A Relevant Framework for Barriers to Entailment

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Abstract

In her recent book, Russell (2023) examines various so-called "barriers to entailment," including Hume's law, roughly the thesis that an 'ought' cannot be derived from an 'is.' Hume's law bears an obvious resemblance to the proscription on fallacies of modality in relevance logic, which has traditionally formally been captured by the so-called Ackermann property. In the context of relevant modal logic, this property might be articulated thus: no conditional whose antecedent is box-free and whose consequent is box-prefixed is valid (for the connection, interpret box deontically). While the deontic significance of Ackermann-like properties has been observed before, Russell's new book suggests a more broad-scoped formal investigation of the relationship between barrier theses of various kinds and corresponding Ackermann-like properties. In this paper, I undertake such an investigation by elaborating a general relevant bimodal logical framework in which several of the barriers Russell examines can be given formal expression. I then consider various Ackermann-like properties corresponding to these barriers and prove that certain systems satisfy them. Finally, I respond to some objections Russell makes against the use of relevance logic to formulate Hume's law and related barriers.

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Ackermann property, Barriers to entailment, Deontic logic, Fallacies of modality, Hume's law, Multimodal logic, Relevance logic.

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1 Introduction

In her recent book [43], Russell examines various so-called *barriers to entailment*, including Hume's law, roughly the thesis that an 'ought' cannot be derived from an 'is' (or, more generally, that no collection of descriptive claims entails any normative claim). The law bears an obvious resemblance to the proscription on fallacies of modality traditionally used to motivate relevant systems of entailment especially (see, e.g., Anderson and Belnap [5, §5.2]). This proscription has formally been captured by the so-called *Ackermann property* (see, e.g., Ackermann [1, §6] and Anderson and Belnap [5, §§5.2, 8.12]), which in the context of relevant modal logic might be articulated thus (cf. Meyer [33, p. 476]): no conditional of the form $\varphi \to \Box\psi$ is valid, where φ is $\Box$-free (for the connection, read $\Box$ deontically).[1] The deontic significance of Ackermann-like properties has been observed before, for example, by Mares [29]. Nevertheless, Russell's new book suggests a more broad-scoped formal investigation of the relationship between barrier theses of various kinds and corresponding Ackermann-like properties.

The primary purpose of this paper is to pursue such a formal investigation. To that end, in Section 2, I present a fairly general relevant bimodal framework in which several (though not all) of the barriers Russell [43] is concerned with can be given formal expression. Routley-Meyer semantics is canvassed for a variety of (bi)modal extensions of **R** and soundness and completeness theorems are sketched in each case.

In Sections 3, 4, and 5, this framework is applied to present relevant systems of modal logic, deontic logic, and tense logic, all of which satisfy versions of the Ackermann property corresponding to particular barrier theses studied by Russell [43]. The exact formulation of the Ackermann property depends on the system and language under consideration, but proofs are in each case given using matrices.[2] Pertinent historical and philosophical issues are discussed as appropriate.

Finally, in Section 6, I close by examining some of the critical remarks Russell [43, pp. 36–41] makes against the use relevance logic to formulate barrier theses. Against Russell, I contend that relevance logic, and the (bi)modal relevant framework propounded in this paper in particular, is a very natural setting in which to express barrier theses of various kinds, and prove that they obtain.

The foregoing notwithstanding, let me add some remarks about what this paper is not. This paper is not intended as a book review or critical notice of [43]. My engagement with Russell's critical comments will be selective, and my engagement

[1] The Ackermann property was first examined (by Ackermann [1, §6]) as a desideratum to be imposed on a primitive entailment connective. Anderson and Belnap [5, §5.2] spell out the property in connection with $\mathbf{E}_{\to}$ thus: $\varphi \to (\psi \to \theta)$ is never a theorem when φ is a propositional variable.

[2] Many (though not all) of these matrices were found using Slaney's program MaGIC [49].

with her positive proposals will be minimal. A more thorough comparison of the relative merits of the proposals made here with her own would be desirable, but such a project is left for future work.

2 A relevant bimodal framework

Fix a countable set of propositional variables $\Pi = \{p_0, p_1, \ldots\}$. The set of formulae, Φ, is built up in the usual way from Π and the connectives $\neg, \vee, \wedge, \to, \Box$, and possibly, $\blacksquare$. (I mostly leave implicit what the language under discussion is.) The usual definition $\Diamond\varphi := \neg\Box\neg\varphi$ (and, if applicable, $\blacklozenge\varphi := \neg\blacksquare\neg\varphi$) is adopted.[3] I write $p, q, \ldots$, for arbitrary propositional variables, and $\varphi, \psi, \ldots$, for arbitrary formulae.

Throughout, I take **R** as the background relevant logic. (I have no special love for **R**; however, it is well-known, and many interesting results that can be proved for **R** carry over, mutatis mutandis, to weaker systems.) For concreteness, let **R** be axiomatized thus:[4]

$$\varphi \to \varphi \tag{I}$$

$$(\varphi \to \psi) \to ((\psi \to \theta) \to (\varphi \to \theta)) \tag{B$'$}$$

$$(\varphi \to (\psi \to \theta)) \to (\psi \to (\varphi \to \theta)) \tag{C}$$

$$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi) \tag{W}$$

$$(\varphi \wedge \psi) \to \varphi \tag{$\wedge$E1}$$

$$(\varphi \wedge \psi) \to \psi \tag{$\wedge$E2}$$

$$((\varphi \to \psi) \wedge (\varphi \to \theta)) \to (\varphi \to (\psi \wedge \theta)) \tag{$\wedge$I}$$

$$\varphi \to (\varphi \vee \psi) \tag{$\vee$I1}$$

[3]Incidentally, because relevant negation is nonclassical, this has the downstream effect that the truth condition for $\Diamond$ (and $\blacklozenge$) is somewhat nonstandard (cf. Fuhrmann [18, pp. 505–506]).

[4]This is the axiomatization given by Anderson and Belnap [5, p. 341], except that I have used C in lieu of their R3 (Assertion) and $\wedge\vee$ in lieu of their R11; the equivalence of these axiomatizations of **R** is well-known.

$$\psi \to (\varphi \vee \psi) \tag{$\vee$I2}$$

$$((\varphi \to \theta) \wedge (\psi \to \theta)) \to ((\varphi \vee \psi) \to \theta) \tag{$\vee$E}$$

$$(\varphi \wedge (\psi \vee \theta)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \theta)) \tag{$\wedge\vee$}$$

$$\neg\neg\varphi \to \varphi \tag{DNE}$$

$$(\varphi \to \neg\psi) \to (\psi \to \neg\varphi) \tag{CP}$$

$$\frac{\varphi, \varphi \to \psi}{\psi} \tag{MP}$$

$$\frac{\varphi, \psi}{\varphi \wedge \psi} \tag{ADJ}$$

I turn now to (bi)modal extensions of **R**. My approach throughout largely follows Fuhrmann [17, 18] who, however, was only concerned with monomodal relevant logic.[5] To obtain the basic regular (bi)modal extension of **R**, $\mathbf{R}^{\Box}$ ($\mathbf{R}^{\Box}_{\blacksquare}$), add the following:

$$\frac{\varphi \to \psi}{\boxdot\varphi \to \boxdot\psi}, \text{ where } \boxdot \text{ is } \Box \text{ (or } \blacksquare\text{)} \tag{RM}$$

$$(\boxdot\varphi \wedge \boxdot\psi) \to \boxdot(\varphi \wedge \psi), \text{ where } \boxdot \text{ is } \Box \text{ (or } \blacksquare\text{)} \tag{$\wedge$C}$$

[5] There is relatively little discussion of relevant bimodal (or multimodal) logic to be found in the literature. Seki [46, 47, 48] invariably treats $\Box$ and $\Diamond$ as primitive connectives with independent accessibility relations (cf. Routley [39, pp. 273–276]), and so in a sense gives a bimodal treatment of what is usually thought of as a monomodal logic. (This is an established approach in modal logic over logics with a nonclassical negation; see, e.g., the intuitionistic system $\mathbf{HK}\Box\Diamond$ from Božić and Došen [9].) Wansing [54] presents a bimodal relevant epistemic logic. Cheng has sketched purely syntactic developments of versions of relevant tense logic and multimodal tense-deontic logic in a series of short papers [11, 12, 13]. Standefer [50, §6.4] devotes some discussion to relevant logics containing combinations of an actuality operator, an alethic modal operator, and a fixedly operator. The work in this section is more substantially anticipated by Routley [39, §2], which I was unfamiliar with when I first drafted this piece. While Routley articulates a general relevant multimodal framework and even discusses a (somewhat peculiar) bimodal relevant tense logic [39, p. 276], his approach and my own nevertheless differ in at least some important respects (e.g., in generally treating diamonds as primitives). Thanks to Shawn Standefer for a number of valuable reference pointers here.

$\mathbf{R}^{\Box}$ is the system $R.C$ from Fuhrmann [18]. Further systems of interest are obtained by adding selections of the following axiom schemata and rules (in general, my interest will be in bimodal logics where different principles are adopted for each of the modal connectives, whence the redundancy):

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \tag{$\Box$K}$$

$$\blacksquare(\varphi \to \psi) \to (\blacksquare\varphi \to \blacksquare\psi) \tag{$\blacksquare$K}$$

$$\Box\neg\varphi \to \neg\Box\varphi \tag{$\Box$D}$$

$$\blacksquare\neg\varphi \to \neg\blacksquare\varphi \tag{$\blacksquare$D}$$

$$\Box\varphi \to \varphi \tag{$\Box$T}$$

$$\blacksquare\varphi \to \varphi \tag{$\blacksquare$T}$$

$$\Box\varphi \to \Box\Box\varphi \tag{$\Box$4}$$

$$\blacksquare\varphi \to \blacksquare\blacksquare\varphi \tag{$\blacksquare$4}$$

$$\varphi \to \Box\Diamond\varphi \tag{$\Box$B}$$

$$\varphi \to \blacksquare\blacklozenge\varphi \tag{$\blacksquare$B}$$

$$\blacklozenge\Box\varphi \to \varphi \tag{$\blacklozenge\Box$}$$

$$\Diamond\blacksquare\varphi \to \varphi \tag{$\Diamond\blacksquare$}$$

$$\Box\neg\varphi \to \neg\blacksquare\varphi \tag{$\Box\blacksquare$D}$$

$$\frac{\varphi}{\Box\varphi} \tag{$\Box$NEC}$$

$$\frac{\varphi}{\blacksquare\varphi} \quad (\blacksquare\text{NEC})$$

The smallest extension of $\mathbf{R}^{\square}$ ($\mathbf{R}^{\square}_{\blacksquare}$) by axioms/rules from some subset $\mathcal{C}$ of the list above will be written $\mathbf{R}^{\square}(\mathcal{C})$ ($\mathbf{R}^{\square}_{\blacksquare}(\mathcal{C})$). Where no particular such extension is intended, I will usually just write $\mathbf{L}$. I adopt the standard useful fiction that a logic $\mathbf{L}$ is a set of formulae and identify theoremhood for $\mathbf{L}$ ($\vdash_{\mathbf{L}}$) with membership. Bootstrapping, $\mathbf{L}$-derivability is defined thus:

Definition 1. A set Δ is $\mathbf{L}$-*derivable* from a set Γ (in symbols, $\Gamma \vdash_{\mathbf{L}} \Delta$) if and only if there are some $\gamma_1, \ldots, \gamma_m \in \Gamma$ and $\delta_1, \ldots, \delta_n \in \Delta$ such that $\vdash_{\mathbf{L}} \bigwedge \gamma_i \to \bigvee \delta_j$.

I turn now to matters semantic. The basic framework I will avail myself to is the (unreduced) Routley-Meyer semantics (see, e.g., Routley et al. [41, Ch. 4]):

Definition 2 (Frame)**.** A *frame* for $\mathbf{R}$ is a structure $\mathfrak{F} = \langle W, N, R, ^* \rangle$, where $\emptyset \neq N \subseteq W$, $R \subseteq W^3$, and $^* : W \to W$, and these components satisfy the following postulates:[6]

p1. $a \leq a$;

p2. $a \leq b$ and $Rbcd$ imply $Racd$;

p3. $a = a^{**}$;

p4. $Rabc$ implies Rac^*b^*;

p5. $R^2(ab)cd$ implies $R^2b(ac)d$;

p6. $R^2(ab)cd$ implies $R^2(ac)bd$;

p7. $Rabc$ implies $R^2(ab)bc$.

Generalizing Fuhrmann [17, 18], to get a frame for $\mathbf{R}^{\square}_{\blacksquare}$, two new components are added to Definition 2 (in all of what follows, for $\mathbf{R}^{\square}$ and its monomodal extensions, just leave out the material specific to $\blacksquare$):

Definition 3 (Modal frame)**.** A *modal frame* for $\mathbf{R}^{\square}_{\blacksquare}$, $\mathfrak{F} = \langle W, N, R, ^*, S_{\square}, S_{\blacksquare} \rangle$, is an $\mathbf{R}$ frame with two new binary relations $S_{\square}, S_{\blacksquare} \subseteq W^2$ which satisfy the further condition:[7]

[6] Abbreviations: $a \leq b := \exists x(x \in N \wedge Rxab)$, $R^2(ab)cd := \exists x(Rabx \wedge Rxcd)$, and $R^2a(bc)d := \exists x(Raxd \wedge Rbcx)$.

[7] Abbreviation: $S_{\boxdot}(a) = \{b \in W : S_{\boxdot}ab\}$, where $\boxdot \in \{\square, \blacksquare\}$.

p8. $a \leq b$ implies $S_{\boxdot}(b) \subseteq S_{\boxdot}(a)$, where $\boxdot \in \{\Box, \blacksquare\}$.

Frames for extensions **L** of $\mathbf{R}^{\Box}$ ($\mathbf{R}^{\Box}_{\blacksquare}$) are obtained by imposing constraints which correspond to the associated axiom schemata and rules (such frames are called *fit for* **L**):[8]

	Axiom/Rule	Frame Condition
p9	$\Box$K	$\exists x(Rabx \wedge S_{\Box}xc) \Rightarrow \exists y \exists z(S_{\Box}ay \wedge S_{\Box}bz \wedge Ryzc)$
p10	$\blacksquare$K	$\exists x(Rabx \wedge S_{\blacksquare}xc) \Rightarrow \exists y \exists z(S_{\blacksquare}ay \wedge S_{\blacksquare}bz \wedge Ryzc)$
p11	$\Box$D	$\exists x(S_{\Box}ax^* \wedge S_{\Box}a^*x)$
p12	$\blacksquare$D	$\exists x(S_{\blacksquare}ax^* \wedge S_{\blacksquare}a^*x)$
p13	$\Box$T	$S_{\Box}aa$
p14	$\blacksquare$T	$S_{\blacksquare}aa$
p15	$\Box$4	$S_{\Box}ab \wedge S_{\Box}bc \Rightarrow S_{\Box}ac$
p16	$\blacksquare$4	$S_{\blacksquare}ab \wedge S_{\blacksquare}bc \Rightarrow S_{\blacksquare}ac$
p17	$\Box$B	$S_{\Box}ab \Rightarrow S_{\Box}b^*a^*$
p18	$\blacksquare$B	$S_{\blacksquare}ab \Rightarrow S_{\blacksquare}b^*a^*$
p19	$\blacklozenge\Box$	$S_{\blacksquare}ab \Rightarrow S_{\Box}b^*a^*$
p20	$\Diamond\blacksquare$	$S_{\Box}ab \Rightarrow S_{\blacksquare}b^*a^*$
p21	$\Box\blacksquare$D	$\exists x(S_{\blacksquare}a^*x \wedge S_{\Box}ax^*)$
p22	$\Box$NEC	$a \in N \wedge S_{\Box}ab \Rightarrow b \in N$
p23	$\blacksquare$NEC	$a \in N \wedge S_{\blacksquare}ab \Rightarrow b \in N$

Definition 4 (Model)**.** A *model* fit for a system **L** extending $\mathbf{R}^{\Box}_{\blacksquare}$ is a structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, where $\mathfrak{F} = \langle W, N, R, ^*, S_{\Box}, S_{\blacksquare} \rangle$ is a frame (Definition 3) fit for **L** and $V : \Pi \to \mathcal{P}(W)$ subject to the heredity condition: $x \in V(p)$ and $x \leq y$ imply $y \in V(p)$.

Given a model $\mathfrak{M} = \langle W, N, R, ^*, S_{\Box}, S_{\blacksquare}, V \rangle$ and $w \in W$, the relation $\models^{\mathfrak{M}}_{w}$ is defined by the following conditions:[9]

p. $\models^{\mathfrak{M}}_{w} p$ iff $w \in V(p)$;

$\neg$. $\models^{\mathfrak{M}}_{w} \neg\varphi$ iff $\not\models^{\mathfrak{M}}_{w^*} \varphi$;

$\wedge$. $\models^{\mathfrak{M}}_{w} \varphi \wedge \psi$ iff $\models^{\mathfrak{M}}_{w} \varphi$ and $\models^{\mathfrak{M}}_{w} \psi$;

$\vee$. $\models^{\mathfrak{M}}_{w} \varphi \vee \psi$ iff $\models^{\mathfrak{M}}_{w} \varphi$ or $\models^{\mathfrak{M}}_{w} \psi$;

[8] Many of these correspondences are noted by Fuhrmann [18, pp. 507–508] (cf. Routley [39, pp. 275–276]).

[9] Abbreviation: $[\varphi]^{\mathfrak{M}} = \{w \in W : \models^{\mathfrak{M}}_{w} \varphi\}$.

$\to$. $\models^{\mathfrak{M}}_{w} \varphi \to \psi$ iff, for all $x, y \in W$, if $Rwxy$ and $\models^{\mathfrak{M}}_{x} \varphi$, then $\models^{\mathfrak{M}}_{y} \psi$;

$\boxdot$. $\models^{\mathfrak{M}}_{w} \boxdot\varphi$ iff $S_{\boxdot}(w) \subseteq [\varphi]^{\mathfrak{M}}$, where $\boxdot \in \{\Box, \blacksquare\}$.

The following lemma, used in the argument for Theorem 1, has a routine proof which I omit.

Lemma 1 (Heredity). *For any model $\mathfrak{M} = \langle W, N, R, ^*, S_{\Box}, S_{\blacksquare}, V\rangle$ and $a, b \in W$, if $a \leq b$ and $\models^{\mathfrak{M}}_{a} \varphi$, then $\models^{\mathfrak{M}}_{b} \varphi$.*

Definition 5 (Validity). A formula φ is valid over a model $\mathfrak{M}$ (in symbols, $\models^{\mathfrak{M}} \varphi$) if for all $w \in N$, $\models^{\mathfrak{M}}_{w} \varphi$. φ is *valid* in the system $\mathbf{L}$ ($\models_{\mathbf{L}} \varphi$) if for all models $\mathfrak{M}$ fit for $\mathbf{L}$, $\models^{\mathfrak{M}} \varphi$.

Theorem 1 (Soundness). *For any of the systems $\boldsymbol{L}$, if $\vdash_{\boldsymbol{L}} \varphi$, then $\models_{\boldsymbol{L}} \varphi$.*

Proof. Just the usual checking of cases. All of the nonmodal and monomodal cases can be found in the literature (see, e.g., Routley et al. [41] and Fuhrmann [17, 18]). I quickly run through the mixed-modality cases here.

Ad $\blacklozenge\Box$: if $\mathbf{L}$ contains $\blacklozenge\Box$ and $\not\models_{\mathbf{L}} \blacklozenge\Box\varphi \to \varphi$, then there is a model $\mathfrak{M} = \langle W, N, R, ^*, S_{\Box}, S_{\blacksquare}, V\rangle$ fit for $\mathbf{L}$ and $a, b \in W$ such that $a \leq b$, $\models^{\mathfrak{M}}_{a} \blacklozenge\Box\varphi$ (i.e., $\models^{\mathfrak{M}}_{a} \neg\blacksquare\neg\Box\varphi$), and $\not\models^{\mathfrak{M}}_{b} \varphi$. Then $\models^{\mathfrak{M}}_{a} \neg\blacksquare\neg\Box\varphi$ implies $\not\models^{\mathfrak{M}}_{a^*} \blacksquare\neg\Box\varphi$ implies $\not\models^{\mathfrak{M}}_{c} \neg\Box\varphi$ for some c such that $S_{\blacksquare}a^*c$, which implies $\models^{\mathfrak{M}}_{c^*} \Box\varphi$. By p3 and p19, $S_{\blacksquare}a^*c$ implies $S_{\Box}c^*a$, whence $\models^{\mathfrak{M}}_{c^*} \Box\varphi$ implies $\models^{\mathfrak{M}}_{a} \varphi$; by Lemma 1, $\models^{\mathfrak{M}}_{a} \varphi$ implies $\models^{\mathfrak{M}}_{b} \varphi$, a contradiction. (The case of $\Diamond\blacksquare$ is entirely analogous to this case and is omitted.)

Ad $\Box\blacksquare$D: if $\mathbf{L}$ contains $\Box\blacksquare$D and $\not\models_{\mathbf{L}} \Box\neg\varphi \to \neg\blacksquare\varphi$, then there is a model $\mathfrak{M} = \langle W, N, R, ^*, S_{\Box}, S_{\blacksquare}, V\rangle$ fit for $\mathbf{L}$ and $a, b \in W$ such that $a \leq b$, $\models^{\mathfrak{M}}_{a} \Box\neg\varphi$, and $\not\models^{\mathfrak{M}}_{b} \neg\blacksquare\varphi$, whence $\models^{\mathfrak{M}}_{b^*} \blacksquare\varphi$. By p21, there is some c such that $S_{\blacksquare}b^*c$ and $S_{\Box}bc^*$. Thus, $\models^{\mathfrak{M}}_{c} \varphi$, and by p8 and $a \leq b$, $S_{\Box}ac^*$, whence $\models^{\mathfrak{M}}_{c^*} \neg\varphi$, and so $\not\models^{\mathfrak{M}}_{c} \varphi$, a contradiction. □

For completeness, a number of preliminary definitions and results concerning theory building are required (cf. Routley et al. [41, pp. 306–307]). Fix a system $\mathbf{L}$. An $\mathbf{L}$-*theory* is a set of formulae Γ closed under ADJ and such that if $\varphi \in \Gamma$ and $\vdash_{\mathbf{L}} \varphi \to \psi$, then $\psi \in \Gamma$. An $\mathbf{L}$-theory Γ is *regular* if $\mathbf{L} \subseteq \Gamma$, and *prime* if whenever $\varphi \vee \psi \in \Gamma$, then either $\varphi \in \Gamma$ or $\psi \in \Gamma$. A pair $\langle\Gamma, \Delta\rangle$ is $\mathbf{L}$-*maximal* iff 1) $\Gamma \not\vdash_{\mathbf{L}} \Delta$ and 2) $\Gamma \cup \Delta = \Phi$.

The following standard results (and others besides), rehearsed in detail in Routley et al. [41, pp. 307–312], carry over wholesale (the key result is Lemma 2, from which much of the rest ultimately follow):

Lemma 2 (Pair extension). *If $\Gamma \not\vdash_{\boldsymbol{L}} \Delta$, then there is an $\boldsymbol{L}$-maximal pair $\langle\Gamma', \Delta'\rangle$ such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.*

Corollary 1 (Priming)**.** *If Γ is an $\boldsymbol{L}$-theory and Δ is a set of formulae disjoint from Γ and such that for any $\varphi, \psi \in \Delta$ (distinct), $\varphi \vee \psi \in \Delta$, then there is a prime $\boldsymbol{L}$-theory Γ' such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \cap \Delta = \emptyset$.*

Corollary 2 (Witness)**.** *If $\nvdash_{\boldsymbol{L}} \varphi$, then there is a prime regular $\boldsymbol{L}$-theory Π such that $\varphi \notin \Pi$.*

Given sets of formulae Γ, Δ, and Σ, define $R^c\Gamma\Delta\Sigma$ iff $\{\psi : \exists\varphi \in \Delta(\varphi \to \psi \in \Gamma)\} \subseteq \Sigma$.

Corollary 3. *For $\boldsymbol{L}$-theories Γ, Δ, Σ, where Σ is prime, if $R^c\Gamma\Delta\Sigma$, then there is a prime $\boldsymbol{L}$-theory Π such that $\Delta \subseteq \Pi$ and $R^c\Gamma\Pi\Sigma$.*

Corollary 4. *For $\boldsymbol{L}$-theories Γ, Δ, Σ, where Σ is prime, if $R^c\Gamma\Delta\Sigma$, then there is a prime $\boldsymbol{L}$-theory Π such that $\Gamma \subseteq \Pi$ and $R^c\Pi\Delta\Sigma$.*

Definition 6 (Canonical model)**.** The canonical model for **L** is the structure $\mathfrak{M}^c = \langle W^c, N^c, R^c, ^{*c}, S^c_{\Box}, S^c_{\blacksquare}, V^c \rangle$ defined as follows:

1. W^c is the set of all prime **L**-theories;
2. N^c is the set of all regular prime **L**-theories;
3. $R^c\Gamma\Delta\Sigma$ iff $\{\psi : \exists\varphi \in \Delta(\varphi \to \psi \in \Gamma)\} \subseteq \Sigma$ (as above);[10]
4. $\Gamma^{*c} = \{\varphi : \neg\varphi \notin \Gamma\}$;
5. $S^c_{\Box}\Gamma\Delta$ iff $\{\varphi : \Box\varphi \in \Gamma\} \subseteq \Delta$;
6. $S^c_{\blacksquare}\Gamma\Delta$ iff $\{\varphi : \blacksquare\varphi \in \Gamma\} \subseteq \Delta$;
7. $V^c(p) = \{\Gamma : p \in \Gamma\}$.

Observe that in the canonical model for **L**, $\mathfrak{M}^c = \langle W^c, N^c, R^c, ^{*c}, S^c_{\Box}, S^c_{\blacksquare}, V^c \rangle$, $\leq^c = \subseteq$.

Lemma 3. *The canonical model for $\boldsymbol{L}$, $\mathfrak{M}^c$, is a model fit for $\boldsymbol{L}$.*

Proof. I ignore the nonmodal cases, and cover the cases of p8, p9, p20, p21, and p22, as representative. (For some of the arguments given below, cf. Fuhrmann [17, pp. 48–50] and Routley [39, pp. 277–278].)

[10] Strictly speaking, as defined above, $R^c \subseteq \mathcal{P}(\Phi)^3$, whereas in compliance with Definition 2, $R^c \subseteq (W^c)^3$ here. It proves convenient and causes no real confusion to employ a wider notion at various points (similarly for $S^c_{\Box}, S^c_{\blacksquare}$).

Ad p8: suppose $\Gamma \subseteq \Delta$, $S^c_{\boxdot}\Delta\Sigma$, and $\boxdot\varphi \in \Gamma$, to show that $\varphi \in \Sigma$; this will suffice to show $S^c_{\boxdot}\Gamma\Sigma$. Then $\boxdot\varphi \in \Gamma \subseteq \Delta$ and $\{\psi : \boxdot\psi \in \Delta\} \subseteq \Sigma$, so in particular, $\varphi \in \Sigma$.

Ad p9: suppose **L** contains $\Box$K and that $R^c\Gamma\Delta\Sigma$ and $S^c_{\Box}\Sigma\Pi$; it must be shown that there are prime **L**-theories Λ and Ξ such that $S^c_{\Box}\Gamma\Lambda$, $S^c_{\Box}\Delta\Xi$, and $R^c\Lambda\Xi\Pi$. Put $\Lambda' := \{\varphi : \Box\varphi \in \Gamma\}$ and $\Xi' := \{\varphi : \Box\varphi \in \Delta\}$. It is clear that Λ' and Ξ' are **L**-theories such that $S^c_{\Box}\Gamma\Lambda'$ and $S^c_{\Box}\Delta\Xi'$. Suppose $\varphi \in \Xi'$ and $\varphi \to \psi \in \Lambda'$; then $\Box(\varphi \to \psi) \in \Gamma$, whence by $\Box$K, $\Box\varphi \to \Box\psi \in \Gamma$. Since also $\Box\varphi \in \Delta$, $\Box\psi \in \Sigma$ by the hypothesis that $R^c\Gamma\Delta\Sigma$, and so $\psi \in \Pi$ by the hypothesis that $S^c_{\Box}\Sigma\Pi$. Therefore, $R^c\Lambda'\Xi'\Pi$. By Corollaries 3 and 4, there are prime **L**-theories Λ and Ξ such that $\Lambda' \subseteq \Lambda$ and $\Xi' \subseteq \Xi$ (whence, $S^c_{\Box}\Gamma\Lambda$ and $S^c_{\Box}\Delta\Xi$), and $R^c\Lambda\Xi\Pi$, as desired.

Ad p20: suppose **L** contains $\Diamond\blacksquare$, that $S^c_{\Box}\Gamma\Delta$, and that $\blacksquare\varphi \in \Delta^{*c}$. Then $\neg\blacksquare\varphi \notin \Delta$, whence $\Box\neg\blacksquare\varphi \notin \Gamma$, whence $\neg\Box\neg\blacksquare\varphi \in \Gamma^{*c}$. But **L** contains $\Diamond\blacksquare$ and Γ^{*c} is an **L**-theory, whence $\varphi \in \Gamma^{*c}$, as desired.

Ad p21: suppose **L** contains $\Box\blacksquare$D and fix an arbitrary prime **L**-theory Γ. It is to be shown that there is some prime **L**-theory Δ such that $S^c_{\blacksquare}\Gamma^{*c}\Delta$ and $S^c_{\Box}\Gamma\Delta^{*c}$. Define $\Delta' := \{\varphi : \blacksquare\varphi \in \Gamma^{*c}\}$; clearly, Δ' is an **L**-theory and $S^c_{\blacksquare}\Gamma^{*c}\Delta'$. Moreover, suppose $\Box\varphi \in \Gamma$; then since **L** contains $\Box\blacksquare$D, $\neg\blacksquare\neg\varphi \in \Gamma$, whence $\blacksquare\neg\varphi \notin \Gamma^{*c}$, whence $\neg\varphi \notin \Delta'$. Now, define $\Sigma := \{\varphi : \Box\neg\varphi \in \Gamma\}$. Observe that $\Delta' \cap \Sigma = \emptyset$; for if $\varphi \in \Sigma$, then $\Box\neg\varphi \in \Gamma$, whence by the foregoing, $\neg\neg\varphi \notin \Delta'$, and so $\varphi \notin \Delta'$. Furthermore, Σ is closed under disjunction. Therefore, by Corollary 1, there is a prime **L**-theory Δ such that $\Delta' \subseteq \Delta$ and $\Delta \cap \Sigma = \emptyset$. Then clearly $S^c_{\blacksquare}\Gamma^{*c}\Delta$. On the other hand, suppose that $\Box\varphi \in \Gamma$; then $\Box\neg\neg\varphi \in \Gamma$, from which it follows that $\neg\varphi \in \Sigma$, $\neg\varphi \notin \Delta$, and finally, that $\varphi \in \Delta^{*c}$, as desired.

Ad p22: suppose **L** contains $\Box$NEC and that $\Gamma \in N^c$ and $S^c_{\Box}\Gamma\Delta$. If $\vdash_{\mathbf{L}} \varphi$, then ex hypothesi, $\vdash_{\mathbf{L}} \Box\varphi$, whence $\Box\varphi \in \Gamma$ and so $\varphi \in \Delta$. But then Δ is regular, that is, $\Delta \in N^c$. □

Lemma 4 (Truth lemma)**.** *In the canonical model for* ***L****,* $\mathfrak{M}^c$*, the following biconditional obtains:* $\models^{\mathfrak{M}^c}_{\Gamma} \varphi$ *iff* $\varphi \in \Gamma$.

Proof. The basis case is by definition (Definition 6). The only induction case I examine is that concerning $\Box$.

Ad $\Box$: suppose $\Box\varphi \in \Gamma$ and $S^c_{\Box}\Gamma\Delta$, that is, $\{\theta : \Box\theta \in \Gamma\} \subseteq \Delta$; then $\varphi \in \Delta$, so $\models^{\mathfrak{M}^c}_{\Delta} \varphi$ by the induction hypothesis, which suffices. Conversely, suppose $\Box\varphi \notin \Gamma$; then observe that $\Delta := \{\theta : \Box\theta \in \Gamma\}$ is an **L**-theory: $\alpha, \beta \in \Delta$ imply $\Box\alpha, \Box\beta \in \Gamma$ implies $\Box(\alpha \wedge \beta) \in \Gamma$ ($\wedge$C) implies $\alpha \wedge \beta \in \Delta$; if $\alpha \in \Delta$ and $\vdash_{\mathbf{L}} \alpha \to \beta$, then since $\Box\alpha \in \Gamma$ and $\vdash_{\mathbf{L}} \Box\alpha \to \Box\beta$ (RM), $\Box\beta \in \Gamma$, whence $\beta \in \Delta$. Now apply Corollary 1 to the pair $\langle \Delta, \{\varphi\} \rangle$; then there is a prime **L**-theory $\Delta' \supseteq \Delta$ such that $\varphi \notin \Delta'$. Clearly, $S^c_{\Box}\Gamma\Delta'$ and $\not\models^{\mathfrak{M}^c}_{\Delta'} \varphi$, which suffices. □

Theorem 2 (Completeness)**.** *For any of the systems* $\boldsymbol{L}$*, if* $\models_{\boldsymbol{L}} \varphi$*, then* $\vdash_{\boldsymbol{L}} \varphi$*.*

Proof. Suppose $\nvdash_{\mathbf{L}} \varphi$; by Corollary 2, there is a prime regular **L**-theory Γ such that $\varphi \notin \Gamma$. By Definition 6, $\Gamma \in N^c$ in the canonical model $\mathfrak{M}^c$. By Lemma 3, $\mathfrak{M}^c$ is fit for **L**. By Lemma 4, $\not\models_{\Gamma}^{\mathfrak{M}^c} \varphi$. Therefore, $\not\models_{\mathbf{L}} \varphi$, as desired. □

3 The modal barrier

The first barrier thesis I examine belongs to alethic modal logic proper, that is, the logic of what is *necessarily so*, or of what *must be*. (Throughout this section, I focus on monomodal logic.) The intuitive idea is that there exists a barrier between what is merely the case on the one hand, and what must necessarily be the case on the other hand. Russell glosses the idea thus:

> According to the *is/must* barrier, no matter what we learn about the way the world is, nothing follows logically about how it must be: no *must* from an *is*. [43, p. 119]

Although Russell [43] does not remark on it, exactly such an intuition as this was what Anderson and Belnap availed themselves to early on in order to motivate a theory of entailment free of the fallacies of modality:[11]

> Modal fallacies arise when it is claimed that entailments follow from, or are entailed by, *contingent* propositions [...] Consider $A \to .A \to A$. Though "snow is white" and "that snow is white entails that snow is white" are both true—the latter necessarily so—it seems implausible that "snow is white" should *entail* that it entails itself. [4, pp. 44–45]

Recall that relevance logic in the Anderson-Belnap tradition was conceived (sinlessly) with the task of rectifying two types of fallacies: those of relevance and those of modality.[12] The ensuing history of the field has shifted the focus almost entirely onto the fallacies of relevance, and the repudiation of such chimeras as $(\varphi \wedge \neg\varphi) \to \psi$, but Anderson and Belnap were at least as (if not more) concerned with the latter category.

How plausible is the is/must barrier? First, it only appears plausible, even in a prima facie way, in the direction from 'is' to 'must;' few would contest the T-axiom

[11] In response to objections such as those of Routley and Routley [40], Anderson and Belnap [5, §5.2.1, §22.1.2] would later be somewhat more circumspect in characterizing the fallacies of modality. More on this below.

[12] See, for example, Anderson and Belnap [4, pp. 42–50], Anderson and Belnap [5, §5], and Meyer [33, p. 472].

($\Box$T) for metaphysical or even much weaker shades of necessity.[13] But even the prima facie plausible direction seems problematic on further reflection. Routley and Routley [40] offer many counterexamples to what they call DC (the Distributivity of Contingency). One particularly acute failure which they note is embodied by $\Box$B:

$$\varphi \to \Box\Diamond\varphi \tag{$\Box$B}$$

If the correct logic of what must be includes $\Box$B, then it would seem that each and every 'is' yields a 'must.'[14]

While it is therefore far from obvious that one should even want the is/must barrier, I will nevertheless now show that one can have it, or a suitable form of the Ackermann property corresponding to it, even in a fairly strong relevant modal logic (which, however, will naturally not contain $\Box$B).

Fix a background logic $\mathbf{L}$ extending $\mathbf{R}^{\Box}$. If $\vdash_{\mathbf{L}} \varphi \to \psi$, then the conditional holds as a matter of logic, and it is natural enough to describe this state of affairs as: φ entails ψ. A *barrier to entailment* or *Ackermann* theorem will then be a result to the effect that $\nvdash_{\mathbf{L}} \varphi \to \psi$ whenever φ and ψ meet certain conditions (in which case, $\mathbf{L}$ will be said to enjoy the barrier or Ackermann property under consideration). In the modal case, it is to be shown that $\nvdash_{\mathbf{L}} \varphi \to \psi$ whenever φ is extensional (an 'is') and ψ is necessary (a 'must'). To give this content, a taxonomy of formulae must be pinned down.[15]

Intuitively speaking, φ is *necessary* if it is $\mathbf{L}$-equivalent to some formula $\Box\psi$,[16] and it is *extensional* if it has no modal content whatsoever (i.e., it is $\Box$-free). I find this taxonomy natural enough, but it does have the rather unattractive feature that the class of necessary formulae is identified by a properly logical property while the class of extensional formulae is characterized by a syntactic property. However, if the result is proved for necessary formulae characterized in this way, it will also

[13] Incidentally, Anderson and Belnap [5, §8.12, pp. 95–96] describe a converse of the Ackermann property, and remark that, "One might wish a system to have this feature if one had the opinion, as we do not, that a non-necessitative could never follow from a necessitative." But a bidirectional barrier is plausible in other cases, for example, in the deontic case discussed below (see Section 4; cf. Weiss [57, p. 396, n. 7]).

[14] Of course, not everyone thinks that $\Box$B ought to be part of the logic of what must be (consult, e.g., Salmon [44]).

[15] Russell [43, pp. 4–7] (cf. [42, pp. 627–628]) helpfully lays out different approaches to setting up a taxonomy or classification of expressions (as normative vs. descriptive, etc.), including a syntax-driven approach—using The List—favored in some form or other and with suitable caveats by, for example, Prior [36, p. 89] and Jackson [24, pp. 89–90], and the model-theoretic approach she favors [38, 42, 43]. My own approach will be, in a sense, syntactic, though I hope not crudely so.

[16] Anderson and Belnap [5, p. 36] call such φ "necessitives," reserving "necessary" for something else.

hold (trivially) for necessary formulae characterized in the most natural syntactic fashion (as formulae *of the form* $\Box\psi$).[17] On the other hand, I cannot see an easy way to formulate a 'properly logical' characterization of the extensional formulae which would be amenable to proving a form of the Ackermann property. So, the definitions given at the start of this paragraph are the definitions that I adopt.

Proposition 1. *$\nvdash_{\boldsymbol{L}} \varphi \to \psi$ whenever φ is extensional and ψ is necessary, for $\boldsymbol{L} \subseteq \boldsymbol{R}^{\Box}(\Box K,\Box T,\Box 4,\Box NEC)$.*

Proof. Let φ be extensional and ψ be necessary; then $\vdash_{\mathbf{L}} \varphi \to \psi$ if and only if $\vdash_{\mathbf{L}} \varphi \to \Box\theta$, for some θ. That $\nvdash_{\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{T},\Box 4,\Box\mathrm{NEC})} \varphi \to \Box\theta$ (for extensional φ and any θ) is a result of Meyer's proved using matrices from [33, p. 476].[18] □

An immediate consequence of Proposition 1 is that the foregoing taxonomy is trichotomous (cf. Russell [43, pp. 26–27]):[19] fixing $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{T},\Box 4,\Box\mathrm{NEC})$ for concreteness, p is extensional, $\Box p$ is necessary, and $q \vee \Box p$ is neither (if $q \vee \Box p$ were equivalent to some $\Box\theta$, then it would follow that $\vdash_{\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{T},\Box 4,\Box\mathrm{NEC})} q \to \Box\theta$, which is impossible). That no formula can be both extensional and necessary is also immediate from the proposition (if there were such a formula φ, it would be the case that $\nvdash_{\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{T},\Box 4,\Box\mathrm{NEC})} \varphi \to \varphi$, contradicting I).

Thus, the is/must barrier is satisfied in relevant modal logics up to and including a relevant analogue of **S4**.[20] It should be pointed out explicitly that classical **S4** does not satisfy this property (e.g., $\vdash_{\mathbf{S4}} p \to \Box(q \vee \neg q)$).

4 Hume's law

I turn now to Hume's law, which is a barrier thesis belonging to deontic logic, the logic of what ought to be. Hume's law, in slogan form, is that you can't derive an 'ought' from an 'is.'[21] Like the modal barrier from Section 3, it has already received

[17]The syntactic characterization of necessary formulae has the untoward consequence that $\Box\varphi \wedge \Box\psi$ is not a 'must,' despite being equivalent (modulo **L**) to $\Box(\varphi \wedge \psi)$.

[18]This requires a slight qualification: Meyer [33] uses a formulation of **R** in which $\neg$ is not primitive (he instead uses a constant f). But the negation table is easily reconstructed (cf. Mares [29, p. 8]).

[19]Thanks to Ed Mares for asking about this.

[20]Meyer [33] refers to $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{T},\Box 4,\Box\mathrm{NEC})$ as **NR**. Is **NR** *the* relevant analogue of classical **S4**? In a certain technical sense, the answer turns out to be 'no' (see, e.g., Mares and Meyer [32]), but this is not important for any of my purposes.

[21]Here is what Hume had to say on the matter. After observing that much philosophical discourse on morality proceeds from constructions concerning 'is' and 'is not' to constructions concerning 'ought' and 'ought not,' he writes, "For as this *ought*, or *ought not*, expresses some new relation

some attention in the relevance logic literature. Mares [29, 31], in particular, has been keen to connect it to relevance logic's rejection of the fallacies of modality and the insistence on Ackermann-like properties.

Mares's work provides a convenient jumping-off point for the project of this section. While he (correctly) connects Hume's law and the fallacies of modality, his approach in [29] is a bit idiosyncratic in that he follows the Andersonian reduction of deontic logic to alethic modal logic.[22] Whatever the philosophical merits of that approach, it does not fit neatly into the framework elaborated in Section 2. Furthermore, [31] is principally intended as a philosophical, rather than as a technical, piece. So, there is work yet to be done.

Below, in Subsections 4.1 and 4.2, I examine two ways of formulating formal Ackermann-like properties corresponding to Hume's law; the ways are, in effect, distinguished by whether the relevant deontic logic in question is monomodal or bimodal. (The rough motivation for going bimodal is to capture certain connections between alethic possibility and deontic necessity, notably, a suitable form of 'ought implies can;' cf. Schurz [45, p. 37].)

Before getting to the technical business, one last remark is in order. Russell [43, pp. 1–2] suggests Hume's law is one of the most controversial barrier theses—it's the sort of thing philosophers have even ended up disagreeing with themselves on (see, especially, Prior [36, p. 88]). I myself am inclined to view it favorably, in no small part because I am favorably inclined towards relevance logic, and many of the arguments against Hume's law trade on principles that are obviously relevantly fallacious (more on this in Section 6). But even if the reader is unconvinced of its desirability, its *attainability* within a suitable relevant framework should suffice to motivate what follows.

4.1 Monomodal deontic logic

Throughout this subsection, I restrict my attention to the monomodal language (no $\blacksquare$). $\Box\varphi$ is interpreted (impersonally) as 'φ ought to be so;' derivatively, $\Diamond\varphi$ is interpreted as 'φ is permissible.' Strong modal logics do not make for plausible deontic logics, and the two systems I focus on in what follows are relatively weak

or affirmation, 'tis necessary that it shou'd be observ'd and explain'd; and at the same time that a reason should be given, for what seems altogether inconceivable, how this new relation can be a deduction from others, which are entirely different from it" [22, §3.1.1, p. 469].

[22] Anderson [2] proposed to reduce deontic logic to alethic modal logic by adding a propositional constant representing a bad state of affairs, and interpreting statements to the effect that φ ought to hold as statements to the effect that φ failing to hold would entail the bad state. For more discussion, see also Prior [35, Ap. D], Anderson [3], and Kanger [25, pp. 53–54].

extensions of $\mathbf{R}^{\Box}$. These systems are $\mathbf{R}^{\Box}(\Box\mathrm{K})$ and $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$ (aliases: **OR.1** and **DR.1** from Goble [19]).[23]

Taking these in reverse order, I will first formulate a form of Hume's law for $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$ and prove that it obtains. I will also show that $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$ is non-normal; in particular, $\Box$NEC is not admissible in it. Both of these results carry over to the weaker system $\mathbf{R}^{\Box}(\Box\mathrm{K})$. However, I will then proceed to show that a more stringent form of Hume's law is enjoyed by the weaker system, as well as a desirable converse of it.

On the paradigm of what was done in Section 3, I am interested in proving a result to the effect that $\nvdash_{\mathbf{L}} \varphi \to \psi$, whenever φ is extensional (an 'is') and ψ is obligatory (an 'ought'), where $\mathbf{L} \subseteq \mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$. The same characterization as from Section 3 may as well be adopted, and it may be stipulated that φ is *obligatory* if it is $\mathbf{L}$-equivalent to some formula $\Box\psi$, and it is *extensional* if it is $\Box$-free.[24] Since $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$ is clearly a subsystem of $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{T},\Box 4,\Box\mathrm{NEC})$, the desired result is really a corollary of Proposition 1.

Nevertheless, I will give a proof of the result using different matrices, based on **RM3**, because the same matrices can also be used to establish other results of interest (cf. Mares [29, pp. 11–13]). The matrices are as follows (2 and 1 are designated):

$\to$	2	1	0
2	2	0	0
1	2	1	0
0	2	2	2

$\wedge$	2	1	0
2	2	1	0
1	1	1	0
0	0	0	0

$\vee$	2	1	0
2	2	2	2
1	2	1	1
0	2	1	0

[23] $\Box$D is arguably what is most characteristic of classical deontic logic (see, e.g., von Wright [52, p. 13], Lemmon [27, §5], and Lemmon [28, p. 40]). Goble [19, §1] argues against requiring $\Box$D, at least if one is using a relevant background logic, hence why the weakest relevant deontic systems he examines omit it. I need not take any position on the matter.

[24] I mention one prima facie peculiar consequence of these definitions: $\Box\varphi \vee \Box\psi$ is not generally obligatory. Suppose it were; then, by definition, $\Box\varphi\vee\Box\psi$ is equivalent—over, say, $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$—to some $\Box\theta$. Then clearly $\vdash_{\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})} \Box\varphi \to \Box\theta$, $\vdash_{\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})} \Box\psi \to \Box\theta$, and $\vdash_{\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})} \Box\theta \to \Box(\varphi \vee \psi)$. A simple syntactic argument shows $\frac{\Box\alpha\to\Box\beta}{\alpha\to\beta}$ to be admissible in $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$ (cf. Chellas [10, pp. 124–125]). Consequently, $\vdash_{\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})} \varphi \to \theta$, $\vdash_{\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})} \psi \to \theta$, and $\vdash_{\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})} \theta \to \varphi \vee \psi$, whence $\vdash_{\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})} \theta \leftrightarrow (\varphi \vee \psi)$, and so $\vdash_{\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})} \Box\theta \leftrightarrow \Box(\varphi \vee \psi)$. Thus, by transitivity, $\Box\varphi\vee\Box\psi$ is equivalent over $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$ to $\Box(\varphi\vee\psi)$, an equivalence which fails for many substitution instances. I believe a more nuanced taxonomy could accommodate such disjunctions (cf. Anderson and Belnap [5, §22.1.2]), but the technical details are formidable and beyond the scope of this paper. Thanks to Rohan French for pressing me to comment on this issue.

$\neg$	
2	0
1	1
0	2

$\Box$	
2	0
1	0
0	0

Proposition 2. $\nvdash_{\boldsymbol{L}} \varphi \to \psi$ *whenever* φ *is extensional and* ψ *is obligatory, for* $\boldsymbol{L} \subseteq \boldsymbol{R}^{\Box}(\Box K,\Box D)$.[25]

Proof. As in Proposition 1, the problem reduces to showing that $\nvdash_{\mathbf{L}} \varphi \to \Box\theta$, whenever φ is $\Box$-free. It is easy to verify that all theorems of $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$ come out designated in the above matrices. Now, consider the valuation ν assigning 1 to all variables; inspection indicates $\nu(\varphi) = 1$ but, clearly, $\nu(\Box\theta) = 0$, whence the conditional takes an undesignated value. □

Proposition 3. *No formula of the form* $\Box\varphi$ *is a theorem of* $\boldsymbol{R}^{\Box}(\Box K,\Box D)$.

Proof. Using the same matrices and any valuation whatsoever, it is clear that $\nu(\Box\varphi) = 0$. □

Before the advent of Kripke semantics, deontic logicians showed a marked preference for nonnormal systems such as **D2** (see Lemmon [27, p. 185]). □NEC seems to have been generally considered to be deontically implausible (see, e.g., Prior [34, pp. 221–222] and Lemmon [27, p. 185]). Indeed, it is deontically implausible: why should *logic* deliver that there are any obligations whatsoever? Fortunately, as Proposition 3 shows, $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$ is not blemished in this way.

Corollary 5. *□NEC is inadmissible in* $\boldsymbol{R}^{\Box}(\Box K,\Box D)$.

Proof. Immediate from Proposition 3 (cf. Mares [29, p. 13]). □

I turn now to the weaker system $\mathbf{R}^{\Box}(\Box\mathrm{K})$. I continue to classify formulae as extensional if they are □-free, but now I classify a formula φ as *deontic* if it's equivalent to either $\Box\psi$ or $\Diamond\psi$ (for some ψ). A natural extension of Hume's line of thought would hold that no deontic propositions (not just obligations) are entailed by extensional propositions.[26] This result demonstrably holds for $\mathbf{R}^{\Box}(\Box\mathrm{K})$.

Proposition 4. $\nvdash_{\boldsymbol{L}} \varphi \to \psi$ *whenever* φ *is extensional and* ψ *is deontic, for* $\boldsymbol{L} \subseteq \boldsymbol{R}^{\Box}(\Box K)$.

[25]Incidentally, Proposition 2 shows that $p, \neg p \nvdash_{\mathbf{L}} \Box q$ (because $\nvdash_{\mathbf{L}} (p \wedge \neg p) \to \Box q$), thereby ruling out a certain counterexample considered by Russell [42, p. 626] on grounds that actually have nothing to do with variable sharing (more on this in Section 6).

[26]A parallel line would not be plausible in the alethic case from Section 3. In the presence of □T—which seems largely uncontroversial—one would expect validities such as $p \to \Diamond p$.

Proof. If $\vdash_{\mathbf{L}} \psi \leftrightarrow \Box\theta$ (for some θ), then $\varphi \to \Box\theta$ (and so $\varphi \to \psi$) can be dispatched using Proposition 2 and its associated matrices. On the other hand, if $\vdash_{\mathbf{L}} \psi \leftrightarrow \Diamond\theta$ (for some θ), then use the same **RM3** matrices except the following matrix for $\Box$:

$\Box$	
2	2
1	2
0	2

It can be verified that every theorem of $\mathbf{R}^{\Box}(\Box\mathrm{K})$ takes a designated value in these matrices. (Incidentally, $\Box$D does *not*.) By inspection, if 1 is assigned to every variable, $\nu(\varphi) = 1$ and $\nu(\Diamond\theta) = 0$; therefore, $\nu(\varphi \to \Diamond\theta) = 0$, as desired. $\Box$

An attractive converse of Hume's law can also be proved for systems contained in $\mathbf{R}^{\Box}(\Box\mathrm{K})$, where extensional and deontic formulae are characterized in the same way as above.

Proposition 5. *$\nvdash_{\boldsymbol{L}} \varphi \to \psi$ whenever φ is deontic and ψ is extensional, for $\boldsymbol{L} \subseteq \boldsymbol{R}^{\Box}(\Box K)$.*

Proof. The argument is symmetric to that given in the proof of Proposition 4. That is, if $\vdash_{\mathbf{L}} \varphi \leftrightarrow \Box\theta$ (for some θ), the pertinent conditional ($\Box\theta \to \psi$) can be dispatched using the **RM3** matrices and the constant 2 $\Box$-matrix from Proposition 4. If, instead, $\vdash_{\mathbf{L}} \varphi \leftrightarrow \Diamond\theta$ (for some θ), the pertinent conditional ($\Diamond\theta \to \psi$) can be dispatched using the **RM3** matrices and the constant 0 $\Box$-matrix from earlier. $\Box$

Propositions 4 and 5 together show that there is a thoroughgoing logical separation of extensional and deontic propositions in $\mathbf{R}^{\Box}(\Box\mathrm{K})$. $\mathbf{R}^{\Box}(\Box\mathrm{K})$ therefore avoids forms of the naturalistic fallacy as well as what might be called the "wishful thinking" fallacy. As to whether $\mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$ enjoys these same properties, I cannot say (the proofs I have do not extend to it, but I do not have counterexamples either).

4.2 Bimodal deontic logic

I turn now to developing a relevant bimodal deontic logic. As in Subsection 4.1, $\Box$ (and, derivatively, $\Diamond$) is used to interpret the deontic modalities. I now also include $\blacksquare$ (and, derivatively, $\blacklozenge$) in the language, and use it to interpret the alethic modalities. The key connecting thesis is 'ought implies can,'[27] or:

$$\Box\neg\varphi \to \neg\blacksquare\varphi \qquad (\Box\blacksquare\mathrm{D})$$

[27] This thesis is sometimes known as 'Kant's law,' although the status and interpretation of the principle in Kant is not unambiguous (for a recent discussion, consult Kohl [26]).

I here consider an extension of $\mathbf{R}^{\Box}_{\blacksquare}$ whose (putative) deontic fragment corresponds to the stronger system examined in the previous subsection, and whose (putative) alethic fragment is **S4**-ish: $\mathbf{R}^{\Box}_{\blacksquare}$(□K,□D,■K,■T,■4,■NEC,□■D).

It is to be noted straightaway that □■D presents problems for framing a barrier based on what might be thought of as the naïve taxonomy: *nondeontic* (in lieu of *extensional*) formulae are those which are □-free, whereas a formula is *deontic* if it is (logically equivalent to something) of the form $\Box\varphi$ or $\Diamond\varphi$. For it is clear that even in a system as weak as $\mathbf{R}^{\Box}_{\blacksquare}$(□■D) one can derive the deontic from the nondeontic as water from a stone (as witnessed by the theorem $\blacksquare\varphi \to \Diamond\varphi$).

So, I will restrict myself to a less ambitious taxonomy. A formula φ is *nondeontic* if it is □-free, whereas it is *obligatory* if it is equivalent to a formula of the form $\Box\psi$ (for some ψ).

Proposition 6. *$\nvdash_{\boldsymbol{L}} \varphi \to \psi$ whenever φ is nondeontic and ψ is obligatory, for $\boldsymbol{L} \subseteq \boldsymbol{R}^{\Box}_{\blacksquare}$(□K,□D,■K,■T,■4,■NEC,□■D).*[28]

Proof. As in earlier cases, the problem comes to showing that $\nvdash_{\mathbf{L}} \varphi \to \Box\theta$ whenever φ is □-free. Once again, I use the **RM3** matrices for the nonmodal connectives, and the following matrices for the modal connectives:

□	
2	0
1	0
0	0

■	
2	1
1	1
0	0

It can be verified that all the theorems of $\mathbf{R}^{\Box}_{\blacksquare}$(□K,□D,■K,■T,■4,■NEC,□■D) come out designated. Now, assigning 1 to all variables, it is clear that $\nu(\varphi) = 1$ while $\nu(\Box\theta) = 0$, whence the pertinent conditional fails. □

As in the case of $\mathbf{R}^{\Box}$(□K,□D), it is clear that □NEC is not admissible in $\mathbf{R}^{\Box}_{\blacksquare}$(□K,□D,■K,■T,■4,■NEC,□■D), and the system has no theorems of the form $\Box\varphi$. (The argument simply reuses the matrices from the proof of Proposition 6.)

5 The time barrier

The last barrier I examine—the past/future barrier—belongs to tense logic. Russell [43, p. 85] glosses it thus:

[28] Schurz [45], interestingly, formulates a bimodal alethic deontic predicate logic and proves a sort of version of Hume's law for it. The formulation of Hume's law he gives might be described as 'relevant-adjacent,' though it is not based on relevance logic (see, especially, [45, pp. 92–93, n. 3]).

> *no set of premises about the past entails a conclusion about the future* [...] Or, in slogan form: No *will* from a *was*.

The unidirectional gloss notwithstanding, the barrier in question seems to be intuitively bidirectional, and in what follows I formulate a system for which statements about the future and statements about the past are, to a large extent, logically separated from one another.

As Russell [43, p. 84] observes, this barrier can also be found in Hume.[29] Prior [37, p. 57] dismisses it, at least in part because, bluntly interpreted, it conflicts with certain intuitive temporal bridge principles (more on this anon). My own position is that, in a suitably restricted form, the barrier is acceptable after all.

Tense logic is inherently bimodal with one set of modalities concerning the future and another set concerning the past. Throughout this section, I use the language with both modal operators, though for ease of exposition and to bring the presentation into conformity with the more typical conventions of tense logic, I redecorate the boxes and diamonds à la Prior [35, 37]:

1. $\mathcal{H}\varphi$ (‘φ has always been the case’) replaces $\square\varphi$;
2. $\mathcal{P}\varphi$ (‘φ was at one time case’) replaces $\Diamond\varphi$;
3. $\mathcal{G}\varphi$ (‘φ is always going to be the case’) replaces $\blacksquare\varphi$;
4. $\mathcal{F}\varphi$ (‘φ will at some time be the case’) replaces $\blacklozenge\varphi$.

The foregoing notwithstanding, I (confusingly) continue to refer to axioms according to the conventions of Section 2. (In fact, it will not be very confusing, because I adopt essentially the same axioms for each modal operator.)

Since there is little previous technical work on relevant tense logic and tense logic is, in any case, not the focus of the present study, I will restrict my attention to a fairly weak system, which is sort of a relevant analogue of the basic classical tense logic $\mathbf{K}_t$.[30] This is the system $\mathbf{R}^{\square}_{\blacksquare}$($\square$K,$\square$NEC,$\blacksquare$K,$\blacksquare$NEC,$\blacklozenge\square$,$\Diamond\blacksquare$), that is, roughly, $\mathbf{K}$ for both $\mathcal{H}$ and $\mathcal{G}$, plus the bridge axioms:

$$\mathcal{F}\mathcal{H}\varphi \to \varphi \qquad (\blacklozenge\square)$$

[29] Hume writes, “all inferences from experience suppose, as their foundation, that the future will resemble the past [...] If there be any suspicion that the course of nature may change, and that the past may be no rule for the future, all experience becomes useless, and can give rise to no inference or conclusion. It is impossible, therefore, that any arguments from experience can prove this resemblance of the past to the future; since all these arguments are founded on the supposition of that resemblance” [23, §4.2, pp. 37–38].

[30] For $\mathbf{K}_t$, see, for example, Prior [37, p. 176] and Blackburn et al. [8, p. 205].

$$\mathcal{PG}\varphi \to \varphi \tag{$\Diamond\blacksquare$}$$

$\blacklozenge\Box$ expresses that if φ will at some time have always been the case, then φ is now the case; and $\Diamond\blacksquare$ expresses that if φ was at some time always going to be the case, then φ is now the case.

As $\Box$B creates obstacles for the alethic barrier considered in Section 3, so $\blacklozenge\Box$ and $\Diamond\blacksquare$ create obstacles for $\mathbf{R}^{\Box}_{\blacksquare}$($\Box$K,$\Box$NEC,$\blacksquare$K,$\blacksquare$NEC,$\blacklozenge\Box$,$\Diamond\blacksquare$). To be more concrete, per $\Diamond\blacksquare$, from at least some statements ostensibly about the past (e.g., $\mathcal{PGF}p$) some statements ostensibly about the future (e.g., $\mathcal{F}p$) *do* follow. Nevertheless, in the bimodal setting under consideration, these obstacles can be overcome if the barrier, and corresponding formula taxonomy, is implemented with sufficient subtlety.

My proposal is this. A formula φ is classified as *pure futuristic* if it is logically equivalent to $\mathcal{G}\psi$ or $\mathcal{F}\psi$ for some ψ which is $\mathcal{H}$-free (e.g., $\mathcal{FG}p$); and it is classified as *simple futuristic* if it is logically equivalent to $\mathcal{G}\psi$ for some ψ which is itself $\mathcal{G}$-free (e.g., $\mathcal{G}p$, but also $\mathcal{GP}p$; not $\mathcal{F}p$). Analogously, a formula φ is classified as *pure past* if it is logically equivalent to $\mathcal{H}\psi$ or $\mathcal{P}\psi$ for some ψ which is $\mathcal{G}$-free (e.g., $\mathcal{H}p \wedge \mathcal{H}q$); and it is classified as *simple past* if it is logically equivalent to $\mathcal{H}\psi$ for some ψ which is itself $\mathcal{H}$-free (e.g., $\mathcal{HF}p$; not $\mathcal{P}p$).

The results to be proved make use of the following matrices found using MaGIC (the designated values are $n \geq 2$):

$\to$	4	3	2	1	0
4	4	0	0	0	0
3	4	3	1	1	0
2	4	3	2	1	0
1	4	3	3	3	0
0	4	4	4	4	4

$\wedge$	4	3	2	1	0
4	4	3	2	1	0
3	3	3	2	1	0
2	2	2	2	1	0
1	1	1	1	1	0
0	0	0	0	0	0

$\vee$	4	3	2	1	0
4	4	4	4	4	4
3	4	3	3	3	3
2	4	3	2	2	2
1	4	3	2	1	1
0	4	3	2	1	0

$\neg$	
4	0
3	1
2	2
1	3
0	4

$\mathcal{H}$	
4	4
3	3
2	3
1	0
0	0

$\mathcal{G}$	
4	4
3	2
2	2
1	2
0	0

Proposition 7. *$\nvdash_{\boldsymbol{L}}$ $\varphi \to \psi$ whenever ψ is pure futuristic and φ is simple past, for $\boldsymbol{L} \subseteq \boldsymbol{R}^{\Box}_{\blacksquare}$($\Box$K,$\Box$NEC,$\blacksquare$K,$\blacksquare$NEC,$\blacklozenge\Box$,$\Diamond\blacksquare$).*

Proof. First, observe that every theorem of $\mathbf{R}^{\Box}_{\blacksquare}$($\Box$K,$\Box$NEC,$\blacksquare$K,$\blacksquare$NEC,$\blacklozenge\Box$,$\Diamond\blacksquare$) is designated in the relevant matrices. Now, the problem falls in two halves: showing that $\mathcal{H}\alpha \to \mathcal{G}\beta$ always fails and that $\mathcal{H}\alpha \to \mathcal{F}\beta$ always fails (where α, β are $\mathcal{H}$-free). The same valuation works for both: assign 2 to every variable. It is clear that $\nu(\mathcal{H}\alpha) = 3$ but $\nu(\mathcal{G}\beta) = \nu(\mathcal{F}\beta) = 2$, whence the conditionals fail. □

Proposition 8. *$\nvdash_{\boldsymbol{L}} \varphi \to \psi$ whenever ψ is pure past and φ is simple futuristic, for $\boldsymbol{L} \subseteq \boldsymbol{R}^{\square}_{\blacksquare}(\square K, \square NEC, \blacksquare K, \blacksquare NEC, \blacklozenge\square, \lozenge\blacksquare)$.*

Proof. The argument is essentially the same as that for Proposition 7, but the matrices for $\mathcal{H}$ and $\mathcal{G}$ given above are swapped, and the relevant subformulae are required to be $\mathcal{G}$-free. □

Propositions 7 and 8 jointly yield a decent bidirectional barrier between certain kinds of statements about the past on the one hand and certain kinds of statements about the future on the other. They preclude from theoremhood $\mathcal{H}\varphi \to \mathcal{G}\varphi$ (strong forwards induction), $\mathcal{H}\varphi \to \mathcal{F}\varphi$ (weak forwards induction), $\mathcal{G}\varphi \to \mathcal{H}\varphi$ (strong backwards induction), $\mathcal{G}\varphi \to \mathcal{P}\varphi$ (weak backwards induction), and a host of related aberrations. But they are silent about such suspicious characters as $\mathcal{P}\varphi \to \mathcal{F}\varphi$ (forwards recurrence) and $\mathcal{F}\varphi \to \mathcal{P}\varphi$ (backwards recurrence).[31]

6 Concluding polemical remarks

In this paper, I have elaborated a relevant bimodal framework in which many particular systems can be implemented and shown to satisfy Ackermann-like properties corresponding to various barrier theses. I focused on three particular barriers, which were drawn from Russell [43]: the modal barrier (is/must), Hume's law (is/ought), and the time barrier (was/will). The framework developed above does not cover two other barriers Russell [43] is concerned with—the particular/universal barrier and the indexical barrier—but this is more indicative of a certain measure of slothfulness on the author's part than of any essential limitation.[32]

In fact, work has already been done on relevant quantified modal logic (not to mention *mere* relevant quantificational logic), and work has also been done on some (putative) indexicals in relevance logic.[33] I naïvely presume that the framework

[31] Actually, over $\mathbf{R}^{\square}_{\blacksquare}(\square K, \square NEC, \blacksquare K, \blacksquare NEC, \blacklozenge\square, \lozenge\blacksquare)$, all of forwards recurrence, strong backwards induction, backwards recurrence, and strong forwards induction are equivalent (cf. Prior [37, pp. 63–64]). While all are thereby ruled out as theorems, Propositions 7 and 8 cannot be appealed to (directly) in every case since the *taxonomic applicability conditions* are not met in every case. Thanks to Katalin Bimbó for drawing my attention to this point.

[32] Russell [43, p. 1] glosses the particular/universal and indexical barriers thus: "no universal claims from particular ones [. . .] no indexical claims from claims which are not indexical."

[33] For relevant quantified modal logic, see, for example, Ferenz [15]. For some discussion of an actuality operator in relevance logic, see Standefer [50].

developed above could be extended to formulate systems—or even formulate one single relevant quantified multimodal system—implementing versions of all of the barriers Russell examines.

Therefore, it would seem (at least prima facie) that relevance logic is a natural home for examining and implementing all of the barriers Russell [43] is interested in. Alas, this is not the approach she favors, though not for lack of consideration. In fact, after surveying a range of purported counterexamples to Hume's law, Russell [43, pp. 36–41] explicitly airs the question of whether adopting a relevant (or paraconsistent) logic to formulate barriers such as Hume's law is advisable, and comes out against the idea.[34] I conclude this paper by examining and responding to Russell's comments on this matter.

By way of situating Russell's comments, a short detour through Prior's dilemma (from [36]; cf. Russell [43, pp. 24–27]) will prove instructive. The dilemma is roughly as follows. Let p be some unambiguous descriptive claim (e.g. 'cats are mammals') and let $\Box q$ be some unambiguous normative claim (e.g., 'cats ought to be petted'). Is $p \vee \Box q$ normative or descriptive?[35] If it is normative, then something normative ($p \vee \Box q$) follows from something descriptive (p) via Addition ($p \vdash p \vee \Box q$); but if it is descriptive, then something normative ($\Box q$) follows from some descriptive claims ($\neg p$, $p \vee \Box q$) via Disjunctive Syllogism ($\neg p, p \vee \Box q \vdash \Box q$).[36] Therefore, it would appear, Hume's law is not correct.

The friend of relevance logic will not be moved. Disjunctive syllogism fails to be valid in all standard relevance logics, so there is, so to speak, no dilemma at all. Mares [31, p. 286] writes:

> Prior's argument too should be rejected by relevantists because it appeals

[34]Let me point out here that the matrices given throughout this paper, while paraconsistent, are not relevant in the sense that none of the finite many-valued logics determined by them satisfy the variable sharing property; in each, $\neg(q \to q) \to (p \to p)$, for example, comes out designated. One might therefore be tempted to conclude that this paper's real thesis is that a *paraconsistent framework* ought to be adopted for implementing the barriers. However, this would be mistaken. Paraconsistency is not doing the interesting work here; paraconsistency is not sufficient, in general, for the Ackermann property (consider once more **RM3** and the constant 2-function interpreting $\Box$; the resulting logic is paraconsistent, but every conditional of the form $\varphi \to \Box\psi$ is designated). I suppose I concede that relevance in the sense of the variable sharing property is also not doing the interesting work here, but as relevance logic traditionally drew significant motivation from the Ackermann property, I do not see that it is inappropriate to describe a framework mainly motivated by it as relevant. In any case, the logics I am really interested in are not these finite many-valued logics, which validate all sorts of problematic theses (e.g., Dugundji-like formulae), though they are nevertheless useful as tools. Thanks to Shay Logan for pressing this issue.

[35]One might be tempted to respond 'neither.' More on that below (cf. Russell [43, pp. 26–27]).

[36]In the general case, the argument assumes that the class of descriptive expressions is closed under negation. In the particular case under consideration, this is clearly uncontroversial.

> to Disjunctive Syllogism for extensional disjunction (henceforth 'EDS'). The rejection of EDS has been treated by non-relevant logicians and philosophers as a serious problem for relevant logic. In fact, I think it is a virtue. The Lewis, and especially the Prior, arguments show that it is a virtue. Because it allows the derivation of irrelevances (such as *ex falso* and the Prior inference), EDS is an unsafe inference form.

As Russell [43, pp. 36–38] observes, many of the proposed counterexamples to Hume's law turn on principles (such as Disjunctive Syllogism, or EDS) which are not relevantly valid. Moreover, in a vein that is consonant with the final sentence quoted from Mares [31] above, she suggests that the relevantist might reject such counterexamples *because* they commit fallacies of relevance:

> There is a link between traditional motivations for relevant logic and a tempting idea about the reason the *is/ought* barrier exists. One motivation for relevant logic is the idea that the premises of a valid argument should be *relevant* to its conclusion, and an informal way of understanding this is as requiring that the conclusion concern the same subject matter as the premises. It is sometimes suggested, with regard to the *is/ought* barrier, that the problem with "of a sudden" drawing conclusions containing *ought*s from premises that contain only *is*es and *is not*s, is that *what ought to be* is a different subject matter from *what is*. [43, p. 38]

The problem with this line of thought, as Section 3 will have made clear, is that the "traditional motivations for relevant logic" include not only the rejection of fallacies of relevance, but also the rejection of fallacies of modality. The latter are not to be explained by the former—they just are taken to be, intuitively, fallacious. While some putative counterexamples are both fallacies of relevance and fallacies of modality—for instance, $(p \wedge \neg p) \rightarrow \Box q$—it is clear that the type of fallacy operative in most of the (formal) counterexamples is in fact the modal variety.[37]

For this reason, several of Russell's subsequent remarks against the use of relevance logic to formulate barrier theses are wide of the mark. Her points in [43, p. 38] that the "line-up between the subject-matter motivation and relevant logic is not as clean as it might seem" and that there "are problems with the subject-matter explanation of Hume's Law" don't address the rejection of the fallacies of modality at all.

[37] In particular, even in Prior's dilemma the operative fallacy can be taken to be one of modality as much as one of relevance. Fix some $\mathbf{R}^{\Box} \subseteq \mathbf{L} \subseteq \mathbf{R}^{\Box}(\Box\mathrm{K},\Box\mathrm{D})$. Then: $\neg p, p \vee \Box q \vdash_{\mathbf{L}} \Box q$ if and only if $\vdash_{\mathbf{L}} (\neg p \wedge (p \vee \Box q)) \rightarrow \Box q$ if and only if $\vdash_{\mathbf{L}} ((\neg p \wedge p) \vee (\neg p \wedge \Box q)) \rightarrow \Box q$ if and only if $\vdash_{\mathbf{L}} (\neg p \wedge p) \rightarrow \Box q$ and $\vdash_{\mathbf{L}} (\neg p \wedge \Box q) \rightarrow \Box q$; but $\nvdash_{\mathbf{L}} (\neg p \wedge p) \rightarrow \Box q$ by Proposition 2.

Russell [43, pp. 38–39] also claims, taking a page from Humberstone [20, pp. 133–135], that relevance logic will not allow for all of the counterexamples to be avoided. The envisioned counterexample, using the same p and $\Box q$ from above, is something like: $\neg p \to \neg(p \wedge \Box q)$. Certainly, this is a theorem of $\mathbf{R}^{\Box}$, $\mathbf{R}^{\Box}_{\blacksquare}$, and all of their extensions, but is it a genuine counterexample?

The line of thought seems to be that since $\Box q$ is entailed by $p \wedge \Box q$, by Hume's law, the latter must itself be normative, and therefore its negation must be as well. This argument is really too quick, though. It's not clear that $p \wedge \Box q$ is normative (it plainly fails to be deontic relative to $\mathbf{R}^{\Box}(\Box\mathrm{K})$ by Proposition 5), nor is it clear that Hume's law, suitably understood, should require that it be so (why isn't it compatible with Hume's law that normative conclusions can be drawn from premises which are neither normative nor descriptive, as one might be tempted to classify $p \wedge \Box q$?).[38]

Russell [43, p. 38] further asserts that relevant consequence is too complicated to capture the intuitive content of Hume's law and of the other barriers. I certainly would not wish to suggest that ordinary reasoners and speakers have the model theory of Section 2 in mind, or that they themselves could formulate technical results such as those given in Propositions 2 and 6 (nor could they, of course, formulate Russell's own results). I think, nevertheless, that there is a clear intuitive connection between what philosophers like Hume have said, and the rigorous formulations of proscriptions on modal fallacies given above. So I do take myself to have met the desideratum on intuitiveness given by Russell [43, p. 41].

The only objection of Russell's that remains is that there is no need to go relevant, since "we can prove each of the barriers in its stronger, classical form" [43, p. 38]. To properly evaluate this claim, I would need to engage at much greater length with Russell's positive proposal than can be done here. But, for the sake of saying something, let me just remark: it is not really clear that strength is a virtue, and even if one could get by with classical logic for the purposes of formulating barrier theses, another argument would be required to show that it is optimal among the possible solution frameworks.

Let me close by raising an objection of my own. In the current proposal, there is not really any substantial explanation of why the barriers hold. To say (for example)

[38] Russell [43, pp. 27–28] considers a problem, attributed to Vranas [53], to the effect that classifying expressions like $p \wedge \Box q$ as 'neither' leads to formulations of Hume's law that are too weak. It is beyond the scope of the present work to engage with these issues in any detail, but it seems to me that the problems raised depend either on taking the conditional as material implication (as no relevantist would) or as interpreting statements that ought to be formalized using a primitive conditional obligation connective as composites of other connectives (on why one should not do this, see, e.g., Chisholm [14] and Chellas [10, p. 201]). On these matters, see also Humberstone [21, pp. 1379ff.].

that you can't get a necessity from a contingency because that's a fallacy of modality and relevance logic rejects the fallacies of modality just does not seem to clarify very much.

What one would like is for the failure of the fallacies of modality to fall out of some deeper logical fact—ideally, in my view, one concerning the interpretation of the different pieces of the semantic apparatus presented in Section 2.[39] The Routley-Meyer semantics and its variations leave much to be desired in terms of naturalness,[40] and I am doubtful that any particularly illuminating interpretation of the machinery developed above will appear quickly. But the situation is not, I now think,[41] entirely hopeless, and I will content myself to leave the matter there.

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[39] One would similarly like for the failure of the fallacies of relevance—captured formally by the variable sharing property (see, e.g., Belnap [7] and Anderson and Belnap [5, §§5.1, 22.1.3])—to fall out of some deeper semantic fact. Elsewhere, I have shown that there is indeed a pleasing correspondence between the formal relevance property and the natural semantic notion of exact verification (cf. Fine [16, p. 558]) within the (positive) semilattice semantics [55, 56].

[40] For some valiant attempts to elucidate aspects of the Routley-Meyer semantics, see, for example, Mares [30] and Beall et al. [6].

[41] In the past, I have been considerably more hostile towards the Routley-Meyer semantics than I am now. Nevertheless, I continue to strongly favor the operational semantics of Urquhart [51] and variations thereof.

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