Ontologies étalées

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> *... What is to come? What will future bring about? I do not know, I don't anticipate anything. When a spider jumps down from a fixed point, toward its consequences, it always sees in front of itself an empty space in which it may not find support regardless of how much it stretches. That same thing happens to me: in front, always empty space; what pushes me forward is a consequence that stands behind me. ...*
>
> Søren Kierkegaard - **Diapsalmata**[11]

> *Philosophy is an art of self-division and self-union—an* art of self-specification *and* self-generation.
>
> Novalis - **Das Allgemeine Brouillon**[15]

1 Mathematics as Ontology, forty years later

Four decades have now passed since Alain Badiou published his monumental book *Being and Event*[1] (hereafter, **BE**). From our temporal perspective, this third decade of a new century, Badiou's work may almost be perceived as the high midpoint of the time interval between Heidegger's magnum opus *Being and Time* and our own perplexing times. Many things have happened in set theory, of course many more in philosophy and mathematics; in the past four decades, the world has witnessed both an absorption of some of Badiou's ideas from the late 1980s and a complete change of references and issues.

In this paper, I address two of the many developments that have happened since **BE** appeared. First, the question of *independence* in mathematics, and some connections with Badiou's ontology. And second, I will propose a *dynamic* reading of Badiou's proposal, through the notion of *étalé ontology* (ontologie étalée), an ontology unfolding over the layers of possible mathematical theories, of possible mathematical classes of structures, leaving the corner of set theory and thereby enriching the original landscape traced by Badiou.

Before plunging into our subject, I want to thank many people for extremely enlightening conversations, for eye-opening discussions: Roman Kossak and Wanda Siedlecka, for their Phenomenology Reading group, originally meeting at the CUNY Graduate Center in New York City, and thanks to the changes of our times, now online[1]. Juan Antonio Nido also helped me streamline some narrative issues in this essay. Innumerable conversations with Fernando Zalamea and Juliette Kennedy have also helped configure and contrast the last part of this essay, the étalé ontology idea[2]. Zalamea's RTHK model[22], combined with Kennedy's emphasis on local analyses of syntactic-semantic dualities and the emergence of logicality in regions apparently devoid of syntax[10], have been powerful constructions; my own proposal of étalé ontologies clearly builds on their constructions. The anonymous referees of this article also drove me to clarify several points: to them, I also extend my gratitude.

And, more than anyone else, I want to thank the editors of this volume, Tzuchien Tho and most especially Mirna Džamonja for their patience with my late submission of this paper[3].

1.1 Novalis and Ontology

I first propose a detour, a temporal detour taking us back more than two centuries, to the work of Novalis who in his very short life wrote an extremely original unfinished draft of an essay dealing with connections between many different subjects, an attempt of blending poetic description with scientific knowledge and philosophical inquiry: *Notes for a Romantic Encyclopaedia,* also known as the *Allgemeine Brouillon*[15]. He proposes there the following

[1]Without our weekly reading of Badiou for a few months, this article would not have been possible the way it is. I also thank Rajesh Kasturirangan, Alfredo Roque Freire and Simon Heller who in that reading group always forced us to contrast and explain what seemed unexplainable.

[2]An important note on two frequently confused words in French: *étale* and *étalé.* The first one is poetic, it was notably used by Paul Valéry in his description of *la mer étale*: the *smooth, flat, unflinching* sea. The second one is almost an antinomy: *étalé* means *spread out* and is notably used in sheaf theory terminology: sheaf spaces are often called *espaces étalés* in French. In our text, this second sense is the correct one. I thank Fernando Zalamea for this clarification of an earlier version of the title. The title of this essay is therefore *Ontologies étalées*; this expression could very well be translated into English as "sheaf ontologies" but I prefer to leave the title in French, and refer to the concept in English as *étalé ontologies.*

[3]Without Mirna's lifeline and open conversation on many aspects dealing with the subject of this paper (ranging from set theory to model theory to politics of our times to ontology and other parts of philosophy), my own belief in the possibilities of modifying Badiou's ontology, my proposal here, would not have seen the light of day!

definition of ontology:

> "ONTOLOGY. Infinities behave like finitenesses, with which they *alternate.* Finiteness is the *integral* of the one (small) infinity—and the differential or the other (large) infinity—which is *one and the same thing.*
>
> The differentials of the infinitely large, behave like the integrals of infinitely small—because they are also one."

Novalis therefore in 1798 starts his ontology directly as a comparison between infinity and the finite, and their alternation. This is a point taken up much later by Badiou in the first meditations of **BE**, in at least two different ways: the interplay between finite and infinite, the special role of nothingness. Novalis immediately draws differentials and integrals of infinities into his description; that aspect differs superficially from Badiou's perspective. However, after a digression on ratios between middle terms and comparisons of convergence[4], he zooms in to themes that reappear much later at the beginning of Badiou's ontology: the *relative Something* identified with emptiness (0), its heterogeneity *in relation to something Else.* And the *process* of homogenization of relative emptinesses, relative 0s becoming realized.

Here is the fragment in question—I add it in part because of the surprising anticipation of Badiou's system almost two centuries before the appearance of **BE**:

> However the relative Something is 0. And—heterogeneous—in relation to something *Else.* It is only through homogenization that the relative 0s become realized with respect to one another—They become

[4]Novalis continues:

$$(1 \times \infty) : 1 :: 1\ 1/\infty$$

The ratios between the *different* units, or the middle terms, are formed in a manner equal to the ratios belonging to the corresponding end terms.

$$(2 \times \infty) : 1 :: 2 : 1/\infty$$

etc.

The products of the heterogeneous constituents *vanish* e.g. the products of quantities of different *orders*—or degrees. They only have a relative worth or value in relation to one *another.* One quantity *vanishes more readily* than another—depending on whether the heterogeneity of the constituents is large or small—Thus the relative quantities—the relative values come into being—Nought has a degree, while each naught receives a relative value depending on the number of different 0s relate to one another—it becomes a relative number, a relative quantity, a relative Something.

(a universal system of annihilation!)

> comparable—factors of a common quantity—by means of the homogenizing principle.

Novalis anticipates here themes eerily similar to part of the beginning of Badiou's **BE**: homogenization of emptiness, relative realization of the different versions of emptiness (the "relative 0s") and most especially the reference to a "universal system" brought about by the realizations of emptinesses, of 0s, *with respect to one another.* The first meditations in **BE** are indeed a description of the formation of such a universal system. Novalis qualifies this system as a *system of annihilation*, with strong emphasis of this word, and then declares the emptinesses comparable *by means of the homogenizing principle.*

Of course, Novalis doesn't quite unfold the possibilities of his definition; rather than formulating a whole philosophical system, he was (at his very young age) registering, classifying, connecting many different aspects of nature, ranging from geological (he earned his living as a mine prospector) to chemical, mathematical, and of course poetic and metaphysic tones. He directed his sensitivity to the web of connections between many aspects of the world—in that sense, he was an heir to 18th Century traditions and a contemporary of the likes of Goethe and Humboldt; his singularity lay perhaps precisely in the incredible lucidity and freedom he brought to his quests. In his sketchy way, he foresaw many themes that crystallized much later; one of them is the version of ontology brought to life by Badiou almost two centuries after Novalis's life.

After this initial detour to the late 18th Century, we will jump into the late 20th Century first, and then to our own times. I step over the long interval between Novalis and Badiou essentially to focus on independence notions as ways of reading the map where I place Badiou's ontology and ultimately describe my proposal of étalé ontology. Of course, the long line of German Idealistic Philosophy, and then the arrival of Husserl and Heidegger was the casting frame for the purely philosophical bases of Badiou's construction.

2 Independence: Model Theoretic and Set Theoretic

2.1 Roots of Independence

The word **independence** includes its own description: the negation of Latin **pendere,** to hang (and its derived words pendule, suspend, etc.). This in turn has the Proto-Indo-European root **(s)pen** (to stretch, to extend). Schwartzman in his work on the etymology of mathematical terms[18] offers the explanation

> "Dependent (adjective): from Latin (...) "down from" (...) stem of pendere "to hang." The connection with Indo-European root (s)pen- "to draw, to stretch, to spin," is apparent in the fact that when a string is allowed to hang down with a weight attached, it is stretched tight. Native English cognates include span and spindle."

And the source[9] adds the following nuance: "Possibly reanalyzed root of $*(s)penh_1$- ("to spin (thread); to stretch") + *-$d^h h_1 eti$, *to do*."

Thus the origin of *independence* is the same as that of the verbs "to stretch, to extend", and the word is an etymological cousin of words such as *spin, spider, ponder, span*, or in French *poids, pensum, pensée, ...* Some of these words reappear later in connection with the mathematical study of independence notions: "span" and the French word *poids*, weight.

This etymological prelude to our matter of interest turns out to be quite revelatory: the original linguistic neighborhood of *independence* includes notions later interwoven with the word: span, spin, *poids* are all central words in the mathematical study of notions of independence, as we shall next see.

2.1.1 Mathematical Independence: von Neumann's ideas

In his Princeton lectures of 1935-1936, John von Neumann addressed the issue of giving an axiomatic *definition* of abstract independence (see *Continuous Geometry*[21]). He had to provide such an axiomatic treatment of independence in order to study issues stemming from Functional Analysis: operator algebras with dimensions taking values in $\mathbb{R}$, the set of real numbers, instead of discrete values. His work contains a whole gamut of possible notions of independence amenable to comparisons between them! The essential point is the presence of a relation he denotes by the sign $\perp$ (usually reserved for perpendicularity). And he defines possible algebraic relations between different objects, in terms of the abstract sign $\perp$. These were the first steps of a long way that would literally explode with Shelah's work in Model Theory after 1970 (generalizing the notion of *dimension* stemming from "Morley Rank," he captured the most central concept governing the *abstract geography* of (first order) mathematical theories: **forking**, usually denoted by the sign $\downarrow$ or $A \underset{B}{\downarrow} C$. The main reference of this work is Shelah's magnum opus *Classification Theory*[19].

Guided by his *intuition* that allowed him to extract axioms for the notion of independence, von Neumann isolated fundamental properties of these notions: monotonicity (being independent from some set A implies being independent from any subset of A), symmetry (if A is independent from B, B should also be independent from A), transitivity (if A is independent from B and B is also independent from C, then A should be independent from C) and some other more technical properties. The important point is that after von Neumann's work we not only had an abstract way of dealing with "being independent" but also a way of comparing different *ways* of being independent.

Decades later, the "Parisian School" of Model Theory, led by Lascar and Poizat[13], proposed different terminologies, different conceptual frameworks to understand independence[5]. In North America, Harrington and Harnik[8], and then Baldwin[3] continued the study of the line originally opened by Shelah.

Some of the difficulties the community seemed to encounter during the first few years after Shelah's original publications on independence notions were gradually streamlined by several authors. Around 2010, Hans Adler reformulated Shelah's notion of forking independence (and other notions of independence that had become important by then[6] and recast it in a spirit akin to von Neumann's original work[21]. Adler adapted to the first order context part of the work of Grossberg and Lessmann[7] from 2000.

2.2 A map of mathematical (first order) theories

The half century after Shelah's initial results, and the completion of his monumental volume *Classification Theory* around 1980 ushered a spectacular development of **model theory** for first order theories. Understanding the hundreds of pages and definitions and notions entails years of study (with little guarantee of success!); however, two motives emerge after the dust settles (so to speak), two motives that have both very specific major consequences in purely mathematical terms but also have deep (and so far only barely explored, at best) philosophical import: *dividing lines*, and their connections with *notions of independence intrinsic to theories,* and their possible properties.

In this sense, Shelah managed to go beyond the most far-fetched dreams: he provided several dividing lines ("cutting through" the map of all theories, with *natural* taxonomies on both sides of the cut), often splitting that world into *one side* that has a very well-behaved notion of independence, and *another side* that has some deep defining feature (for example, a definable infinite ordering, in the emblematic case of the dividing line called "stability/instability")[7].

The model theorist Gabriel Conant has created a useful, interactive online representation of the map of first order theories[5] (he calls it "map of the universe"; the use of the word

[5]They introduced the dual notions of *heir* and *coheir*; inheriting essential properties described in a language, or not inheriting them, as ways of characterizing abstract independence.

[6]Among these, mainly *thorn-forking*, *splitting*, etc.

[7]Shelah in my interview to him[20] offers the contrast between these dividing lines, where both sides are clearly significant and well-defined with "non-dividing lines" where only one side has significance. He often quotes the example CH/¬CH as an example of one "side" of the line having clear significance (CH) while the other side is essentially meaningless per se.

universe in the website's title is quite misleading, as it only refers to first order theories, a rather small albeit very important fragment of the currently known mathematical universe). Conant's map is extremely useful, both as a research and synthesis tool, and as a strong metaphor-generating place. See in Figure 1 the state of the map in February 2023.

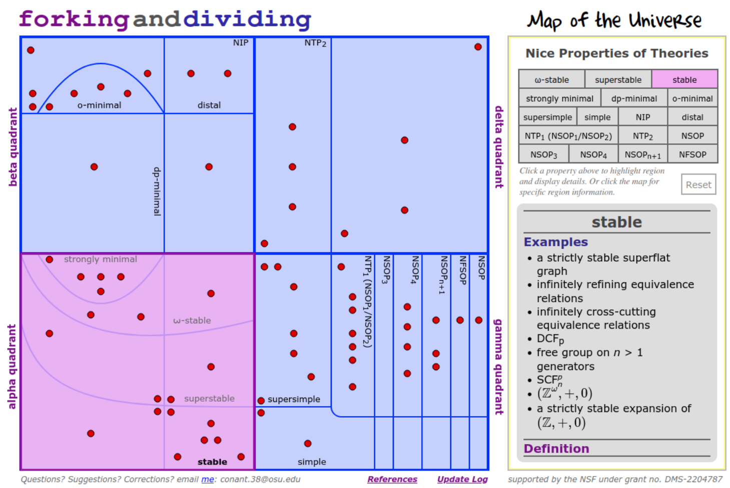

Figure 1: Conant's "Map of the Universe" (as of February 2023)

The zones of this map correspond to areas where different notions of independence derived from Shelah's original forking have specific properties; it illustrates the major taxonomies and dividing lines derived from many theorems, the work of many model theorists for more than a half century now. In principle, we could place on this map any first order theory, and look for other important theories in the same zone. This taxonomy was built mainly on the question of the *presence* (or properties) of independence notions, and it includes *all* first order theories. A whole system, a whole web of *dividing lines*, enriching themselves historically and continuously, emerges from concrete properties of independence notions!

Independence is directly related to the possibility of defining *degrees of freedom*, **dimensions.** At one corner of the map (the "SouthWest"), possible models tend to be unidimensional, and often are completely determined by their (infinite) cardinality. In the ideal case of "categorical theories," this is precisely the case. Those are the theories that realize in the most extreme way Aristotle's (or Leibniz's) dream of a "perfect correspondence" between models of the world, instances of the world, and their descriptions through language[6].

As we start wandering (slowly, gradually) toward the "East" or "North" of the map, we start abandoning this unidimensionality; the "geometrical behaviour" of independence begins to gain complexity. Classical theorems due to Shelah allow us to characterize in a precise and unique way the kind of independence notions (linked to language, to logic) defining the "tamer" regions of the map—the South West, so to speak—and gradually disappear as we get farther from that zone, toward the so-called *unstable*, or *unsuperstable* theories.

Reaching the other extreme of the map, the "North East," independence notions similar to those classifying models of theories at the tame end of the map have almost completely disappeared. There may remain *traces* of those independence notions, but the wealth of structures, the variety they have goes beyond all hope of classifying them by means of independence notions. Such is the case of two emblematic theories: Arithmetic (usually axiomatized by systems called PA, Peano Arithmetic); and Set Theory (most usually axiomatized by the Zermelo-Fraenkel system, called ZF).

Having reached that far corner of the map, all hope of classifying the possible models of theories by the Shelahian-derived notions of independence has evaporated, and one lands in a world rich in another sort of independence, that one may call *logical,* by opposition to *mathematical* independence derived from Shelah (and ultimately, connected to von Neumann's ideas). Forking configurations here do not play a central role, and they are replaced by a totally different world, a bit as if reaching a zone of the universe where the more usual laws of gravity are replaced by situations charged with brutal dilemmas and infinite torsions. We now check this world a bit closer.

This stormy corner of the map is where Badiou's ontology takes place. In the next section, we start exploring that corner, through Badiou's lens.

2.3 Logical Independence: the "stormy corner" of the map

We now leave the area where independence notions such as forking $\downfree$, geometric/algebraic independence notions dominate and control the behaviour of models, and we describe very quickly what we have called the "stormy corner" of the map.

For the theories ZFC and PA (and those that share their properties in terms of the taxonomies given by stability theory), the kind of relevant independence notions is quite different: *Logical Independence* phenomena now dominate the treatment of these theories[8]. I only touch here in an extremely general way the line of results that went from Cantor to Cohen, by way of Gödel, to establish the **independence** of the Continuum Hypothesis (CH) and of the Axiom of Choice (AC) with respect to the ZFC axioms of set theory. The work of Cohen[4] in 1963/1964 ushered a complete methodological *revolution*, by placing the focus of study of independence in the *construction of models* via forcing, adding "generic"

[8]An important historic distinction between logical independence for arithmetic and for set theory: forcing methods have been extremely fruitful for set theory, ever since what Badiou calls an ontological revolution. The corresponding revolution for arithmetic has not happened, or at least not in the same all-encompassing way. There is so far no general "arithmetic forcing" method.

elements to the base universe of set theory (or to a countable version of it); those generic elements may be named from the base universe, however—named, but never specified. As Badiou stresses in the last meditations of **BE,** Cohen's discovery was much more than a methodological device; it constituted a complete ontological revolution, through the control of the *generic* elements, the *unnameable* elements from the base universe, and the irruption of those individuals in possible extensions of the universe, mixing definability aspects with indiscernability properties.

The richness of this method therefore cannot be overstressed; the final meditations (34, 35) of Badiou's **BE** are completely centred in a description of the connections between forcing, Leibniz, and even Lacan, from an ontological perspective. Badiou even links forcing with theories of the subject. He deems the discovery of forcing the most important revolution in ontology, and attempts a project of explaining the intellectual achievement in terminologies more accessible to the non-specialists. It is a subject of debate whether he manages to achieve that aspect of his project; four decades later, this part of **BE** seems still largely absent from discussions on ontology.

2.4 Logical Independence and Semantic Verification

One of the strongest (and in some sense strangest) aspects of logical or axiomatic independence is that it relies on *semantic* verification. Independence of this kind is not (unlike mathematical independence, forking $\downarrow$ notions) verified in algebraic terms[9], but rather in very semantic terms: building universes (relatively, locally, controlled by "names" visible from the base model) through partial descriptions of properties desired, and pushing the possibilities of making this universe-building process crystallize to their extreme. This gives rise to a whole gamut of *modulating principles*: minimality, maximality, naturalness, etc. In **BE**, Badiou places forced universes (built through forcing), objects nameable from the base universe, properties captured by *indiscernibility* and *genericity.* The nod to Leibniz is explicit (in the central role of indiscernibility of objects nameable in the base universe and intermonadic relations in Leibniz's theory of possible worlds, in the maximality principle invoked) and implicit in the meditations (34 and 35, chiefly) dealing with forced universes in **BE.** Some families of axioms, and therefore some notions of independence, result from these modulating principles. Among these, the most notable started appearing in the first decade after Cohen's work: so called "forcing axioms" (the first and most emblematic of these is called Martin's Axiom MA) are maximality principles of the kind *everything that may happen, without obvious contradiction with the theory, will happen.* As may be expected,

[9]So called Boolean-Valued models, or sheaf models, have deep connections with algebraic logic, and a reduction of forcing to the study of algebraic questions on those models is possible. However, set theoretical practice has favoured semantic methods, for reasons that in my opinion are not mere coincidence.

the notions "everything" and "without obvious contradiction" require fine-tuning, and the technicalities of axioms such as MA and other forcing axioms (called Proper Forcing Axiom PFA, Martin's Maximum MM, and many variants MM^{+}, MM^{+++}, etc.) take care of the many details of what these notions may mean. However, what remains is a whole series of maximality principles provided by some of the modulating principles natural to semantic verification.

What I called above the "nod to Leibniz" in Badiou's description of forcing and maximality principles may be linked to the following fragment (quoted by Rescher in his essay *Leibniz on the World's Contingency*):

> *We must also say that God makes the maximum of things he can, and what obligates him to seek simple laws is precisely the necessity of finding room for as many things as can be put together: if he made use of other laws, it would be like trying to make a building with round stones, which make us lose more space that they occupy.*
>
> Leibniz, in a letter to Malebranche[17, p.60]

Another important by-product of semantic verification, connected with a sociological aspect of scientific endeavour, is the brutal *freedom* of construction of "alternative universes", once the many details of the technique are mastered. This is familiar to the work of mathematicians, and the relative naturalness or artificiality of such constructions is often discussed, in more (or often less) explicit ways. In the corner of the map where these notions of independence happen (ZFC/PA - set theory, arithmetic) this sort of freedom is perhaps pushed to more extreme degrees than in the rest of mathematics. In some sense, explorations of the so-called *Set Theoretic Multiverse* tackle aspects of the freedom afforded by these semantic methods; the freedom and its possible dangers.

The many ramifications of forcing since its inception are now the subject of as many philosophical interpretations, in the work of Maddy, Koellner, Kennedy, among others (see, for example, [14], [12] and [10]). Ontological variants, the import of set-theoretic multiverses, what one may call the "second forcing revolution" (Shelah's invention of *Proper Forcing*, a unifying principle reducing the need for combinatorial devices and making them more semantically controlled, and at the same time giving rise to strong maximality principles), and lately the tension between "Gödelian" inner models (Ultimate-L being the most famous perhaps, but many other constructions intend to continue Gödel's idea of tight linguistic control) and strong maximality principles, all of these are theatres of interaction where Badiou's ontological proposal, and especially the later meditations, may be played again.

But I now propose to continue in **a different direction:** modulating Badiou's original equation through what I will call *Étalé Ontology,* namely, allowing the comparisons between independence notions describe here to be now lifted off the "main layer" of the map, onto many additional leaves of a foliation.

3 Étalé Ontology?

3.1 The "foliated" map

The world is not flat. Just as our physical earth, the map of possible mathematical theories whose geography is determined by the sort of independence notions living on them is not flat either; an interesting shape has been emerging from the work of the past two or three decades. It takes, however, a series of steps to realize this. An extension of the great map we have just studied, an extension of Model Theory, now going well beyond first order theories, is now in the works, in sometimes quite surprising ways. In slightly more "technical" lingo, this extension is called *stability theory for abstract elementary classes.* This includes many classes of structures important in mathematics that are out of reach of first order axiomatizability, and therefore "off the chart", outside of the standard (and flat) map we described in the previous section. Many of these structure classes are axiomatizable using various kinds of infinitary logics[10], but the main extension of the taxonomic classification lines has happened, historically (during the past quarter century), in the world of abstract elementary classes, first introduced by Shelah around the same time that Badiou wrote **BE**).

Abstract elementary classes are classes of models that attempt to capture *in a purely semantic way* the *usually syntactic* notion of a class axiomatized by a theory: instead of focusing in the axiomatization, they consist of coherent systems of structures where the emphasis is on "how well embedded" is one structure in another. Instead of starting from a theory T and the basic notion of satisfaction of a formula $M \models \varphi$, as in usual settings in logic, the starting point is an abstract notion of "strong embedding", of how "correctly placed" is a small structure within a larger one, a concept usually denoted as $M \prec^* N$. Abstract elementary classes thus consist of classes of structures, together with a "strong embedding" relation $\prec^*$ between them, in such a way that these notions [AEC1] respect basic form (preserve isomorphism), have a version of coherence [AEC2] enabling them to deal with an abstract version of what we call "existential closure" in mathematics (but with no reference to existential quantifiers $\exists$), [AEC3] allow taking limits of constructions (a crucial

[10] One reason for the importance of the first-order map until now was the original development of stability theory, the ideas underlying the taxonomy, first for that particular logic. Finding logics with "good model theory" but different from first order logic, is not an easy task. However, in the past 25 years, the taxonomic classification, and the presence of good independence notions has been partially extended way beyond the first order situation. Naturally, this calls for an extension of the map, and as we propose here, an extension of the ontology originally set forth by Badiou.

step for the development of "model theory": the possibility of taking limits of increasing models is a landmark consequence of the classical Gödelian completeness theorem linking syntax to semantics, and its cognate, the *compactness theorem*; here in the absence of an anchoring in syntax, we invoke one of the most useful semantic consequences of those notions, in the form of the possibility of taking limits of increasing chains of models) and [AEC4] incorporate a notion of "internal small closure": inside any structure N and given any small subset A of N, there is some "small and well-placed" version of N, N_0, containing A.

The axioms may thus be seen as ways of placing models (structures) in a general situation where they may incorporate new phenomena [AEC3], while having tools to localize and capture small situations [AEC4] (usually called the "Löwenheim-Skolem axiom of AECs" since it encapsulates the usual theorem of the same name in first order logic), while respecting general form [AEC1], in a way coherent with *abstract existential* behaviour [AEC2].

The amount of consequences that may be obtained from these seemingly sparse and very general notions ([AEC1] to [AEC4]), the fact that the resulting foliated map not only extends the classical, flat one, but allows to capture many of the essential dividing lines, with a shift of emphasis from syntax to semantics, is a contemporary instance of the classical ontological problem of logicality, the aptness and limitations of language to capture a notion of the world[10].

A possible description of the bold move to model theory in abstract elementary classes is the move (inside logic!) from an emphasis on λόγος, on formulas and axiomatizations, on the syntactic, to an emphasis on τὸ ἁρμόττον, *harmotton,* the Greek root of our notion of harmony (fitting well, being well-positioned) as in the abstract $\prec^*$ relation we are formulating. An aesthetic criterion (a notion of beauty different from that given by τὸ καλὸς), as Patočka describes in his essay on the genesis of European reflection on the Beautiful in Ancient Greece[16, pp. 50 and ff.], becomes the forefront of logicality in the "foliated map", and the place where Badiou's ontology could most naturally evolve!

These classes include all those in the flat map, and many others, some of them axiomatizable in "infinitary logics" (logics allowing infinite conjunctions and disjunctions, and to some degree infinite quantification), and have allowed the development of a very serious body of work of model theory.

What matters at this point for our purposes is that those Shelahian taxonomies that determined the original map's geography in such a refined and detailed way are now being prolonged, sometimes in entangled ways, sometimes smoothly, toward much larger zones. In the original map, the definitions given by language, by formulas, seemed essential; the role of independence properties could at times seem to take a back seat to descriptions in terms of formulas, to definability.

In the larger, not flat, universe of abstract elementary classes or of infinitary logics, these taxonomies place the notions of independence, their configuration diagrams, their "internal geometry" and *semantic* descriptions, at the very forefront!

This has been the work started by Shelah in the mid-1980s, and continued in the past 25 years by Grossberg and a group of scholars formed by him at Carnegie Mellon[11], and by a few other groups notably in Helsinki and Bogotá. Their work places immediately the question of the tension between logical independence and algebraic/geometric independence under a surprising angle.

The stability map may, **for some theories,** entail a rather extreme displacement from the more set theoretic end of the map to the more algebraic/model theoretic corner, from the "North East" of the map to the "South East".

Here is an example of this displacement: Logical independence may be present in the theories of certain structures in principle very different from ZF or PA. This is notably the case of the structure central to Complex Analysis since the 19th Century:

$$(\mathbb{C}, +, \cdot, 0, 1, \exp)\,;$$

the complex numbers, together with the exponential function $(z \mapsto e^z)$. This structure is connected with classical questions in mathematics, notably in arithmetic. Models of arithmetic, with all their complexity, may be *recovered* as definable predicates, in this structure; therefore, the complex numbers with exponentiation are placed in the "stormy corner" of the first order map, close to ZF and PA.

The displacement happens with a *passage* from one corner of the map almost to the opposite extreme, at the price of "changing the leaf" of the map: Zilber, led by his original conjectures and his intuitions with respect to the role of *categoricity* of theories (or of structures)[12], posited a new way of understanding complex exponentiation, in terms of infinitary logic (in order to avoid, with the expressive power of this logic, the problem of the presence of set theoretic independence phenomena). The complex exponentiation structure "recast" by Zilber (called pseudoexponentiation or the Zilber field) in the infinitary logic $\mathbb{L}_{\omega_1\omega}$ has been shown to be categorical in all uncountable cardinalities. This places it at the tame extreme of the map, but outside the first order map. Notions of independence connected to geometric *abstract closure* notions, occurring in that "upper leaf" of the foliated map (leaves being given by different logics) is the reason of this theorem. The most important point is the passage from the "wild" end of the map to the "tame" one, by way of a change of leaf.

Étalé Ontology is therefore a blend of Badiou's ontology (happening mostly at the set theoretical corner of the first order map) with awareness of the global map and the role of

[11] Lessmann, VanDieren, Kolesnikov, Boney and Vasey, among others.

[12] Categoricity is also ontologically charged and central to model theory. Model theory eschews total categoricity for infinite models, by the Löwenheim-Skolem theorem, but makes strong use of categoricity in power. Baldwin has explored these connections, not from the point of view of ontology but from a rather more epistemologic/methodologic perspective.

different independence notions on it. The ontology is therefore now dynamic: moving across the foliated map, sometimes changing leaves in order to reexamine a class of structures—and draw the consequences of these, just as Badiou does for genericity, forcing and constructibility in set theory, and of the movement from leaf to leaf.

There is in Book VI of *Logiques des mondes*[2] (the continuation/response Badiou wrote to **BE**) an approach that in various ways connects with the sort of questions we pose here: an emphasis on *locality* (especially in his analysis of the topological structure of Points in the World), as well as Badiou'sdirect attention to the unfolding of beings, bodies and objects, his engaging Husserl's phenomenological approach as opposed to the ontological focus in **BE.** Badiou's *Logiques des mondes* is anchored in logicalities (in plural), in the quest for the correct logical conditions of unfolding of being. The étalé ontology we propose has certain points in common with Badiou's approach there; however, an important difference lies in the layering, the foliation we propose based on the "flat map" of **BE** via the internal logic of abstract elementary classes. In that sense, our proposal of étalé ontologies is an organic extension of Badiou's ontological proposal from **BE**. It extends in a natural way, we believe, the anchoring in set theory, through the model theoretic emphasis on abstract elementary classes.

4 Some conclusive remarks (personal and philosophic)

Kierkegaard in his *Diapsalmata* asked *What is to come? What will future bring about?* and then describes the role of empty space in front of a hypothetical spider, the *empty space in which it may not find support regardless of how much it stretches.* I quoted this phrase at the opening of this paper, for two reasons: one personal, one philosophical. In many ways, Kierkegaard's was a very vivid description of my own feeling with respect to Badiou's ontology, with respect to his initial equation **Mathematics** = **Ontology** for a long time. As a trained set theorist, I first found his use of forcing both displaced and extremely interesting. For a long time, I was intrigued by this emptiness in my own approach to a subject that I knew well (forcing) and at the same was being used in a totally different by an amazing mind. I had read and absorbed parts of the work, perusing some of them, making some critique of various points. When conversations with the editors of this volume started, I felt ready to reread whole parts of the book and have my say. Yet, despite long sessions of rereading, of carefully going through the first parts, those not dealing with forcing, in a reading group (thanks to Roman Kossak and Wanda Siedlecka, to Rajesh Kasturirangan and Simon Heller for all those long conversations on the first meditations of **BE**!), when the time came to write down the paper, I felt exactly like Kierkegaard's spider, jumping into absolute emptiness. Philosophically, in many ways the move to ontology is very aptly captured by

Kierkegaard's image, by *may not find support regardless of how much it stretches.* I slowly unravelled this essential first part of the Meditations, the appearance of **events** essentially and ultimately from emptiness, emptiness linked by Badiou to the appearance of elements, and elements being linked to the whole, to the universe. Novalis's onthology captures, in its own semi-poetic, fragmented, late 18th century way, this essential point. Badiou then creates the system and we are now, forty years later, with the problem of continuing his work, with the challenge of really understanding what he proposed.

In this paper, I went then from Novalis's original ontology of 1798—drawing the role of emptiness, of infinity, of composition of different emptinesses, of the system thereby extracted, all the way through a quick mention of Badiou's ontology in **BE** as the midpoint between Heidegger's Being and Time and our own times. These suggest a whole description of ontologies shifting on the map given by independence, from its more "model theoretic" (tame) corner to its more "set theoretic" (wild) one. And then, the possibility of shifting variants of the *same* (possibly ontologically laden, such as a structure coding arithmetic, or even set theory) structure, from one corner of the map to its opposite. A full-fledged development of the étalé ontology stemming from these considerations remains to be completed.

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