutative diagram satisfying $1_E (f) \cdot \theta_A = \theta_B \cdot 1_E (f)$. (**B**) Since the natural transformation is an endomorphism of identity functors with $1_E (f) = f$, we need a commutative diagram satisfying $f \cdot \theta_A = \theta_B \cdot f$. (**C**) Central idempotent θ assigns to each object (A, α) in the category ***E*** of idempotents a map θ_A: (A, α) –> (A, α), and to each morphism *f*: (A, α) –> (B, β) a commutative diagram satisfying $f \cdot \theta_A = \theta_B \cdot f$. (**D**) With idempotent endomaps as components of the natural transformation, i.e. $\theta_A = \alpha$: A –> A, $\theta_B = \beta$: B –> B, we obtain a commutative diagram satisfying $f \cdot \alpha = \beta \cdot f$.

Figure 6: Composite endofunctor discrete · points on the category of reflexive graphs. Each graph X = (s_x: X_A –> X_D, t_x: X_A –> X_D, i_x: X_D –> X_A) in the category ***R*** of reflexive graphs is mapped to a discrete graph X_D = (1_{XD}: X_D –> X_D, 1_{XD}: X_D –> X_D, 1_{XD}: X_D –> X_D) in ***R***, and each graph morphism *f*: X –> Y i.e. each *f* = <f_A: X_A –> Y_A, f_D: X_D –> Y_D> in ***R*** is mapped to the graph morphism f_D = <f_D: X_D –> Y_D, f_D: X_D –> Y_D> in ***R*** by the composite endofunctor **discrete · points: *R* –> *S* –> *R***.

Figure 7: Natural transformation from the composite endofunctor discrete · points to the identity functor on reflexive graphs. Since $f_A \cdot i_X = i_Y \cdot f_D$ (from the definition of graph morphism), we take the inclusion i_X: X_D –> X_A of dots in X_D into X_A as preferred loops as components of the natural transformation *n*: **discrete · points** –> $1_{\boldsymbol{R}}$. With $1_{\boldsymbol{R}}$ (<f_A: X_A –> Y_A, f_D: X_D –> Y_D>) = <f_A: X_A –> Y_A, f_D: X_D –> Y_D> and **discrete · points** (<f_A: X_A –> Y_A, f_D: X_D –> Y_D>) = <f_D: X_D –> Y_D, f_D: X_D –> Y_D>, and taking n_X = <n_{XA}: X_D –> X_A, n_{XD}: X_D –> X_D> = <i_X: X_D –> X_A, 1_{XD}: X_D –> X_D> and n_Y = <n_{YA}: Y_D –> Y_A, n_{YD}: Y_D –> Y_D> = <i_Y: Y_D –> Y_A, 1_{YD}: Y_D –> Y_D>, we have all the commutativity conditions satisfied and hence a natural transformation from the composite endofunctor **discrete · points** to the identity functor $1_{\boldsymbol{R}}$ on the category of reflexive graphs.

Figure 1



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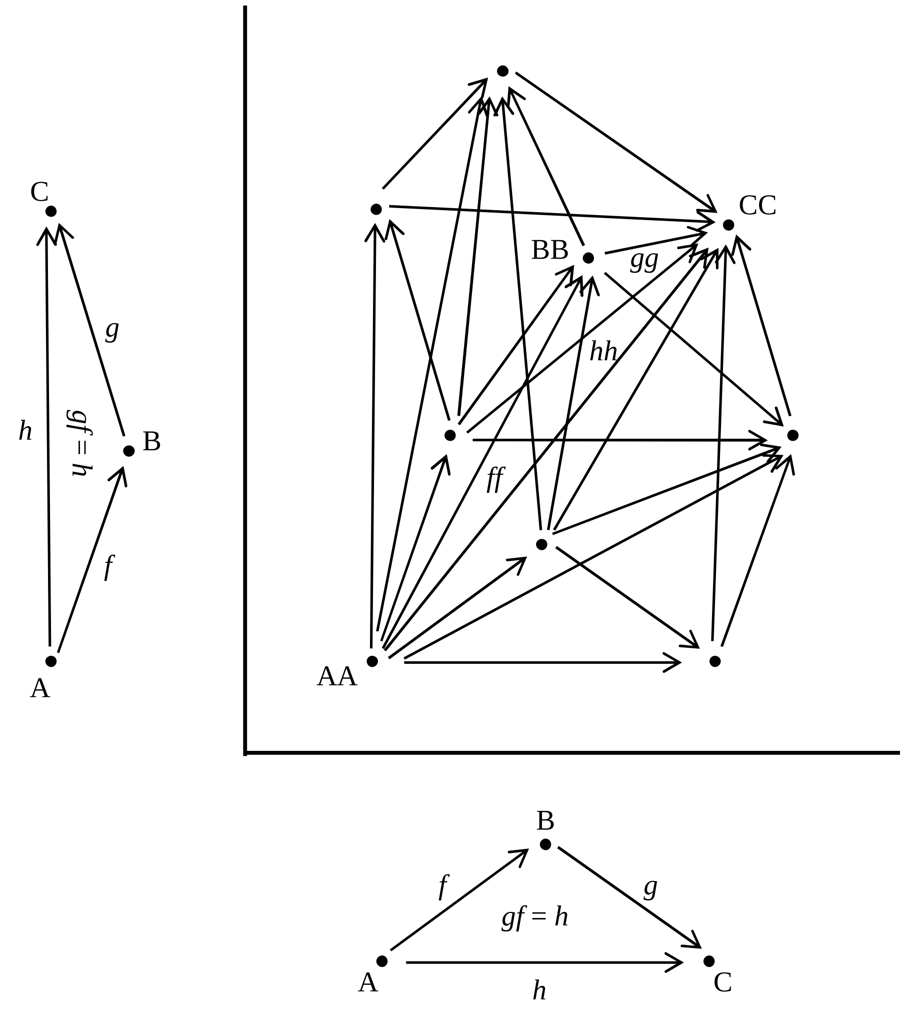

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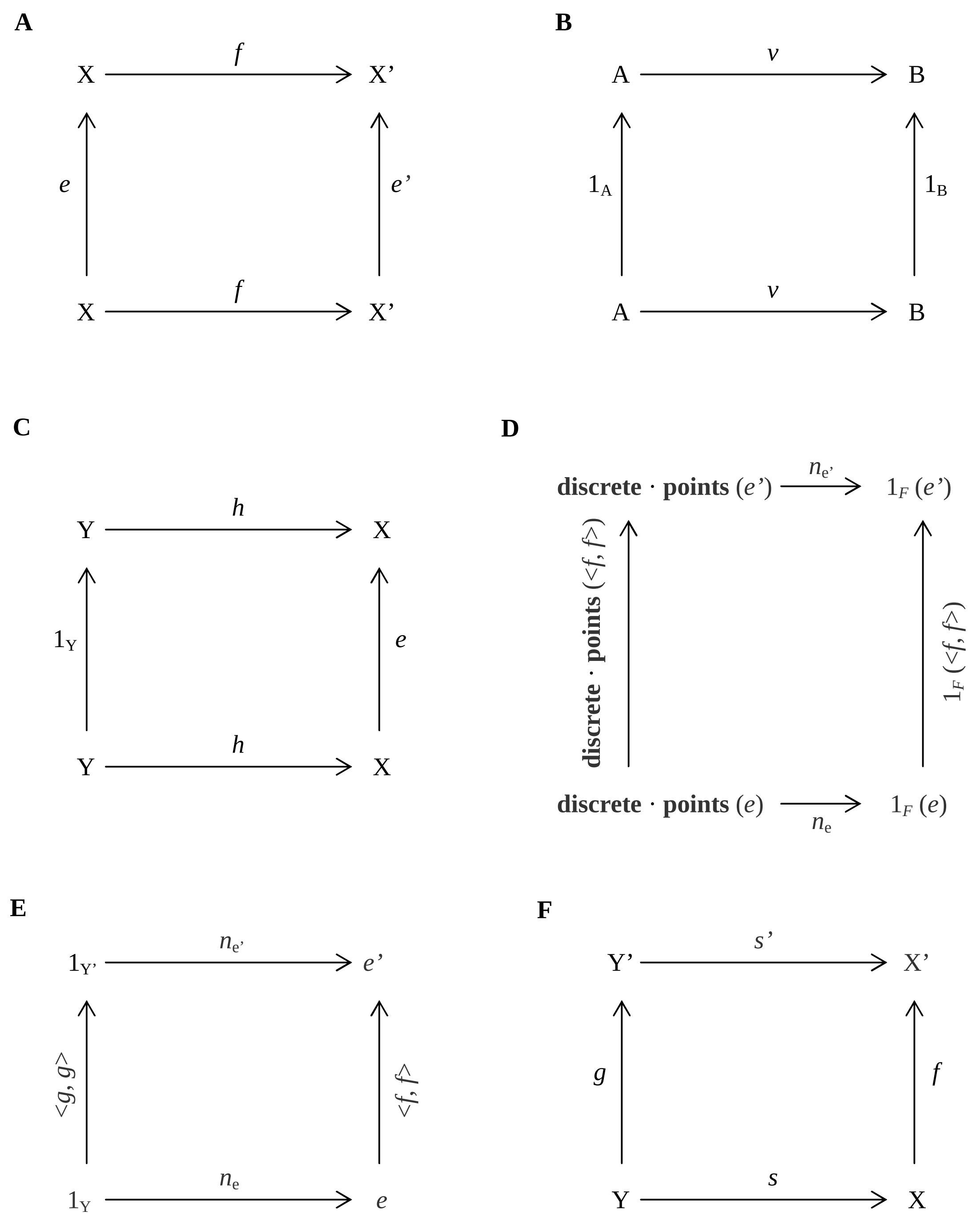
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Figure 4

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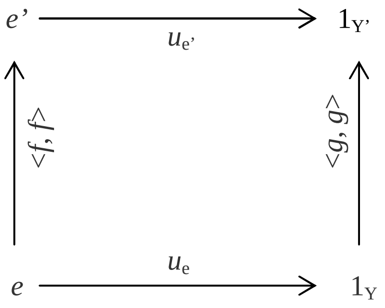

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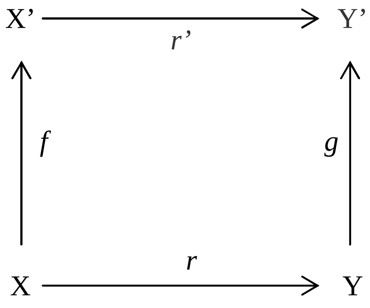
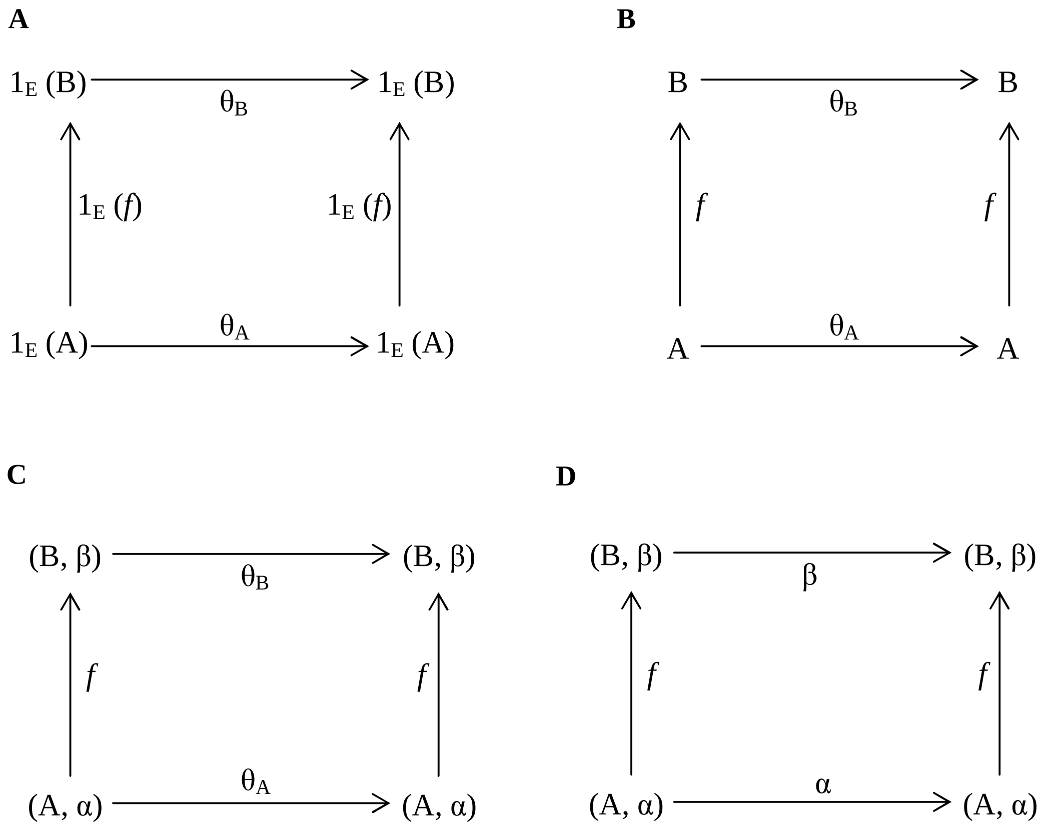


Figure 5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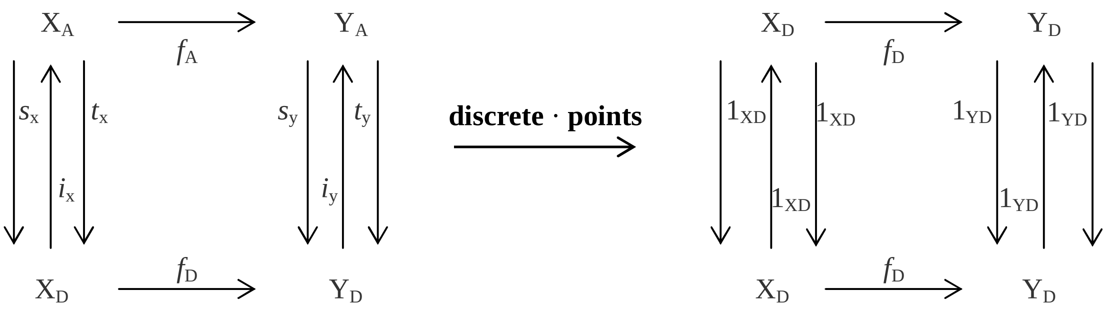
B
B
θB
B
f
f
A
θA
A
C
(B, β)
θB
(B, β)
f
f
(A, α)
θA
(A, α)
D
(B, β)
β
(B, β)
f
f
(A, α)
α
(A, α)

Figure 6



X_A
f_A
Y_A
s_x
t_x
i_x
s_y
t_y
i_y
X_D
f_D
Y_D
discrete · points
X_D
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1_XD
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1_YD
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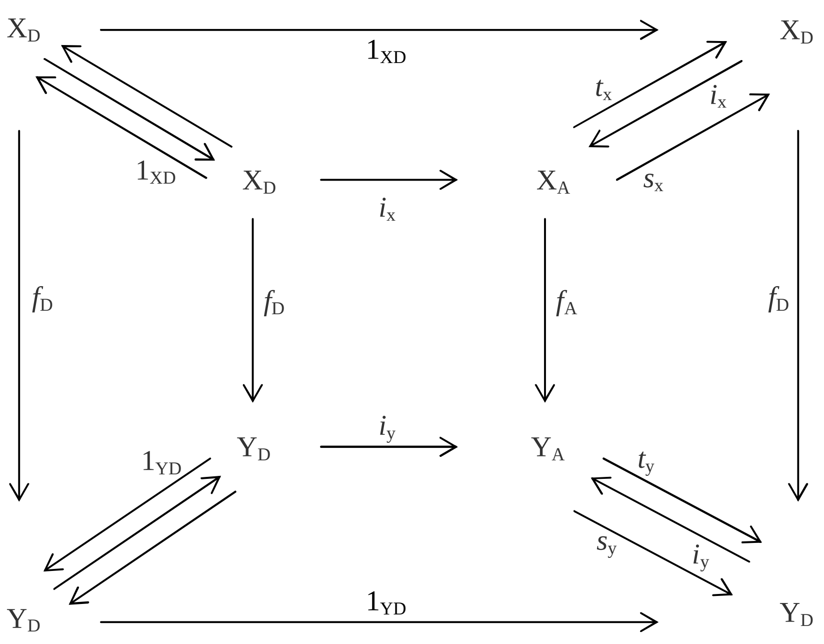
X_D
f_D
Y_D

Figure 7


X_D
1_XD
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t_x
i_x
1_XD
X_D
i_x
X_A
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f_A
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i_y
Y_D
Y_A
1_YD
t_y
s_y
i_y
Y_D
1_YD
Y_D

JMP-15-92: Final Decision

From: JMP (ELS) jmp@elsevier.com, Date: Wed, Aug 19, 2015 at 1:31 PM, Subject: JMP-15-92: Final Decision, To: posinavrayudu@gmail.com

Ms. No.: JMP-15-92

Title: QUALIA SPACE AND CATEGORY THEORY

Corresponding Author: Mr. Venkata Rayudu Posina

Authors: Sisir Roy, Ph.D

Dear Mr. Posina,

Thank you for submitting your manuscript to the Journal of Mathematical Psychology. Your paper, referenced above, has been reviewed by experts in the field. Based on the comments of these reviewers, we regret to inform you that we are unable to accept your manuscript for publication in the Journal of Mathematical Psychology.

The comments of the reviewers are included below in order for you to understand the basis for our decision, and we hope that their thoughtful comments will help you in your future studies.

Please note that if the decision letter and reviews are not provided below because they have been uploaded as separate pdf attachments, you will be able to view this information by logging onto

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While you may be disappointed by this decision, I would like to urge you to continue to consider the Journal of Mathematical Psychology for publication of future manuscripts.

Sincerely,

Jun Zhang

Associate Editor

Journal of Mathematical Psychology

E-mail: jmp@elsevier.com

Editor and Reviewers' comments:

"Qualia Space and Category Theory" by Mr. Venkata Rayudu Posina (JMP-15-92)

SUMMARY

This paper attempts to put forth a formal framework, using the language of category theory, to deal with conjunction of features (or as the author called it "qualitative universals", such as black or bitter) in our conscious experience of objects. In particular, the framework attempts to address the question of "cohesiveness" of different qualities, that is, the co-existence of "black and bitter" that "combine into the seamless unity of our conscious experience."

CRITIQUE

The paper models the qualia space as a "categorical product of qualities", where each quality itself is modeled as a commutative triangle with functors as edges and categories as vertices. The author invoked the category of idempotents and reflexive graphs; this is beyond my mathematical expertise. My limited understanding is that the author modeled the category of "qualities" as a commutative triangle, such that to model composite qualities, one has to calculate the product of two generic triangles. The outcome is a "product-of-qualities", which essentially is a product space construction of underlying features. The author worked out a calculation for the product of two commutative triangles. However, the interpretation of the entities is not clear. The author put is as future work ("In subsequent work, we plan to interpret the vertices and edges as categories and functors, respectively …" (line 167). So the author merely "defined composite qualities of our conscious experience as category products of qualities" (line 327).

This paper is very difficult to read, as it is based on a mathematical framework relatively unfamiliar to the mathematical psychology community. Category theory is heavy machinery, with laborious overheads (functor, commutative triangle, adjoint, retract, split, etc) whose meaning to cognition/psychology is not apparent. The basic problem of this paper is that the author has not succeeded in translate category-theoretic explanation in plain psychological terms. Hence, "things to be explained" and "the explanation itself" has not been clearly distinguished. One crucial question is what, if any, new insights that is generated by category-theoretic analysis. I am sure there are, once correspondence between category-theoretic notions and psychological concepts are flushed out and made explicit.

Below, I list a sample of the places where connections to psychology/cognition should be made explicit:

a) "Qualities … are commutative triangles with functors as edges and categories as vertices". What is the analogy of "functor" in psychology?

b) "Extensive qualities" (component-preserving functor) and "intensive qualities" (points-preserving functor): can you translate those into plain language in terms of features?

c) "Basic shapes of a category are those objects of the category in terms of which every object of the category can be completely determined." What is "basic shape", and what do they correspond in psychology?

d) In section 3, "In this section we show that the functor discrete: S -> F (from the category S of sets to the category F of idempotents) has a right adjoint points: F -> S, which is also left adjoint to the discrete functor, and hence makes the category F of idempotents a quality type over the category S of sets (Lawvere, 2007)." Can you translate those terms in plain language for consciousness?

e) "Our product-of-qualities formalization can be thought of as a refinement of the basic idea of feature conjunctions, wherein the percept of 'red square' is associated with the activation of 'red' neuron and 'square' neuron. The theoretical refinement is essentially in taking into account the structure of qualities (intensive colors vs. extensive shapes) and of their composition into the cohesiveness of consciousness" (line 330) I don't see any implications to neuroscience offered by author's "theoretical refinement". What does the distinction of intensive vs extensive mean for psychology/neuroscience?

f) "Extensive and intensive qualities are functors, from the domain category (reflexive graphs) of one functor (defining cohesion) to the domain category (idempotents) of another functor (defining quality type). Cohesion and quality type are defined relatively i.e. relative to the category of sets, which is the common codomain of these two functors. Hence qualities (both extensive and intensive) are commutative triangles. With extensive and intensive qualities as commutative triangles, composite qualities are products of commutative triangles." (line 338-342). Please translate those into plain language in the context of consciousness. In particular, what does cohesiveness requirement mean for neuroscience of consciousness?

g) From what is stated in the manuscript, basically Lawvere the study of qualities resulting from the variation of quantities within cohesive spaces (Lawvere, 2014), led to axiomatization of the hitherto vague notions of cohesion and quality (Lawvere, 2007; Lawvere & Menni, 2015). Can you provide a plain language explaination of Lauvere's theory and work on cohesive space? And how they are important for explaining psychological consciousness?

EDITORIAL DECISION

Based on my reading of the manuscript, I find it to fall short of making a solid contribution to theorizing a fundamental process of psychology, namely, that of consciousness. I applaud the effort of the author for trying to bringing in category theory to psychology; however, the current manuscript remains a paper of interest to mathematician community. The proposed explanation of coherency of conscious experience remains an idea, with no details worked out or at least no mapping to psychological concepts have been achieved (vertices/categories, edges/functors). Therefore, I have no choice but to reject this manuscript. If the author can overcome the above objections and make explicit the psychological interpretations, then the author is encouraged to resubmit it to JMP for re-consideration.

Gmail

Venkata Rayudu Posina <posinavrayudu@gmail.com>

Invitation to revise manuscript BIO_2017_135

Gary Fogel (BioSystems) <EviseSupport@elsevier.com> Fri, Sep 8, 2017 at 8:29 AM
Reply-To: gfogel@natural-selection.com
To: posinavrayudu@gmail.com

Ref: BIO_2017_135
Title: Category Theoretic Formalization of Qualia Space
Journal: BioSystems

Dear Mr. Posina,

Thank you for submitting your manuscript to BioSystems. We have completed the review of your manuscript. A summary is appended below. While revising the paper please consider the reviewers' comments carefully. We look forward to receiving your detailed response and your revised manuscript.

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I look forward to receiving your revised manuscript as soon as possible.

Kind regards,

Dr Fogel
Editor-in-Chief
BioSystems

Comments from the editors and reviewers:
-**Reviewer 1**

-

I think this manuscript needs a major revision before it is published, for two reasons. First, contrary to its promise, it is not accessible to cognitive neuroscientists due to its heavy mathematical terminology. Second, I am afraid that the phenomena of neuroscience and consciousness that the authors claim to account for are not presented in sufficient depth. I raise three points to illustrate these problems.

(1)

In what ways the brain might be a measuring device (p4) has been debated by philosophers and also by experimental psychologists with a good sense to mathematics. It is not an unpopular view that sensation is at most a fairly biased measuring device, because that is what serves organisms' adaptive interests. This issue may deserve some attention; see

> Akins, K. (1996). Of Sensory Systems and the “Aboutness” of Mental States, *Journal of Philosophy*, **93**, 337-372.
>
> Matthen, M. (1999). The Disunity of Color. *Philosophical Review*, **108**, 47-84.
>
> Wuerger, S., M., Maloney, L., T., Krauskopf, J. (1995). Proximity Judgments in Color Space: Tests of a Euclidean Color Geometry. *Vision Research*, **35**, (6), 827-835.

(2)

Offering a common formal framework for the unity of consciousness and the unity of mathematics (p16) raises some questions. The two fields are distant indeed, and the authors' account addresses few issues (exactly one, in fact) that are of interest in philosophy of mind and cognitive science. Take for example the claim that extensive and intensive qualities are functors of certain sort. Functors are abstract objects, therefore in this account the 'are' cannot be taken to express type identity. Conscious experiences are not abstract objects like square root two. A more plausible intrerpretation of this claim is that qualities have intrinsic or relational structure that is properly described by certain functors. Unfortunately not even this idea is spelled out in much detail; the only issue addressed, to a limited extent, is the unity of consciousness. However, within philosophy of mind unity of consciousness is intimately related to questions about the fundamental nature of mental representation. Andrew Brook, Paul Raymont, and Uriah Kriegel have written useful pieces on this topic. The SEP entry *Unity of consciousness* is a good starting point to assess these issues. Apparently the authors are offering a different approach to the unity of consciousness; I think they should place their new account in the context of earlier research.

Throughout the paper the authors make little effort show that their mathematical model is useful for philosophers – or cognitive neuroscientists for that matter. The manuscript has been written in purely mathematical jargon; I'm afraid that most neuroscientists would need to take a course in category theory before they can understand the relevance of the mathematical ideas presented here. Take an example: on p16 the authors say "Our product-of-qualities formalization can be thought of as a refinement of the basic idea of feature conjunctions, wherein the percept of ‘red square’ is associated with the activation of ‘red’ neuron and ‘square’ neuron." This refinement might indeed be an important contribution, provided that the mathematocal language is to some extent translated to terms that are accessible to neuroscientists or philosophers. This, however, does not happen; as it stands, the above citation would leave all non-mathematician readers wondering what is gained by such a complex formal theory of something they already understand fairly well, namely neuronal coactivation. Again, feature binding in cognitive science is a complex and important issue, but at some places the authors appear to present a rather simplistic view of it and back this apparently simplified view with a general theory of qualities (p3).

(3)

Extensive and intensive qualities are defined on p7, and this may appear interesting for more philosophically-minded readers. There is an age-old distinction between primary and secondary qualities in theories of perception which still has an important influence today, although in rephrased forms. This is something specific to mental states in particular, not qualities in general.

How is the extensice-intensive distinction related to primary vs secondary qualities? Color is intensive and secondary; shape is extensive and primary. However, temperature is intensive and primary, whereas it is arguable that pleasantness is extensive but secondary (because it is mind-dependent just like odor or color). Thus the two divisions seem to cut across one another, therefore the primary-secondary distinction is by no means accounted for by the extensive-intensive distinction. But then what is the importance of the extensive-intensive distinction from the point of view of cognitive theorizing?

Part of the reason why the primary-secondary distinction os not captured by the extensive-intensive one is that the authors' account is a general theory of qualities, and as such it does not make a distinction between intentional vs non-intentional properties. Next, the primary-secondary quality distinction crucially depends on how experience represents the environment, that is, on its intentional properties. On this issue see:

Byrne, A., Hilbert, D., R. (2003). Color Realism and Color Science. *Behavioral and Brain Sciences* 26, 3-64

McLaughlin, B., P. (2003). Color, consciousness, and color consciousness. In Quentin Smith (Ed.): *New Essays on Consciousness.* Oxford: Oxford University Press, 97-152.

Jakab, Z. (2012). Reflectance physicalism about color: the story continues. *Croatian Journal of Philosophy*, Vol. XII. No. 36 463-488. (https://philpapers.org/rec/JAKRPA)