Fundamental and emergent geometry in Newtonian physics

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Abstract

Using as a starting point recent and apparently incompatible conclusions by Simon Saunders (*Philosophy of Science* 80 (2013) pp.22-48) and Eleanor Knox (*British Journal for the Philosophy of Science* 65 (2014) pp.863-880), I revisit the question of the correct spacetime setting for Newtonian physics. I argue that understood correctly, these two theories make the same claims both about the background geometry required to define the theory, and about the inertial structure of the theory. In doing so I illustrate and explore in detail the view — espoused by Knox, and also by Harvey Brown (*Physical Relativity*, OUP 2005) — that inertial structure is defined by the dynamics governing subsystems of a larger system. This clarifies some interesting features of Newtonian physics, notably (i) the distinction between using the theory to model subsystems of a larger whole and using it to model complete Universes, and (ii) the scale-relativity of spacetime structure.

1 Introduction

Long after it was superseded as a fundamental theory, the spacetime setting for Newtonian physics remains contested. To Newton himself, it was an absolute space, and an absolute time, and so an absolute standard of rest and of motion. Later pre-relativistic developments (see Barbour 2001, chapter 12) gave a central role to the concept of *inertial frame*, thus abandoning an absolute standard of rest but maintaining a distinction between accelerated and nonaccelerated motion. The theory of relativity led philosophers, for a while, to regard spacetime background as a straightforward philosophical error (cf Earman 1989, pp.6-7); in modern times it has been rehabilitated (Anderson (1964, 1967, 1971), Stein (1967), Trautman (1966), Earman (1970, 1974), Earman and Friedman (1973)), and the theory formulated in a 'Galilean' spacetime with absolute affine structure but no standard of absolute rest (a spacetime, that is, which embeds the notion of 'inertial frame', and the relations between different inertial frames, as part of its structure). More recently still (Malament 1995, Norton 1995), considerations of cosmology have led to linear acceleration —

but not rotation — also to be regarded as relative, and have brought to the fore Cartan and Friedrich's formulation of Newtonian gravity ('Newton-Cartan gravity') as a theory of dynamical geometry (Cartan 1923; Friedrichs 1928; see Malament (2012) for a modern presentation), in analogy to the general theory of relativity.

My starting point in this paper is a pair of papers by Simon Saunders (2013) and Eleanor Knox (2014). Both argue for a spacetime setting different from Galilean spacetime in which linear acceleration is relative, even outside the cosmological context. But from this common starting point they reach — at first sight — strikingly different conclusions. To Knox, Newtonian gravity — cosmological or not — must properly be understood as a theory where the natural motions of particles under gravity are taken as inertial, and so it is a theory of dynamical spacetime geometry even before Cartan's and Friedrich's reformulation. To Saunders, acceleration is relative throughout Newtonian physics (not just in Newtonian *gravity*), and the theory properly understood needs no notion of inertial structure, dynamical or otherwise. It seems, therefore, that these two authors are led to profoundly different conclusions about the geometry of spacetime in Newtonian physics, and indeed different conclusions about which spacetime that is: for Saunders, 'Maxwellian' spacetime; for Knox, Newton-Cartan spacetime.

This is not the case: understood correctly, the two versions of Newtonian physics are the same, or nearly so, both conceptually and mathematically, and so the arguments of Knox and Saunders can both be seen as pointing us towards the true geometry of Newtonian physics. But the path required to establish this is winding and requires consideration of the definition of inertial frames, of the distinction between a theory of an isolated subsystem of a larger universe and of a universe as a whole, and of the distinction between the background geometric notions required to define a theory and the emergent facts about its geometry that derive from its dynamics. Central in the argument is Knox's *spacetime functionalist* view (a close relative of views developed by Harvey Brown; cf Brown (1997, 2005, 2009)) that what defines spacetime structure is the inertial motion of bodies and more generally the local form of the physics in inertial frames.

In sections 2–5 I begin with traditional Galilean-covariant formulations of Newtonian mechanics and show in full generality that such theories, regarded as theories of the Universe as a whole, are unstable: they are empirically equivalent to theories in which linear acceleration and not just velocity is relative. Saunders' reformulation of Newtonian mechanics, which I call 'vector relationism', exemplifies this general argument and, as such, may be formulated on a spacetime whose background geometry is too impoverished to support a preferred standard of unaccelerated motion. Yet when we consider the physics of *subsystems* of a Newtonian universe, a notion of inertial motion reemerges, and indeed it is precisely the notion of inertial frame, inertial trajectory and inertial geometry that Knox takes to be what defines spacetime structure. So Saunders' theory, understood via Knox's spacetime functionalism, is after all a theory with inertial structure, and that structure is local, and dynamical, whenever the forces involved in the theory are, like gravitation, universal. And as a corollary, the geometric setting of a Newtonian subsystem of the universe is, at least for some purposes, best taken after all to be Galilean spacetime.

In sections 6–9 I turn to Knox's approach to Newtonian physics. I eschew the traditional differential-geometry presentations of Newtonian and Newton-Cartan gravity to develop an account of Maxwellian spacetime, and of connections on Maxwellian spacetime, in terms of the preferred coordinatisations of that spacetime, and, derivatively, in terms of affine spaces and bundles of those spaces. From this perspective, the distinction between 'standard' potential-based formulations of Newtonian gravity and the Newton-Cartan theory is just a matter of a preferred boundary condition, and that when that condition is lifted the Newtonian potential can be identified via its transformation properties as a component of a Maxwellian connection.The inertial structure previously identified in Saunders' vector relationism is also naturally represented as a Maxwellian connection, and its dynamical equation as a geodesic deviation equation for that connection, so that the mathematical as well as conceptual distance between Knox's and Saunders' positions largely evaporates. The machinery of potential-based Newtonian mechanics can now be applied to enrich our understanding of inertial structure; it turns out that Knox's spacetime functionalism entails that inertial structure ought properly to be understood as scale-relative, so that systems of different sizes experience different spacetime geometries. (The formal, mathematical parts of these equivalence are discussed in a coordinate-free differential-geometric framework byWeatherall (2016b).)

A word on notation: Greek-letter subscripts and superscripts range over $(0,1,2,3)$; Roman-letter subscripts and superscripts i,j range over $(1,2,3)$. In both cases these are actual subscripts, not abstract indices (as found in, e. g. , Wald (1984)). As such, an equation like $x^i = \partial_i V$ is well-formed, and simply abbreviates $x^1 = \partial_1 V, x^2 = \partial_2 V, x^3 = \partial_3 V$. For Roman indices in particular, by definition I take $x_i = x^i$; hence, Roman indices can be raised and lowered freely. I adopt the Einstein summation convention; x schematically labels the triple (x^1, x^2, x^3), and $|x| \equiv \sqrt{x^i x_i}$.

2 Newtonian mechanics and Galilean spacetime

Newtonian N-particle dynamics, expressed in coordinate form, has these dynamical equations:

$$\ddot{x}^i_n(t) = -\frac{1}{m_n}\sum_{k\neq n}\partial_i V_{nk}(|x_n(t) - x_k(t)|), \tag{1}$$

where $V_{nk}(r) = V_{kn}(r)$. The dynamical symmetry group of these equations is the *Galilei group*,

$$t' = t + \tau; \quad x'^i(t') = R^i_j x^j(t) + a^i + b^i t \tag{2}$$

for arbitrary vectors a^i, b^i, arbitrary scalar τ and arbitrary rotation matrix R^i_j. (Spatial translation symmetry is entailed by the fact that the forces $\partial_i V_{nk}$

depend only of differences of absolute positions; rotational symmetry, by the fact that the force is a gradient of a rotationally invariant function; time translation symmetry, by the time-invariance of the forces and masses.)

The natural spacetime setting for a theory with dynamical symmetry group $\mathcal{G}$ also has symmetry group $\mathcal{G}$. In this paper I will take the spacetime structure for a theory to be defined by the coordinate-transformation laws of its dynamical symmetries: given a group $\mathcal{G}$ of bijections of R^4, a *$\mathcal{G}$-structured space* is a set $\mathcal{S}$ together with a nonempty set of bijections from $\mathcal{S}$ to R^4 (the 'coordinatisations' of $\mathcal{S}$), such that (i) if f is a coordinatisation and $\varphi \in \mathcal{G}$ then $\varphi \cdot f$ is also a coordinatisation; (ii) conversely, if f and f' are coordinatisations then $f' \cdot f^{-1} \in \mathcal{G}$. (I develop this way of characterising spacetime structure, which differs from the more familiar differential-geometric methods used in philosophy of physics but is widely used in mainstream physics, in Wallace (2016b); note that at least locally, this is the standard way to characterise the structure of a differentiable manifold, taking $\mathcal{G}$ to be the group of diffeomorphisms of R^4.)

In particular, Galilean spacetime is a set equipped with coordinatisations related by elements of the Galilei group. We can pick out invariant properties of this spacetime simply by looking for features unaffected by Galilean coordinate transforms. For instance, the *temporal distance* between two points (t, x^1, x^2, x^3) and (t', x'^1, x'^2, x'^3) is $(t - t')$ and can readily be seen to be invariant under Galilean transformations. The (squared) *spatial distance* between those two points, $|x - x'|^2 = (x_i - x'_i)(x^i - x'^i)$, is invariant only if $t = t'$, so only spatial distances between simultaneous events are well-defined. Similarly, the notion of a straight line is well-defined, as a set S of points can be characterised as

$$f(S) = \{x^\mu : x^\mu = x_0^\mu + v^\mu \lambda, \lambda \in \mathrm{R}\} \tag{3}$$

(for some x_0^μ, v^μ) with respect to coordinatisation f only if it can be so characterised by any coordinatisation. So the family of straight lines is also an invariant structure — one way to characterise the *affine structure* of the spacetime.

Since the group of Galilei transformations (2) is a subgroup of the diffeomorphism group, we can consistently regard Galilean spacetime as a manifold by treating the Galilean coordinatisations as charts, which lets us define differential-geometric entities like vectors and tensors, but the additional structure defined by the preferred Galilean coordinatisations allows us to make distinctions between vector fields that are finer than the differential structure alone permits. In particular, we can define a vector as *spatial* if its coordinatisation is $(0, v^i)$ in any Galilean coordinate system, since these vectors transform like

$$(0, v^i) \to (0, R^i_j v^j) \tag{4}$$

under Galilean transformations. Similarly, the *timelike vectors*, with coordinatisation $(1, v^i)$, transform like

$$(1, v^i) \to (1, R^i_j v^j + b^i) \tag{5}$$

and so again form a well-defined subset. (The choice here of $+1$ for the zero component is just for convenience.)

In fact, we can use these transformation properties to *define* spatial and timelike vectors: A spatial (resp.timelike) vector is an assignment of a triple v^i to each coordinate system, such that triples in coordinate systems related by a Galilei transformation (2) are related by $v^i \to R^i_j v^j$ (resp. $v^i \to R^i_j v^j + b^i$).

In differential geometry, vectors are localised to points of the manifold (geometrically speaking they are elements of the tangent space to a point). But since the transformation rules (4–5) are independent of (t, x), it is well-defined (i. e., independent of a choice of Galilean coordinate system) whether two vectors have the same coordinates, and so we can use this to consistently identify vectors at different spacetime points, and so drop the need to regard vectors as spatiotemporally localised. (This points to a coordinate-free way to define Galilean spacetime without differential geometry: it can be defined as an *affine space* with additional structure on the space of vectors.)

3 Vector relationism and Maxwellian spacetime

Here are three commonplaces about Newtonian physics. Firstly, it is *translation-invariant*, which means — loosely — that a system's position in absolute space is irrelevant to the rest of its dynamics. Secondly, it is *boost-invariant*, which means — equally loosely— that a system's velocity with respect to absolute space is likewise irrelevant. And thirdly, it is governed by *second-order equations of motion*, which means that the system's instantaneous configuration and the rate of change of that configuration, both at some fixed time, determine the system's entire dynamical history.

A system with translation but not boost invariance might be a system in which absolute space remained relevant even as a system's location in space was impossible to discern. A world with a universal frictional force, for instance, is a world where a system's velocity relative to absolute space has a direct dynamical role to play. And a system governed by dynamics higher than second order might be one in which absolute space remained relevant even though a system's instantaneous absolute-space position and absolute-space velocity were both dynamically irrelevant. But these three facts about Newtonian dynamics, collectively, tell us that a system's location (though not necessarily its orientation) in absolute space is ultimately dispensable.

Spelling this out in a fairly general setting: the instantaneous configuration of a spatially located system can be decomposed as (α, r), where r encodes the location of the system in absolute space (say, via the location of the centre of mass, or of some arbitrarily selected particle), and α encodes the relational facts about the system (say, via the vector displacements from the centre of mass, or from the arbitrarily selected particle, to all the other particles). Then since the equations of motion are second-order, there must be functions $\mathcal{F}, \mathcal{G}$ such that

$$\ddot{\alpha} = \mathcal{F}(\alpha, \dot{\alpha}, r, \dot{r}); \quad \ddot{r} = \mathcal{G}(\alpha, \dot{\alpha}, r, \dot{r}). \tag{6}$$

Since both translations and boosts are symmetries, and since accelerations transform trivially under both, we must have

$$\mathcal{F}(\alpha, \dot{\alpha}, r + a, \dot{r} + b) = \mathcal{F}(\alpha, \dot{\alpha}, r, \dot{r}) \tag{7}$$

for quite arbitrary a, b — which is to say that $\mathcal{F}$ is quite independent of r or $\dot{r}$. So there is a self-contained dynamics for the relational information that makes no reference to the dynamics of the system as a whole in absolute space: no information about the absolute-space location of the system is needed to study the remaining degrees of freedom.

(Notice that this does not follow *solely* from the translation symmetry. Trying the analogous move for, say, rotational symmetry allows us to write down a dynamics for the relational degrees of freedom, but where that dynamics depends on the overall angular momentum of the system. A self-contained dynamics arises only if some further stipulation is made about that angular momentum — such as by setting it to zero, as in the relational dynamics developed by Barbour and co-workers (Barbour and Bertotti 1982; Barbour 1982; Barbour 2012). It is the invariance of the dynamics *both* under a time-dependent symmetry, *and* under a constant 'boost' with respect to that symmetry, that gives rise to the result.)

We can apply this recipe fairly directly in the case of Newtonian mechanics: the centre of mass of the system,

$$x^i_{COM} \equiv \frac{\sum_n m_n x^i_n}{\sum_n m_n}, \tag{8}$$

can readily be seen to satisfy

$$\ddot{x}^i_{COM}(t) = 0. \tag{9}$$

and so the relational quantities $x^i_n - x^i_{COM}$ satisfy

$$\ddot{x}^i_n(t) - \ddot{x}^i_{COM}(t) = \frac{1}{m_n} \sum_{k \neq n} \nabla_i V_{nk}(|x_n(t) - x_k(t)|). \tag{10}$$

(Note that these N equations determine only $N-1$ *independent* equations, since x_{COM} is a function of $x_1, \ldots x_N$). As predicted, (10) is a closed system for the relative vector positions (since $x_n - x_k = (x_n - x_{COM}) - (x_k - x_{COM})$), and equation (9) can be dropped as irrelevant to the empirically accessible dynamics.

Saunders (2013) reaches this dynamical system by considering instead the $N(N-1)$ (non-independent) equations obtained by taking pairwise differences of the Newtonian equations:

$$\ddot{x}^i_n(t) - \ddot{x}^i_m(t) = -\frac{1}{m_n} \sum_{k \neq n} \partial_i V_{nk}(|x_n(t) - x_k(t)|) + \frac{1}{m_m} \sum_{k \neq m} \partial_i V_{mk}(|x_m(t) - x_k(t)|). \tag{11}$$

In this format, it is transparent that this is a well-defined dynamics for the vector displacements between particles; hence we might call it *vector relationism.*

Saunders' formulation also has the advantage of being well defined in certain contexts — like Newtonian cosmology — in which the number of particles is infinite and the centre of mass is not well defined.

These equations (in either Saunders' formulation or the centre-of-mass version) are invariant under the larger *Maxwell group*[1] of symmetries,

$$t' = t + \tau; \quad x'^i(t') = R^i_j x^j(t) + a^i(t), \tag{12}$$

where now $a^i(t)$ is an arbitrary smooth vector function of time, generating arbitrary time-dependent translations. So the natural spacetime setting for vector relationism is Maxwellian spacetime,[2] which can be defined via the Maxwell group in exactly the same way that Galilean spacetime was defined in section 2, via a collection of coordinatisations related by Maxwell transformations. This spacetime has less structure than Galilean spacetime: the temporal metric, and same-time spatial metric, remain well-defined, but straight lines are not invariant under generic time-dependent translations and so do not pick out invariant structure. It still permits a distinction between spatial vectors (with coordinatisation $(0, v^i)$) and timelike vectors (with coordinatisation $(1, v^i)$); their transformation properties are now

$$\text{Spatial: } v^i \to R^i_j v^j; \quad \text{Timelike: } v^i \to R^i_j v^j + \dot{a}^i(t). \tag{13}$$

Notice that the spatial vector transformation rule remains time-independent, so that we can continue to say of vectors at distinct spacetime points whether or not they are the same in a well-defined manner; the timelike vector transformation rule is dependent on time, so that timelike vectors at simultaneous events can be meaningfully compared but timelike vectors at different times cannot. So timelike vectors need to be thought of as defined at particular times but not particular locations in space.

(For context, if we further weakened the spacetime structure by moving from the Maxwell to the Leibniz group — with transformation rule

$$t' = t + \tau; \quad x'^i(t') = R^i_j(t) x^j(t) + a^i(t) \tag{14}$$

for arbitrary smooth function $R^i_j(t)$ taking values in the rotation matrices, the spatial and timelike vectors would transform like

$$\text{Spatial: } v^i \to R^i_j(t) v^j; \quad \text{Timelike: } v^i \to R^i_j(t) v^j + \dot{R}^i_j(t) x^j + \dot{a}^i(t), \tag{15}$$

[1]Terminology varies here; Saunders (2013) calls it the 'Newtonian group'; Duval (1993) calls it the 'Milne group' after McCrea and Milne (1934) (cf (Bain 2004, p.351)); Ehlers (1999) calls it the 'Heckmann-Schücking group' after Heckmann and Schücking (1955, 1956). My terminology follows Earman (1989); it is not intended to imply any priority claim for the historical Maxwell.

[2]Again, terminology proliferates: Saunders prefers *Newton-Huygens spacetime*, reserving 'Maxwellian spacetime' for a specifically differential-geometric characterisation of the space (and arguing that Newton and Huygens deserve priority, a claim that lies outside the scope of this paper); Weatherall (2016b) compromises on 'Maxwell-Huygens spacetime', although he does have in mind a differential-geometric characterisation of the space.

so that spatial vectors at different times, and timelike vectors at different spacetime points, cannot be invariantly compared.)

To sum up: given a dynamics which is Galilei-covariant, there is always another dynamics that is Maxwell-covariant and which agrees with the Galilei-covariant theory on everything except the motion of the centre of mass — and the latter is in principle undetectable from within the system. So there is something unstable about the idea that *any* dynamical theory could have Galilean spacetime as its natural spacetime setting: once such a theory's dynamical symmetries include both time-independent translations and boosts, nothing empirical can argue against further extending it to consider arbitrary time-dependent translations.[3]

4 Recovering the Galilei group: dynamics of subsystems

In traditional formulations of Newtonian physics, a central role is played by *inertial frames*: unaccelerated, non-rotating reference frames. The Galilean symmetry of the theory is said to describe the fact that we can move from one inertial frame to another by — but only by — applying a translation, a velocity boost, or a rotation. And the inertial structure is coded in Galilean spacetime by means of the affine structure of that spacetime, which as we have seen distinguishes straight from curved spacetime trajectories.

Maxwellian spacetime has no such distinction between straight and curved lines, no distinction between accelerated and non-accelerated reference frames; all it can do is distinguish the rotating from the non-rotating reference frames. On this basis Saunders argues that Newtonian physics needs no concept of inertial frame:

> I am suggesting a reading [of Newtonian physics] . . . in which the concept of inertial motion in the usual sense may not even be defined . . . Corollary VI frees the theory from the need to give any operational significance to the notion of inertial frame. (Saunders 2013, p.25)

(Here 'Corollary VI' refers to Newton's result that the motions of bodies among themselves is unaffected if the bodies are all subjected to the same acceleration.)

We have seen that this is quite correct if we are considering the motion of the Universe as a whole, and indeed if we are considering the movements among themselves of the bodies of any isolated system. It is not quite correct in general: once we consider the movements of isolated systems *relative to one another*, the concept of an inertial frame recurs, as does the Galilean symmetry group.

To spell this out, suppose we consider some subsystem of the Universe: say, that consisting of the first K particles, where the total number N of particles is much larger than K.(For simplicity I assume a Universe of finitely many

[3] Knox (2014) identifies the same instability from a somewhat different starting point.

particles, though my conclusions can be carried over to the infinite case.) I will say that the subsystem is

- *dynamically autonomous* if there is a closed set of equations for the coordinates $x_1^i, \ldots x_K^i$;
- *dynamically isolated* if, in addition, that closed set of equations is the same set that would apply if those K particles were alone in the Universe.[4]

For instance, consider electrostatics: K charged particles interacting under the Coulomb force. The first K particles might be isolated if they were located at a very great distance from all other charged matter. If they were not so distant, but were located in the vicinity of a distribution of charge so large as to be scarcely affected by the N particles, the system would not be isolated but might be autonomous: the electric field of the remaining particles could be treated as a (perhaps time-dependent) background field. Clearly, autonomy and isolation are approximate notions and depend, to any degree of approximation, upon the positions of the particles not straying outside some region; only in idealisation could we treat their dynamical equations as exact and applicable irrespective of the particles' positions.

When can either occur? Adopting a vector notation, and labelling the subsystem particles as particles 1 through K, the acceleration of a body in the subsystem relative to the centre of mass $\mathbf{X}_{COM}$ of the entire Universe is

$$\ddot{\mathbf{x}}_n(t) - \ddot{\mathbf{X}}_{COM}(t) = \frac{1}{m_n}\left(\mathbf{F}_n^{int} + \mathbf{F}_n^{ext}\right) \tag{16}$$

where

$$\mathbf{F}_n^{int} = -\sum_{k=1, k\neq n}^{k=K} \nabla V_{nk}(|x_n - x_k|); \quad \mathbf{F}_n^{ext} = -\sum_{k>K} \nabla_{nk}(|x_n - x_k|). \tag{17}$$

Dynamical autonomy is then the requirement that the evolutions of the bodies outside the system are sufficiently independent of those inside the system that they may be taken as a fixed background, so that $\mathbf{F}_{nm}^{ext}$ may be regarded as a function of $\mathbf{x}_n$.

From (16) we deduce

$$\ddot{\mathbf{x}}_{COM} - \ddot{\mathbf{X}}_{COM} = \frac{\sum_{k\leq K} \mathbf{F}^{ext}(\mathbf{x}_k)}{\sum_{k\leq K} m_k} \tag{18}$$

and, subtracting,

$$\ddot{\mathbf{x}}_n - \ddot{\mathbf{x}}_{COM} = \frac{1}{m_n}\mathbf{F}_n^{int} + \left(\frac{1}{m_n}\mathbf{F}_n^{ext}(\mathbf{x}_n) - \frac{\sum_{k\leq K} \mathbf{F}^{ext}(\mathbf{x}_k)}{\sum_{k\leq K} m_k}\right). \tag{19}$$

[4]Autonomy is much the more general notion, and remains well-defined even after arbitrary changes of coordinates; non-equilibrium statistical mechanics, for example, is to a large extent the search for autonomous dynamics for the collective coordinates that describe many-body systems averaged over large numbers of their consituents. I develop this point in Wallace (2015).

Dynamical isolation is then the further requirement that the second term in (19) approximately vanishes,

$$\frac{1}{m_n}\mathbf{F}_n(\mathbf{x}_n) - \frac{\sum_{k\leq K}\mathbf{F}^{ext}(\mathbf{x}_k)}{\sum_{k\leq K} m_k} \simeq 0, \tag{20}$$

for the whole range of dynamically relevant values for the x_m. We can then idealise this term as vanishing exactly for all x_n, and the dynamics of the subsystem will be the same as they would be if that subsystem were alone in the Universe. The condition for this term to vanish is

$$m_n\mathbf{F}_n^{ext} \simeq \text{constant, independent of } n. \tag{21}$$

To get some insight into when this might plausibly occur, we can write the Newtonian potential (with only slight loss of generality) as

$$V_{nk}(r) = m_n m_k \mathcal{V}_U(r) + q_n q_k \mathcal{V}_N(r) \tag{22}$$

where we assume the ratio q_n/n_n is not constant. The potential term $\mathcal{V}_U$ determines *universal* forces: the relative acceleration of a particle under the universal acceleration alone is independent of its mass. (There could be multiple non-universal forces; I omit them for simplicity.) Then

$$\mathbf{F}_n^{ext} = -m_n \sum_{k>K} m_k \nabla\mathcal{V}_U(|x_n - x_k|) - q_n \sum_{k>K} q_k \nabla\mathcal{V}_N(|x_n - x_k|) \tag{23}$$

and the condition for a dynamically isolated subsystem is

$$\sum_{k>K} m_k \nabla\mathcal{V}_U(|x - x_k|) \simeq \text{ constant} \equiv \mathbf{F}_U; \quad \sum_{k>K} q_k \nabla\mathcal{V}_N(|x_n - x_k|) \simeq 0. \tag{24}$$

The subsystem equations of motion are then

$$\ddot{\mathbf{x}}_n - \ddot{\mathbf{x}}_{COM} = \mathbf{F}_n^{int}; \quad \ddot{\mathbf{x}}_{COM} - \ddot{\mathbf{X}}_{COM} = \mathbf{F}_U, \tag{25}$$

with the equations for the subsystem degrees of freedom alone being the same as if the subsystem were alone in the Universe.

Now suppose that observers not part of the subsystem nonetheless wish to study its dynamics, and in particular its dynamical symmetries. That is: they are interested in which transformations of the coordinates of the bodies in the subsystem alone leave the subsystem equations of motion invariant. We know that the equations for $\mathbf{x}_n - \mathbf{x}_{COM}$ are invariant under the Maxwell group, but we need to preserve not only these equations but the equation for $\mathbf{x}_{COM} - \mathbf{X}_{COM}$. And this latter equation is invariant only under the Galilei group: to a good approximation $\mathbf{X}_{COM}$ is unaffected by transformations of the subsystem coordinates alone, as is $\mathbf{F}_U$, so we need $\ddot{\mathbf{x}}_{COM}$ to be invariant, which requires constant or linear-in-time translations. (We could read the same conclusion directly off the equations (16).)

The dynamical symmetries of the subsystem include time-independent translations and rotations, time translation, and velocity boosts. But they do not include arbitrary time-dependent translations. Corollary VI notwithstanding, these are not dynamical symmetries of isolated subsystems of a larger universe. The dynamics of that isolated subsystem, as studied from without, is a dynamics with a well-defined and dynamically relevant notion of inertial frame.

So: the notion of an inertial frame, though useless when applied to the Universe as a whole, has a clear operational significance when applied to an isolated subsystem of the Universe. There is no need for such frames in the formulation of the theory — no need, that is, for Galilean spacetime as a background for the theory — but they emerge naturally for subsystems, defined dynamically by the distribution of everything *not* in the subsystem.

Now consider *two* isolated subsystems, and assume that neither has an appreciable dynamical effect on the other, so that we can consider both evolving against the background dynamical effects of the rest of the Universe. For the systems to be isolated, the non-universal forces must be negligible, and the universal forces must be constant, across the spatial extent of both subsystems. Their centres of mass $\mathbf{x}_{1,COM}$ and $\mathbf{x}_{2,COM}$ will then satisfy

$$\ddot{\mathbf{x}}_{i,COM} - \ddot{\mathbf{X}}_{COM} = \mathbf{F}_U(\mathbf{x}_{i,COM}) \tag{26}$$

where $\mathbf{F}_U(\mathbf{x})$ is the universal relative acceleration from the rest of the Universe on a system at location $\mathbf{x}$, and hence

$$\ddot{\mathbf{x}}_{1,COM} - \ddot{\mathbf{x}}_{2,COM} = \mathbf{F}_U(\mathbf{x}_{1,COM}) - \mathbf{F}_U(\mathbf{x}_{2,COM}). \tag{27}$$

If there are no universal forces, then, the concept of an inertial frame is global: a coordinate system is inertial for one isolated system iff it is inertial for another. In the presence of universal forces, inertial frames become locally defined: an inertial frame for one system may be accelerating (though may not be rotating) relative to an inertial frame for another.

Though it lies rather outside the main thrust of this paper, pretty much the same story applies for inertial structure in Barbour and Bertotti's relational dynamics, in which time-dependent rotations as well as translations are among the symmetries of the equations of motion. The difference is that whereas in vector relationism there is an absolute standard of rotation, so that inertial frames can be defined simply by the motion of their centres of mass, in Barbour-Bertotti theory the standard of rotation, as well as that of inertial motion, is determined by the matter distribution.

Nonetheless, it *is* so determined: for a subsystem to be isolated in Barbour-Bertotti theory, it must be nonrotating relative to the reference frames in which the angular momentum of the Universe is zero. The inertial frames are then those in which (i) the origin is moving along an inertial trajectory, i. e. a trajectory of a test particle that feels only the universal forces, and (ii) the axes are nonrotating in this dynamically-defined sense.

And this is, of course, pretty much what Mach (1883) had in mind: not a universe with *no* inertial structure, but a universe where the matter distribution determines the inertial structure. With respect to the linear-acceleration part of that structure, the determination may (if there are universal forces) be local; with respect to the rotational part, it is always global, with two subsystems that are nonrotating relative to the inertial frame also nonrotating relative to one another.

5 Knox on inertial structure

The picture of inertial structure in Saunders' version of Newtonian dynamics that I have developed is simply a playing out, in the Newtonian context, of the general account of spacetime structure developed by Knox and Brown. It is most explicitly developed in Knox (2013), and goes as follows:

A Identify the inertial frames in a theory:

> In Newtonian theories, and in special relativity, inertial frames have at least the following three features:
>
> 1. Inertial frames are frames with respect to which force free bodies move with constant velocities.
> 2. The laws of physics take the same form (a particularly simple form) in all inertial frames.
> 3. All bodies and physical laws pick out the same equivalence class of inertial frames (universality).
>
> (Knox 2013, p.349)

By "the laws of physics", Knox means the non-gravitational (i. e., non-universal) interactions (here she follows Brown (1997, p.76): "Ultimately, it is because of certain symmetry properties of the non-gravitational interactions that ...the metric means operationally what it means."). By 'force free bodies' she means those bodies that do not feel the non-universal interactions.

B Define the inertial trajectories as the trajectories of the force-free bodies.

C Define 'spacetime structure' functionally, as by definition that structure picked out by those inertial trajectories.

I digress to make two observations about Knox's (and, indirectly, Brown's) framework in the light of the detailed case study that Saunders-style Newtonian mechanics has provided. Firstly, Knox writes that

> [O]ne may worry that frames fitting [Knox's] definition may not exist in regions of spacetime that are not well-behaved (near black holes, for example). I don't intend here for the fulfilment of [my] definition of inertial frame everywhere to be a necessary condition for our empirical access to spacetime. Rather, the definition of an inertial frame here should fix our interpretation of the theory, thus allowing

> us to determine the force-free trajectories. Once determined, we can acknowledge inertial trajectories in regions where our full notion of inertial structure breaks down. (Knox 2013, p.349)

This plays out in the Newtonian context, never mind black holes. Some inertial trajectories may be inertial only by courtesy, if the universal force field varies so rapidly in the vicinity of the trajectory that no realistic subsystem could actually by isolated. More specifically, a given inertial trajectory determines an inertial frame for a given test subsystem if (a) that subsystem is small enough that $\mathbf{F}_U$ does not vary significantly over its volume; (b) that subsystem is dynamically insignificant enough, relative to its surroundings, that back-reaction on those surroundings can be neglected. (And indeed, this scale-relativity of the validity of inertial frames plays out in the relativistic contexts Knox considers, too: spacetime near (astrophysical) black holes has tidal forces sufficient to rip apart most macroscopic bodies, but negligible on the scale of, e. g. , radioactive particles.)

Secondly, the analysis of section 4 plays out perfectly well even in the absence of non-universal forces. A pair of gravitationally co-orbiting bodies, for instance, can perfectly well survey the inertial structure defined by the remainder of the matter distribution, provided that inertial structure is defined by universal forces whose variation across the inter-body distance can be neglected relative to the bodies' mutual attraction. Indeed, the Earth-Moon system thus surveys the inertial structure defined by the Sun. Newtonian gravity — let alone general relativity — is rich enough that even its own interactions in microcosm can define inertial frames and inertial trajectories.

(Brown (1997, pp.76–77) questions the coherence of the vacuum solutions of general relativity on the grounds that they lack the non-gravitational interactions he believes necessary to define local inertial structure, and thus make operational sense of the metric field. But I suspect that this (i) conflates 'no non-gravitational interactions' with 'no non-gravitational matter' — dust or, less phenomenologically, a free field, lacks non-gravitational *interactions* but could provide operational significance to the metric via subsystem dynamics; (ii) underestimates the complexity of general relativity and the availability of gravitational waves, black holes and the like that can themselves survey the inertial structure.)

Returning to the main theme, Knox calls her approach *spacetime functionalism* (Knox 2015), and the name is apt: as is standard with functionalism, it identifies a property by its functional / structural properties, and is entirely neutral as to how they are instantiated. In particular, there is no reason that the spacetime structure picked out by her account need be ontologically self-subsistent. If the movements of test particles are determined by some independent dynamical entity (as is typically supposed for the metric field of general relativity) then that independent entity is the realiser of the spacetime role, but in a theory where material trajectories supervene entirely on the distribution of matter, the realiser will be a certain structural feature of the matter distribution.

The latter is the case in Saunders' version of Newtonian dynamics. There, no spacetime structure at all is needed in the *formulation* of the theory; all that is needed is relative vector accelerations. So the inertial structure discovered in section 4 is a spacetime structure instantiated entirely by the relational dynamics of the matter particles: no additional structure needs to be *posited* (beyond those features required to formulate Saunders' relational dynamics, such as a standard of rotation and an instantaneous spatial geometry).

This suggests that the gulf between Saunders' and Knox's conceptions of Newtonian gravity may not be as large as it might appear. To see the details, however, we will need to move beyond the essentially approximate, emergent characterisation of inertial structure developed in section 4 and connect vector relationism to the sharp, precise — if formal — notion of inertial structure found in Newton-Cartan theory.

6 Connections on Maxwellian spacetime

Galilean and Maxwellian spacetime, characterised as I have done in sections 2–3 via classes of coordinatisations, are also differential manifolds, and so in both cases we can define on them an affine connection, with a particular choice of connection characterised, relative to a coordinate system, by a collection of Christoffel symbols $\Gamma^{\mu}_{\nu\tau}$. However, the mathematical function of a connection is to define a parallel transport rule — recall that the change in components of a vector V^{μ} parallel-transported along an infinitesimal displacement δx^{μ} is

$$\delta V^{\mu} = -\Gamma^{\mu}_{\nu\tau} V^{\nu} \delta x^{\tau} \tag{28}$$

— and the background geometry of these spacetimes already allows a parallel-transport rule to be defined in many cases.

In Galilean spacetime, indeed, we have seen that the question of whether two vectors at different spacetime points are the same — that is, have the same components in a given Galilean coordinate system — is coordinate-system-independent. So we can naturally define a vector field along a line as parallel-transported iff vectors at different points on the line have the same coordinates. The connection thus determined has Christoffel symbols $\Gamma^{\mu}_{\nu\tau} = 0$. As a sanity check, we can apply the general transformation law for Christoffel symbols — As a sanity check on our analysis, we can apply the general transformation rule for the Christoffel symbols under a coordinate transformation $(x^0, x^1, x^2, x^3) \rightarrow (X^0, X^1, X^2, X^3)$,

$$\Gamma'^{\alpha}_{\beta\gamma} = \frac{\partial X^{\alpha}}{\partial x^{\sigma}} \frac{\partial x^{\mu}}{\partial X^{\beta}} \frac{\partial x^{\nu}}{\partial X^{\gamma}} \Gamma^{\sigma}_{\mu\nu} - \frac{\partial^2 X^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial x^{\mu}}{\partial X^{\alpha}} \frac{\partial x^{\nu}}{\partial X^{\beta}}, \tag{29}$$

in the particular case of the Galilei group, whereupon we find that if all the Christoffel symbols vanish in one Galilean coordinate system, they vanish in all such systems. So we can equally well *define* the natural connection on Galilean spacetime as the unique connection with vanishing Christoffel symbols.

The same trick does not *quite* work in Maxwellian spacetime, because as we have seen, the transformation rule for timelike vectors is time-dependent, and so timelike vectors at different times cannot be invariantly compared. But:

1. The spatial translation group is a well-defined subgroup of the Maxwell group and can be used to transport any vector in an arbitrary purely-spatial direction: in a Maxwell coordinate system, the components of a vector are unchanged by this transport, so (from the parallel-transport rule (28) a connection compatible with this rule satisfies $\Gamma^{\mu}_{\nu i}$ for arbitrary spacetime indices μ, ν and spatial index i.

2. Similarly, since purely spacelike vectors can be invariantly compared even if they are at different times, there is a well-defined sense of parallel transport of spacelike vectors; reproducing this rule imposes $\Gamma^{\mu}_{i\nu} = 0$.

3. Finally, the temporal component of any vector field is invariant under Maxwell transformations. We can then consistently require that this component of a parallel-transported vector field is constant, which imposes $\Gamma^{0}_{\nu\tau} = 0$.

Putting these three together, the only nonvanishing components of an affine connection compatible with the existing parallel-transport structure are the three components Γ^{i}_{00}, and we define a *Maxwellian connection* as a connection where all other Christoffel symbols vanish. Again as a sanity check, we can apply (29) in the particular case of the Maxwell group: if the connection is Maxwellian with respect to the first coordinate system, we find that in the second coordinate system

$$\Gamma'^{i}_{00} = R^{i}_{j}\Gamma^{j}_{00} - \ddot{a}^{i} \tag{30}$$

and all other $\Gamma'^{\mu}_{\sigma\nu}$ vanishing. So the condition "all Christoffel symbols vanish except possibly Γ^{i}_{00}" is invariant under the action of the Maxwell group, and so defines a *coordinate-independent* property of a connection on Maxwellian spacetime.

In fact, this rather general Christoffel formalism is overkill. Stepping back for a moment: an affine connection is the right geometric object to define parallel transport on a general differentiable manifold; that is, on a space whose Kleinian symmetry group is the full diffeomorphism group. And requiring that the parallel-transport equation (28) holds in arbitrary coordinate systems in the manifold's atlas suffices to derive the transformation law (29). But we are concerned only with a parallel-transport rule on Maxwellian spacetime, and such a rule need only hold in arbitrary *Maxwellian* coordinate charts. So we could, directly, define a *Maxwellian* connection as a parallel-transport rule specifically for timelike vectors v^i:

$$v^i(x + \delta x, t + \delta t) = v^i(x) + \delta t A^i(x, t) \tag{31}$$

or equivalently as a derivative operator on timelike vector fields

$$(\nabla v)^i(x, t) = \frac{\partial v^i}{\partial t}(x, t) + A^i(x.t) \tag{32}$$

Requiring the left hand side of (31) to transform as a vector under the Maxwell group gives the transformation rule for the connection coefficients A^i:

$$A'^i(x', t') = R^i_j A^j(x, t) - \ddot{a}^i(t); \tag{33}$$

this could also be read off from the identification

$$\Gamma^i_{00} = A^i. \tag{34}$$

Just as the Christoffel coefficients appear to be the coordinates of a tensor but actually have a different transformation law, so the Maxwellian connection coefficients appear to be, but are not, the coordinates of a spatial vector field.

We can divide the geodesics of a Maxwellian connection into spatial and timelike. The former are simply the straight lines with respect to the affine structure of the instantaneous spaces. For the latter, we can reparametrise any such geodesic $\gamma^\mu(\tau)$ so that $\gamma^0(\tau) = t$. The geodesic equation is then simply

$$\ddot{\gamma}^i + A^i = 0 \tag{35}$$

and the family of such geodesics fully characterise the connection.

Returning temporarily to the generally covariant formalism, the curvature $\omega^\alpha_{\beta\gamma\delta}$ of the connection is readily determined to be

$$\omega^i_{0j0} = -\omega^i_{00j} = \partial_j \Gamma^i_{00}, \tag{36}$$

all other components vanishing. The *geodesic deviation equation*, which gives the relative acceleration of two test particles initially comoving with velocity V^μ a distance δx^ν from one another to be $\omega^\sigma_{\mu\nu\mu} V^\mu \delta x^\nu$, lets us interpret $\omega^i_{0j0}\delta x^j$ as the relative acceleration of two non-spatial geodesics separated by spatial distance δx.

Again, the full curvature is overkill to describe the local curvature of a connection on Maxwellian spacetime. We may as well define that directly as

$$\Omega_{ij}(x, t) = \partial_j A^i(x, t). \tag{37}$$

Its geometric significance is entirely characterised by the geodesic deviation equation: given two geodesics through infinitesimally close points (x, t) and $x + \delta x, t)$, their relative acceleration is $-\Omega^i_j \delta x^j$. (This can be read directly off (35), without needing to go via the generally-covariant formalism.)

The Maxwellian curvature transforms under Maxwell transformations like

$$\Omega'^i_j(x', t') = R^i_k \Omega^k_l(x, t)(R^{-1})^l_j \tag{38}$$

—exactly like a spatial tensor. It satisfies the Bianchi identity

$$\partial_k \Omega^i_j - \partial_j \Omega^i_k = 0 \tag{39}$$

and it is easy to verify that it the curvature at a given time determines the Maxwellian connection at that same time up to an arbitrary constant, and indeed that any spatial 2-tensor field satisfying the Bianchi identity is the curvature of a connection fixed up to that constant.

If Ω^i_j is symmetric, the connection is *irrotational*, so-called because a group of initially co-moving test particles may spread out or be sheared by geodesic deviation but will not be placed into rotation. In this case the vector field $\partial_j A^i$ is curl-free and so the connection can be written (uniquely up to an arbitrary constant factor) as

$$A^i(x,t) = \partial_i \Phi(x,t). \tag{40}$$

The potential term Φ transforms under the Maxwell group like

$$\Phi'(x',t') = \Phi(x,t) - \ddot{a}_i(t)x^i + f(t) \tag{41}$$

where $a_i = a^i$ and f is arbitrary. Just as A^i looks like, but is not, a vector field, so Φ looks like, but is not, a scalar field. (Geometrically, it is helpful to think of Φ as an arbitrary representation of a curl-free connection rather than as an entity in itself.)

7 Knox on Newtonian gravity

Following Knox (2014), let's consider the familiar potential-based form of Newtonian gravity, which in coordinate-based notation would be written

$$\dot{v}^i(x,t) + \partial_i V(x,t) = 0 \tag{42}$$

$$\partial^i \partial_i V(x,t) = 4\pi\rho(x,t) \tag{43}$$

where ρ is the mass density and $\dot{v}^i$ is the acceleration of a particle at spacetime point (x,t). For this to be a closed dynamical theory we also need equations for ρ; one possibility is to include discretely many particles with positions $x^i_n(t)$ and masses m_n moving on test-particle trajectories, so that

$$\rho(x,t) = \sum_n m_n \delta(x - x_n(t)) \tag{44}$$

but this has the awkward feature that V is singular at all the particle positions, so an alternative is to the continuum limit of a dust of infinitesimal-mass particles, whose velocities are represented by a vector field $v(x,t)$; elementary fluid dynamics now gives

$$\partial_i(v^i\rho) + \frac{\partial\rho}{\partial t} = 0 \tag{45}$$

and

$$\dot{v}^i = \frac{\partial v^i}{\partial t} + v^j \partial_j v^i. \tag{46}$$

In either case, the dynamical symmetry group for this dynamical system is the Maxwell group.[5] ρ transforms like a scalar, and v^i like a spatial vector, under

[5]Actually, it is slightly larger, and also includes certain correlated scale transformations of space and time; for simplicity I set these aside.

these transformations; V transforms like

$$V'(x',t') = V(x,t) + \ddot{a}_i(t)x^i + f(t) \tag{47}$$

for some arbitrary f. This is the transformation law of the potential for an irrotational Maxwell connection, and so — on the coordinate-based way of representing a physical theory, in which to be a geometric object of type X is to transform like an object of type X — V *is* the potential for an irrotational Maxwell connection. Once this is seen, we can recognise $\partial^i\partial_i V$ as the scalar part of the curvature of that connection, and rewrite (42–43) as

$$\nabla v^i = 0 \tag{48}$$

$$\Omega[\nabla]^i_i = 4\pi\rho \tag{49}$$

where ∇ is an irrotational Maxwell-affine connection and $\Omega[\nabla]^i_j$ is its curvature. The theory written in this way is *Newton-Cartan gravity.*

In an island Universe (that is: one with a finite number of particles, or with finite total mass in the continuum case) we can impose a boundary condition

$$\lim_{|x|\to\infty} V(x,t) = 0. \tag{50}$$

This reduces the symmetry group of the theory to the Galilei group, since if V is constant under (47) then $\ddot{a}^i(t) = 0$; as such, it reintroduces absolute inertial structure to the theory in at least a formal sense. (Mathematically speaking, it also fixes V to be

$$V(x,t) = \int \mathrm{d}x'^{\,3} \frac{\rho(x')}{|x-x'|} \tag{51}$$

and so establishes equivalence with the force-based formalism.) Knox makes this absolute structure explicit by continuing (in effect) to work with the larger Maxwellian symmetry group but introducing a split $V = V_0 + V_1$, where V_0 is required to be flat:

$$\partial_j\partial^i V_0 = 0. \tag{52}$$

As a consequence, V_1 transforms as a scalar (any two irrotational Maxwell connections differ by the spatial gradient of a scalar field, just as any two affine connections differ by a tensor field), satisfies

$$\partial^i\partial_i V_1(x,t) = 4\pi\rho(x,t), \tag{53}$$

and may be interpreted (at least formally) as generating a force which causes test particles to deviate from the trajectories defined by the flat connection. If the matter density tends to zero sufficiently quickly we can also require that V_1 tends to zero at spatial infinity. (This is incompatible with (53) if the mass density does not drop off sufficiently quickly.)

Knox makes two — persuasive — criticisms of this decomposition, and (equivalently) of imposing the boundary condition (50):

1. What does it actually gain us? Sure, we *can* decompose the connection into a flat and a curved part; in island-universe contexts we can even do it uniquely. But why bother? (She analogises it to our freedom to choose electromagnetic gauges, and our unwillingness to single out one gauge as preferred.)

2. The total connection defined by V has operational significance: it determines the inertial trajectories and the inertial frames. The connection defined by V_0 alone, by contrast, is operationally irrelevant and cannot be determined empirically short of surveying the entire mass distribution (and not even then, in cosmological contexts where the mass distribution is asymptotically nonzero). Knox does not make this point — although Brown and Pooley (2006, fn.25) say something similar — but there is a close analogy with the splitting of the general-relativistic metric into flat and non-flat parts, in the spin-2-field approach to general relativity (Feynman 1964). [6]

(There is a purely *technical* problem with removing the boundary condition, not generally acknowledged in the philosophy-of-spacetime literature but well known in cosmology: with no boundary condition at all, equation (49) radically underdetermines Φ and renders the theory viciously indeterministic. I discuss this issue, and the threat it poses to cosmological interpretations of Newtonian physics, in Wallace (2016a); for the purposes of this paper, I assume that some weaker boundary condition is specified that fixes a unique solution to (49) up to Maxwellian transformations.)

Knox's argument for dropping the boundary condition does not presuppose her functionalist analysis of spacetime structure, but fits naturally with it. Once it is observed that the absolute structure is not (modulo the technical problem above) required to formulate the theory, then the residual case for retaining it is that inertial structure plays some necessary conceptual role in understanding Newtonian gravitation. But whatever that role is, it is not (on Knox's functionalism) to represent *spacetime*'s inertial structure; that role is played by the Newton-Cartan connection, and the residual role for the unobservable background connection is obscure.

[6] It's worth noting that my treatment of the theory is somewhat heterodox in its use of coordinate-transform methods. Conventional presentations, including Knox's, formulate potential-based Newtonian gravity on a spacetime equipped with a flat connection, so that V generates deviations from that connection; the move to a spacetime with a nonflat connection and no separate V field is then a move to a different sort of mathematical object (whether it is the same physical theory is a moot point; cf Weatherall (2016a)). The coordinate-transform approach has the virtue of making fewer physics-opaque mathematical distinctions; formulated this way, potential-based Newtonian gravity was a theory with a nonflat connection all along, and Newton-Cartan gravity is not a reformulation of that theory but a mere relabelling.

8 Vector relationism and Newton-Cartan theory

Let's now return to Saunders' vector-relationism theory. We saw in sections 4–5 that it contains an emergent inertial structure, which we can probe by means of subsystems small compared to the lengthscales over which the universal force from the rest of the Universe varies significantly and located in regions where non-universal forces are negligible. This can be idealised by introducing *inertial test particles*, which do not interact via the non-inertial forces and which have negligible mass, and so negligible back-effect on the relative motions of other particles.

Writing $v^i(x,t)$ for the velocity of a test particle at spacetime point (x,t),[7] we have from Saunders' relative-acceleration equation (11)

$$\dot{v}^i(x,t) - \dot{v}^i(y,t) = \sum_n m_n \left\{ (\partial_i \mathcal{V}_U(|x - x_n(t)|) - \partial_i \mathcal{V}_U(|y - x_n(t)|) \right\} \tag{54}$$

for the relative acceleration of any two test particles. (This equation needs to be treated carefully (i. e. , distributionally) if, as in the case of Newtonian gravity, the potential function diverges at the origin.)

The acceleration of a (non-test) particle also at (x,t) differs from the acceleration of a test particle at that location by the action of the non-universal force, again via Saunders' relative-acceleration equation:

$$\ddot{x}^i_n(t) - \dot{v}^i(x_n(t),t) = q_n \sum_{k \neq n} q_k \partial_i \mathcal{V}_N(|x^n(t) - x^k(t)|). \tag{55}$$

So we can see dynamics in Saunders' theory as breaking down into two components: the first determines the relative motions of the test particles under the universal force; the second determines the motions of the actual particles relative to the test particles. We can think of the test particles as defining an idealisation of the dynamically-emergent family of inertial frames, relative to which the particles move. In the limit as the universal forces become negligible, the relative accelerations of any two test particles tend to zero and this dynamically-emergent inertial structure becomes global; if there are non-negligible universal forces, the inertial structure is local and widely separated inertially-moving particles can undergo relative acceleration.

We can get further insight into the inertial structure by (i) defining the potential function

$$\mathcal{W}_U(x,t) = \sum_n m_n \mathcal{V}_U(|x - x_n(t)|) \tag{56}$$

so that the relative acceleration of test particles is given by

$$\dot{v}^i(x,t) - \dot{v}^i(y,t) = \partial_i \mathcal{W}_U(x,t) - \partial_i \mathcal{W}_U(y,t) \tag{57}$$

[7] This is a mild abuse of notation; we can perfectly well have multiple test particles at the same point, with different velocities. However, we are interested in their acceleration, which is velocity-independent.

and (ii) taking the infinitesimal limit of the relative acceleration,

$$\partial_j \dot{v}^i(x,t) = \partial_i \partial_j \mathcal{W}_U(x,t) \equiv \Omega_{ij}(x,t). \tag{58}$$

The matrix Ω_{ij} is clearly symmetric and satisfies

$$\partial_j \Omega_{ik} - \partial_i \Omega_{jk} = 0, \tag{59}$$

and transforms under Maxwell transformations as a spatial 2-tensor. Via (58) it encodes the entire universal-force part of the theory, and the potential function $\mathcal{V}_U$ can be recovered from it up to a physically irrelevant linear term. We can then think of a given vector-relationist dynamics as given by such a tensor together with an expression for the non-universal force.

But we have seen that any such tensor has a geometric interpretation: it is the curvature of an irrotational connection, and indeed (58) can be interpreted as the geodesic deviation equation for that connection. So universal forces in Saunders' framework can be thought of as inducing a spacetime curvature on the Maxwell spacetime on which the theory is defined, such that bodies move on the geodesics of the irrotational connection with that curvature in the absence of non-universal forces. The indeterminacy of the linear term in that connection simply reflects the Maxwellian symmetry of Saunders' dynamics. (This result is established in a coordinate-free differential-geometric setting by Weatherall (2016b).)

If we specialise to the case of pure gravity, Saunders' theory becomes

$$\partial_j \dot{v}^i(x,t) = \int \mathrm{d}x'^{\,3} \partial_j \partial^i \frac{\rho(x')}{|x-x'|} \tag{60}$$

where

$$\rho(x) = \sum_k m_k \delta(x - x_k). \tag{61}$$

Carrying out the differentiation in (60),[8] we get

$$\partial_j \dot{v}^i(x,t) = 4\pi\rho(x,t)\delta^i_j + \int \mathrm{d}x'^{\,3} \rho(x',t) \left(\frac{\delta^i_j}{|x-x'|^3} - 3\frac{(x^i - x'^i)(x_j - x'^j)}{|x-x'|^5} \right). \tag{62}$$

Saunders' theory is therefore specified completely by the geodesic-deviation equation (58) (plus, in the presence of non-universal forces, a Newtonian equation for the deviation of real particles' trajectories from geodesic motion).

But now the difference between Newton-Cartan gravity and Saunders' theory fades into invisibility. Introducing a potential V to define the connection whose

[8] It helps to regularise the singularity in $1/|x|$ by inserting a factor $\mathrm{e}^{-\epsilon/r}$ and take $\epsilon \rightarrow 0$ after the differentiation has been performed. The validity of this manipulation can be established either by delicate considerations of the topology on an appropriate distribution space, or just by confirming directly that the integrand of (58) is indeed the distributional derivative of $\rho(x')/|x-x'|$.

curvature is $\partial_j \dot{v}^i$ (that is, $\partial_i \partial_j V = \partial_j \dot{v}^i$), we obtain

$$\partial_j \partial^i V(x,t) = 4\pi\rho(x,t)\delta^i_j + \int \mathrm{d}x'^{\,3} \rho(x',t) \left(\frac{\delta^i_j}{|x-x'|^3} - 3\frac{(x^i - x'^i)(x_j - x'^j)}{|x-x'|^5} \right) \tag{63}$$

which specifies the connection uniquely up to the usual arbitrary linear and constant terms. That connection satisfies $\partial_i \partial^i V = 4\pi\rho$, and so it defines a legal Newton-Cartan connection. And conversely (as I show in Wallace (2016a)), if we choose as boundary condition for Newton-Cartan gravity

$$\lim_{|x|\to\infty} \frac{V(x,t)}{|x|^2} = \frac{2}{3}\pi\bar{\rho}(t), \tag{64}$$

where $\bar{\rho}(t)$ is the spatially averaged mass density at time t, then any solution to Poisson's equation satisfies (63).

8.1 Inertial structure in Newton-Cartan gravity

Mathematically speaking, there is no real distinction between Newton-Cartan gravity (or Newtonian potential-based gravity, which in the formalism of this paper is the same thing) and vector relationism: both are built using Maxwellian spacetime as a background; both have dynamics which can be expressed as a set of inertial trajectories defined by the matter distribution and in turn constraining the matter distribution via a matter dynamics according to which material particles follow those trajectories except when acted on by non-gravitational forces.

As such, my analysis of inertial structure as explored via subsystems of the Universe, carried out in section 4 for vector relationism, now carries over directly to Newton-Cartan gravity; in that context it can be seen as filling in the details — outlined by Knox — of how it is that the Newton-Cartan connection defines the spacetime geometry in that theory. (Recall that for Knox, it is insufficient that it has the *mathematical form* of a connection, and insufficient that we simply *declare* that the theory is one with dynamical spacetime: to count as spacetime the connection needs to describe the inertial frames.)

The machinery of the Newton-Cartan connection actually allows us to reformulate some of the earlier discussion in a helpful way (from here on I write "gravitational" synonymously with my earlier "universal"). To begin with, recall that a subsystem is isolated if (i) the external non-gravitational forces are negligible, and (ii) the external gravitational forces are approximately constant. The latter condition amounts to requiring that $\partial_j \partial^i V \simeq 0$ across the system, which in idealisation (where we remove the environmental matter to infinity and impose this requirement everywhere in space) this is the boundary condition

$$\lim_{|x|\to\infty} V(x,t) = b_i(t)\cdot x^i + c(t) \tag{65}$$

for some given functions b^i and c. In particular, if we work in locally inertial coordinates for the subsystem, this becomes

$$\lim_{|x|\to\infty} V(x,t) = c(t), \tag{66}$$

which (up to a dynamically irrelevant constant) is exactly the boundary condition normally placed on Newtonian potentials, and exactly the condition that Knox rejected in her move from Galilean-covariant to Maxwell-covariant formulations of gravity. And either condition suffices to make the split between inertial and gravitational structure *for the subsystem* perfectly well-defined: the inertial structure is specified by a flat connection satisfying that boundary condition.

This in no way undermines Knox's (and Malament's (1995), and Norton's (1995)) objection to any such split. But those objections applied when we were considering Newtonian gravity as a theory *of the whole Universe*. The split is perfectly well defined when we are considering the theory of a subsystem embedded in a wider Universe. Furthermore, in that context the flat part of the connection has a straightforward operational meaning: it is the background inertial structure against which the interactions internal to the subsystem play out. That background inertial structure is invariant under Galilean, though not more general Maxwellian, transformations; in that sense, Galilean spacetime remains the right setting for Newtonian mechanics where it is used to analyse an isolated subsystem of a larger system.

(Note that once gravitationally interacting systems are included in the determination of inertial frames, there is more to this observation than the simple mathematical fact that a Maxwellian connection is locally Galilean, or the physical fact that the non-gravitational laws are Galilei-covariant. The existence of solutions of the gravitational field equations that satisfy $\lim_{|x|\to\infty} V(x) = 0$ is also required. (Something similar occurs in general relativity; cf Wallace (2009).)

This idea of a 'background inertial structure' may seem odd. Shouldn't spacetime just have an inertial structure, once and for all? Not if 'spacetime structure' is to be understood in terms of inertial frames, and inertial frames are to be understood in turn in terms of the dynamics of subsystems. The Earth-Moon system, for instance, is not usefully thought of as evolving in the inertial structure set by the whole Universe, including Earth and Moon themselves; rather, the inertial frames relevant for that system are determined by the gravitational effects of the rest of the matter in the Universe (most saliently the Sun).

This might be called a *scale relativity* of inertial structure (and thus, on Knox's functionalism, of spacetime geometry). The inertial structure seen by (i. e., relevant to) the Earth-Moon system does not include the effects on geodesics of the Earth and Moon themselves: that is an additional interaction between Earth and Moon layered on top of the geometry defined by the Sun. But the inertial structure seen by a system *on* the surface of the Earth does include the Earth's (and the Moon's) contribution.

Scale relativity of inertial structure is actually quite common in applications of Newtonian gravity. Consider, for instance, the dynamics of the stars in the

Galaxy.[9] To first approximation we treat them as a continuous dust, as in section 7, and thus deduce a smoothly varying Galactic gravitational potential. But in reality, there are finitely many stars, and so the potential of the Galaxy is much spikier. And close encounters between stars — "collisions" in the terminology of galactic dynamics, though there is no physical collision — cause deviation in the evolution of the actual Galactic distribution away from what the dust theory would predict.

A standard way to handle this (in somewhat crude form) is to break the gravitational potential $m/|x|$ of each star into two parts: a near-field part covering (say) the potential within 10 parsecs of the star, and a long-range part covering the rest. The total potential V_{LR} generated by the long-range terms for each star is taken as a background potential, determining a background Maxwellian connection; it is, in effect, a smoothing out of the exact potential V. The near-field terms then allow for scattering events between close stars, against that overall background: the trajectory of a star is given by an expression like

$$\ddot{x}^i(t) = -\partial_i V_{LR}(x) + \text{collision terms}. \tag{67}$$

With some care, this approach can be parleyed up into the *Boltzmann-Vlasov* equation, which governs the evolution of a stellar distribution under both long-range interaction with its own smoothed-out connection and short-range collisions. Knox's functionalist recipe tells us that it is the averaged connection determined by V_{LR}, not the exact potential V, that plays the role of inertial structure for this system.

The same thing happens on a still larger scale. A galaxy is a huge, complicatedly structured system with a very deep gravity well; from the point of view of a star in that galaxy, the galaxy's contribution to the inertial structure dominates that of the rest of the Universe. But to the cosmologist, the galaxy is a point particle in a gas of galaxies. From the point of view of the galaxy as a whole, the relevant inertial structure is the connection defined by the distribution of matter across the whole Universe: averaged, that is, not over parsecs but over megaparsecs.

Something similar also occurs in general relativity, although there caution is needed because the equations are nonlinear and so we cannot simply decompose the metric field into components and attribute them to different subsystems. In particular, consider gravity waves. It is absolutely standard in astrophysical applications to talk about a gravity wave as propagating through some region of spacetime, in exactly the same way that an electromagnetic wave so propagates. If spacetime is definitionally given by the metric field, this does not literally make sense: the gravity wave is a disturbance *of* spacetime, not *in* spacetime. What is meant *mathematically* speaking is that we have linearised the Einstein field equations around some solution g_0:

$$g(x) = g_0(x) + \delta g(x) \tag{68}$$

[9] For technical references see Binney and Tremaine (2008) and references therein.

so that the perturbation δg satisfies a wave equation. The spacetime through which the wave is taken to propagate is that defined by g_0, not that defined by g — and again, a functionalist account licenses us to regard g_0 and not g as determining the true inertial structure for the gravity wave, and for any matter scattering off it. (I discuss the embedding of self-gravitating subsystems into larger systems in general relativity in more detail in Wallace (2009).)

Scale-relativity of spacetime structure is not mentioned explicitly in Knox's, and Browns', own discussions of spacetime structure, probably because (as noted previously) they confine their analyses to non-gravitational interactions, which are typically extremely short-range, and so tend to probe the finest-grained features of the gravitational potential. But when that restriction is dropped, it seems to be a natural consequence of Knox's spacetime functionalism — and a welcome consequence, fitting naturally with usage in contemporary astrophysics.

9 Reconciling Knox and Saunders

I have argued that there is essentially no difference between Newton-Cartan theory (Knox's preferred understanding of Newtonian dynamics) and Saunders' relational version of Newtonian dynamics: at the formal level the latter can be reformulated as the former; at the substantive level, the inertial structure of Saunders' theory is well-defined and coincides with that defined by the Newton-Cartan connection.

Knox and Saunders themselves disagree:

> Saunders and I arrive at different conclusions (Knox 2014, p.875)

> What is the relation between a theory of gravity (and other forces) formulated in Maxwell space-time, and one based on Newton-Cartan spacetime? In the latter there is a notion of parallelism for time-like as well as spacelike vectors. That notion, we must conclude, is dispensable, to be derived, if at all, by fixing of gauge. Saunders 2013, p.46

The appearance of disagreement has, I think, two sources. The first is technical: the standard presentations of Newton-Cartan spacetime and Maxwellian spacetime look very different, partly because the former is standardly cast in the language of differential geometry whereas the latter (in particular, its rotation standard) is very awkward in that language, and partly because in Newton-Cartan theory, the connection does double duty, imposing both the rotation standard (a piece of absolute structure) and the inertial structure (something dynamical and contingent). One purpose of my somewhat idiosyncratic presentation of Newton-Cartan theory is to emphasise the fact that the Newton-Cartan connection is naturally understood as an additional piece of structure added to Maxwellian spacetime; indeed, as the Maxwellian version of the affine connection.

The second is more important: Knox and Saunders are operating with very different conceptions of the physical content of Newton-Cartan spacetime. In Saunders' view, to include spacetime structure in a theory is to add that structure as part of the theory's basic posits. And since Newtonian dynamics can be formulated in an empirically adequate way without any reference to spacetime structure, that structure should not be reified: it is 'unneeded surplus structure' (Saunders 2013, p.46). Similarly, Saunders suggests that Newton-Cartan gravity allows the drawing of unphysical distinctions between possibilities:

> Take possible worlds each with only a single, structureless particle. Depending on the connection, there will be infinitely-many distinct trajectories, infinitely-many distinct worlds of this kind. But in [Saunders' version of Newtonian dynamics], as in Barbour-Bertotti theory, there is only one such world — a trivial one, in which there are no meaningful predications of the motion of the particle at all. Only for worlds with two or more particles can distinctions among motions be drawn. From the point of view of the latter theories, the fault lies with introducing a non-trivial connection curvature without any source, unrelated to the matter distribution. At a deeper level, it is with introducing machinery — a standard of parallelism for time-like vectors, defined even for a single particle — that from the point of view of a relationalist conception of particle motions is unintelligible.

These comments presuppose a conception of spacetime like that of Friedman (1983), Earman (1989), or Maudlin (2012) — a conception according to which the spacetime structure is conceptually and ontologically independent of matter, and where the connections between the two — the equations that link the connection to the mass distribution, and that state that test particles follow geodesics except when deflected by non-gravitational forces — are laws of nature and not conceptual truths. On this conception, a "Newton-Cartan spacetime" is just an arbitrary distribution of matter on Maxwellian spacetime, plus an arbitrary Maxwellian connection. Some of those spacetimes — the ones where the scalar curvature is proportional to the mass density and where the motion of particles is given in terms of geodesics — are nomically possible, and others are not, but they are equally legitimate as metaphysical possibilities. And then, indeed, that spacetime structure looks like surplus structure, and these claims about the spacetime background for a single particle look inappropriate.

But this is not Knox's (nor Brown's) interpretation of the Newton-Cartan connection. To Knox, the connection's interpretation as inertial structure is derivative on the matter dynamics; the geodesic equation for particles *defines* the connection rather than asserting a lawlike relationship between independent entities. So the Newton-Cartan connection is defined in the single-particle world — if at all — by the counterfactuals as to how other particles would move if they were added to that world. Insofar as these counterfactuals are indeterminate (perhaps because a Humean view of laws (Lewis 1973, 1986) is assumed), so is the Newton-Cartan connection; in any event, Newton-Cartan theory *a la* Knox draws its distinctions no finer than Newtonian mechanics *a la* Saunders.

10 Conclusions

Newtonian physics, regarded as the physics for the Universe as a whole (or for an isolated subsystem as observed from within that subsystem) needs only Maxwellian spacetime as background geometry; in particular, no affine structure need be specified in advance to make sense of its dynamics.

However, a Newtonian dynamical system does determine an inertial structure, which gets its operational significance from the behaviour of isolated subsystems of that system and its mathematical representation as a connection compatible with the Maxwellian background. If we adopt Knox's functionalist approach to spacetime structure, the spacetime geometry of Newtonian physics is then just this connection — albeit the effective spacetime geometry experienced by a given system may instead be a coarse-graining of that geometry.

Furthermore, the dynamics of those isolated subsystems, as expressed relative to that inertial structure, is Galilean- rather than Maxwell-covariant, and so Galilean spacetime remains the right setting for the Newtonian dynamics of isolated systems as studied from outside those systems.

These conclusions follow whether we begin with Newton's theory in long-range-force form and follow Saunders in reformulating it to speak only of relative accelerations, or instead begin with Newton's theory in potential form and follow Knox in relaxing its boundary conditions and so dropping its background inertial structure. As such, the apparent discrepancy between Knox's and Saunders' conception of Newtonian physics masks underlying, implicit agreement.

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