A Modal Extension of the Quantified Argument Calculus

Simon D. Vonlanthen [1]

Institute of Philosophy II
Ruhr University Bochum
Universitätsstraße 150
44801 Bochum, Germany

Abstract

The quantified argument calculus (Quarc) is a novel logic that departs in several ways from mainstream first-order logic. In particular, its quantifiers are not sentential operators attached to variables, but attach to unary predicates to form arguments – quantified arguments – of other predicates. Furthermore, Quarc includes devices to account for anaphora, active-passive-voice distinctions, and sentence- versus predicate-negation. While this base system has already been shown to be sound and complete, modal extensions still lack such results. The present paper fills this lacuna by developing a modal extension of Quarc, including identity. The semantics will invalidate the Quarc-analogues of the Barcan-formula and its converse, as well as treat identity as contingent by default. Furthermore, an unlabelled Gentzen-style natural deduction system will be presented, which includes the full expressive power of Quarc. It will be shown to be strongly sound and complete with respect to relational frames. The paper closes off with considerations on how to extend the system to cover other normal modal logics as well as extensions to suitable three-valued semantics that capture relevant types of presupposition-failure.

Keywords: Quantified argument calculus, quantified modal logic, substitutional quantification, contingent identity, natural deduction.

1 Introduction

The quantified argument calculus (Quarc) is a novel logic first presented in [2]. It departs in several ways from mainstream first-order logic, chiefly in the way it handles quantification. Instead of being sentential operators that attach to variables, the quantifiers attach to unary predicates. These *quantified arguments* then function as arguments for other predicates. It also contains devices to represent anaphora, passive-active-voice distinctions and sentence-versus predicate-negation. Last, but not least, it distinguishes between particular quantification, predication of existence and 'instantial sentences' of the form

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There is/are Fs. In all these respects, it is arguably closer to natural language than the more widely employed predicate calculus.

The present paper does not focus on the justification of this claim nor the motivations behind the design choices of Quarc. Instead, it explores a modal extension of Quarc from the technical point of view. Such extensions have already been studied in [4], but while the proof system there was argued to be sound, its completeness was not demonstrated. Moreover, said paper employed the model-theoretic semantics of Quarc (developed in [9] and further used in [11]). However, since its inception, Quarc has standardly employed a version of substitutional semantics called *truth-valuational semantics*. It generalises the familiar approach of Boolean valuations in propositional logic to the case of quantification, by *directly* assigning truth-values to basic formulas, without the employment of model-theoretic tools. This, in turn, generates the truth-values of quantified formulas (cf. [5]). An exploration of modal extensions of Quarc on the basis of such truth-valuational semantics is still outstanding.

The present paper fills this lacuna by developing and presenting a modal extension of Quarc under truth-valuational semantics. It will also incorporate identity, which is treated as contingent by default, following the suggestion in [3]. Furthermore, while there is no semantic *de dicto*/*de re* distinction for more basic formulas lacking quantification, the Quarc-analogues of the Barcan-formula and its converse are shown to be invalid. The paper then presents the first natural deduction proof system for the Quarc-analogue of the modal logic K and demonstrates its strong soundness and completeness with respect to relational frames. Finally, it closes off with some considerations on capturing other normal modal logics and incorporating the three-valued semantics previously considered for Quarc (cf. [9] and [15]).

2 The Syntax

We follow the basic set-up of [15] and start by defining the languages of *mQuarc*:

Definition 2.1 A language $\mathcal{L}$ of *mQuarc* consists of the following symbols:

(a) Denumerably many singular arguments: $a_1, a_2, a_3, ...$
(b) Denumerably many predicates, for each arity $k \in \mathbb{N}$, $1 \leq k$: $P_1^k, P_2^k, P_3^k, ...$
(c) A special predicate of arity 2, called *identity*: $=$
(d) Connectives: $\wedge$, $\neg$, $\rightarrow$
(e) Quantifiers: $\forall$, $\exists$
(f) Denumerably many anaphora: $x_1, x_2, x_3, ...$
(g) One modal operator: $\Box$
(h) Auxiliary symbols: brackets ((,)), a comma (,) and denumerably many numerals $1, 2, 3, ...$

Remark 2.2 $P, Q, R, ...$ will be used to denote arbitray predicates, while $a, b, c, ..., o, ...$ will denote arbitrary singular arguments. Lastly, the lower-case letter q will be used to denote either $\forall$ or $\exists$. Moreover, all languages will be $mQuarc$-languages from here on out, with $\mathcal{L}$ denoting an arbitrarily given language.

Remark 2.3 The requirement of having denumerably many singular arguments and anaphora could technically be dropped, but having it in place allows for more straightforward soundness and completeness proofs. It is also not unique to either Quarc or the truth-valuational approach. For even in the predicate calculus, allowing only finitely many variables creates issues for completeness, too. For example, in the presence of only two variable symbols – x and y – the formula $\forall x \forall y Rxy \rightarrow \forall y \forall x Ryx$, while a tautology, would be rendered unprovable in the standard proof systems. [2]

Having defined what counts as a language of $mQuarc$, we must now define several further notions before we move on to the set of formulas.

Definition 2.4 Let P be a unary predicate. Then $\forall P$ and $\exists P$ will be called *universally* and *particularly quantified arguments*, respectively.

Definition 2.5 Let P be some predicate symbol of arity $1 < k$. Let π be a non-identity permutation on $\{1, ..., k\}$. Then P^{π} (or explicitly: $P^{\pi 1, ..., \pi k}$) is called a *reordered form* of P. The special predicate $=$ is the only predicate of arity > 1 that has no reorders.

Definition 2.6 Let a be some singular argument and qP a quantified one. Let x be some anaphora. Then a_x and qP_x are called x*-labelled*, where the x will be considered neither an anaphora nor a part of any argument. If an anaphora x appears in a string of symbols, its *source* will be the nearest x-labelled argument (singular or quantified) to its left, and x will be said to be an *anaphora of* said argument.

Remark 2.7 We shall use $\equiv$ exclusively to denote *syntactic* identity, reserving '=' for either a symbol in the semantics (cf. 3 below) or the object-language special predicate $=$.

Definition 2.8 A *predicate operator* Δ is always either $\neg$ or $\Box$, never any other symbol. A string of k-many predicate operators will be denoted by Δ^k.

Definition 2.9 Let $\mathcal{L}$ be given. The set of *formulas* of $\mathcal{L}$, denoted by $FORM_{\mathcal{L}}$, is defined recursively as follows:

(a) **Basic formulas**: Let $a_1, ..., a_k$ be unlabelled singular arguments and P_i^k be a k-ary predicate that is not reordered. Then $a_1...a_k P_i^k$ is a *basic formula*. If a and b are unlabelled singular arguments, then $ab =$ is a basic formula.
(b) **Reorders**: If $a_1...a_k P_i^k$ is a basic formula and π is a non-identity permutation on $\{1, ..., k\}$, with $1 < k$, then $a_{\pi 1}...a_{\pi k} P_i^{\pi 1, ..., \pi k}$ is a formula, where $P_i^{\pi 1, ..., \pi k}$ is a reordered form of P.
(c) **Modes of predication**: If $a_1...a_k P_i^k$ is a formula (P can be reordered) and Δ^n a string of n-many predicate operators, $1 \leq n$, then $a_1...a_k \Delta^n P_i^k$ is a formula.

[2] Such systems are indeed incomplete for finite axiom schemas, cf. [10].

(d) **Modals**: If ϕ is a formula, then $\Box\phi$ is a formula.
(e) **Connectives**: If ϕ and ψ are formulas, then so are $(\phi \wedge \psi)$ and $(\phi \rightarrow \psi)$. [3] If ϕ is a formula, then so is $\neg\phi$.
(f) **Led formulas**: Let ϕ be a formula that contains k-many distinct occurrences $o_1, ..., o_k$, numbered from left to right, of a singular argument o, $1 < k$, none of which are labelled. There can be more occurrences of o in ϕ, and o_1 does not have to be the leftmost occurrence of o in ϕ. Assume further that the anaphora x does not occur in ϕ. If there is no labelled singular or a quantified argument to the left of o_1 and ϕ does not contain a (proper) substring ψ that is a formula, contains all $o_1, ..., o_k$ *and* contains all anaphora of all arguments in ψ, then $\phi[o_x/o_1, x/o_2, ..., x/o_k]$ is a formula, where we substitute o_1 with o_x and the other occurrences o_i, $1 < i \leq k$, with x. We say that it is *led* by the labelled singular argument o_x.
(g) **Governed formulas**: Let ϕ be a formula containing an occurrence of the (potentially labelled) singular argument o, and let qP be an unlabelled quantified argument. If there is no quantified argument or labelled singular argument to the left of o, and ϕ contains no (proper) substring ψ which is a formula, contains o and all anaphora of any arguments in ψ, then $\phi[qP/o]$ (replacing *that* occurrence of o only) is a formula. It is said to be *governed* by that occurrence of qP.
(h) **Closure**: Nothing else is a formula.

Remark 2.10 In the following, we shall not write $ab =$, but $a = b$ instead, as is customary. However, this should not obscure the fact that its proper form *is* $ab =$. Similarly, we shall write $a\Delta^n = b$ instead of $ab\Delta^n =$, etc.

Example 2.11 $(\forall P_x \forall Q_y \Box \neg P_1^2 \rightarrow xyP_3^{2,1})$ is a formula governed by $\forall P$ and generated from a led formula of the form $(o_x \forall Q_y \Box \neg P_1^2 \rightarrow xyP_3^{2,1})$ for some singular argument o. $\Box\forall P_x = x$ (technically: $\Box\forall P_x x =$) is also a formula, but not governed by $\forall P_x$, due to the position of $\Box$.

Remark 2.12 While it is an idealization with respect to natural language to allow countably many predicate operators in clause (c), this idealization is on par with allowing equally many such symbols in front of a formula, as in clauses (d) and (e).

As was proven in [15], a similar syntax, although lacking $=$, $\Box$ and modes of predication beyond $\neg$, has unique parsing. The same result applies to the syntax of *mQuarc* as well, which in turn allows us to define the complexity of a formula:

Proposition 2.13 *Let $\mathcal{L}$ be given. Then, for each formula $\phi \in FORM_{\mathcal{L}}$, exactly one of the following holds:*

(a) *ϕ is a basic formula.*
(b) *ϕ is a reorder.*

[3] In the sequel, we will drop these outer brackets if no ambiguity arises, following the standard conventions about binding-strength.

(c) *ϕ is a mode of predication.*
(d) *ϕ has the form $\Delta\psi$, for some formula ψ, and with Δ being a predicate operator.*
(e) *ϕ has the form $(\psi \wedge \chi)$ or $(\psi \rightarrow \chi)$, and is neither led nor governed.*
(f) *ϕ is a formula led by a labelled singular argument, and is immediately generated from another formula as in 2.9(f).*
(g) *ϕ is a formula governed by a quantified argument qP, and is immediately generated from another formula as in 2.9(g).*

Furthermore, from (b) to (f), the formula(s) from which ϕ has been generated is/are uniquely determined in each case. In (g), the formula from which ϕ has been generated is unique up to the identity of the singular argument that is replaced by qP.

Proof. The proof bases itself on the analogous result in [15], given that the present syntax is merely an extension of the one discussed there. Thus, it suffices to ensure our additions to the syntax do not break unqiue parsing. Observe that each string Δ^n of predicate operators is uniquely determined, since $\Box \not\equiv \neg$. By the same token, there cannot be a lack of unique parsing for formulas of the form $\Delta\phi$. Lastly, the presence of = cannot break unique parsing either, since it is merely an additional binary predicate. □

Definition 2.14 The complexity of formulas of an *mQuarc*-language $\mathcal{L}$ is a function $c : FORM_{\mathcal{L}} \rightarrow \mathbb{N}$, defined as follows:

- If ϕ is basic, then $c(\phi) = 0$.
- If ϕ is a reorder, then $c(\phi) = 1$.
- If ϕ is of the form $a_1...a_k\Delta^n P_i^k$ (P potentially reordered), then $c(\phi) = c(a_1...a_k P_i^k) + n$.
- If ϕ is of the form $\Delta\psi$, then $c(\phi) = c(\psi) + 1$.
- If ϕ is of the form $(\psi \otimes \chi)$ and neither led nor governed, for $\otimes \in \{\wedge, \rightarrow\}$, then $c(\phi) = max\{c(\psi), c(\chi)\} + 1$.
- If ϕ was generated immediately from a formula ψ as in 2.9(f), then $c(\phi) = c(\psi) + 1$.
- If ϕ was generated immediately from ψ as in 2.9(g), then $c(\phi) = c(\psi) + 1$.

3 The Semantics

As announced earlier, we shall employ the truth-valuational semantics introduced in [2] and developed further in [5]. There are many advantages to this approach, which are discussed elsewhere (e.g. [1] and [5]). Since this paper focusses only on the technical aspects, we shall only note that the semantics for the modal extension of Quarc is straightforward, as previously observed in [4], and that it allows for a particularly smooth completeness proof. Before demonstrating this, we first define the basic semantic machinery.

Definition 3.1 Let W be a non-empty set of indices, where arbitrary indices will be denoted by w, v, etc. Let $R \subseteq W \times W$ be a binary relation on W. Then, the ordered pair $< W, R >$ is a *frame*. We denote arbitrary frames with $\mathcal{F}$.

Definition 3.2 Let $\mathcal{L}$ be given, and let $\mathcal{F} =< W, R >$ be a frame. We define a *valuation* $\mathcal{V}$ as a function from $FORM_{\mathcal{L}} \times W$ to the set $\{T, F\}$, whose members we call *truth-values*. Let ϕ be a formula of $\mathcal{L}$ and $w \in W$. $\mathcal{V}$ is defined recursively as follows:

(i) **Basic formulas**: For every basic ϕ: $\mathcal{V}(\phi, w) = T$ or $\mathcal{V}(\phi, w) = F$, never both. Moreover, the following two constraints must be satisfied:
 (a) For all $w \in W$ and all singular arguments a: $\mathcal{V}(a = a, w) = T$.
 (b) If $\mathcal{V}(a = b, w) = T$ and $\mathcal{V}(\phi(a_1, ..., a_n), w) = T$, where $\phi(a_1, ..., a_n)$ has (at least) n-many distinct occurrences of the singular argument a, then $\mathcal{V}(\phi[b/a_1, ..., b/a_n], w) = T$.

(ii) **Reorders**: If ϕ is of the form $a_{\pi 1}...a_{\pi k} P_i^{\pi 1,...,\pi k}$ for some basic formula $a_1...a_n P_i^k$ and a non-identity permutation π on $\{1, ..., k\}$, then $\mathcal{V}(a_{\pi 1}...a_{\pi k} P_i^{\pi 1,...,\pi k}, w) = \mathcal{V}(a_1...a_n P_i^k, w)$.

(iii) **Modes of predication**: If ϕ is of the form $a_1...a_k \Delta^n P_i^k$ for some formula $a_1...a_n P_i^k$, where the a_i are unlabelled and P is potentially reordered, and a string of predicate operators $\Delta^n \equiv \Delta\Delta^{n-1}$, then $\mathcal{V}(a_1...a_k \Delta\Delta^{n-1} P_i^k, w) = \mathcal{V}(\Delta a_1...a_k \Delta^{n-1} P, w)$.

(iv) **Connectives**:
 (a) If ϕ is of the form $(\psi \wedge \chi)$, then $\mathcal{V}((\psi \wedge \chi), w) = T$ iff $\mathcal{V}(\psi, w) = \mathcal{V}(\chi, w) = T$. $\mathcal{V}((\psi \wedge \chi), w) = F$ otherwise.
 (b) If ϕ is of the form $(\psi \to \chi)$, then $\mathcal{V}((\psi \to \chi), w) = F$ iff $\mathcal{V}(\psi, w) = T$ and $\mathcal{V}(\chi, w) = F$. $\mathcal{V}((\psi \to \chi), w) = T$ otherwise.
 (c) If ϕ is of the form $\neg\psi$, then $\mathcal{V}(\neg\psi, w) = T$ iff $\mathcal{V}(\psi, w) = F$. $\mathcal{V}(\neg\psi, w) = F$ otherwise.

(v) **Modals**: If ϕ is of the form $\Box\psi$, then $\mathcal{V}(\Box\psi, w) = T$ iff for every $v \in W$ with $< w, v >\in R$, $\mathcal{V}(\psi, v) = T$. $\mathcal{V}(\Box\psi, w) = F$ otherwise.

(vi) **Anaphora**: Let ϕ be of the form $\psi[a_x]$, where ψ is led by the labelled singular argument a_x. Let $\psi[a]$ be the formula from which ϕ was immediately generated, as per 2.9(f). Then: $\mathcal{V}(\psi[a_x], w) = \mathcal{V}(\psi[a], w)$.

(vii) **Particular quantification**: Let ϕ be of the form $\psi(\exists P)$, where $\exists P$ governs ψ. Then: $\mathcal{V}(\psi(\exists P), w) = T$ iff for some singular argument a, we have both $\mathcal{V}(aP, w) = T$ and $\mathcal{V}(\psi[a/\exists P], w) = T$, where a replaces only the governing occurrence of $\exists P$. Otherwise, $\mathcal{V}(\psi(\exists P), w) = F$.

(viii) **Universal quantification**: Let ϕ be of the form $\psi(\forall P)$, where $\forall P$ governs ψ. Then: $\mathcal{V}(\psi(\forall P), w) = T$ iff for every singular argument a: if $\mathcal{V}(aP, w) = T$, then $\mathcal{V}(\psi[a/\forall P], w) = T$, where a replaces only the governing occurrence of $\forall P$. Otherwise, $\mathcal{V}(\psi(\forall P), w) = F$.

(ix) **Instantiation**: For every $w \in W$ and every unary predicate P, there is some singular argument a such that $\mathcal{V}(aP, w) = T$.

Remark 3.3 As is customary, we shall write $w \vDash \phi$ if a valuation $\mathcal{V}$ assigns T to ϕ at w, and $\mathcal{V}$ is clear by context. Similarly for formulas assigned F, where we write $w \nvDash \phi$ instead.

Remark 3.4 Clause (ix) is obviously a simplification for formal purposes, giving us a two-valued semantics. When we entertain alternative possibilities, it is not required that every unary predicate must have instances vis-à-vis these possibilities. This requirement could be dropped, yielding a three-valued version of *mQuarc* similar to the semantics in [15]. We shall return to this point towards the end.

Remark 3.5 Thanks to proposition 2.13, we can deduce that each valuation is uniquely determined by the assignment of truth-values to basic formulas.

Having our basic semantic apparatus in place, we can now move on to define the validity of sequents of formulas,[4] as well as satisfiability. However, as already noted in [8], defining validity under substitutional semantics requires us to vary the list of singular arguments of the relevant language:

Definition 3.6 Let $\mathcal{L}$ be given. Let Γ be a set of $\mathcal{L}$-formulas and $\phi \in FORM_{\mathcal{L}}$. Then, Γ *entails* ϕ iff for every frame $\mathcal{F} =< W, R >$, every index $w \in W$, every language $\mathcal{L}'$ – which is like $\mathcal{L}$, except that it may contain a different list of singular arguments *apart* from those occurring in Γ and ϕ – and every $\mathcal{L}'$-valuation $\mathcal{V}$ on $\mathcal{F}$, we have that: if $\mathcal{V}$ assigns T at w to every formula in Γ, then $\mathcal{V}$ assigns T at w to ϕ. If this is the case, we write $\Gamma \vDash \phi$. In the special case that $\Gamma = \varnothing$, we simply write $\vDash \phi$ and say that ϕ is a *validity.*

Definition 3.7 Let Γ once again be a set of $\mathcal{L}$-formulas. Then, we say that Γ is *satisfiable* iff there is a frame $\mathcal{F} =< W, R >$, an index $w \in W$, a language $\mathcal{L}'$ – having the same properties as the languages in the definition of validity – and an $\mathcal{L}'$-valuation $\mathcal{V}$ on $\mathcal{F}$ such that $\mathcal{V}$ assigns T to every formula in Γ at w, written as $w \vDash \Gamma$. Otherwise, Γ is said to be *unsatisfiable.* A single $\mathcal{L}$-formula ϕ is satisfiable iff $\{\phi\}$ is.

Remark 3.8 In the following, we shall sometimes only speak of a 'suitable language $\mathcal{L}'$', implicitly understood to be a language containing all singular arguments occurring in some relevant formulas of another given language $\mathcal{L}$, all other symbols of $\mathcal{L}$, yet (possibly) different singular arguments otherwise.

With these semantic notions in hand, we can prove the following result:

Proposition 3.9 *Let $\Delta^n a_1 ... a_m \Delta\Delta^k P$ be a formula of a given language $\mathcal{L}$, where each Δ is a predicate operator, the singular arguments a_i are unlabelled, and P is an m-ary predicate. Let $\mathcal{F} =< W, R >$ be a frame, $w \in W$ and $\mathcal{V}$ an $\mathcal{L}'$-valuation on $\mathcal{F}$, for a suitable language $\mathcal{L}'$. Then, $w \vDash \Delta^n a_1 ... a_m \Delta\Delta^k P$ iff $w \vDash \Delta^n \Delta a_1 ... a_m \Delta^k P$.*

Proof. By induction on n, keeping k fixed. Cf. appendix for details. □

In other words, we can always shift the singular arguments completely to the left or the right inside such formulas while retaining the assigned truth-value.

[4] We could also define validity on a single frame and validity at an index, but these notions play no role in the remainder of this paper. Similarly, a generalisation to classes of frames would be straightforward, but equally irrelevant for present purposes.

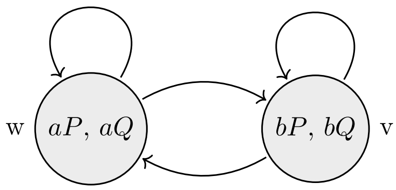
As such, the semantic distinction between *de re* and *de dicto* is non-existent for such formulas. The same does not hold for formulas containing quantifiers, as was already noted in [4]. Indeed, the Quarc-analogues of the Barcan-formula and its converse are easily proven to be invalid:

Proposition 3.10 *The Quarc-analogues of the Barcan-formula and its converse are not valid:*

(i) $\nvDash \Box \forall PQ \to \forall P \Box Q$

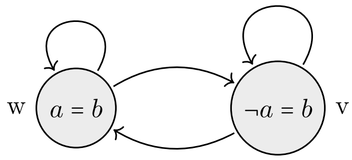
(ii) $\nvDash \forall P \Box Q \to \Box \forall PQ$

Proof. We shall only prove (i), as the other case is similar. Consider the following frame including $\mathcal{L}$-valuation, for some given language $\mathcal{L}$:



where at both indices, no other singular arguments satisfy P or Q. Thus, $w \vDash \forall PQ$ and $v \vDash \forall PQ$, hence $w \vDash \Box \forall PQ$. However, since $v \nvDash aQ$, $w \nvDash \Box aQ$ and therefore $w \nvDash a \Box Q$, given 3.2(iii). As such, $w \nvDash \forall P \Box Q$.[5] □

As a result, the truth-valuational semantics for *mQuarc* both neatly alleviates issues surrounding the usual design choices when studying quantified modal logic, such as whether to allow for expanding or decreasing domains, as well as worries about the validity of the Barcan-formulas on more general grounds.[6] Furthermore, analogous observations also establish the invalidity of $a = b \to \Box a = b$:



There is nothing in 3(i) that prevents us from changing the truth-value of $a = b$ across indices, hence we can easily falsify $a = b \to \Box a = b$.

Thus, identity is naturally treated as contingent, with the exception of *self-identity*. That is, formulas of the form $a = a$ are always rendered true as per 3.2(i). Thus, $\Box a = a$, $\Box \forall P_x = x$ and similar formulas will all be validities, as is easily verified.

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[5] Since the frame happens to be an equivalence frame, we can see that this also holds for many more restricted classes of frames. Similarly for $a = b \to \Box a = b$ below.

[6] The invalidity also obtains when adopting model-theoretic semantics (cf. [4]), and thus is a feature of Quarc, not of the truth-valuational semantics.

4 Proof Theory

In the following, we present an unlabelled Gentzen-style natural deduction system for *mQuarc*. It will be our analogue of the usual basic modal logic K. Adopting an unlabelled proof system has the advantage of allowing combinations with different semantics (e.g. [2]), apart from the standard relational ones used in the previous section. For the precise formal set-up for the proof trees, we follow [14] and [13]. We fix a language $\mathcal{L}$.

Definition 4.1 A *proof* is a rooted tree where every vertex is named by an element $\phi \in FORM_{\mathcal{L}}$, and each edge is named by one of the rules given below. The root is the *conclusion* of the proof, and the leaves are assumptions that are either *discharged* or *undischarged*, as determined by the applications of the rules below. The conclusion is said to *depend* on the undischarged assumptions. We follow [13] by explicitly allowing *empty* assumption classes.

Assuming this basic graph-theoretic set-up, we consider now the logic $N_{\Box Q=}$. The rules for the $\Box$-free fragment are imported from [15] and [5], whereas the single rule for $\Box$ is taken from [12]:

Definition 4.2 The logic $N_{\Box Q=}$ is given by the following rules:

- **Connectives**: Let ϕ and ψ be arbitrary formulas.
  - We adopt the standard introduction and elimination rules for $\wedge$ and $\rightarrow$.
  - Negation:

$$\dfrac{\begin{matrix}[\phi]^i & & [\phi]^i \\ \vdots & & \vdots \\ \psi & & \neg\psi\end{matrix}}{\neg\phi}\ \neg\mathrm{I}_i \qquad\qquad \dfrac{\begin{matrix}[\neg\phi]^i & & [\neg\phi]^i \\ \vdots & & \vdots \\ \psi & & \neg\psi\end{matrix}}{\phi}\ \neg\mathrm{E}_i$$

- **Natural Logic**: Let $a_1, ..., a_m$ be unlabelled singular arguments and P an m-ary predicate.
  - Reorders:

$$\dfrac{a_1...a_n P}{a_{\pi 1}...a_{\pi m} P^{\pi}}\ \mathrm{RI} \qquad\qquad \dfrac{a_{\pi 1}...a_{\pi m} P^{\pi}}{a_1...a_n P}\ \mathrm{RE}$$

  where π is a non-identity permutation on $\{1, ..., m\}$ and $a_1...a_m P$ is basic.
  - Modes of predication:

$$\dfrac{\Delta a_1...a_n \Delta^k P}{a_1...a_n \Delta\Delta^k P}\ \mathrm{SP} \qquad\qquad \dfrac{a_1...a_n \Delta\Delta^k P}{\Delta a_1...a_n \Delta^k P}\ \mathrm{PS}$$

  where P can be reordered and k can be 0.
  - Anaphora: Let $\phi[a_x]$ be a formula led by the labelled singular argument a_x and let $\phi[a]$ be the formula from which $\phi[a_x]$ is immediately generated, as per 2.9(f):

$$\frac{\phi[a]}{\phi[a_x]}\ \text{AI} \qquad\qquad \frac{\phi[a_x]}{\phi[a]}\ \text{AE}$$

- **Quantifiers**: Let $\phi(qP)$ be a formula governed by the quantified argument qP, and let $\phi[a/qP]$ be the formula obtained by replacing the governing occurrence with the singular argument a:
- Universal quantification:

$$\frac{\begin{matrix}[aP]^i \\ \vdots \\ \phi[a/\forall P]\end{matrix}}{\phi(\forall P)}\ \forall\text{I}_i,\ * \qquad\qquad \frac{\phi(\forall P) \qquad aP}{\phi[a/\forall P]}\ \forall\text{E}$$

where the side condition $*$ requires a to not occur in any undischarged assumptions or $\phi(\forall P)$.
- Particular quantification:

$$\frac{aP \qquad \phi[a/\exists P]}{\phi(\exists P)}\ \exists\text{I}$$

- Import:

$$\frac{\phi(qP) \qquad \begin{matrix}[aP]^i, [\phi[a/qP]]^i \\ \vdots \\ \psi\end{matrix}}{\psi}\ \text{Imp}_i,\ *$$

where the side condition $*$ requires a to not occur in any undischarged assumptions, $\phi(qP)$ or ψ.

- **Identity**: Let a and b be singular arguments, and let $\phi(a_1, ..., a_n)$ be a basic formula with (at least) n-many distinct occurrences of the singular argument a:

$$\frac{}{a = a}\ {=}\text{I} \qquad\qquad \frac{a = b \qquad \phi(a_1, ..., a_n)}{\phi[b/a_1, ..., b/a_n]}\ {=}\text{E}$$

where in =I, the conclusion does not depend on any assumptions.

- **Modality**: Let ϕ be a formula:

$$\frac{\phi}{\Box\phi}\ \Box\text{I},\ *$$

where $*$ says that $\Box\phi$ now depends on $\Box\gamma$, for any γ that was an assumption on which ϕ depended.

Remark 4.3 We explicitly allow multiple applications of $\Box$I. We keep track of the added $\Box$s by putting a $\Box$ in the superscript of an undischarged assumption γ each time $\Box$I is applied, yielding assumptions of the form $\gamma^{\Box^k}$. We add the following convention: if we discharge an assumption of the form $\gamma^{\Box^k}$, we discharge $\Box^k\gamma$.

Remark 4.4 The set of derivations for $\mathcal{L}$ could now be explicitly and recursively defined in the usual way (cf. [14]), but we omit the details. That said, we do include the above convention related to □I.

Definition 4.5 Let Γ be a set and ϕ a single formula of $\mathcal{L}$. We say that Γ *proves* ϕ, or that ϕ is a *syntactic consequence* of Γ, iff there are finitely many $\gamma_1, ..., \gamma_k \in \Gamma$ such that there is a proof of ϕ with every undischarged assumption being a γ_i. In such a case, we write $\Gamma \vdash \phi$. If $\Gamma = \varnothing$, we say that ϕ is a *theorem* and write $\vdash \phi$.

Definition 4.6 A set of $\mathcal{L}$-formulas Γ is *consistent* iff there is no $\mathcal{L}$-formula ϕ such that $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$. It is *inconsistent* otherwise.

As an immediate consequence of these rules, the following fact can be straightforwardly established, and hence will not be proven:

Fact 4.7 *$N_{\Box Q=}$ is explosive and allows for double negation-elimination.*

Lastly, just as on the semantic side, we can prove that we can also 'syntactically' move singular arguments to the left and right in salient formulas:

Proposition 4.8 *The following two rules are admissible in $N_{\Box Q=}$, where the singular arguments a_i are unlabelled and P can be reordered:*

$$\frac{\Delta^n a_1 ... a_m \Delta\Delta^k P}{\Delta^n \Delta a_1 ... a_m \Delta^k P}\, PS* \qquad\qquad \frac{\Delta^n \Delta a_1 ... a_m \Delta^k P}{\Delta^n a_1 ... a_m \Delta\Delta^k P}\, SP*$$

Proof. Again by induction on n, keeping k fixed. Cf. appendix for details. □

5 Soundness

In order to prove the strong soundness of $N_{\Box Q=}$, we need several lemmas centred around the possibility of adding or replacing singular arguments within given valuations, without changing certain truth-values. First, two straightforward results about entailment/provability and one about valuations:

Lemma 5.1 (Containment) *Let Γ and Δ be sets of formulas of some language $\mathcal{L}$ such that $\Delta \subseteq \Gamma$. Furthermore, assume $\Delta \vDash \phi$ for some $\mathcal{L}$-formula ϕ. Then: $\Gamma \vDash \phi$.*

Proof. Follows directly from the definition of entailment. □

Lemma 5.2 (Negation) *Let $\mathcal{L}$ be given. Then, for all sets of $\mathcal{L}$-formulas Γ and all such formulas ϕ, the following hold:*

(a) $\Gamma \vdash \phi$ *iff* $\Gamma \cup \{\neg\phi\}$ *is inconsistent.*
(b) $\Gamma \vdash \neg\phi$ *iff* $\Gamma \cup \{\phi\}$ *is inconsistent.*
(c) $\Gamma \vDash \phi$ *iff* $\Gamma \cup \{\neg\phi\}$ *is unsatisfiable.*
(d) $\Gamma \vDash \neg\phi$ *iff* $\Gamma \cup \{\phi\}$ *is unsatisfiable.*

Proof. Follows directly from the relevant definitions and the rules for $\neg$. □

Lemma 5.3 (Replacement) *Let $\mathcal{L}$ be a language and $\mathcal{F} =< W, R >$ a frame with an $\mathcal{L}$-valuation $\mathcal{V}$. Let δ be a singular argument of $\mathcal{L}$, and let ϵ be a singular* *argument outside of $\mathcal{L}$. We define $\mathcal{L}^r$ to be the language $\mathcal{L}$ with δ replaced by ϵ. This induces a translation between $FORM_{\mathcal{L}}$ and $FORM_{\mathcal{L}^r}$, where each formula not containing δ is sent to itself, and each formula containing δ is sent to its counterpart formula where all occurrences of δ have been replaced by ϵ. By abuse of notation, let us denote these counterparts by $\phi[\epsilon/\delta]$, for some formula $\phi(\delta)$ originally containing δ. We now define a new valuation $\mathcal{V}^r$ on $\mathcal{F}$ for $\mathcal{L}^r$ as follows: $\mathcal{V}^r(\phi[\epsilon/\delta], w) := \mathcal{V}(\phi(\delta), w)$ for all basic formulas $\phi(\delta)$ of $\mathcal{L}^r$ that contain δ, and $\mathcal{V}^r(\phi, w) := \mathcal{V}(\phi, w)$ otherwise, for all $w \in W$. Then:*

a) For all ϕ containing ϵ: $\mathcal{V}^r(\phi[\epsilon/\delta], w) = \mathcal{V}(\phi(\delta), w)$, for all $w \in W$.
b) For all ϕ not containing ϵ: $\mathcal{V}^r(\phi, w) = \mathcal{V}(\phi, w)$, for all $w \in W$.

Proof. While a detailed induction proof can be given, it is clear that the two valuations and sets of formulas are mere typographic variants of each other. Thus, the proof shall be omitted. □

Lemma 5.3 allows us to replace a singular argument in a valuation with a fresh one, without changing the truth-values *modulo* an induced translation. In particular, however, the truth-values for δ-free formulas do not change. A similar result allows us to extend a language of a valuation with new singular arguments while retaining the truth-values for the formulas of the original language:

Lemma 5.4 (Extension) *Let $\mathcal{L}$ be a language and $\mathcal{F} = < W, R >$ a frame with an $\mathcal{L}$-valuation $\mathcal{V}$. Let $\mathcal{L}^+$ be the language obtained from $\mathcal{L}$ by adding countably many new singular arguments. We now define a valuation $\mathcal{V}^+$ for $\mathcal{L}^+$ on $\mathcal{F}$ as follows: If ϕ is a basic formula of $FORM_{\mathcal{L}}$, then $\mathcal{V}^+(\phi, w) := \mathcal{V}(\phi, w)$, for all $w \in W$. If ϕ is a basic formula in $FORM_{\mathcal{L}^+} \setminus FORM_{\mathcal{L}}$ of the form $a = a$, then $\mathcal{V}^+(a = a, w) := T$, for all $w \in W$. For all other new basic formulas $\phi \in FORM_{\mathcal{L}^+} \setminus FORM_{\mathcal{L}}$, we define $\mathcal{V}^+(\phi, w) := F$ for all $w \in W$. Then: for all $\phi \in FORM_{\mathcal{L}}$: $\mathcal{V}^+(\phi, w) = \mathcal{V}(\phi, w)$, for all $w \in W$.*

Proof. By induction on $c(\phi)$. Cf. appendix for details. □

This result has an important corollary:

Corollary 5.5 *Let Γ be a set of $\mathcal{L}$-formulas that entails a basic $\mathcal{L}$-formula of the form aP. Then, for all languages $\mathcal{L}'$, if an $\mathcal{L}'$-valuation $\mathcal{V}$ on some frame $\mathcal{F}$ satisfies Γ at some index, then that $\mathcal{L}'$ contains a. If Γ entails a basic $\mathcal{L}$-formula of the form $a = b$, where $a \not\equiv b$, then any such $\mathcal{L}'$ contains both a and b.*

Proof. Cf. appendix. □

As a last ingredient to prove soundness, we shall also show that we can always add a fresh singular argument to a language of a given valuation and let it 'mimic' an already present singular argument. This creates a new valuation that is once again conservative over the formulas of the original language: [7]

[7] I am indebted to Hanoch Ben-Yami for this lemma. A version of it can also be found in his [1].

Lemma 5.6 (Swapping) *Let $\mathcal{L}$ be a language, $\mathcal{F} = < W, R >$ a frame and $\mathcal{V}$ an $\mathcal{L}$-valuation on $\mathcal{F}$. Let δ be a singular argument new to $\mathcal{L}$ and pick some singular argument a already in $\mathcal{L}$. We define $\mathcal{L}^s$ to be the language obtained from $\mathcal{L}$ by adding δ to it. Define a new valuation $\mathcal{V}^s$ for $\mathcal{L}^s$ as follows: for all $w \in W$, $\mathcal{V}^s(\phi(\delta), w) := \mathcal{V}(\phi[a/\delta], w)$ for all basic $\phi \in FORM_{\mathcal{L}^s}$ containing δ and where in $\phi[a/\delta]$ a replaces all occurrences of δ. For all other basic formulas ϕ, we define $\mathcal{V}^s(\phi, w) := \mathcal{V}(\phi, w)$ for all $w \in W$. Then, for all $w \in W$:*

(a) *For all formulas ϕ containing δ: $\mathcal{V}^s(\phi(\delta), w) = \mathcal{V}^s(\phi[a/\delta], w)$.*
(b) *For all formulas ϕ not containing δ: $\mathcal{V}^s(\phi, w) = \mathcal{V}(\phi, w)$.*

Proof. By induction on $c(\phi)$. Cf. appendix for details. □

With these tools in hand, we can finally prove soundness:

Theorem 5.7 (Soundness) *Let $\mathcal{L}$ be a language. Then, for all sets of $\mathcal{L}$-formulas Γ and all $\mathcal{L}$-formulas ϕ: $\Gamma \vdash \phi$ entails $\Gamma \vDash \phi$.*

Proof. Cf. appendix. □

This proof also refines the original one in [2], which contained an error in the step concerning Imp (there called Ins), and is therefore the first fully polished soundness result for the truth-valuational semantics of Quarc.

6 Completeness

While the soundness proof was more involved, completeness follows straightforwardly by adapting the proof found in [15]. We show that if a set of formulas Γ does not prove ϕ, then Γ also does not entail ϕ, via the construction of a canonical *valuation.* While most of the proof is adopted from [15], we list the most important steps and definitions again, for completeness' sake. [8] We fix a language $\mathcal{L}$, and let Γ be any set of such formulas:

Definition 6.1 Γ is *maximally consistent* iff it is consistent (cf. definition 4.6) and for every formula ϕ not in Γ, $\Gamma \cup \{\phi\}$ is inconsistent.

Definition 6.2 Γ is *instance-complete* iff for every unary predicate P of $\mathcal{L}$ there is a singular argument a such that $aP \in \Gamma$.

Definition 6.3 Γ is *witness-complete* iff for every formula $\phi(\exists P)$ governed by some particularly quantified argument $\exists P$, there is a singular argument a such that $aP, \phi[a/\exists P] \in \Gamma$.

Let Γ^* be some maximally consistent set of $\mathcal{L}$-formulas. Let ϕ and ψ be some such formulas. The following facts about arbitrary Γ^*, ϕ and ψ are all straightfoward to establish:

Fact 6.4 *For all ϕ: $\Gamma^* \vdash \phi$ entails $\phi \in \Gamma^*$.*

Fact 6.5 *For all ϕ: either $\phi \in \Gamma^*$ or $\neg\phi \in \Gamma^*$.*

Fact 6.6 *For all singular arguments a: $a = a \in \Gamma^*$.*

---
[8] Pun intended.

Fact 6.7 *Let $\phi(a_1, ..., a_n)$ be a basic formula with (at least) n-many distinct occurrences of a. Then: $a = b, \phi(a_1, ..., a_n) \in \Gamma^*$ entails $\phi[b/a_1, ..., b/a_n] \in \Gamma^*$.*

Fact 6.8 *For all ϕ and ψ: $\phi \wedge \psi \in \Gamma^*$ iff $\phi, \psi \in \Gamma^*$.*

Fact 6.9 *For all ϕ and ψ: $\phi \rightarrow \psi \notin \Gamma^*$ iff $\phi \in \Gamma^*$ and $\psi \notin \Gamma^*$.*

Fact 6.10 *Let $a_1...a_m P$ be a basic formula and π a non-identity permutation on $\{1, ..., m\}$. Then: $a_{\pi 1}...a_{\pi m} P^{\pi} \in \Gamma^*$ iff $a_1...a_m P \in \Gamma^*$.*

Fact 6.11 *Let $a_1...a_m \Delta\Delta^k P$ be a mode of predication. Then: $a_1...a_m \Delta\Delta^k P \in \Gamma^*$ iff $\Delta a_1...a_m \Delta^k P \in \Gamma^*$.*

Fact 6.12 *Let $\phi[a_x]$ be led by the labelled singular argument a_x and let it be immediately generated from $\phi[a]$ as in 2.9(f). Then: $\phi[a_x] \in \Gamma^*$ iff $\phi[a] \in \Gamma^*$*

Fact 6.13 *Let Γ^* also be witness-complete, and let $\phi(\exists P)$ be governed by $\exists P$. Then: $\phi(\exists P) \in \Gamma^*$ iff for some singular argument a, $aP, \phi[a/\exists P] \in \Gamma^*$.*

The proof of the following fact is more involved, but can be found in [15]:

Fact 6.14 *Let Γ^* also be witness-complete. Let $\phi(\forall P)$ be governed by $\forall P$. Then: $\phi(\forall P) \in \Gamma^*$ iff for every singular argument a: $aP \in \Gamma^*$ entails $\phi[a/\forall P] \in \Gamma^*$.*

With these tools in hand, we can prove the Quarc-analogue of Lindenbaum's lemma:

Lemma 6.15 (Lindenbaum) *Let Γ be a consistent set of $\mathcal{L}$-formulas. Let $\mathcal{L}^*$ be the language obtained from $\mathcal{L}$ by adding denumerably many new singular arguments to $\mathcal{L}$. Then, there is a maximally consistent, witness- and instance-complete set of $\mathcal{L}^*$-formulas Γ^* such that $\Gamma \subseteq \Gamma^*$.*

Proof. Since we do not need to change anything in the construction, we can directly adopt the proof found in [15]. □

In analogy to the usual strategy in modal logic, we shall now use the set of all such maximally consistent, witness- and instance-complete sets to construct a frame:

Definition 6.16 The *canonical frame* $\mathcal{F}^C =< W^C, R^C >$ has as its set of indices all maximally consistent, witness- and instance-complete sets of $\mathcal{L}^*$-formulas over a given base language $\mathcal{L}$. Furthermore, $\Gamma^* R^C \Delta^*$ is defined to hold iff for every $\mathcal{L}^*$-formula ϕ: $\Box\phi \in \Gamma^*$ entails $\phi \in \Delta^*$.

We can now proceed to define the *canonical valuation* $\mathcal{V}^C$:

Definition 6.17 For all basic formulas of $\mathcal{L}^*$ and all $\Gamma^* \in W^C$, we define $\mathcal{V}^C$ as follows: $\Gamma^* \vDash \phi$ iff $\phi \in \Gamma^*$.

This, in turn, allows us to finally prove a salient version of the *truth-lemma*:

Lemma 6.18 (Truth) *For all formulas $\phi \in FORM_{\mathcal{L}^*}$ and all $\Gamma^* \in W^C$: $\Gamma^* \vDash \phi$ iff $\phi \in \Gamma^*$.*

Proof. By induction on $c(\phi)$. Cf. appendix for some details. □

These preceding results now entail the strong completeness of $N_{\Box Q=}$:

Theorem 6.19 *For all sets of $\mathcal{L}$-formulas Γ and all such formulas ϕ: $\Gamma \vDash \phi$ entails $\Gamma \vdash \phi$.*

Proof. The proof is the standard argument by contraposition, using the negation-lemma (5.2), Lindenbaum's lemma (6.15) and the truth-lemma (6.18). □

Thus, we have shown that $N_{\Box Q=}$ is both strongly sound *and* complete with respect to all relational frames. In particular, we note the straightforwardness of the proof, especially with respect to identity, which usually requires more elaborate constructions. While we lack the space to demonstrate these assertions in detail, we claim it would now also be possible to prove analogous results for a whole family of systems, differing only with respect to the rules for $\Box$. It is possible to prove such results for analogues of D, T, S4 and S5, by adopting the other rules discussed in [12] and devising further ones. Moreover, one can also show that if we drop clause (ix) in definition 3.2 and make the step towards a strong-Kleene three-valued version of Quarc (cf. [9] and [15]), these systems remain strongly sound and complete with respect to such semantics, too, assuming strict-to-tolerant validity (cf. [7]). Such versions would be of special interest, since the modelled presupposition-failure with respect to quantification would be especially relevant in the modal context, as noted in remark 3.4.

7 Conclusion

We have presented the first strongly sound and complete natural deduction system for *mQuarc*, the modal extension of Quarc. The system $N_{\Box Q=}$ was a counterpart to the usual modal logic K, given its completeness with respect to all relational frames. We have also seen that in *mQuarc*, there is no semantic distinction between *de re* and *de dicto* for more basic, quantifier-free formulas. However, the Barcan-formula and its converse are invalid, and identity is treated as contingent by default. This alleviates a whole range of issues and design choices otherwise present in quantified modal logic. This combines with the already beneficial distinctions present in Quarc, such as keeping existence and particular quantification apart, further circumventing philosophically difficult terrain. Thus, we submit that *mQuarc* is a particularly attractive and elegant approach to quantified modal logic. Unfortunately, we must leave its further exploration along the lines suggested above to another occasion.

Appendix

A Proof of Proposition 3.9

Proof. We prove the claim by induction on n, keeping k fixed. We first observe that the base case, $n = 1$, is just 3.2(iii). Thus, we assume for the induction hypothesis that the claim holds for strings of Δ of length $\leq n$.
Consider now a formula of the form $\Delta\Delta^n a_1...a_m\Delta^k P$, and assume for some frame $\mathcal{F} =< W, R >$ with an index $w \in W$ and an $\mathcal{L}'$-valuation $\mathcal{V}$ that $w \vDash \Delta\Delta^n a_1...a_m\Delta^k P$. We must distinguish two cases: either the leftmost Δ is $\neg$ or $\Box$.

- $\Delta \equiv \neg$: $w \vDash \neg\Delta^n a_1...a_m\Delta^k P$ iff $w \nvDash \Delta^n a_1...a_m\Delta^k P$, by 3.2(iv). By the induction hypothesis, this holds iff $w \nvDash \Delta^{n-1} a_1...a_m\Delta\Delta^k P$. This, by 3.2(iv), is the case iff $w \vDash \neg\Delta^{n-1} a_1...a_m\Delta\Delta^k P$.
- $\Delta \equiv \Box$: $w \vDash \Box\Delta^n a_1...a_m\Delta^k P$ iff for every $v \in W$ such that wRv, $v \vDash \Delta^n a_1...a_m\Delta^k P$, by 3.2(v). By the induction hypothesis, this holds iff $v \vDash \Delta^{n-1} a_1...a_m\Delta\Delta^k P$. This, by 3.2(v), is the case iff $w \vDash \Box\Delta^{n-1} a_1...a_m\Delta\Delta^k P$.

□

B Proof of Proposition 4.8

Proof. We prove the claim by induction n, keeping k fixed. Just as in the semantic case, the base case of $n = 1$ will just be an instance of either PS or SP. Thus, assume for the induction hypothesis that the claim holds for strings of Δ of length $\leq n$. We only show the admissibility of PS*; the proof for SP* proceeds analogously. We immediately distinguish two cases for $\Delta^n a_1...a_m\Delta\Delta^k P$: either the leftmost Δ is $\neg$ or $\Box$.

- $\Delta \equiv \neg$:

$$\dfrac{\dfrac{[\Delta^{n-1}\Delta a_1...a_m\Delta^k P]^1}{\Delta^{n-1} a_1...a_m\Delta\Delta^k P}\text{IH} \qquad \neg\Delta^{n-1} a_1...a_m\Delta\Delta^k P}{\neg\Delta^{n-1}\Delta a_1...a_m\Delta^k P}\neg\text{I}_1$$

Here, the induction hypothesis guarantees a proof from $\Delta^{n-1}\Delta a_1...a_m\Delta^k P$ to $\Delta^{n-1} a_1...a_m\Delta\Delta^k P$.

- $\Delta \equiv \Box$
We use the induction hypothesis directly to assume the existence of a derivation from $\Delta^{n-1} a_1...a_m\Delta\Delta^k P$ to $\Delta^{n-1}\Delta a_1...a_m\Delta^k P$ and then construct the following proof:

$$\dfrac{\dfrac{\Delta^{n-1} a_1...a_m\Delta\Delta^k P^{\Box}}{\Delta^{n-1}\Delta a_1...a_m\Delta^k P}\text{IH}}{\Box\Delta^{n-1}\Delta a_1...a_m\Delta^k P}\Box\text{I}$$

which is a derivation from $\Box\Delta^{n-1} a_1...a_m\Delta^k P$ to $\Box\Delta^n a_1...a_m\Delta^k P$, as desired.

□

C Proofs of Lemmas in Section 5

C.1 Proof of Lemma 5.4

Proof. We prove the claim by induction on $c(\phi)$. While the base case holds by definition, we must still make sure our newly defined valuation $\mathcal{V}^+$ satisfies the constraints (a) and (b) found in 3.2(i).

- It satisfies (a) by construction directly, since we assign T to each new basic formula of the form $a = a$ for all $w \in W$. Together with the fact that $\mathcal{V}$ was already a proper valuation, we deduce that (a) is satisfied.
- To see (b), assume that for some singular arguments a and b and for some basic formula $\phi(a_1, ..., a_n)$ of $\mathcal{L}^+$, which contains (at least) n-many distinct occurrences of a, that $\mathcal{V}^+(a = b, w) = \mathcal{V}^+(\phi(a_1, ..., a_n), w) = T$ for some given $w \in W$. If $a \equiv b$, there is nothing to prove, since then $\phi(a_1, ..., a_n) \equiv \phi[b/a_1, ..., b/a_n]$. Thus, assume $a \not\equiv b$. In that case, however, neither a nor b can be a newly added singular argument, since $a = b$ is true. Similarly, $\phi(a_1, ..., a_n)$ cannot contain any new singular arguments either. For otherwise, it could only be true by being of the form $a = a$, which would mean that a would have to be fresh, which we already ruled out. Thus, all formulas under consideration belong to $\mathcal{L}$, and we conclude, by construction and the fact that $\mathcal{V}$ satisfies (b), that $\mathcal{V}^+(\phi[b/a_1, ..., b/a_n], w) = T$.

For the inductive steps, we show only one (interesting) case, which is for formulas $\phi(\forall P) \in FORM_{\mathcal{L}}$ governed by some universally quantified argument $\forall P$: Assume that $\mathcal{V}^+(\phi(\forall P), w) = T$. This holds iff for all singular arguments a in $\mathcal{L}^+$, if $\mathcal{V}^+(aP, w) = T$, then $\mathcal{V}^+(\phi[a/\forall P], w) = T$. Observe that this conditional immediately holds for all singular arguments in $\mathcal{L}$, too. However, by construction, there are no fresh singular arguments c such that $\mathcal{V}^+(cP, w) = T$. The conditional thus holds for all singular arguments in $\mathcal{L}^+$ iff it holds for all singular arguments in $\mathcal{L}$. Applying the induction hypothesis to the formulas of the form $\phi[a/\forall P]$, this in turn is the case iff $\mathcal{V}(\phi(\forall P), w) = T$. □

C.2 Proof of Corollary 5.5

Proof. We prove the case where $\Gamma \vDash aP$, the other one is analogous. Assume the converse, i.e. there is such a language $\mathcal{L}'$ that satisfies Γ at some index w of some frame $\mathcal{F}$ for some $\mathcal{L}'$-valuation $\mathcal{V}$. Then, since a is not in $\mathcal{L}'$, we use the extension-lemma to generate $\mathcal{L}'^+$ by adding a and $\mathcal{V}^+$ as per the construction above. Then, by said construction, we still have $\mathcal{V}^+(\gamma, w) = T$ for all $\gamma \in \Gamma$, yet also $\mathcal{V}^+(aP, w) = F$. Thus, $\Gamma \nvDash aP$. □

C.3 Proof of Lemma 5.6

Proof. The proof is done by induction on $c(\phi)$. The base case holds by construction. We omit the proof that the valuation satisfies the constraints of 3.2(i), as the proof is straightforward, if a bit tedious, given the many case distinctions that arise in relation to the potential occurrences of δ.
The inductive steps are trivial for almost all cases as far as sub-claim (b) is concerned. Establishing sub-claim (a) is mostly a matter of carefully account-

ing for the substitutions of δ-containing formulas. We only demonstrate two cases, each merely a part of establishing (a) or (b), respectively:

- Assume $\phi(\delta)$ is of the form $\phi(\exists P)(\delta)$, which is governed by $\exists P$. Assume $\mathcal{V}^s(\phi(\exists P)[a/\delta], w) = T$. Then, there is a singular argument c such that $\mathcal{V}^s(cP, w) = T$ and $\mathcal{V}^s((\phi(\exists P)[a/\delta])[c/\exists P], w) = T$. We must distinguish two cases. If $c \equiv \delta$, then $\mathcal{V}^s(\delta P, w) = T$ and $\mathcal{V}^s((\phi(\exists P)[a/\delta])[\delta/\exists P], w) = T$. Thus, in particular, by construction, $\mathcal{V}^s(aP, w) = T$. In this case, $(\phi(\exists P)[a/\delta])[\delta/\exists P]$ is a formula that contains exactly one occurrence of δ. By the induction hypothesis, this formula has the same truth-value as the formula $\phi(a)[a/\delta]$, which is the formula with the single occurrence of δ also substituted with a. Applying the induction hypothesis again, this formula has the same truth value as $\phi(a)(\delta)$, i.e. the formula where δ is introduced back where it was replaced by a. Thus, a is a verifying instance, and we conclude $\mathcal{V}^s(\phi(\exists P)(\delta), w) = T$. If $c \not\equiv \delta$, we can use the induction hypothesis directly to reason from $\mathcal{V}^s(\phi[c/\exists P][a/\delta], w) = T$ to $\mathcal{V}^s(\phi[c/\exists P](\delta), w) = T$. For in this case, $(\phi(\exists P)[a/\delta])[c/\exists P]$ is the same as $\phi[c/\exists P][a/\delta])$. Thus, we can conclude that $\mathcal{V}^s(\phi(\exists P)(\delta), w) = T$, since $\mathcal{V}^s(cP, w) = T$. Therefore, either way, $\mathcal{V}^s(\phi(\exists P)(\delta), w) = T$.
- We assume that ϕ is of the form $\phi(\forall P)$, where $\forall P$ governs ϕ, and does not contain δ. Assume that $\mathcal{V}(\phi(\forall P), w) = T$, i.e. for all singular arguments c in $\mathcal{L}$, $\mathcal{V}(cP, w) = T$ entails $\mathcal{V}(\phi[c/\forall P], w) = T$. Now, assume $\mathcal{V}^s(\phi(\forall P), w) = F$. Then, there is a singular argument b in $\mathcal{L}^s$ such that $\mathcal{V}^s(bP, w) = T$ and $\mathcal{V}^s(\phi[b/\forall P], w) = F$. If that b is not δ, then it must have been in $\mathcal{L}$. This would immediately contradict the assumption. Thus, assume b is δ. Then, we have $\mathcal{V}^s(\delta P, w) = T$ and $\mathcal{V}^s(\phi[\delta/\forall P], w) = F$. But $\mathcal{V}^s(\delta P, w) = T$ iff $\mathcal{V}^s(aP, w) = T$ (by construction), and this is the case iff $\mathcal{V}(aP, w) = T$, also by construction. Since $\mathcal{V}(\phi(\forall P), w) = T$, must thus have $\mathcal{V}(\phi[a/\forall P], w) = T$ (δ does not appear in ϕ, after all). But by the induction hypothesis, this is the case iff $\mathcal{V}^s(\phi[a/\forall P], w) = T$. Thus, again by the induction hypothesis, this is the case iff $\mathcal{V}^s(\phi[\delta/\forall P], w) = T$. So δ cannot be a counterinstance either. We conclude: $\mathcal{V}^s(\phi(\forall P), w) = T$.

□

D Proof of Theorem 5.7

Proof. We first prove the weaker claim that soundness holds for all *finite* sets of formulas Γ. The reason for this will become apparent further below. After we have done this, we shall cover the case where Γ is infinite.

We follow [14] and prove the stronger claim that *any* finite set Γ that contains all undischarged assumptions of a proof of ϕ also entails ϕ. We do so by induction on the structure of the proof. We shall only showcase three of the inductive steps, under the general induction hypothesis that any finite set containing all undicharged assumptions of any of the subproofs is such that it also entails the conclusion of said subproof.

- $\exists$I: Let Γ'' contain all undischarged assumptions of $\dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ aP & \phi[a/\exists P]\end{matrix}}{\phi(\exists P)}\ \exists\text{I}$.

  Let Γ and Γ' contain all and only the undischarged assumptions of $\mathcal{D}$ and $\mathcal{D}'$, respectively. Thus, $\Gamma \cup \Gamma' \subseteq \Gamma''$. By the induction hypothesis, $\Gamma \vDash aP$ and $\Gamma' \vDash \phi[a/\exists P]$, hence, by the containment-lemma (5.1), $\Gamma'' \vDash aP$ and $\Gamma'' \vDash \phi[a/\exists P]$. Let a suitable language $\mathcal{L}'$ for Γ'' and $\phi(\exists P)$ be given, and let $\mathcal{F}$ be a frame with an index w and an $\mathcal{L}'$-valuation $\mathcal{V}$ such that $w \vDash \Gamma''$. By corollary 5.5, a is in $\mathcal{L}'$. Thus, $w \vDash aP$ and $w \vDash \phi[a/\exists P]$. By 3.2(vii), $w \vDash \phi(\exists P)$. Generalizing yields $\Gamma'' \vDash \phi(\exists P)$.

- Let Γ' contain all undischarged assumptions of $\dfrac{\begin{matrix}[aP]^i \\ \mathcal{D} \\ \phi[a/\forall P]\end{matrix}}{\phi(\forall P)}\ \forall\text{I}_i$, and choose Γ to contain all and only the undischarged assumptions of $\mathcal{D}$. Then, $\Gamma \subseteq \Gamma' \cup \{aP\}$, and by induction hypothesis, $\Gamma \vDash \phi[a/\forall P]$. Observe that a does not occur in any undischarged assumptions of the proof nor in $\phi(\forall P)$. Assume now that $\Gamma' \nvDash \phi(\forall P)$. Then, there is a suitable language $\mathcal{L}'$ with a singular argument c, a frame with an index w and an $\mathcal{L}'$-valuation $\mathcal{V}$ such that $w \vDash \Gamma'$ and $w \vDash cP$, yet $w \nvDash \phi[c/\forall P]$. Let δ be a singular argument of $\mathcal{L}$ that occurs neither in any formula of the proof nor in Γ'. Such a singular argument is always available, since we have denumerably many of them, whereas only finitely many could occur in the proof and Γ'.[9]

  We now have two cases. Either δ is already in $\mathcal{L}'$ or it is not. Consider the latter case first. We apply the swapping-lemma (5.6) and create a new valuation $\mathcal{V}^s$ for $\mathcal{L}'^s$, which is $\mathcal{L}'$ plus δ, in which δ mimics c. We still have $w \vDash \Gamma'$, but now also $w \vDash \delta P$ and $w \nvDash \phi[\delta/\forall P]$ under $\mathcal{V}^s$. Next, we replace every occurrence of a in the initial proof with δ. This does not change Γ', since all undischarged assumptions were a-free to begin with. Moreover, by the choice of δ, the resulting proof is still a correct derivation, and the induction hypothesis still applies. We thus choose a new Γ that contains all undischarged assumptions of the new subproof, which, via the induction hypothesis and the containment-lemma (5.1), gives us $\Gamma' \cup \{\delta P\} \vDash \phi[\delta/\forall P]$. Now, $\mathcal{L}'^s$ contains all relevant singular arguments, and $w \vDash \Gamma'$ and $w \vDash \delta P$ under $\mathcal{V}^s$. Thus, we must also have $w \vDash \phi[\delta/\forall P]$. This yields a contradiction, hence we conclude that $\Gamma' \vDash \phi(\forall P)$ after all.

  If δ would already occur in $\mathcal{L}'$, we choose an additional singular argument ϵ which is neither in $\mathcal{L}$ nor in $\mathcal{L}'$. We then replace δ with ϵ first, via an application of the replacement-lemma (5.3). We then proceed as before, since no formula in the proof or Γ' would be affected, given our choice of δ.

[9] The reason we do this is because c might not belong to $\mathcal{L}$, thus we need to use some δ of $\mathcal{L}$. Furthermore, the fact that Γ' is finite is crucial here, for otherwise all singular arguments of $\mathcal{L}$ could occur in Γ'.

- $\Box$I: Let Γ' contain all undischarged assumptions of $\dfrac{\begin{matrix}\mathcal{D}\\ \phi\end{matrix}}{\Box\phi}\,\Box\text{I}$. Choose Γ to contain all and only the undischarged assumptions of $\mathcal{D}$. Thus, $\Box\Gamma \subseteq \Gamma'$. By standard modal reasoning, we can establish the following result: for any set of formulas Γ and formulas ϕ, if $\Gamma \vDash \phi$, then $\Box\Gamma \vDash \Box\phi$. Now, by induction hypothesis, $\Gamma \vDash \phi$, hence $\Box\Gamma \vDash \Box\phi$. Thus, by the containment-lemma (5.1), $\Gamma' \vDash \Box\phi$.

Other cases, such as =E and Imp, work analogously to the cases of $\exists$I and $\forall$I, respectively. All other cases are straightforward with the lemmas introduced in section 5.

Now that we have established soundness for finite sets of $\mathcal{L}$-formulas, we can deduce *strong* soundness as follows. Assume for an infinite set of $\mathcal{L}$-formulas Γ that it proves some $\mathcal{L}$-formula ϕ. Then, by definition 4.5, there is a finite set $\Gamma_0 := \{\gamma_1, ..., \gamma_n\}$, a subset of Γ, such that $\Gamma_0 \vdash \phi$. By the now established weak soundness, $\Gamma_0 \vDash \phi$. Thus, since $\Gamma_0 \subset \Gamma$, by the containment-lemma (5.1), $\Gamma \vDash \phi$. □

E Proof of Lemma 6.18

Proof. The claim is proven by induction on $c(\phi)$. Since most of the steps are identical to the proof found in [15], we only cover those specific to *mQuarc*.
Thus, while the base case holds by construction, we must still ensure we satisfy both constraints found in 3.2(i). However, by fact 6.6, $a = a \in \Gamma^*$ for every singular argument a in $\mathcal{L}^*$. Thus, $\Gamma^* \vDash a = a$, for every such a and Γ^*. Also, if $\Gamma^* \vDash a = b$ and $\Gamma^* \vDash \phi(a_1, ..., a_n)$, then $a = b, \phi(a_1, ..., a_n) \in \Gamma^*$. By fact 6.7, $\phi[b/a_1, ..., b/a_n] \in \Gamma^*$, hence $\Gamma^* \vDash \phi[b/a_1, ..., b/a_n]$, by construction.
For the inductive steps, we only cover modes of predication and formulas of the form $\Box\psi$. However, observe that the case of modes of predication, via fact 6.11, reduces to the cases of $\neg\psi$ and $\Box\psi$. Thus, we only discuss the case where $\phi \equiv \Box\psi$. The argument can be adopted directly from the usual completeness proofs of K for the propositional case (e.g. [6]):

$\Rightarrow$ Assume $\Box\psi \notin \Gamma^*$. Define $\Delta := \{\phi : \Box\phi \in \Gamma^*\} \cup \{\neg\psi\}$. This set is consistent: if it were not, then there would be a χ such that $\Delta \vdash \chi$ and $\Delta \vdash \neg\chi$. Thus, by the negation-lemma, $\Delta \setminus \{\neg\psi\} \vdash \psi$. Thus, there are $\gamma_1, ..., \gamma_n \in \{\phi : \Box\phi \in \Gamma^*\}$ such that $\gamma_1, ..., \gamma_n \vdash \psi$. By $\Box$I, $\Box\gamma_1, ..., \Box\gamma_n \vdash \Box\psi$. Since $\Box\gamma_i \in \Gamma^*$, we would have $\Box\psi \in \Gamma^*$. This cannot be, so Δ is consistent. Thus, by Lindenbaum's lemma, there is a $\Delta^* \supseteq \Delta$ that is maximally consistent, instance- and witness-complete. By construction: $\Gamma^* R^C \Delta^*$ and $\neg\psi \in \Delta^*$, thus $\psi \notin \Delta^*$, by fact 6.5. By the induction hypothesis, $\Delta^* \nvDash \psi$, thus $\Gamma^* \nvDash \Box\psi$.

$\Leftarrow$ Assume $\Box\psi \in \Gamma^*$ and let Δ^* be such that $\Gamma^* R^C \Delta^*$. It follows that $\psi \in \Delta^*$, hence by induction hypothesis: $\Delta^* \vDash \psi$. Generalizing over Δ^* yields $\Gamma^* \vDash \Box\psi$. □

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