Macroscopic oil droplets mimicking quantum behavior: How far can we push an analogy?

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Abstract. We describe here a series of experimental analogies between fluid mechanics and quantum mechanics recently discovered by a team of physicists. These analogies arise in droplet systems guided by a surface (or pilot) wave. We argue that these experimental facts put ancient theoretical work by Madelung on the analogy between fluid and quantum mechanics into new light. After re-deriving Madelung's result starting from two basic fluid-mechanical equations (the Navier-Stokes equation and the continuity equation), we discuss the relation with the de Broglie-Bohm theory. This allows to make a direct link with the droplet experiments. It is argued that the fluid-mechanical interpretation of quantum mechanics, if it can be extended to the general N-particle case, would have an advantage over the Bohm interpretation: it could rid Bohm's theory of its strongly non-local character.

1. Introduction

Historically analogies have played an important role in understanding or deriving new scientific results. They are generally employed to make a new phenomenon easier to understand by comparing it to a better known one. Since the beginning of modern science different kinds of analogies have been used in physics as well as in natural history and biology (for a recent historical study, see Gingras and Guay 2011). With the growing mathematization of physics, mathematical (or formal) analogies have become more frequent as a tool for understanding new phenomena; but also for proposing new interpretations and theories for such new discoveries. Einstein, for example, used formal analogies in several of his papers to reveal the corpuscular nature of light (Gingras 2005, Norton 2006) and the wave-particle duality (Gingras 2011). Indeed, Einstein's theory of light was prompted by the mathematical identity between the formulae for the entropies of radiation and of an ideal gas. Also, there is an important philosophical literature devoted to discussing the general validity of analogical inferences (cf. e.g. Hesse 1966, Bartha 2010, Norton 2014). One general conclusion of the latter works is that although analogies between phenomena are rarely perfect, this mode of inference has, to the least, an essential heuristic value.

In this article we analyze a striking case of experimental analogies, that may shed new light on the ancient problem of the interpretation of quantum mechanics, in particular the wave-particle duality. Over the last ten years, a group of French physicists led by Yves Couder has shown, through a series of original experiments, that many properties typical of quantum systems can also be observed in classical systems. The team investigates an experimental fluid-dynamical system essentially composed of a thin film of fluid (a special oil), made to vertically vibrate, on which oil droplets are deposited; the dynamics of the system is such that under specific conditions such droplets can horizontally 'walk' over the oil surface for indefinite time. The Paris group showed in particular that walking droplets can exhibit double-slit interference, quantization of angular momentum, and the analogue of tunneling and Zeeman splitting (Couder et al. 2005, 2006, Fort et al. 2010, Eddi et al. 2011, 2012). Other researchers have already confirmed and extended these results (Molacek and Bush 2013a, 2013b). These analogies are striking because macroscopic fluid mechanics and microscopic quantum mechanics are usually thought to be quite disjoint. At the same time they suggest that, contrary to what is generally believed, an intuitive understanding of quantum mechanics is maybe not beyond reach. These *experimental* analogies also point to the possibility that *formal* analogies between hydrodynamics and quantum mechanics could exist and be further revealed.

The first objective of this article is to present these experiments in such a way as to make apparent the foundational issues they raise. The second objective is to link these experiments to a few of the key articles of the foundations of quantum mechanics, again paying a little more attention to conceptual issues than is usually done. The description of the most relevant experimental results, revealing the analogies, is given in Section 2. In Section 3 we will recall that a formal (mathematical) analogy between fluid and quantum mechanics had already been proposed by the German physicist Erwin Madelung, right at the birth of quantum mechanics (Madelung 1927). Since this theoretical result gains new import in the context of the Paris experiments, we will re-derive it in Section 3. We will do so in a somewhat more detailed manner than in (Madelung 1927) and in other reference works (Jammer 1974 and the articles cited there on pp. 33-38 and 50-54). We will explicitly start from classic fluid-mechanical equations, and enumerate all hypotheses made. This will prove necessary for our arguments in the next Sections. In Section 4 we will investigate the link between Madelung's 'fluid-mechanical' or 'hydrodynamic' interpretation of quantum mechanics and the better-known interpretation proposed by David Bohm (1952) based on de

Broglie's work. We will argue that an upgraded 'stochastic' version of Bohmian mechanics, presented in Bohm and Vigier (1954), allows to make a direct connection with the droplet-experiments. As Bohm and Vigier state themselves (and several authors after them), this upgraded version should rather be seen as a *fluid-mechanical* model.

In view of the striking analogies studied by the Couder, Fort et al., and in view of theoretical work by Madelung (1927), Bohm and Vigier (1954) and others, the remainder of the article (Section 5) is devoted to the question whether a fluid-mechanical interpretation of quantum mechanics may have advantages over Bohm's interpretation. This part is speculative: while the de Broglie – Bohm theory (or Bohmian mechanics) is a quite mature theory elaborated by Bohm and many others (cf. e.g. Bohm 1952, Holland 1993, Duerr and Teufel 2009, Oriols and Mompart 2012), its hydrodynamic counterpart is not as well developed – the hydrodynamic interpretation appears to be investigated by an even smaller community. Bohmian mechanics can easily be extended to N particles (in configuration space), but it has not (yet) been shown how to derive the general case of N interacting quantum particles from fluid-mechanical laws (in 3D space). Yet it deserves to be noted that publications exist showing the potential of the hydrodynamic model, not only for interpreting the Schrödinger but also e.g. the Pauli equation, and possibly for the N-particle case (cf. e.g. the references in Jammer 1974 p. 33-38 and 50-54, Wilhelm 1970, Kuzmenkov and Maksimov 1999, Tsekov 2012 and references therein). Recently 'quantum hydrodynamics' has gained interest for the interpretation and numerical modelling of experiments, notably in plasma and chemical physics (see e.g. Wyatt 2002, Sanz et al. 2002, Sanz 2015 and references). Some authors have more or less explicitly argued for the superiority of the Madelung interpretation over the Bohmian one (Tsekov 2012, Sanz et al. 2002, Sanz 2015). Here we will not review these – interesting and important – articles, but rather focus on what seems to be a new argument, which we can immediately derive from our conceptual analysis of Sections 3-4, based on the original articles (Madelung 1927, Bohm 1952 and Bohm and Vigier 1954). Indeed, it seems that there are cogent indications that hydrodynamic theories should be termed *local* (even if fluids are extended or delocalized). If that is correct, the hydrodynamic interpretation could rid Bohmian mechanics of its (alleged) strongly non-local character. This point is made in Section 5. Although it is independent, this argument can be compared and completed with a recent publication (Vervoort 2015a), as we will briefly explain in Section 5.

By suggesting relations between different processes or object domains, analogies can contribute to the unification of what appear as radically distinct phenomena. Based on the experimental and mathematical analogies presented in Sections 2 and 3, one could infer, as a kind of 'maximal working hypothesis' that quantum mechanics might, in the end, *be* nothing else than a fluid-dynamical theory. Needless to say, the latter 'maximal induction' from the theoretical results of Madelung and others and from the experimental results of Couder, Fort et al. is highly speculative for the time being. Here we will only advance a few arguments for the idea that Madelung's program deserves renewed interest.

2. The experimental analogies

2.1. Double-slit interference

Couder et al. have succeeded in creating 'hovering' oil droplets, by depositing them on a vertically vibrating oil film (Couder et al. 2005, 2006, Fort et al. 2010, Eddi et al. 2011, 2012). Under stringent experimental conditions, the millimeter-size droplets bounce rapidly on the vibrating film, and simultaneously 'walk' horizontally over it. However, this stable walking regime only occurs in well-defined experimental conditions, i.e. for precise values of the physical parameters of the system, essentially the frequency and amplitude of the external vibration, the size of the droplet, the geometry of the oil film and bath, and the viscosities of film and droplet. If these parameters lie within the precise ranges of values documented by the researchers, the droplets walk horizontally; outside these ranges the movement becomes erratic and/or the droplet is captured by the film. In the walking regime various experiments can be performed on a statistical ensemble of (identical) droplets. For instance, one can send a series of identical droplets (one after the other) through a slit, and measure the deflection angle α after the slit (Fig. 1a), and its probabilistic distribution.

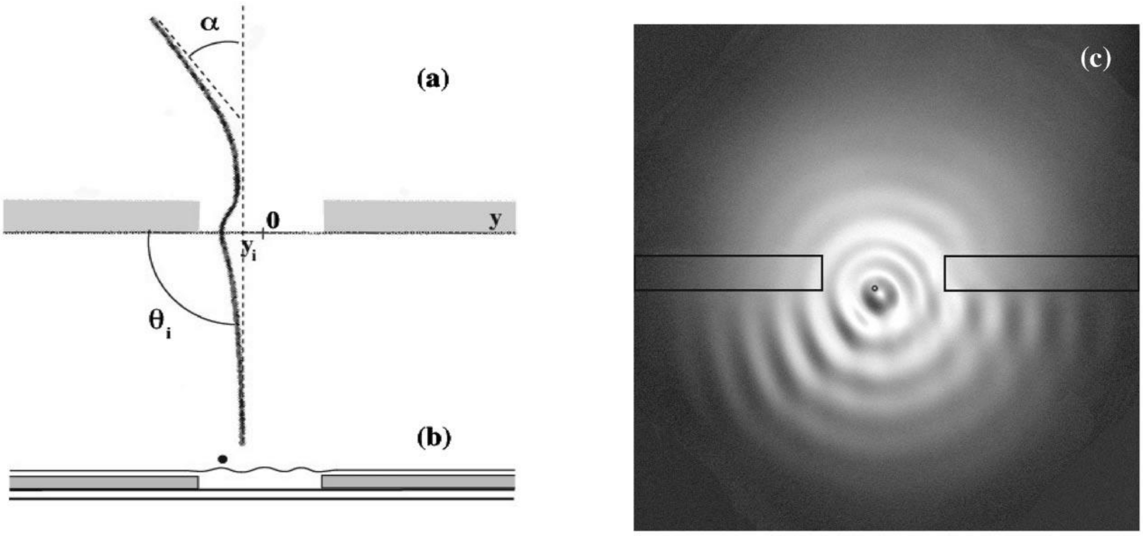

Fig. 1a-b. A droplet passes through a slit (formed by barriers in the oil film, cf. Fig. 1b). Fig. 1c is a photograph of a droplet and its accompanying 'pilot wave'. Reprinted from Couder and Fort (2006).

Fig. 1c shows such a droplet at the moment it passes through the slit. An essential characteristic of this fluid-dynamical system is the (roughly circular) surface wave that accompanies the droplet (Fig. 1c). The vibrating film gives kinetic energy to the droplet, but the bouncing droplet back-reacts, i.e. periodically hits the film and partly determines the shape and characteristics of the surface wave on the oil film. So the circular waves, of which the center approximately follows the walking droplet, are created by the external vibration plus the back-reaction of the droplet[1]. Given the observed experimental behavior, this system has therefore been termed the first experimental "particle + pilot-wave". This experimental system, then, can be used to reproduce features typical of light and electrons. Thus when a series of droplets is sent (one at the time) towards a double-slit barrier, the droplets deflect after the barrier under an angle α. Fig. 2 shows the probabilistic distribution of the deflection angle α measured on 75 droplets. Clearly, each droplet passes through only one of the two slits, as can be seen with a camera; but the pilot-wave passes through both slits, and its shape is influenced by the geometry of the slits. (Notice that the pilot-wave in Fig. 1c is not perfectly symmetric – this is due to the diffraction of the wave on the barrier + slit.) So the trajectory and deflection angle of each droplet are probabilistically determined by the

[1] In somewhat more detail, the wave field itself results from the superposition of the waves generated by the periodic impacts of the droplet on the film. It thus contains a memory of the past trajectory of the particle – a mild form of non-locality. A related type of non-locality in the system stems from the fact that the detailed characteristics of the wave field depend on the parameters of the *whole* experimental set-up, including the precise geometry of the bath. (These are mild forms of non-locality because *per se* they obviously do not invoke faster-than-light forces – which amount to strong, pathological non-locality.)

diffracted pilot-wave, that guides the droplet. This conclusion is made cogent by the Paris group by providing a theoretical model, describing the classical (Newtonian) interaction of a particle and a surface wave (Couder and Fort 2006, Eddi et al. 2011). Of course, the remarkable feature of the histogram in Fig. 2 is that it shows a series of maxima and minima, typical of an interference pattern. The system is thus analogous to Young's experiment using light waves or quantum particles (electrons etc.) passing through a double slit. Here is the first, and quite spectacular, analogy between a classical fluid-dynamical system and quantum systems.

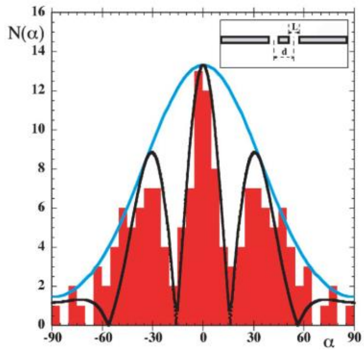

Fig. 2. Distribution (histogram) of the deflection angle α after a double-slit barrier, for 75 droplets. Reprinted from (Couder and Fort 2006).

Of course, these results obtained with macroscopic droplets suggest a way out of typical quantum riddles related to Young's experiment. This experiment is generally interpreted in a (very counterintuitive) way by stating that each photon or electron passes through *both* slits and interferes with *itself* (cf. e.g. Dirac 1958). The Paris experiments suggest another option: namely that the system produces a particular probabilistic distribution for the particles, which pass through one slit at the time while guided by a wave that passes through both slits and interferes. This is also how Bohmian mechanics interprets this experiment (cf. Section 4).

2.2. Quantization of angular momentum

The same experimental system was used to explore what happens when one makes the oil bath rotate, in addition to vertically vibrate. Under specific conditions, the droplet will now rotate (Fig. 3). More precisely, in the laboratory frame the droplet's motion will be epicycloidal (Fig. 3D), while in a frame fixed to the rotating bath it will describe circles (Fig. 3E (these circles are reconstructed by image processing from the trajectories as in 3D). To each constant angular velocity of the bath (Ω) corresponds a constant mean radius R, which has some dispersion as Fig. 3E shows.

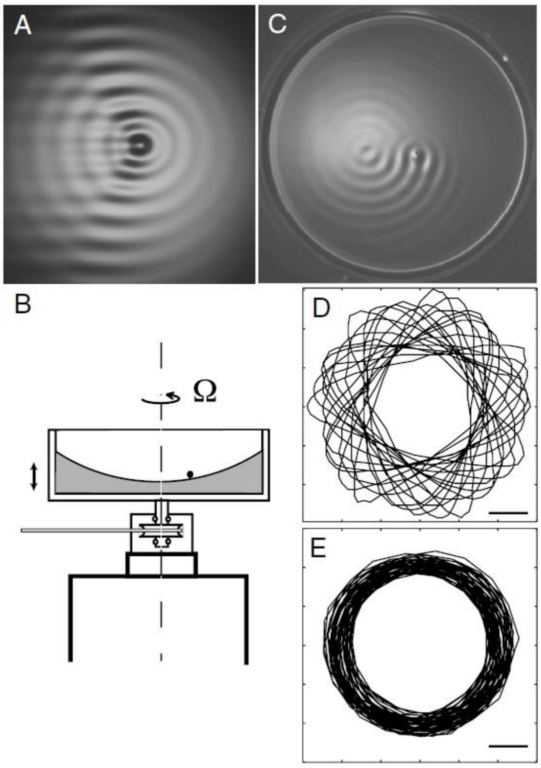

Fig. 3. The oil bath is rotated with a constant angular velocity **Ω**. Instead of walking in a straight line (A), the droplet now rotates (C). In the laboratory frame, it describes epicycles, while in a rotating frame fixed to the bath, it describes circles (E). Reprinted from (Fort et al. 2010).

The remarkable result obtained by Fort et al. (2010) is that when the rotation velocity **Ω** is gradually increased, the droplet *abruptly* jumps to a smaller radius. For a continuous range of Ω-values, there is only a discrete set of radiuses available for the droplet's motion. This 'quantization' of angular momentum is usually associated with quantum systems. In particular, such a discretization phenomenon is analogous to the discrete set of energy levels that an electron can assume in a magnetic field – the so-called Landau levels[2]. At the same time the researchers can again well explain the observed trajectories by a purely classical calculation describing the Newtonian action of a surface wave field – the pilot-wave – on the droplet (Fort et al. 2010).

[2] In the semi-classical Bohr-Sommerfeld approximation, these discrete energy levels also lead to discrete radiuses on which the electron can move, just as in the droplet's case.

It is important to acknowledge that the Paris group pushes the analogy between the discrete radiuses and the magnetic Landau levels further than described above. For the time being one might think that the analogy is merely qualitative and superficial, without a formal basis. However, they note that indications of some (more or less profound) formal analogy between (i) a charged (quantum or classical) particle in a magnetic field and (ii) a particle in a rotating fluid, are already known. First, there is an obvious formal similarity between the electromagnetic expression $\mathbf{B} = \nabla \times \mathbf{A}$ and the fluid-mechanical expression $2\,\mathbf{\Omega} = \nabla \times \mathbf{u}$ (here **B** is the magnetic field, **A** the vector potential, $2\mathbf{\Omega}$ the vorticity and **u** the fluid velocity field). So the role of **B** is taken here by $2\mathbf{\Omega}$. This analogy between electromagnetism and fluid mechanics goes further. For instance, a charge q moving at velocity **V** in a homogeneous external magnetic field **B** experiences a Lorentz force $\mathbf{F} = q(\mathbf{V} \times \mathbf{B})$. In a fluid-mechanical system rotating at an angular velocity $\mathbf{\Omega}$, the Coriolis force on a mass m moving with velocity **V** is $\mathbf{F} = -\, m(\mathbf{V} \times 2\mathbf{\Omega})$, where $2\,\mathbf{\Omega}$, again taking the role of **B**, is the vorticity due to solid-body rotation (this is noted by Fort et al. 2010). Both these forces lead to circular motions in planes perpendicular to **B** and $\mathbf{\Omega}$ respectively. Now these analogies between electromagnetism and fluid mechanics hold for classical electromagnetic systems; can they be extended to quantum systems ? The strongest indication comes from results by Berry et al. (1980), who experimentally and theoretically showed that the phase of a surface wave on a fluid can be altered by a point-like vortex in the fluid. This is the equivalent of the celebrated Aharonov – Bohm effect, a quantum effect involving the electromagnetic vector potential **A**. So the formal and experimental analogies are more general than one might think, as noted by the Paris group.

The strongest semi-formal analogy drawn by Fort et al. in their (2010) is the following. Given their results (Fig. 2 and Fig. 3) and the already known analogies between a (quantum) particle in a magnetic field and fluid systems, they conjecture that the de Broglie wavelength (λ_{dB}) of a quantum particle in a magnetic field might correspond to the characteristic wavelength of the pilot-wave (λ_F) in the fluid. Now replacing the de Broglie wavelength in the expression of the radius of the Landau-levels[3] by λ_F, leads to radiuses that fit well to those measured on the droplets ! Here is a bold analogical conjecture that is confirmed – a result that strengthens the idea that the droplet's pilot-wave might be more fundamentally related to de Broglie's pilot-wave than meets the eye.

[3] In the Bohr-Sommerfeld approximation, cf. former footnote.

In sum, the Paris group quite convincingly shows that the analogy between fluid and quantum mechanics is not only qualitative (some variables are quantized), but also formal – there is a partial parallelism in the laws describing both types of systems, at least at the phenomenological level. Of course this parallelism is not perfect and the obvious question is: how far can it be pushed ?

2.3. Zeeman splitting

The next analogy was discovered by making two droplets interact, as reported in Eddi et al. (2012). It turns out that under specific conditions two identical walkers coming close enough begin to rotate around each other. Each droplet is trapped in the wave-field of its partner; the center-of-mass of the droplets describes circles while the droplets describe epicycles. The circular diameter (d_n) of such a bound state is again discrete and scales with the characteristic wavelength of the pilot-wave (λ_F):

$$d_n = (n - \varepsilon_0).\lambda_F \,. \qquad (1)$$

Here ε_0 (= 0.21) is an experimentally determined constant. Thus the orbit diameters are indexed by a discrete number n – which is reminiscent of quantum systems, in particular the bound energy states of atoms. There are two families of bound states: both droplets may vertically bounce in phase, or in anti-phase. The experiments show that in the first case n is integer (n = 1,2,3,…) while in the second it is half-integer (n =1/2, 3/2,…).

In quantum systems atomic energy levels can be split by applying an external magnetic field – the Zeeman effect. Could an analogous phenomenon exist for bound droplet states ? Based on the results described in the former Section (Fort et al. 2010), we already inferred that the analogue of the magnetic field **B** is the vorticity **Ω**. Thus in a next step, which in hindsight appears as quite logical, the experimenters applied an external rotation **Ω** to the bath containing the two walkers (Eddi et al. 2012). Here two types of motion can be created: the pair can rotate in the same direction as the external rotation **Ω**, say counterclockwise (such 'corotating' states are denoted as n^+-states); or in opposite direction (n^--states or 'contrarotating' states). Together with the integer and half-integer states, this combines to four families of states. The essential experimental result from Eddi et al. (2012) is reproduced in Fig. 4.

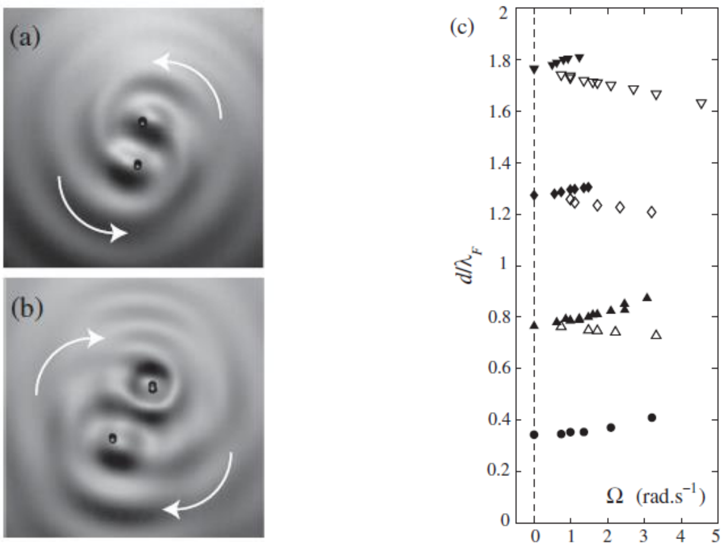

Fig. 4. (a) A snapshot of a pair of droplets orbiting counterclockwise, in the same direction as **Ω**, with $n^+ = 1^+$. (b) A clockwise rotating pair with $n^- = (3/2)^-$. (c) The diameters d_n from Eq. (1) as a function of **Ω**. Open symbols correspond to n^--states (contrarotating) and black ones are n^+-states (corotating). Reproduced from (Eddi et al. 2012).

The lowest state in Fig. 4c corresponds to n = 1/2, the second pair of states to n = 1, the third pair to n = 3/2 and the last one to n = 2. Thus we see that while the n^+ and n^--states coincide for Ω = 0, they are split when Ω increases. Needless to say, this shows a striking similarity to the lifting of the 'quantum degeneracy' (i.e. the coincidence) of atomic energy levels in a magnetic field. As for the results discussed in the previous Section, the Paris group succeeds in giving a formal backbone to what is at least a qualitative analogy. Using their experimental findings, in particular those of Fig. 4, and applying straightforward classical mechanics they show that the angular velocity of the n^{th} bound level can be expressed as:

$$\delta\omega_n^{\pm} = \pm\alpha_n.\Omega \quad . \tag{2}$$

Here $\delta\omega_n^{\pm}$ is the change in angular velocity induced by the imposed rotation Ω; it differs in sign for n^+ and n^--states. Further, α_n is a factor that only depends on n and certain constants of the fluid system. Now formula (2) is formally identical to the expression of the energy shift of an electron rotating in a magnetic field. More precisely, if we replace in (2) $\delta\omega_n^{\pm}$ by $\delta E_n^{\pm}$ and Ω by B, we obtain the correct expression for the Zeeman effect. Also in the latter effect states with opposite angular momentum will be split. However, not surprisingly, the analogy is again not perfect in all quantitative detail: the factor α_n has a different dependence on n in the quantum case. Still, these results of Eddi et al. (2012) push the analogy between quantum mechanics and fluid mechanics quite far. The authors conclude their article as follows:

> "Two walkers can form quantized orbiting bound states when they interact. Here we have probed the properties of this set of orbits when submitted to an external field. We have shown that the wave mediated interaction is dominant over inertial effects, and that the bound states present a strong analogy with atomic systems. An external field induces a level splitting of degenerate states, reminiscent of Zeeman splitting, and is also able to force transitions between successive levels. The present results have thus extended the range of the documented analogies [read: in our previous publications] between phenomena due to wave-particle duality at classical and quantum scales, respectively. The existence of these behavioral similarities opens a new perspective for the understanding of both types of dualities" (Eddi et al. 2012).

As cautiously expressed in the last phrase of this quote, the ensemble of experimental and theoretical analogies obtained by the Paris group incite one to consider the question *how far the analogy goes*. Even if this question is largely open at this point of investigation, we will see in the following Section that the above experimental analogies might have a deeper-lying foundation based on a formal analogy between quantum mechanics and fluid mechanics – this time at the fundamental level. We will first simply describe the formal analogy, and make in Section 4 a closer link with the droplet experiments.

3. Madelung's fluid-mechanical interpretation of quantum mechanics

Already in 1926 the German physicist Erwin Madelung had observed that there seems to be a link – an analogy – between the 1-particle Schrödinger equation and basic equations from fluid dynamics (Madelung 1927; for early related articles, see Jammer 1974). Since the Paris experiments put the theoretical work of Madelung in a new perspective, and since Madelung's article (published in German) is rather cryptic and the result not broadly known, we re-derive it here. We will be explicit about the assumptions made and start from the Navier-Stokes and continuity equations of fluid dynamics. This will allow us a detailed discussion of the degree of analogy and a tailored comparison with the de Broglie – Bohm interpretation of quantum mechanics, which is much better known.

The Navier-Stokes equation is the fundamental equation of fluid mechanics describing the flow of fluid substances; it represents the second law of Newton applied to an infinitesimal fluid element that continuously deforms under the action of pressure and forces (Batchelor 2000, Kundu and Cohen 2008). The Navier-Stokes equation reads:

$$\rho_M \left(\frac{\partial \overline{u}}{\partial t} + (\overline{u} \cdot \overline{\nabla}) \overline{u} \right) = -\overline{\nabla} P + \mu . \nabla^2 \overline{u} + \rho_M . \overline{F} \; . \tag{3a}$$

Here $\rho_M = dM\,/\,dV$ is the mass density of an elementary fluid element with mass *dM* and volume *dV*; $\overline{u}$ the fluid element's velocity (a vector field); *P* the thermodynamic pressure (related to ρ_M and the temperature *T*); $\overline{\nabla} P$ the pressure gradient; $\overline{F}$ the body force per unit of mass acting on the fluid element (typically: gravity); and μ the dynamic viscosity depending on the thermodynamic state (the temperature).

The Navier-Stokes equation is only valid under a list of certain conditions – conditions which are however satisfied by most 'normal' fluids, hence its capital importance for fluid mechanics. The conditions of validity of Eq. (3a) are: (C1) the fluid is incompressible, (C2) the fluid is in thermodynamic equilibrium or nearly so; (C3) the fluid's stress-strain tensor is linear and isotropic; (C4) the thermodynamic pressure in the fluid is equal to the mean of the normal stresses (Stokes' assumption); (C5) the fluid is 'Newtonian' (this is implied by (C2-C4)); (C6) the viscosity μ is spatially constant in the fluid (the temperature differences in the fluid are small enough).

Note that, since the left side of Eq. (3a) is equal to $\rho_M d\overline{u}/dt$, i.e. ρ_M times the total or material derivative of $\overline{u}$, one can still more or less recognize Newton's law in (3a). The term $\mu . \nabla^2 \overline{u}$, a Laplacian, corresponds to viscous diffusion of momentum. If one moreover supposes (C7) that there are no viscous losses in the fluid, so $\mu . \nabla^2 \overline{u} = 0$ or small enough, then the Navier-Stokes equation reduces to the Euler equation.

The second fundamental equation of fluid mechanics we will use is the mass conservation or 'continuity' equation:

$$\frac{\partial \rho_M}{\partial t} + \overline{\nabla} \cdot (\rho_M \overline{u}) = 0. \tag{3b}$$

This equation expresses, in differential form, that the rate of increase of mass within a fixed volume must equal the rate of inflow through the boundaries.

To bring about the analogy between Eq. (3a-b) and the Schrödinger equation, we will further suppose that our fluid consists of 'particles' or entities of constant mass 'm'; so that a mass *M* of the fluid can be understood as composed of *N* masses *m* (this is one of the hypotheses that is not explicitly mentioned in Madelung's paper). This allows to introduce a *probability* density ρ for the 'fluid particles' as follows:

$$\rho_M = \frac{dM}{dV} = \frac{d(N.m)}{dV} = m.\frac{dN}{dV} \equiv \rho.m \,. \tag{4}$$

Eq. (3b) can then be interpreted as the conservation of probability. So by a classic and innocuous change of variables, replacing the mass density ρ_M by the probability density ρ (times m), we see that *pure deterministic fluid mechanics is compatible with the physics of probabilistic systems*. (Note that also in a conceptual sense this helps us to make the link with the Schrödinger equation, the empirical content of which is probabilistic.) In sum, it appears that we need, for our derivation, the hypothesis that the fluid elements following the streamlines are filled with 'something' (particles,…) having constant mass m. For further reference and in a self-explaining manner, let us call this the 'stochastic hypothesis'.

We now re-write the equations (3a-b), making further simplifying hypotheses. The second term on the left in Eq. (3a) can be written as follows, using standard vector differential calculus:

$$(\bar{u}\cdot\bar{\nabla})\bar{u} = \bar{\nabla}\frac{u^2}{2} - [\bar{u}\times(\bar{\nabla}\times\bar{u})] \,, \tag{5}$$

where $\bar{\nabla}\times\bar{u} \equiv \bar{\varpi}$ is the vorticity which underlies vortex formation in the fluid. If one assumes (assumption C8) that the fluid is irrotational, i.e. that its vorticity $\bar{\varpi} = \bar{\nabla}\times\bar{u}$ is zero everywhere, then one can suppose that the velocity derives from a scalar function (a field) S:

$$\bar{u} \equiv \frac{1}{m}\bar{\nabla}S \,. \tag{6}$$

Using Eq. (4-6), it is straightforward to transform the Navier-Stokes equation (3a) into:

$$\frac{\partial}{\partial t}\bar{\nabla}S + \frac{1}{2m}\bar{\nabla}(\bar{\nabla}S)^2 = -\frac{1}{\rho}\bar{\nabla}P + m.\bar{F} \,. \tag{7}$$

Fluid dynamics usually assumes that the body forces $\bar{F}$ are conservative (C9), so that $\bar{F}$ (or m$\bar{F}$) derives from a potential U, as is the case of gravity. If one finally assumes (C10) that the flow is barotropic, i.e. that ρ is a function of P only, then one easily proves (Kundu and Cohen 2008 p. 118):

$$\frac{1}{\rho}\bar{\nabla}P = \bar{\nabla}\int\frac{dP}{\rho} \,. \tag{8}$$

If one now defines:

$$\frac{1}{\rho}\bar{\nabla}P = \bar{\nabla}\int\frac{dP}{\rho} \equiv \bar{\nabla}Q \,, \tag{9}$$

then one readily obtains from (7):

$$\boxed{\frac{\partial}{\partial t}\overline{\nabla}S+\frac{1}{2m}\overline{\nabla}(\nabla S)^2+\overline{\nabla}(U+Q)=0\,.} \tag{10a}$$

Now, with the *same* assumption and definition as introduced in (6), the second fundamental equation, the continuity equation (3b) becomes:

$$\boxed{\frac{\partial\rho}{\partial t}+\overline{\nabla}\cdot(\rho\frac{\overline{\nabla}S}{m}) \quad =0.} \tag{10b}$$

Thus, our assumptions transformed Eqs. (3a-b) into (10a-b).

It is then straightforward to relate the Eqs. (10a-b) to the 1-particle Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi+U\psi \ . \tag{11}$$

Indeed, any solution to (11) can be written as:

$$\psi = R.\exp(iS/\hbar) = \sqrt{\rho}.\exp(iS/\hbar), \tag{12}$$

where R and S are real functions, and where $\rho = R^2$ is the probability density of the particle with mass m, according to the Born interpretation of the wave function (S is the phase of the wave function). By inserting (12) into (11) and separating real and imaginary parts, we obtain two equations for ρ and S, equivalent to the Schrödinger equation for ψ. The first equation is precisely (10b), the (transformed) continuity equation of hydrodynamics. The second equation is:

$$\frac{\partial S}{\partial t}+\frac{(\nabla S)^2}{2m}+U+Q \ =0, \tag{13}$$

if we define:

$$Q \equiv \frac{\hbar^2}{2m}\left[\left(\frac{\nabla\rho}{2\rho}\right)^2-\frac{\nabla^2\rho}{2\rho}\right] = -\frac{\hbar^2\nabla^2 R}{2mR}. \tag{14}$$

Finally, if we take the gradient of Eq. (13) we precisely obtain the (transformed) Navier-Stokes equation (10a). Alternatively, we could integrate Eq. (10a) and obtain Eq. (13) (if we put the integration constant equal to zero).

Madelung concludes (our translation): "Also this equation [(10a) obtained by taking the gradient of (13)] precisely corresponds to a hydrodynamic one, namely that of an irrotational flow under the action of conservative forces." From our explicit derivation also follows (compare (9) and (14)) that the 'quantum potential' Q in (14) corresponds to $\int \frac{dP}{\rho}$,

which, in Madelung's words, "one could term the potential of the 'inner' forces of the continuum". Thus Madelung has proven what the abstract of his article had promised: "It will be shown that one can transform the Schrödinger equation for the electron problem under the form of hydrodynamic equations" (Madelung 1927).

In sum, there is a formal analogy between the 1-particle Schrödinger equation and basic equations of fluid dynamics; at least if we make certain assumptions for the fluid, namely C1- C10 above. Clearly, even if assumptions C1 – C10 are 'natural' and describe a wide variety of fluids, the analogy is not perfect. We would have a perfect analogy if we were able to *derive* the precise form of the quantum potential Q given in (14) by using fluid-mechanical arguments only; i.e. if we were able to show that in certain fluids $\int \frac{dP}{\rho} = Q$ in (14)[4]. As far as we know, no conclusive fluid-mechanical derivation of this expression has been published; on the other hand there seem to be no physical laws prohibiting this as a matter of principle.

Another conceptual worry might be the following: we have an analogy between the *one*-particle Schrödinger equation and fluid-mechanical equations describing an *ensemble* of particles (recall the 'stochastic hypothesis'). Isn't this a bit awkward ? Maybe, but this worry can be removed by recalling that the 1-particle Schrödinger equation is equivalent to the N-particle equation for identical non-interacting particles.

Needless to say, even in view of Madelung's result we are still far from the maximal induction, the hypothesis that quantum mechanics is, in the end, a type of fluid-mechanical theory. In order to definitely advance Madelung's program, we should, to start with, show how to derive the N-particle Schrödinger equation for interacting particles from fluid-mechanical principles, without making extravagant assumptions. That is an unfinished program, but recall e.g. the references (Wilhelm 1970, Kuzmenkov and Maksimov 1999, Wyatt 2002, Sanz et al. 2002, Tsekov 2012) for reasons for hope. With this as an important caveat, Madelung's result seems to constitute at least the nucleus for an interpretation of quantum mechanics that could be a competitor of the Copenhagen interpretation, in a quite similar manner as the de Broglie – Bohm interpretation is an alternative interpretation of

[4] To that end it would *not* be necessary to derive the numerical value of $\hbar$; it would suffice to show which combination of fluid-mechanical constants formally takes the role of $\hbar$.

quantum mechanics. As is well-known to researchers investigating de Broglie-Bohm theory, there is a strong link between Madelung's and Bohm's interpretation of quantum mechanics, as recalled below. Importantly, exploring this link in some detail will allow us to make a connection with the Paris experiments, i.e. to understand why Madelung's interpretation might be directly relevant for the droplet experiments.

4. Relating Madelung to Bohm, and then to Couder et al.

From a historical point of view it is quite remarkable that Madelung proposed his interpretation already in 1926, while the Schrödinger equation was just discovered. Intriguing is also that his work went quite unnoticed until it was rediscovered by Bohm in 1952 (for a more detailed historical review, see Jammer 1974). As is well known, the derivation of Eqs. (13-14) and (10b) based on the Schrödinger equation and the decomposition (12) is precisely what Bohm in 1952 proposed in his seminal article (1952), laying the basis of the de Broglie – Bohm theory. Bohm refers to the article of Madelung and comments: "Madelung has proposed a similar interpretation of the quantum theory, but like de Broglie he did not carry this interpretation to a logical conclusion" (Bohm 1952).

Although Bohm's and Madelung's approach are formally identical at the start – both exhibit equations as (12), (10b), (13) –, their interpretation, in particular of Eq. (13), is very different. In Bohm's article the quantity $\overline{\nabla} S(\bar{x})$/m (cf. (6)) is interpreted as the velocity of a particle with mass m passing through point $\bar{x}$; so this interpretation associates *with each individual quantum particle* (say an electron) precisely defined and continuously varying values of position and momentum. However in a fluid-mechanical interpretation $\overline{\nabla} S(\bar{x})$/m is the fluid velocity; with the stochastic hypothesis this means that the particles with mass m follow *on average* the streamlines defined by $\overline{\nabla} S(\bar{x})$/m. Interestingly, this interpretation is adopted by Bohm and Vigier (1954) (cf. below), apparently in order to upgrade the initial model; but it is (more or less implicitly) part of Madelung's theory, as argued in the preceding Section. Other conceptual differences are the following. While Madelung interprets (13) as an equation describing the dynamics of a fluid, Bohm interprets it as a 'modified Hamilton-Jacobi' equation. If $\hbar \to 0$ and therefore $Q \to 0$, it coincides with the classical Hamilton-Jacobi equation. Moreover, Bohm regards the wave function ψ as an 'objectively real field', exerting a force on the electron itself, in a manner that is somewhat analogous to, but not identical with, the way in which an electromagnetic field exerts a force on a charge (more on this below). In Madelung's interpretation, or our understanding of it, ψ is just a *short-hand notation* representing by a complex-valued quantity *two* physical quantities (the probability density ρ and the velocity potential *S*). This is the so-called 'complex number notation' that is ubiquitous in classical (wave) physics. Note that in classical physics complex quantities do not 'exist' in the same way as their real-valued components exist – i.e. as directly measurable quantities: they are just a symbol handy for faster calculation.

Bohm applied his theory to a series of examples to illustrate it, and gave a full interpretation of the emblematic quantum experiments and concepts (e.g. the double-slit experiment, complementarity etc.) in terms of "the assumption that an electron is a particle following a continuous and causally defined trajectory with a well-defined position, accompanied by a physically real wave field" (Bohm 1952). As said, in at least one later publication Bohm seems to have largely adopted the fluid approach, or rather integrated it in his theory (Bohm and Vigier 1954). Certain of his theoretical results had led him to hypothesize that the particles of mass *m*, intervening in his upgraded interpretation and in the stochastic hypothesis of Madelung's interpretation (cf. text above Eq. (4)), are *singularities of some nature* that follow the 'Madelung fluid' while undergoing stochastic fluctuations – fluctuations about the mean values ρ and $\bar{u}(\bar{x},t)$ described by the Madelung equations. As Bohm puts it:

> "We therefore complete [our] model by postulating a particle, which takes the form of a highly localized inhomogeneity that moves with the local fluid velocity [$\bar{u}(\bar{x},t)$]. The precise nature of this inhomogeneity is irrelevant for our purposes. It could be, for example, a foreign body, of a density close to that of the fluid, which was simply being carried along with the local velocity of the fluid as a small floating body is carried along the surface of the water at the local stream velocity of the water. Or else it could be a stable dynamic structure existing in the fluid; for example, a small stable vortex or some other stable localized structure, such as a small pulse-like inhomogeneity. Such structures might be stabilized by some nonlinearity that would be present in a more accurate approximation to the equations governing the fluid motions than is given by [Eq. (13)] and [Eq. (10b)]" (Bohm and Vigier 1954, p. 209).

This passage shows that Bohm and Vigier come very close to Madelung's interpretation. From many respects, it is almost futile to distinguish both frameworks (cf. e.g. the above quote). This sentiment seems widely shared in modern publications – especially by authors having a sympathy for the fluid-mechanical interpretation (cf. e.g. Wilhelm 1970, Kuzmenkov and Maksimov 1999, Wyatt 2002, Sanz et al. 2002, Tsekov 2012). We will therefore in the following often refer to the 'Madelung-Bohm' or 'Bohm-Madelung' interpretation. However, from some respects it will appear useful to distinguish them, as we will see further. Let us here emphasize that there seems to be a growing literature that enriches the Madelung-Bohm theory with probabilistic formalisms stemming from the physics of Brownian motion, diffusion, etc. As we have seen, such elaborations most naturally fit into the hydrodynamic framework (and one could see them as corroborations of the latter). As one example, in (Wilhelm 1970) it is shown that in the hydrodynamic picture the uncertainty relations have an observer-*in*dependent interpretation, and arise due to momentum fluctuations related to internal stresses within the Madelung fluid; thus they are akin to uncertainty relations in Brownian motion. In (Tsekov 2012) a (succinct) argument is given that such a stochastic theory can even explain the origin of entanglement: this would be due to spatial correlations of the vacuum fluctuations. (Note that the Madelung fluid is often identified with the physical vacuum that can fluctuate.)

Most importantly, the above passage allows to make a direct link with the droplet-experiments of Couder, Fort et al. (Section 2), or to hypothesize such a link. Indeed, if we now transpose the above passage, describing the Madelung fluid, to the classical fluid used in the Paris experiments (a special oil in a basin), then the singularity Bohm and Vigier refer to corresponds to the droplet guided by the surface wave on the oil film. If one prefers, the droplet movement follows and reveals *the movement of a singularity (the point of impact) in the oil film*. If such a transposition is indeed allowed, then the Madelung-Bohm equations could also describe the droplet-experiments, notably double-slit interference (if $\hbar$ is replaced by the appropriate fluid-mechanical constant, cf. footnote 4). This hypothesis is explored in a recent publication by Sbitnev (2014). The Bohm trajectories for double-slit interference, which can be calculated for electrons within Bohm's theory (e.g. Holland 1993), are calculated in this reference for the macroscopic droplets of the Paris experiments, fitting the experimental results (e.g. Fig. 2) quite well. Of course, supposing that both electrons and fluid systems are formally subject to analogous basic equations (essentially (10a-b)), it is to be expected that their calculated trajectories are similar in shape. Moreover, the intuitive mechanisms underlying electron and droplet motion are clearly similar too: both can be explained in terms of a guiding pilot-wave (Couder et al. 2005, 2006, Sbitnev 2014). Thus it seems that eq. (10a-b) could represent the basis of a model for certain experiments on droplets. Of course, for fundamental physics the converse inference is more important: the droplet experiments seem to give credit to the analogy between fluid and quantum mechanics revealed by Madelung in 1927. In the droplet experiments the interference patterns after a double slit arise because the droplet is guided by a diffracted pilot wave; similarly Madelung-

Bohm theory explains the quantum version of this experiment by a pilot wave guiding each particle through *one* slit – in clear contrast to the Copenhagen interpretation, which professes that the particle self-interferes and passes through *both* slits (Dirac 1958). Another recent and interesting theoretical investigation of why the bouncing oil droplets might be a good model for quantum particles is given by Brady and Anderson (2014), who expose a series of other formal analogies. In the following Section we will suggest further arguments why we believe the fluid-mechanical analogy deserves renewed interest. These arguments can be added to those published by others (Wilhelm 1970, Kuzmenkov and Maksimov 1999, Wyatt 2002, Sanz et al. 2002, Tsekov 2012, Sbitnev 2014, Brady and Anderson 2014), and to those we have exposed in (Vervoort 2015a).

5. Arguments in favor of the hydrodynamic interpretation. Do we really need nonlocality ?

Even if it appears fair to use the term 'Madelung-Bohm' interpretation in many contexts, it seems there are reasons to believe that the fluid interpretation (supposing it can be generalized) might be an improvement of the de Broglie – Bohm interpretation, as we will now argue. Two preliminary remarks are worth making. First, since Bohm's interpretation has become a full-blown field of physical research backed-up by a full-fledged theory, one heuristic strategy would be to use Bohm's theory *as a mathematical theory* but interpret it *à la Madelung* (see arguments below). Second, as an experimental hint for the import of the Bohm – Madelung theory, let us highlight a recent experimental result obtained by Steinberg and collaborators. This result gives, in our view, at least indirect support to the Bohm-Madelung interpretation (Kocsis et al. 2011). Steinberg's group has succeeded in measuring, through so-called 'weak measurements', the *average trajectories of single photons* in a two-slit interferometer. Each such trajectory passes through *one* of both slits (cf. Fig. 3 in Kocsis et al., 2011). Since the concept of a well-defined trajectory passing through one slit is not part of the Copenhagen interpretation (Dirac 1958), the only theory that allows to interpret the measured trajectories *in a natural way* is Bohm's or Madelung's. This fact begins now to be acknowledged in the literature (e.g. in Braverman and Simon 2013, Sanz 2015). Of course, this is an interpretational, conceptual argument: the trajectories can also be calculated by standard quantum theory, which is mathematically equivalent.

Therefore, for those dissatisfied with the Copenhagen interpretation, there seems quite an incentive to use Bohm's theory. Why then interpret it à la Madelung ? Or more precisely, why should we be motivated to push Madelung's program further ? We will here only advance three arguments (A1-A3) that follow immediately from our analysis in Section 3. Of these arguments (A3) seems new and clearly the most important point.

(A1) In Madelung's interpretation *both* basic equations (10a) and (10b) (or (13) and (10b)) have a fluid-dynamical origin, while in Bohm's interpretation (10b) stems from fluid dynamics and (13) from Hamilton-Jacobi theory. In this sense Bohm's interpretation is less homogeneous.

(A2) As we have seen, Bohm interprets "the wave function of an individual electron as a mathematical representation of an objectively real field. This field exerts a force on the particle in a way that is analogous to, but not identical with, the way in which an electromagnetic field exerts a force on a charge, and a meson field exerts a force on a nucleon. In the last analysis, there is, of course, no reason why a particle should not be acted on by a ψ-field, as well as by an electromagnetic field, a gravitational field, a set of meson fields, and perhaps by still other fields that have not yet been discovered" (Bohm 1952, p. 170). That may be so, but still there is something very unique about the ψ-field. It somehow emanates from the particle in question (no particle implies no ψ-field) but also acts on the particle itself, via the quantum potential Q. Indeed, the force $\overline{F}$ on the particle is $-\overline{\nabla}(U+Q)$ and Q is derived from the amplitude $\rho = |\psi|^2$ (cf. Eq. (14); see also Eq. (8a) in Bohm 1952). Now self-forces are very unusual in physics[5]. No such self-forces exist in the Madelung interpretation: *Q* is now a force potential that results from a (new but classical) type of inner (elastic) stresses in the Madelung fluid (cf. Section 3). So *Q* results from the pressure from the *other* particles that constitute the Madelung fluid – recall the stochastic hypothesis.

(A3) By far the most important problem of Bohm's theory is the following; it is related to (A1) and (A2). Although 'non-locality' is not explicitly mentioned in Bohm (1952) – the concept became a hot topic only after John Stewart Bell's work in the sixties (Bell 1964) – in virtually all works it is asserted that Bohm's theory is non-local. This is easily understood by the following well-known argument. The force on a particle depends on the quantum potential *Q*, which in turn depends on the wave amplitude $\rho = |\psi|^2$ (Eq. (14). Therefore if *anything* changes in the system, e.g. a boundary condition at a very large distance of the

[5] Another argument that is sometimes advanced is that in Bohm's theory the ψ-field acts on the particle, but not the particle on the field (in the sense that the field is the same for different trajectories); which is again highly unusual.

particle in question, then the wave amplitude $\rho = |\psi|^2$ changes *instantaneously*; therefore the force on the particle changes instantaneously. This is the paradigmatic description of a strong non-local effect, in Bell's sense (Bell 1964, cf. footnote in Section 2.1), and indeed the kind of superluminal non-locality that conflicts with relativity theory. This non-local interpretation of Bohm's theory, which is generally accepted, is likely the main reason of its classification as a highly non-standard theory in the wider physics community.

It seems however that such a non-local interpretation, immanent in (Bohm 1952), does not apply to Madelung's interpretation of the 1-particle Schrödinger equation. In Madelung's interpretation the potential $Q = \int \frac{dP}{\rho}$, and depends on how the bulk pressure within the Madelung fluid is related to the (mass / particle) density distribution in it. Sure, a fluid has an extended, 'delocalized' character, but the internal pressure may well build up *without in any sense implying strongly non-local, i.e. superluminal, forces*. In other words, in a full-blown theory *à la* Madelung the force *Q* arises from a new but classical – and therefore Bell-local – mechanism. Moreover one understands that this fact is well compatible with the fact that the fluid-mechanical characteristics, the streamlines and stream velocity, do depend on the global boundary conditions of the fluid – as happens in the most classic fluid mechanics. The point is that in the fluid-mechanical picture there is no need to invoke superluminal forces. In sum, *there seems to be no ground to term the Madelung interpretation non-local*. In view of the strong link between the Madelung and the de Broglie-Bohm interpretation, this conclusion may by extension hold for (a revisited version of) the de Broglie – Bohm theory. This, we believe, is an essential point.

The first counterargument to (A3) that comes to mind is, of course, that a Bell-local hidden-variable theory as Madelung's must be rejected by Bell's theorem[6] (Bell 1964). Such theories should satisfy the Bell-inequality, while quantum mechanics and the experiments violate it. However, elsewhere we have argued that fluid-mechanical theories, and more generally 'background-based' theories, can escape from Bell's no-go predicament (Vervoort 2015a; see also Vervoort 2013). This is a highly surprising result which we cannot explain in detail here; but the essential elements of our model, which is again inspired by the Paris droplet experiments, are the following. To derive Bell's inequality, one needs to assume,

[6] In the hydrodynamic framework, the 'hidden variables' of Bell's theorem are the (initial) positions of the streamlines, just as in Bohm's framework.

besides locality, a much more subtle premise, namely the hypothesis that the hidden variable distribution is *independent* of the analyzer settings. One usually calls this premise ‘measurement independence’ (MI), and justifies it by noting that violation of MI would amount to superdeterminism or ‘conspiracy’. While this argument may be convincing in the case the hidden variables describe the particle pair, it appears it generally fails *when these variables describe a background medium that can interact with the analyzers*. In the latter case it is quite intuitive, and one can formally show, that violation of MI has nothing to do with superdeterminism, but simply arises through the interaction of the analyzers with a background field (Vervoort 2015a). This phenomenon is beautifully illustrated in the Couder experiments, where the ‘background field’ is the fluid’s surface or pilot wave that interacts with the droplets and all surrounding boundaries or measurement apparatuses. If one of the premises of Bell’s theorem (MI) is violated in background-based models, then these can potentially violate the Bell inequality; and indeed it can be shown that they can reproduce the quantum correlation of the Bell-experiment (Vervoort 2015a). It is also argued in the latter article that fluid-mechanical theories as those of Madelung are natural candidates for such background-based theories: within Madelung’s theory the ‘stochastic hypothesis’ suggests that particles are dragged along with the Madelung fluid, interacting with it (cf. also Bohm and Vigier 1954). Of course, this background medium, as any normal field or fluid, is extended, delocalized; but if we are looking for physically acceptable hidden-variable theories the essential point is that its dynamics does not need to involve any strongly nonlocal, superluminal interaction to violate the Bell-inequality. (Analogously, in the droplet experiments strong large-scale correlations between far-apart subsystems clearly exist; but they do not arise through superluminal (or here supersonic) interactions. All interactions in these systems are perfectly local.) For a full treatment, we refer to the original article.

Thus our argument (A3), making the case for a local interpretation of Madelung’s theory, gains in cogency in combination with other work (Vervoort 2015a); but let us emphasize that A3 is an independent argument. Let us also note that other physical systems and theories have been studied in which a stochastic background medium plays a decisive role. For instance, in (Vervoort 2015b) and (Vervoort 2013) we showed that in spin-lattices the Bell-inequality can be violated, again through mediation with a stochastic background (here an ensemble of spins in Boltzmann equilibrium). But the theory that immediately comes to mind is Nelsonian mechanics (Nelson 1966); an interesting conceptual analysis of this theory is provided in (Bacciagaluppi 2005). Nelson’s mechanics comes rather close to the stochastic version of de Broglie-Bohm theory, even if (the formal approach to) the underlying physics is different. In particular, it is as non-local as de Broglie-Bohm theory is (cf. Bacciagaluppi 2005). So here one could point, a priori, to a meaningful difference with the hydrodynamic picture as we conceive it. On the other hand, since Nelsonian mechanics conceptually fits well to the idea of quantum particles carried by a stochastic (Brownian) background field (e.g. the physical vacuum), it may well be that our argument (A3) could, mutatis mutandis, be modified so as to apply to Nelsonian mechanics; a question we can however not answer here. Let us here end our succinct review of (Vervoort 2015a) with following remark. As said, a recent publication (Tsekov 2012) has argued that within a hydrodynamic picture entanglement can be explained by spatial correlations of the vacuum fluctuations. It is interesting that our abstract analysis related to Bell's theorem in (Vervoort 2015a), inspired by the droplet experiments of Couder and Fort et al., comes to the same conclusion: in our model, initially correlated particles can make a background field vibrate (they interact with it), resulting in a symmetric field that correlates the particles also when they are far apart.

In conclusion, since some experimental properties of quantum systems are analogous to the corresponding properties of fluid-dynamical systems (Section 2), and since some formal characteristics (equations) of quantum mechanics are analogous to the corresponding characteristics of fluid-dynamics (Section 3), one may wonder how far the analogy goes. Some may intuit it is likely that other analogies exist; most researchers will assume a cautious "wait and see" attitude – comforted by the skepticism that the Copenhagen interpretation has advocated in these matters since almost a century. The experimental analogies exhibited by Couder et al. and the formal analogy of Madelung both implicitly contain as a maximal (though highly speculative) possibility the hypothesis that the whole of quantum mechanics and fluid-mechanics would be analogical – can be unified. Clearly, stark conjectures as these should be treated with extreme caution. This cautiousness is apparent in the way Couder et al. carefully also mention dissimilarities (cf. Section 2), the most obvious being the fact that their fluid system needs to be sustained by an external energy source to counterbalance friction losses. No such thing is part of quantum physics at face value (but one could refer to zero-point fluctuations as a possible equivalent).

5. Conclusion

The goal of this article was to present the analogies between fluid and quantum mechanics recently revealed by experiments by a French research team; to show that they can be linked to Madelung's theoretical analogy between quantum and fluid mechanics; and to advance a few arguments for why the fluid-mechanical interpretation of quantum mechanics deserves renewed interest. We focused on conceptual arguments, rather than on new mathematical developments, by revisiting seminal work by Madelung (1927), Bohm (1952) and Bohm and Vigier (1954). First we described the analogies in sufficient detail so as to make them suitable for a foundational analysis. Then we derived Madelung's result explicitly starting from the Navier-Stokes and continuity equations and precisely stated all assumptions under which these equations transform into the Schrödinger equation. This immediately showed, through the stochastic hypothesis, that a deterministic description of a fluid is compatible with stochastic movement of 'entities' following, on average, the streamlines. We recalled that Bohm and Vigier (1954) integrated these concepts from Madelung's theory into their interpretation, leading them to see quantum particles as *singularities in a Madelung fluid*. We argued that it is this hypothesis (essentially the stochastic hypothesis) that allows to connect the droplet-experiments to the Madelung – Bohm theory. Finally and importantly, we argued that *within the hydrodynamic picture there is no ground to term the Schrödinger equation 'non-local'* – in stark contrast with Bohmian mechanics. This suggests several avenues for further research. The first is to revisit the de Broglie-Bohm theory, closely related to the Madelung theory but more mature, in order to investigate whether it can be made local. Of course, fluids are extended, 'delocalized'; but that is harmless compared to the strong nonlocality that has all too often discredited Bohm's theory in the wider physics community.

Therefore, although we cannot answer the title of this article, we hope to motivate research that further explores the analogy between fluid and quantum mechanics.

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