Revisiting Constructive Mingle: Algebraic and Operational Semantics*

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Abstract

Among Dunn's many important contributions to relevance logic was his work on the system **RM** (**R**-mingle). Although **RM** is an interesting system in its own right, it is widely considered to be too strong. In this chapter, I revisit a closely related system, **RM0** (sometimes known as 'constructive mingle'), which includes the mingle axiom while not degenerating in the way that **RM** itself does. My main interest will be in examining this logic from two related semantical perspectives. First, I give a purely operational bisemilattice semantics for it by adapting previous work of Humberstone. Second, I examine a more conventional algebraic semantics for it and discuss how this relates to the operational semantics. A novel operational semantics for **J** (intuitionistic logic) as well as its conventional Heyting algebraic semantics emerge as special cases of the corresponding semantics for **RM0**. The results of this chapter suggest that **RM0** is a more interesting logic than has been appreciated and that Humberstone's operational semantic framework similarly deserves more attention than it has received.

Keywords: Bisemilattices, Intuitionistic logic, Mingle, Operational semantics, Relevance logic.

1 Introduction

Among Mike Dunn's many important contributions to relevance logic was his work on the system **RM** (**R**-mingle) [9, 13, 10].[1] **RM**, which results

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[1]Indeed, with Storrs McCall, Dunn is one of the system's 'parents'. Some of the history is recounted in [12, §7.3].

by adding $\varphi \to (\varphi \to \varphi)$ to **R**, is one of the best behaved systems in the broader family of (quasi-)relevance logics and, not unrelatedly, also rather a disappointment.[2] On the one hand, it is semantically natural, possessing both elegant binary relational and algebraic semantics, is decidable, and prima facie looks like an eminently reasonable axiomatic extension of **R**. On the other hand, it is just way too strong, producing such unsavory theorems as $(\varphi \to \psi) \vee (\psi \to \varphi)$ (sometimes called the 'chain theorem') and ultimately tilting into the abyss of irrelevance.

The cognoscenti have long appreciated that the original sin of **RM** has less to do with the innocuous seeming mingle axiom than to do with the negation postulates of **R**:

> But the breakdowns that afflicted **RM** rested on **R**-style negation, which [...] is not as transparent as the other truth-functional connectives. Accordingly, further pursuit of the original Dunn-McCall insights, dropping the **R**-style negation [...] appears an interesting present alternative. (Meyer, in Anderson and Belnap, Jr. [1, §29.3, p. 394].)

This seems to me—and has seemed to others—to be an eminently reasonable suggestion.[3] The result of adding the mingle axiom to the pure implicational fragment of **R** yields a system, $\mathbf{RM0}_{\to}$, which does *not* in fact coincide with the pure implicational fragment of **RM** but which is, on any reasonable understanding, relevant.[4] Anderson and Belnap, Jr. [1, §8.15, pp. 98–99] call this "constructive mingle", as it is a subsystem of the implicational fragment of **J** (intuitionistic logic). I will extend this name to all of **RM0**, which I take to be $\mathbf{RM0}_{\to}$ extended with conjunction, disjunction, and the constant $\perp$—all governed by their usual axioms—and potentially some further connectives, though *not* the negation of **R** (Section 2).

[2]That **RM** is a disappointment is, as far as I am aware, the consensus view, though it is not universal. In any case, Dunn [12, p. 143] suggests (pace Meyer [1, §29.3, pp. 393–394]) that **RM** is superior to **R** "when all things are considered".

[3]For example, see Méndez [23] for a discussion of how various sorts of alternative negations might be added to the standard axiomatic (not actual) positive fragment of **RM** (the article also provides ternary relational—though not algebraic or operational—semantics for some of these variations on **RM**).

An alternative idea is pursued by Avron [2, 3], who considers and advocates for an implication-negation system—the standard axiomatic (not actual) fragment of **RM** in that language—in which intensional versions of conjunction and disjunction can be defined. This project certainly has its interest, though it is quite different from the project which I shall pursue here.

[4] In particular, $\mathbf{RM0}_{\to}$ (as well as its extension with the usual axioms for disjunction and conjunction) satisfies the *variable sharing property* (i.e., $\varphi \to \psi$ is never a theorem when φ and ψ do not share propositional variables) [23, p. 286].

This chapter is primarily devoted to a study of **RM0** from two semantical perspectives. In Section 3, I give a purely operational *bisemilattice semantics* (cf. the semilattice semantics of Urquhart [36]) for **RM0** by adapting previous work of Humberstone [18]. An operational semantics for **J** then emerges as the special case in which the bisemilattices—which here play the role of frames—are lattices. In Section 4, I examine a more conventional algebraic semantics for **RM0** and relate it to the previously developed operational semantics; here, the familiar Heyting algebraic semantics for **J** emerges as the special case.

Let me emphasize that my main interest in this chapter is not so much novelty (though there will be some novelty) as it is in reframing existing ideas and situating them in a more abstract, broadly lattice-theoretic context. I will point out a number of connections and conceptual links which do not appear to have been adequately appreciated and also highlight certain ways in which Humberstone's ideas, properly situated, have anticipated subsequent developments (e.g., in inquisitive semantics). Some concluding remarks on such morals and outstanding problems are offered in Section 5.

2 Axiomatics

In this section, I present an axiom (Hilbert) system for **RM0** as well as certain extensions thereof. In what follows, the basic propositional language contains a countable set of propositional variables Π, the propositional constant $\bot$, and the binary connectives $\{\to, \wedge, \vee\}$. Formulae, etc., are defined as usual. I will use $p, q, \ldots$ for arbitrary propositional variables and $\varphi, \psi, \ldots$ for arbitrary formulae. I denote the set of all formulae in this language by Φ.

The axioms for **RM0** are just those of positive **R** (see, e.g., Dunn and Restall [14, §1.3]), together with the mingle axiom M and $\bot$.[5]

DEFINITION 1. The system **RM0** is the smallest set of formulae containing all instances of the following axiom schemata and closed under the following rules:

$$\varphi \to \varphi \tag{I}$$

$$(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi)) \tag{B}$$

$$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) \tag{C}$$

[5]It bears emphasis that this is *not* the fragment of **RM** in this language. The easiest way to see this is to note that **RM0**, so formulated, is a subsystem of **J**, whereas **RM**, which contains the chain theorem [1, §29.3.1, p. 397], clearly is not.

$$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi) \tag{W}$$

$$\varphi \to (\varphi \to \varphi) \tag{M}$$

$$(\varphi \wedge \psi) \to \varphi \tag{$\wedge$E1}$$

$$(\varphi \wedge \psi) \to \psi \tag{$\wedge$E2}$$

$$((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi)) \tag{$\wedge$I}$$

$$\varphi \to (\varphi \vee \psi) \tag{$\vee$I1}$$

$$\psi \to (\varphi \vee \psi) \tag{$\vee$I2}$$

$$((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi) \tag{$\vee$E}$$

$$(\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee \chi) \tag{DIS}$$

$$\bot \to \varphi \tag{$\bot$}$$

$$\frac{\varphi, \psi}{\varphi \wedge \psi} \tag{ADJ}$$

$$\frac{\varphi, \varphi \to \psi}{\psi} \tag{MP}$$

Theoremhood ($\vdash_{\mathbf{RM0}}$) is defined as usual.[6] This axiomatization of **RM0** contains some redundancy (e.g., I easily follows from M and W by MP), but it has the benefit of making clear the relationship between **RM0** and **R**. Also, note that $\top$ is definable as $\bot \to \bot$ and, so defined, it is clear that $\vdash_{\mathbf{RM0}} \varphi \to \top$.

For certain purposes, I will be interested in extensions of **RM0** with the propositional constant t as well as the binary connective $\circ$ (for intensional conjunction or fusion). If I need to refer to the set of formulae formulated in

[6] One could of course also define a suitable consequence relation, holding between sets of formulae and formulae, though I will not pursue this here.

a language containing either or both of these additional connectives, I will refer to it by Φ'. Where these are included in the language, the corresponding axioms for them are as follows, where $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \to \psi) \wedge (\psi \to \varphi)$:

$$\varphi \leftrightarrow (t \to \varphi) \tag{t}$$

$$(\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \circ \psi) \to \chi) \tag{$\circ$}$$

In Subsection 3.4, I will have occasion to make special use of **RM0** extended by t. For emphasis, I will sometimes designate this system by $\mathbf{RM0}^t$.

It is clear that **J**, intuitionistic logic, is axiomatized by extending **RM0** with the weakening axiom schema:

$$\varphi \to (\psi \to \varphi) \tag{K}$$

Of course, this system has a number of redundancies, but that is alright. One could also add to **J**, formulated in the appropriate language, t and $\circ$, but the result would be that t and $\circ$ are equivalent (in the obvious sense) to $\top$ and $\wedge$, respectively, so there is little point (though see Lemma 9).

Finally, note that constructive negation ($\neg$) is definable in both **RM0** and **J** in the usual way: $\neg\varphi$ abbreviates $\varphi \to \bot$.[7] Incidentally, it may be complained that **RM0** is not really a relevance logic as, for example, $(\varphi \wedge \neg\varphi) \to \psi$ will come out a theorem. Without wishing to digress for too long on what makes a logic relevant, let me nevertheless state that I do not view this as a serious objection to the relevant credentials of **RM0**. In any case, the reader should note that the positive fragment of **RM0** *does* satisfy the variable sharing property (see Footnote 4), standard relevance logics like **R** are themselves not infrequently presented with constants including $\bot$, and **RM0** does not have as theorems 'bad guys' like the chain theorem or K.[8]

3 Operational Semantics

In this section, I present a purely operational bisemilattice semantics for **RM0** as well as **J**. All of the essential features of this semantics were already isolated by Humberstone [18], however, his focus was on different systems and my own presentation will reframe the material by placing it in a broadly lattice-theoretic context, the benefits of which will become clear shortly.

[7]For a recent study of various logics with intuitionistic-type negations from a broadly relevant perspective (i.e., using ternary relational semantics), consult Robles and Méndez [33].

[8]Omission of this last is how Bimbó [5, p. 723] characterizes relevance logics.

In Subsection 3.1, I review some essential concepts from lattice theory and the theory of bisemilattices. In Subsection 3.2, I present the formal semantics and discuss its relationship to some other frameworks, including inquisitive semantics. I sketch the proofs of soundness and completeness in Subsection 3.3. Finally, in Subsection 3.4, I illustrate an application of this semantics and results concerning it by giving an embedding of $\mathbf{J}$ in $\mathbf{RM0}^t$.

3.1 Lattice-Theoretic Preliminaries

I begin by briefly reviewing some familiar and less familiar algebraic structures and definitions. The lattice-theoretic material is standard (consult, for example, Davey and Priestley [7] and Grätzer [16]). The material on bisemilattices should also be fairly standard, though I will only be concerned with elementary results concerning them (for additional background and some more advanced results, the reader might consult Balbes [4], Romanowska [34], and Ledda [22], for example).

DEFINITION 2 (Semilattice)**.** A *semilattice* is a structure $\langle S, \bullet \rangle$ where S is a set and $\bullet : S \times S \to S$ satisfies the following equations:

AS) $(x \bullet y) \bullet z = x \bullet (y \bullet z)$;

CO) $x \bullet y = y \bullet x$;

ID) $x \bullet x = x$.

A semilattice $\langle S, \bullet \rangle$ can be used to define a partial order in two ways. In a *meet-semilattice*, the semilattice will generally be written as $\langle S, \wedge \rangle$ and the partial order $\langle S, \leq_\wedge \rangle$ is defined by putting $x \leq_\wedge y$ if and only if $x \wedge y = x$. Dually, in a *join-semilattice*, the semilattice will generally be written as $\langle S, \vee \rangle$ and the partial order $\langle S, \leq_\vee \rangle$ is defined by putting $x \leq_\vee y$ if and only if $x \vee y = y$.

There are notions of distributivity for both kinds of semilattice. So as not to overburden a limited terminology, however, I will follow Humberstone [18, p. 67] in describing these semilattice-distribution properties as decomposition properties. A join-semilattice $\langle S, \vee \rangle$ is said to be *join-decomposable* if $z \leq_\vee x \vee y$ implies $\exists x', y'$ such that $x' \leq_\vee x$, $y' \leq_\vee y$, and $z = x' \vee y'$. Dually, a meet-semilattice $\langle S, \wedge \rangle$ is said to be *meet-decomposable* if $x \wedge y \leq_\wedge z$ implies $\exists x', y'$ such that $x \leq_\wedge x'$, $y \leq_\wedge y'$, and $z = x' \wedge y'$. How decomposability relates to distribution will be discussed below.

There are also notions of bounds for both semilattices. A join-semilattice $\langle S, 0, \vee \rangle$ has a *least element (bottom)* 0 if for any x, $x \vee 0 = x$. A meet-semilattice $\langle S, 1, \wedge \rangle$ has a *greatest element (top)* 1 if for any x, $x \wedge 1 = x$.

DEFINITION 3 (Bisemilattice)**.** A *bisemilattice* is a structure $\langle S, \vee, \wedge\rangle$ where $\langle S, \vee\rangle$ and $\langle S, \wedge\rangle$ are semilattices.

A bisemilattice will be called join-decomposable (meet-decomposable) just when the underlying join-semilattice (meet-semilattice) is. It will simply be called *decomposable* if it is both join-decomposable and meet-decomposable. A *bounded bisemilattice* is a bisemilattice $\langle S, 0, 1, \vee, \wedge\rangle$ with both least and greatest elements. Let it be emphasized that 'least' and 'greatest' are relative to the orders $\leq_\vee$ and $\leq_\wedge$, respectively; what is greatest (least) in one order need not be greatest (least) in the other. A bounded bisemilattice in which $x \vee 1 = 1$ holds will be called *top respecting* and a bounded bisemilattice in which $x \wedge 0 = 0$ holds will be called *bottom respecting*.

A bisemilattice $\langle S, \vee, \wedge\rangle$ is *meet-distributive* if its operations satisfy the equation $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and *join-distributive* if they satisfy the equation $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. If a bisemilattice is both meet-distributive and join-distributive, it will be called *distributive*.

If $\langle S, \vee, \wedge\rangle$ is a bisemilattice, a set $\emptyset \neq T \subseteq S$ is called a *filter* if $x, y \in T$ if and only if $x \wedge y \in T$. Thinking in terms of the induced partial order, a filter is a nonempty set which is upwards-closed under $\leq_\wedge$ and closed under meet. I will call a filter T *join-closed* if whenever $x, y \in T$, $x \vee y \in T$. The following result will frequently be used (mostly implicitly) in the sequel:

LEMMA 1. *If $\langle S, \vee, \wedge\rangle$ is either a meet-distributive or join-distributive bisemilattice and T is a filter in it, T is join-closed.*

Proof. Suppose that $\langle S, \vee, \wedge\rangle$ is meet-distributive. Then $(x \wedge y) \wedge (x \vee y) = ((x \wedge y) \wedge x) \vee ((x \wedge y) \wedge y) = (x \wedge y) \vee (x \wedge y) = x \wedge y$, so $x \wedge y \leq_\wedge x \vee y$. Clearly, then, if $x, y \in T$, $x \vee y$ is as well by upwards-closure and the fact that $x \wedge y \in T$. Alternatively, suppose that $\langle S, \vee, \wedge\rangle$ is join-distributive. Then $(x \wedge y) = (x \wedge y) \vee (x \wedge y) = ((x \wedge y) \vee x) \wedge ((x \wedge y) \vee y) = ((x \vee x) \wedge (x \vee y)) \wedge ((x \vee y) \wedge (y \vee y)) = (x \wedge y) \wedge (x \vee y)$, that is, $x \wedge y \leq_\wedge x \vee y$, which suffices by parallel reasoning. □

If $\mathfrak{B}$ is a bisemilattice, I write $\mathcal{F}(\mathfrak{B})$ for the set of all filters in $\mathfrak{B}$ and I write $\uparrow x$ for the *principal filter* generated by x, i.e., $\{y : x \leq_\wedge y\}$. Ideals, meet-closed ideals, and principal ideals are defined dually, though I will have little use for them in this chapter.

DEFINITION 4 (Lattice)**.** A *lattice* is a bisemilattice $\langle S, \vee, \wedge\rangle$ in which $\vee$ and $\wedge$ satisfy the absorption equations:

A1) $x \vee (x \wedge y) = x$;

A2) $x \wedge (x \vee y) = x$.

In any lattice $\langle S, \vee, \wedge \rangle$, the partial orders $\langle S, \leq_\wedge \rangle$ and $\langle S, \leq_\vee \rangle$ coincide. Consequently, where $\langle S, \vee, \wedge \rangle$ is a lattice, the unambiguous induced partial order will generally be written as $\langle S, \leq \rangle$. Over bisemilattices, all of join-decomposability, meet-decomposability, join-distributivity, and meet-distributivity are independent.[9] On the other hand—and this illustrates how strong the absorption laws really are—all of these properties are equivalent over lattices (consult, e.g., Grätzer [16, p. 167]). Any filter T in a lattice, regardless of whether it is distributive, is join-closed (indeed, satisfies the stronger property that if $x \in T$, $x \vee y \in T$ for any y). Finally, any bounded lattice is both top and bottom respecting.

Remark. What separates lattices from bisemilattices are the absorption postulates (A1) and (A2). A weakening of the absorption postulates, that $x \vee (x \wedge y) = x \wedge (x \vee y)$, is sometimes known as *Birkhoff's equation*, and bisemilattices which satisfy this are known as *Birkhoff systems* (see, e.g., Harding and Romanowska [17, p. 46]). It is obvious that any join-distributive or meet-distributive bisemilattice is a Birkhoff system.[10]

Before rounding out this subsection by giving some examples of various of the foregoing algebraic structures, I will note two more facts concerning bisemilattices and lattices which will turn out to play an important role in semantically distinguishing (and relating) **RM0** and **J**:

LEMMA 2. *If $\langle S, 0, 1, \vee, \wedge \rangle$ is a bounded join-distributive bisemilattice, it is a lattice if and only if it is bottom respecting.*[11]

Proof. For the easy direction, if $\langle S, 0, 1, \vee, \wedge \rangle$ is a lattice, then by (A2): $0 \wedge x = 0 \wedge (0 \vee x) = 0$. Conversely, suppose that $\langle S, 0, 1, \vee, \wedge \rangle$ is bottom respecting. It must be shown that the absorption equations from Definition 4 are satisfied. Ad (A2): $x = x \vee 0 = x \vee (y \wedge 0) = ((x \vee y) \wedge (x \vee 0)) = x \wedge (x \vee y)$. Ad (A1): $x \vee (x \wedge y) = ((x \vee x) \wedge (x \vee y)) = x \wedge (x \vee y) = x$, by (A2). $\Box$

LEMMA 3. *If $\langle S, 0, 1, \vee, \wedge \rangle$ is a bounded join-distributive bisemilattice, $\langle \uparrow 0, 0, 1, \vee, \wedge \rangle$ is a bounded distributive lattice (where these operations are restricted to $\uparrow 0$).*

Proof. In view of Lemma 2, it suffices to show that $\langle \uparrow 0, 0, 1, \vee, \wedge \rangle$ is bottom respecting (which is obvious, since if $x \in \uparrow 0$, $0 \leq_\wedge x$, i.e., $x \wedge 0 = 0$) and closed

[9] I am not sure if this exact fact is stated anywhere in the literature, but various parts of this independence result can be found (e.g., in Romanowska [34, p. 37]) and the rest can be shown without too much difficulty.

[10] I am grateful to H. P. Sankappanavar for suggesting that Birkhoff systems may be relevant to the subject of this chapter.

[11] Cf. Płonka [28, p. 195, Theorem 2].

under the relevant operations (and so, a sub-bisemilattice of $\langle S, 0, 1, \vee, \wedge\rangle$). The only case that requires thought involves $\vee$: if $x, y \in {\uparrow}0$, by the assumption that $\langle S, 0, 1, \vee, \wedge\rangle$ is join-distributive, $x \vee y \in {\uparrow}0$ by Lemma 1. □

I now briefly give some examples. The first two, reducts of the strong and weak Kleene algebras [20, §64, p. 334], are among the best-known lattices and bisemilattices in logic. The third, which I believe is original to this chapter, combines them; this last structure turns out to be a (non-degenerate) frame for **RM0**.

Example 1 (Strong Kleene)**.** Consider the structure $\langle\{0, .5, 1\}, 0, 1, \vee, \wedge\rangle$ where the operations $\wedge$ and $\vee$ are defined by the following strong Kleene tables:

$\wedge$	0	.5	1
0	0	0	0
.5	0	.5	.5
1	0	.5	1

$\vee$	0	.5	1
0	0	.5	1
.5	.5	.5	1
1	1	1	1

It is of course well-known that these tables determine a bounded distributive lattice.

Example 2 (Weak Kleene)**.** Consider the structure $\langle\{0, .5, 1\}, 0, 1, \vee, \wedge\rangle$ where the operations $\wedge$ and $\vee$ are defined by the following weak Kleene tables:

$\wedge$	0	.5	1
0	0	.5	0
.5	.5	.5	.5
1	0	.5	1

$\vee$	0	.5	1
0	0	.5	1
.5	.5	.5	.5
1	1	.5	1

This is easily shown to be a bounded join-distributive meet-decomposable bisemilattice, but it is *not* a lattice: $0 \wedge (0 \vee .5) = .5$, contradicting (A2). It is also neither top respecting ($.5 \vee 1 = .5$) nor bottom respecting ($.5 \wedge 0 = .5$).

Example 3 (Moderate Kleene)**.** Consider the structure $\langle\{0, .5, 1\}, 0, 1, \vee, \wedge\rangle$ where the operations $\wedge$ and $\vee$ are defined by the weak and strong Kleene tables, respectively:

$\wedge$	0	.5	1
0	0	.5	0
.5	.5	.5	.5
1	0	.5	1

$\vee$	0	.5	1
0	0	.5	1
.5	.5	.5	1
1	1	1	1

This is another example of a bounded join-distributive meet-decomposable bisemilattice that's not a lattice and is not bottom respecting. However, this one *is* top respecting.

3.2 Bisemilattice Models

In this subsection, I present bisemilattice frames and models for **RM0** and **J** and prove some basic results about the semantics which will be required in later parts of the chapter. I also discuss connections between this semantics and the semantics of Humberstone [18] as well as Punčochář [30].

As I have already indicated, the semantics to be presented here is directly inspired by, and largely follows, Humberstone [18]. Nevertheless, there are important differences. Humberstone's focus is on positive **R** and the frames he proposes for it are structures of the form $\langle S, 1, 0, \cdot, +\rangle$ where $\langle S, 1, \cdot\rangle$ is an Abelian (commutative) monoid, $\langle S, 0, +\rangle$ is a join-decomposable join-semilattice, $\cdot$ distributes over $+$, $0 \cdot x = 0$, and $\cdot$ and $+$ satisfy "pseudo-idempotence", i.e., $x \cdot (x + 1) = x \cdot x = x^2$ [18, pp. 66–67].

The condition of pseudo-idempotence is particularly aesthetically and otherwise unfortunate (which Humberstone [18, p. 67] actually concedes), but Humberstone also considers, if only briefly, what occurs if you adopt the real thing: you get bisemilattice frames which suffice to characterize **RM0** [18, pp. 75–76].[12] I will use the following bisemilattices to furnish a semantics for **RM0**:

DEFINITION 5 (Mingle Frame)**.** A *mingle frame* is a bounded, top respecting, join-distributive, meet-decomposable bisemilattice $\mathfrak{F} = \langle S, 0, 1, \vee, \wedge\rangle$.

It must be emphasized that the bisemilattice frames described by Definition 5 are still not exactly the same as those which Humberstone considered for **RM0**. The central distinction is that, in my proposal, everything is, as it were, flipped (thus, I have meet-decomposability where Humberstone has join-decomposability, etc.). The motivation for this is narrowly technical and has to do with the naturalness of certain constructions yet to come.

Concrete instances of mingle frames are given in Examples 1 and 3, though the first is degenerate in the sense that it is a lattice.[13] It turns out that the class of lattice mingle frames characterizes intuitionistic logic.

DEFINITION 6 (Intuitionistic Frame)**.** An *intuitionistic frame* is a structure $\mathfrak{F} = \langle S, 0, 1, \vee, \wedge\rangle$ where $\mathfrak{F}$ is a mingle frame which is a lattice (equivalently, in view of Lemma 2, which is bottom respecting). More succinctly, an intuitionistic frame is just a bounded distributive lattice.

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[12]Humberstone [18, pp. 75–76] does not actually use the word 'bisemilattice' or talk about **RM0** by that name, but this is effectively what he describes.

[13]It is worth remarking that, while combining weak Kleene conjunction with strong Kleene disjunction yields a mingle frame, it would not do to combine weak Kleene disjunction with strong Kleene conjunction. The resulting structure would be bottom respecting, but not top respecting.

Definition 6 marks a considerable departure from the frames used to characterize **J** in Humberstone's own semantics. For Humberstone, frames for (positive) **J** are just frames for positive **R** (as described above) which satisfy the added condition that $x + 1 = 1$ [18, p. 66]. Flipping, this amounts to the condition which I have called bottom respect. But, over the relevant class of bisemilattice structures, this turns out to be equivalent to being a lattice, per Lemma 2.

It is here, in the formal apparatus for **J**, that the real conceptual clarity afforded by the bisemilattice semantics shines. It allows us to mark the difference between relevant (**RM0**) and irrelevant (**J**) logics by those properties which distinguish bisemilattices from lattices: the absorption laws. As my principal interest in this chapter is not philosophical, I will not dwell long on this, but allow me to point out that, of all the laws defining distributive lattices, these are the only *non-regular identities* (i.e., identities in which the variables on the sides of '=' are mismatched)—a strong whiff of irrelevance, indeed.[14]

DEFINITION 7 (Model). A *mingle (intuitionistic) model* is a structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ where $\mathfrak{F} = \langle S, 0, 1, \vee, \wedge \rangle$ is a mingle (intuitionistic) frame and $V : \Pi \to \mathcal{F}(\mathfrak{F})$.

Thus, a model is obtained by assigning filters to propositional variables in the underlying frame; note that, by Lemma 1, all such filters must be join-closed. As would be expected from what has been said so far, in Humberstone's own semantics, one gets a model by assigning ideals to variables (Humberstone [18, p. 68] proposes something a bit more convoluted, but this is what it would come to in a bisemilattice framework).

Turning now to the truth conditions, which are essentially those of Humberstone [18, pp. 63–65, 72] (cf. Urquhart [36, §§2, 4]) modulo 'flipping', with respect to a mingle model $\mathfrak{M} = \langle S, 0, 1, \vee, \wedge, V \rangle$ where $x \in S$, the relation $\models_x^{\mathfrak{M}}$ is defined as follows:[15]

(1) $\models_x^{\mathfrak{M}} p$ if and only if $x \in V(p)$;

(2) $\models_x^{\mathfrak{M}} \perp$ if and only if $x = 1$;

(3) $\models_x^{\mathfrak{M}} t$ if and only if $0 \leq_\wedge x$;

[14] For more on regular identities and their importance, consult Padmanabhan [27].

[15] Note that all of the truth conditions are in fact purely operational. In particular, $\leq_\wedge$ is a *defined* relation. Therefore, the truth condition for t, for example, could instead have been given as: $\models_x^{\mathfrak{M}} t$ if and only if $0 \wedge x = 0$. This feature of the semantic framework distinguishes it from Fine's hybrid partial order-operational framework which postulates a primitive relation $\leq$ [15].

(4) $\models_x^{\mathfrak{M}} \varphi \wedge \psi$ if and only if $\models_x^{\mathfrak{M}} \varphi$ and $\models_x^{\mathfrak{M}} \psi$;

(5) $\models_x^{\mathfrak{M}} \varphi \vee \psi$ if and only if $\exists y, z \in S$ such that $x = y \wedge z$, $\models_y^{\mathfrak{M}} \varphi$, and $\models_z^{\mathfrak{M}} \psi$;

(6) $\models_x^{\mathfrak{M}} \varphi \to \psi$ if and only if for all $y \in S$, $\not\models_y^{\mathfrak{M}} \varphi$ or $\models_{x \vee y}^{\mathfrak{M}} \psi$;

(7) $\models_x^{\mathfrak{M}} \varphi \circ \psi$ if and only if $\exists y, z \in S$ such that $y \vee z \leq_\wedge x$, $\models_y^{\mathfrak{M}} \varphi$, and $\models_z^{\mathfrak{M}} \psi$.

With reference to a given model $\mathfrak{M} = \langle S, 0, 1, \vee, \wedge, V \rangle$ and formula φ, define $[\varphi]^{\mathfrak{M}} = \{x \in S : \models_x^{\mathfrak{M}} \varphi\}$. $[\varphi]^{\mathfrak{M}}$ may intuitively be thought of as the *proposition* expressed by φ in $\mathfrak{M}$.

The following two results (Lemma 4 and Corollary 1) are versions of Humberstone's Plus and Zero lemmata [18, pp. 68–69] though, in the present framework, the second is a mere corollary of the first:

LEMMA 4 (Propositional Filters)**.** *For any formula φ and any mingle model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, $[\varphi]^{\mathfrak{M}} \in \mathcal{F}(\mathfrak{F})$.*

Proof. The result holds by Definition 7 for propositional variables. Since $\uparrow 1 = \{1\}$ is obviously a filter (indeed, the smallest one), $[\bot]^{\mathfrak{M}} \in \mathcal{F}(\mathfrak{F})$. It is also obvious that $\uparrow 0 = [t]^{\mathfrak{M}}$ is a filter. The other cases follow by induction. □

I have been rather brief with Lemma 4 because I will effectively cover some of the primary inductive cases as part of a more general and related result concerning the algebra of propositions below (Lemma 10).

COROLLARY 1. *For any formula φ and any mingle model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, $1 \in [\varphi]^{\mathfrak{M}}$.*

Proof. Immediate from Lemma 4, noting that 1 is an element of any filter. □

DEFINITION 8 (Validity)**.** Where $\mathfrak{M} = \langle S, 0, 1, \vee, \wedge, V \rangle$ is a mingle model, φ is valid in $\mathfrak{M}$ ($\models^{\mathfrak{M}} \varphi$) if $0 \in [\varphi]^{\mathfrak{M}}$. Where $\mathfrak{F} = \langle S, 0, 1, \vee, \wedge \rangle$ is a mingle frame, φ is valid in $\mathfrak{F}$ ($\models^{\mathfrak{F}} \varphi$) if $\models^{\mathfrak{M}} \varphi$ for every model $\mathfrak{M}$ over $\mathfrak{F}$. φ is *valid* in **RM0** ($\models_{\mathbf{RM0}} \varphi$) if $\models^{\mathfrak{F}} \varphi$ for every mingle frame $\mathfrak{F}$ and valid in **J** ($\models_{\mathbf{J}} \varphi$) if $\models^{\mathfrak{F}} \varphi$ for every intuitionistic frame $\mathfrak{F}$.

Before concluding this subsection, I wish to touch upon the relation of this semantics to inquisitive semantics or, in any case, the sort of 'generalization' of inquisitive semantics developed for **J** by Punčochář [30]. Punčochář [30] shows (among other things) that **J** is characterized by all *distributive information models*, where a distributive information frame (algebra) is a join-decomposable join-semilattice with a least element and a model is obtained by assigning to each propositional variable an ideal in the algebra.

The truth conditions proposed by Punčochář [30, p. 1648] for $\bot$, $\wedge$, and $\vee$ are identical to those of Humberstone [18], that is to say, to flipped versions of the conditions presented above. The condition for $\to$ offered by Punčochář [30, p. 1648] is superficially different. Taking the liberty to flip things as appropriate, it amounts to the following:

(6′) $\models^{\mathfrak{M}}_{x} \varphi \to \psi$ if and only if for all $x \leq_{\vee} y$, $\not\models^{\mathfrak{M}}_{y} \varphi$ or $\models^{\mathfrak{M}}_{y} \psi$.

In fact, though, this condition is just equivalent to (6) over the lattice frames given for **J** above. For suppose condition (6) obtains and $x \leq y$ (subscripts may be ignored in a lattice frame as there is only one unambiguous partial order) and $\models^{\mathfrak{M}}_{y} \varphi$; then $\models^{\mathfrak{M}}_{x \vee y} \psi$, that is, $\models^{\mathfrak{M}}_{y} \psi$, given that $y = x \vee y$, as required for (6′). Conversely, suppose condition (6′) obtains and $\models^{\mathfrak{M}}_{y} \varphi$; then as $x \leq x \vee y$ and $\models^{\mathfrak{M}}_{x \vee y} \varphi$—since $y \in [\varphi]^{\mathfrak{M}}$ and $[\varphi]^{\mathfrak{M}}$ is upwards closed—it follows that $\models^{\mathfrak{M}}_{x \vee y} \psi$, as required for condition (6).

It is clear, then, that there is significant overlap between the inquisitive semantic approach to **J** developed by Punčochář [30], as well as related work by other inquisitive semanticists, and the decades-earlier but unfortunately not well-known work of Humberstone [18] and my own presentation of that material here. Since the work of Punčochář [30] and other inquisitive semanticists is, however, quite independent as far as I can tell,[16] the recurrence of these ideas should be taken to speak to their quality.

3.3 Soundness and Completeness

In this subsection, I prove that **RM0** and **J** (Section 2) are sound and complete with respect to their operational semantics from Subsection 3.2. The arguments straightforwardly adapt results of Humberstone [18], but are worth including in some detail to make this chapter self-contained.

THEOREM 1 (Soundness). *If* $\vdash_{\textbf{RM0}} \varphi$*, then* $\models_{\textbf{RM0}} \varphi$*.*

Proof. I survey just a couple representative cases. Suppose that the mingle axiom M fails, i.e., that $\not\models_{\textbf{RM0}} \psi \to (\psi \to \psi)$; then there is a mingle model $\mathfrak{M} = \langle S, 0, 1, \vee, \wedge, V \rangle$ and some $x, y \in S$ such that $x, y \in [\psi]^{\mathfrak{M}}$ and $x \vee y \notin [\psi]^{\mathfrak{M}}$. But $[\psi]^{\mathfrak{M}}$ is a join-closed filter by Lemmata 1 and 4, so $x \vee y \in [\psi]^{\mathfrak{M}}$,

[16] In fact, in a recent article, Punčochář and Tedder [31, p. 357] *do* note the connection to Humberstone's condition for $\vee$ in any case. In another fairly recent article, Humberstone [19] himself discusses various accounts of disjunction including his own from [18] as well as inquisitive views.

a contradiction. Suppose for contradiction that axiom $\bot$ fails, i.e., that $\not\models_{\mathbf{RM0}} \bot \to \psi$; then there is a mingle model $\mathfrak{M} = \langle S, 0, 1, \vee, \wedge, V \rangle$ and an $x \in S$ such that $x \in [\bot]^{\mathfrak{M}}$ and $x \notin [\psi]^{\mathfrak{M}}$. But then $x = 1$, so by Corollary 1, $x \in [\psi]^{\mathfrak{M}}$, a contradiction. □

THEOREM 2 (Soundness)**.** *If* $\vdash_{\boldsymbol{J}} \varphi$*, then* $\models_{\boldsymbol{J}} \varphi$*.*

Proof. There is only one further case to consider. To show the validity of axiom K, suppose for contradiction that $\not\models_{\mathbf{J}} \psi \to (\theta \to \psi)$. Then this fails in some intuitionistic model $\mathfrak{M} = \langle S, 0, 1, \vee, \wedge, V \rangle$ which must be a lattice. So there are $x, y \in S$ such that $x \in [\psi]^{\mathfrak{M}}$ and $y \in [\theta]^{\mathfrak{M}}$ and $x \vee y \notin [\psi]^{\mathfrak{M}}$. But $[\psi]^{\mathfrak{M}}$ is a filter in a *lattice*, whence $x \in [\psi]^{\mathfrak{M}}$ implies $x \vee y \in [\psi]^{\mathfrak{M}}$, which gives the desired contradiction. □

To prove completeness, I construct a canonical model for **L** (I will use **L** to refer ambiguously to **RM0** or **J** in what follows, and disambiguate where it becomes relevant). A set of formulae Γ is a **L** *theory* if the following conditions are satisfied:

1. $\varphi \in \Gamma$ and $\psi \in \Gamma$ imply $\varphi \wedge \psi \in \Gamma$;
2. $\varphi \in \Gamma$ and $\vdash_{\mathbf{L}} \varphi \to \psi$ imply $\psi \in \Gamma$.

I write $\mathrm{Th}(\Gamma)$ for the smallest theory containing the set of formulae Γ, or just $\mathrm{Th}(\varphi)$ if $\Gamma = \{\varphi\}$.[17] By $\mathbb{TH}$, I denote the set of all theories; $\mathbb{TH} \setminus \{\emptyset\}$ is, then, obviously the set of all nonempty theories. Define $\Gamma \cdot \Delta = \{\psi : \exists \varphi \in \Delta(\varphi \to \psi \in \Gamma)\}$ (cf. Fine [15, p. 353]).

DEFINITION 9. The *canonical model* for **L** is the structure $\mathfrak{M}^c = \langle \mathbb{TH} \setminus \{\emptyset\}, \mathbf{L}, \Phi, \cdot, \cap, V^c \rangle$ where $V^c(p) = \{\Gamma \in \mathbb{TH} \setminus \{\emptyset\} : p \in \Gamma\}$.[18]

Remark. One reason for my preference for the flipped, filter semantics rather than Humberstone's ideal semantics is that the canonical model construction is more natural. In Humberstone's construction, $\cap$ counterintuitively plays the role of join with Φ as semilattice bottom [18, pp. 70–71].

LEMMA 5. *The structure* $\mathfrak{M}^c = \langle \mathbb{TH} \setminus \{\emptyset\}, \boldsymbol{RM0}, \Phi, \cdot, \cap, V^c \rangle$ *is a mingle model.*

[17] In the interest of rigor, I really ought to write something like $\mathrm{Th}_{\mathbf{L}}(\Gamma)$ for the smallest **L** theory containing Γ, but I will generally suppress what system **L** I am talking about when talking about theories.

[18] Technically, depending on the language, Φ' should be used instead of Φ. For the purposes of this subsection, I just intend by Φ the set of all formulae of whatever the language is. Incidentally, nothing in the basic completeness proof requires the use of the constants or $\circ$.

Proof. The argument is essentially that given by Humberstone [18, pp. 70–72] (cf. Fine [15, §3]). For the flavor, I show that $\cdot$ is idempotent, sketch the main ideas required for proving meet-decomposability and join-distributivity, and verify that V^c meets the condition required by Definition 7, i.e., that each $V^c(p)$ is a filter.

To show that $\cdot$ is idempotent, suppose that $\varphi \in x \cdot x$; then $\exists\psi \in x$ such that $\psi \to \varphi \in x$. Since x is closed under ADJ, $\psi \wedge (\psi \to \varphi) \in x$ whence $\varphi \in x$ by the fact that $\vdash_{\mathbf{RM0}} (\psi \wedge (\psi \to \varphi)) \to \varphi$ (note that the proof of this makes indispensable use of W). Conversely, suppose that $\varphi \in x$; then since $\vdash_{\mathbf{RM0}} \varphi \to (\varphi \to \varphi)$ by M, $\varphi \to \varphi \in x$, which suffices to show $\varphi \in x \cdot x$. Therefore, $x = x \cdot x$, as required by idempotence. To show that $\langle \mathbb{TH} \setminus \{\emptyset\}, \Phi, \cap \rangle$ is meet-decomposable, on the supposition that $x \cap y \subseteq z$, put $x' = \mathrm{Th}(x \cup z)$ and $y' = \mathrm{Th}(y \cup z)$. This immediately delivers everything that is needed except for the property that $x' \cap y' \subseteq z$, which follows making use of DIS. Ad join-distributivity, the difficult direction is showing that $(x \cdot y) \cap (x \cdot z) \subseteq x \cdot (y \cap z)$. Suppose $\varphi \in (x \cdot y) \cap (x \cdot z)$; then $\exists\psi \in y$ such that $\psi \to \varphi \in x$ and $\exists\theta \in z$ such that $\theta \to \varphi \in x$. By ADJ and $\vee$E, $(\psi \vee \theta) \to \varphi \in x$, and by $\vee$I1 and $\vee$I2, $\psi \vee \theta \in y \cap z$. Hence, $\varphi \in x \cdot (y \cap z)$, as required. Finally, to show that $V^c(p)$ is a filter, note that it must be nonempty since $\Phi \in V^c(p)$ and $x, y \in V^c(p)$ if and only if $p \in x, y$ if and only if $p \in x \cap y$ if and only if $x \cap y \in V^c(p)$. $\square$

LEMMA 6. *The structure* $\mathfrak{M}^c = \langle \mathbb{TH} \setminus \{\emptyset\}, \boldsymbol{J}, \Phi, \cdot, \cap, V^c \rangle$ *is an intuitionistic model.*

Proof. The proof is identical to that of Lemma 5, except it also has to be shown that $\mathfrak{M}^c$ is a lattice. By Lemma 2, it suffices to show that $\mathfrak{M}^c$ is bottom respecting. Obviously, $\mathbf{J} \cap x \subseteq \mathbf{J}$, so, for the converse, suppose that $\varphi \in \mathbf{J}$; then, as there is some $\psi \in x$ and $\vdash_{\mathbf{J}} \psi \to \varphi$ (by K), $\varphi \in x$, which suffices to show $\mathbf{J} \subseteq \mathbf{J} \cap x$, as desired. $\square$

LEMMA 7 (Truth Lemma)**.** *If* $\mathfrak{M}^c = \langle \mathbb{TH} \setminus \{\emptyset\}, \boldsymbol{L}, \Phi, \cdot, \cap, V^c \rangle$ *is the canonical model for* $\boldsymbol{L}$*, then for any* $x \in \mathbb{TH} \setminus \{\emptyset\}$*,* $x \in [\varphi]^{\mathfrak{M}^c}$ *if and only if* $\varphi \in x$*.*

Proof. By induction on the complexity of φ. The result holds by definition when φ is a propositional variable and is obvious when φ is t, $\bot$, or of the form $\psi \wedge \theta$. I will just consider the cases in which φ is either of the form $\psi \to \theta$ or $\psi \vee \theta$, supposing the result holds for ψ and θ. (The arguments for $\to$ and $\vee$ are essentially the same as those found in Fine [15, p. 355] and Humberstone [18, p. 72], respectively.)

Suppose $\psi \to \theta \in x$ and $y \in [\psi]^{\mathfrak{M}^c}$; by the induction hypothesis, $\psi \in y$, therefore, $\theta \in x \cdot y$, i.e., $x \cdot y \in [\theta]^{\mathfrak{M}^c}$, which suffices to show $x \in [\psi \to \theta]^{\mathfrak{M}^c}$.

Conversely, suppose that $\psi \to \theta \notin x$ and put $y = \mathrm{Th}(\psi)$. Then $\theta \notin x \cdot y$; for otherwise, there would be a formula χ such that $\vdash_{\mathbf{L}} \psi \to \chi$ and $\chi \to \theta \in x$, which would imply that $\psi \to \theta \in x$ (by suffixing), a contradiction. Thus, by the induction hypothesis, $y \in [\psi]^{\mathfrak{M}^c}$ and $x \cdot y \notin [\theta]^{\mathfrak{M}^c}$, which suffices.

Suppose $\psi \vee \theta \in x$ and put $y = \mathrm{Th}(\psi)$ and $z = \mathrm{Th}(\theta)$. Then $y \cap z \subseteq x$, for if $\chi \in y \cap z$, then $\vdash_{\mathbf{L}} \psi \to \chi$ and $\vdash_{\mathbf{L}} \theta \to \chi$, whence $\vdash_{\mathbf{L}} (\psi \vee \theta) \to \chi$ by $\vee$E, so $\chi \in x$. By meet-decomposability, there are $y \subseteq y' \in \mathbb{TH} \setminus \{\emptyset\}$ and $z \subseteq z' \in \mathbb{TH} \setminus \{\emptyset\}$ such that $x = y' \cap z'$. By the induction hypothesis, $\psi \in y \subseteq y' \in [\psi]^{\mathfrak{M}^c}$ and $\theta \in z \subseteq z' \in [\theta]^{\mathfrak{M}^c}$, which yields the result. Conversely, suppose $x \in [\psi \vee \theta]^{\mathfrak{M}^c}$; then there are y, z such that $x = y \cap z$, $y \in [\psi]^{\mathfrak{M}^c}$, and $z \in [\theta]^{\mathfrak{M}^c}$. By the induction hypothesis, $\psi \in y$ and $\theta \in z$, whence it follows that $\psi \vee \theta \in y \cap z = x$ by $\vee$I1 and $\vee$I2. $\square$

THEOREM 3 (Completeness)**.** *If* $\models_{\boldsymbol{RM0}} \varphi$*, then* $\vdash_{\boldsymbol{RM0}} \varphi$*.*

Proof. Suppose $\nvdash_{\mathbf{RM0}} \varphi$; then $\varphi \notin \mathbf{RM0}$ and so, by Lemma 7, $\mathbf{RM0} \notin [\varphi]^{\mathfrak{M}^c}$, i.e., $\not\models^{\mathfrak{M}^c} \varphi$. Moreover, by Lemma 5, $\mathfrak{M}^c$ is a mingle model, so $\not\models_{\mathbf{RM0}} \varphi$, which suffices. $\square$

THEOREM 4 (Completeness)**.** *If* $\models_{\boldsymbol{J}} \varphi$*, then* $\vdash_{\boldsymbol{J}} \varphi$*.*

Proof. The proof is essentially that for Theorem 3, except the role of Lemma 5 is played by Lemma 6. $\square$

3.4 An Embedding of J in RM0^t

Using a well-known translation scheme (see, e.g., Meyer [24, pp. 198ff.] and Dunn and Meyer [13, pp. 229–230]), I shall now give an embedding of **J** into **RM0**^{t}. The result (if I may say so) gives a nice illustration of an application of the foregoing semantics and some of the results concerning it.

DEFINITION 10 (Translation)**.** Define the function $\tau : \Phi \to \Phi'$ as follows:

1. $\tau(p) = p$;
2. $\tau(\bot) = \bot$;
3. $\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$;
4. $\tau(\varphi \vee \psi) = \tau(\varphi) \vee \tau(\psi)$;
5. $\tau(\varphi \to \psi) = (\tau(\varphi) \wedge t) \to \tau(\psi)$.

LEMMA 8. *For any* $\varphi \in \Phi$*: If* $\vdash_{\boldsymbol{J}} \varphi$*, then* $\vdash_{\boldsymbol{RM0}^t} \tau(\varphi)$*.*

Proof. Suppose $\nvdash_{\mathbf{RM0}^t} \tau(\varphi)$. By Theorem 3, there is a mingle model $\mathfrak{M} = \langle S, 0, 1, \vee, \wedge, V\rangle$ such that $\not\models_0^{\mathfrak{M}} \tau(\varphi)$. Define $\mathfrak{M}' = \langle \uparrow 0, 0, 1, \vee, \wedge, V'\rangle$, where $V'(p) = V(p) \cap \uparrow 0$ and the operations are likewise restricted. $\langle \uparrow 0, 0, 1, \vee, \wedge\rangle$ is an intuitionistic frame by Lemma 3 and, as intersections of filters are filters, $V'(p)$ is a filter for every p. Thus, $\mathfrak{M}'$ is an intuitionistic model.

It is to be shown by induction that, for all formulae $\psi \in \Phi$ and $x \in \uparrow 0$, $\models_x^{\mathfrak{M}'} \psi$ if and only if $\models_x^{\mathfrak{M}} \tau(\psi)$. The basis cases are immediate, so suppose the result holds for θ and χ. I examine just the cases concerning $\vee$ and $\to$.

Suppose $\models_x^{\mathfrak{M}} \tau(\theta \vee \chi)$, i.e., $\models_x^{\mathfrak{M}} \tau(\theta) \vee \tau(\chi)$. Then $\exists y, z \in S$ such that $x = y \wedge z$, $\models_y^{\mathfrak{M}} \tau(\theta)$, and $\models_z^{\mathfrak{M}} \tau(\chi)$. By the induction hypothesis and the fact that $y, z \in \uparrow 0$ since $y \wedge z = x \in \uparrow 0$, $\models_y^{\mathfrak{M}'} \theta$ and $\models_z^{\mathfrak{M}'} \chi$, i.e., $\models_x^{\mathfrak{M}'} \theta \vee \chi$. Conversely, if $\models_x^{\mathfrak{M}'} \theta \vee \chi$, then $\exists y, z \in \uparrow 0$ such that $x = y \wedge z$, $\models_y^{\mathfrak{M}'} \theta$, and $\models_z^{\mathfrak{M}'} \chi$, which immediately yields the result by the induction hypothesis.

Suppose $\models_x^{\mathfrak{M}} \tau(\theta \to \chi)$, i.e., $\models_x^{\mathfrak{M}} (\tau(\theta) \wedge t) \to \tau(\chi)$, and suppose $\models_y^{\mathfrak{M}'} \theta$. By the induction hypothesis and the fact that $0 \leq_\wedge y$, $\models_y^{\mathfrak{M}} \tau(\theta) \wedge t$, whence $\models_{x \vee y}^{\mathfrak{M}} \tau(\chi)$. $x, y \in \uparrow 0$ implies $x \vee y \in \uparrow 0$ (Lemma 1), so by the induction hypothesis, $\models_{x \vee y}^{\mathfrak{M}'} \chi$, which suffices to show $\models_x^{\mathfrak{M}'} \theta \to \chi$. Conversely, suppose $\not\models_x^{\mathfrak{M}} \tau(\theta \to \chi)$, i.e., $\not\models_x^{\mathfrak{M}} (\tau(\theta) \wedge t) \to \tau(\chi)$. Then $\exists y \in S$ such that $\models_y^{\mathfrak{M}} \tau(\theta) \wedge t$ and $\not\models_{x \vee y}^{\mathfrak{M}} \tau(\chi)$. Then $0 \leq_\wedge y$ so, by the induction hypothesis, $\models_y^{\mathfrak{M}'} \theta$ and $\not\models_{x \vee y}^{\mathfrak{M}'} \chi$, that is, $\not\models_x^{\mathfrak{M}'} \theta \to \chi$.

Then $\not\models_0^{\mathfrak{M}'} \varphi$ follows from $\not\models_0^{\mathfrak{M}} \tau(\varphi)$. Therefore, by Theorem 2, $\nvdash_{\mathbf{J}} \varphi$, which was to be proved. □

LEMMA 9. *For any $\varphi \in \Phi$: If $\vdash_{\mathbf{RM0}^t} \tau(\varphi)$, then $\vdash_{\mathbf{J}} \varphi$.*

Proof. Let $\mathbf{J}'$ be $\mathbf{J}$ formulated in the language with t and the axiom t. Then it is clear that $\mathbf{RM0}^t$ is a subsystem of $\mathbf{J}'$, so if $\vdash_{\mathbf{RM0}^t} \tau(\varphi)$ (ex hypothesi), we have $\vdash_{\mathbf{J}'} \tau(\varphi)$. By induction, $\tau(\varphi)$ and φ are provably equivalent in $\mathbf{J}'$, thus $\vdash_{\mathbf{J}'} \varphi$. Lastly, it must be shown that $\mathbf{J}'$ is a conservative extension of $\mathbf{J}$, i.e., that for any $\psi \in \Phi$, $\vdash_{\mathbf{J}'} \psi$ only if $\vdash_{\mathbf{J}} \psi$. But this clearly holds since in any proof in $\mathbf{J}'$ of such a ψ, t can be replaced with any theorem of $\mathbf{J}$ (e.g., $p \to p$) thereby yielding a proof of ψ in $\mathbf{J}$. Thus, $\vdash_{\mathbf{J}} \varphi$, as desired. □

THEOREM 5. *For any $\varphi \in \Phi$: $\vdash_{\mathbf{J}} \varphi$ if and only if $\vdash_{\mathbf{RM0}^t} \tau(\varphi)$.*

Proof. Immediate from Lemmata 8 and 9. □

4 Algebraic Semantics

In this section, I present an algebraic semantics for **RM0**. The kind of algebraic structure used for modeling **RM0** is the obvious extension of what

Meyer [25, p. 39] and Meyer and Routley [26, p. 408] call a *Dunn monoid*, in honor of the pioneering work of Dunn [8] (published as [11]).[19] Whereas Dunn monoids furnish an algebraic semantics for positive **R**, what I will call *Dunn semilattices* furnish an algebraic semantics for **RM0**.[20] The name is, in a sense, unfortunate, since Dunn semilattices are also bisemilattices and, indeed, lattices (under different operations). However, I hope the reader will indulge my penchant for semilattice nomenclature, if only because the name highlights that the pertinent (commutative) monoids are now required to be fully idempotent.

DEFINITION 11 (Dunn Semilattice)**.** A *Dunn semilattice* is a structure $\mathbf{D} = \langle D, \mathbf{1}, \mathbf{0}, \bullet, \Rightarrow, \sqcup, \sqcap \rangle$ where $\mathbf{0}, \mathbf{1} \in D$ and the binary operations $\bullet$, $\Rightarrow$, $\sqcup$, and $\sqcap$ satisfy the properties that:

1. $\langle D, \mathbf{0}, \sqcup, \sqcap \rangle$ is a distributive lattice with least element $\mathbf{0}$;[21]

2. $\langle D, \mathbf{1}, \bullet \rangle$ is a meet-semilattice with greatest element $\mathbf{1}$;

3. $a \bullet \mathbf{0} = \mathbf{0}$;

4. $a \bullet (b \sqcup c) = (a \bullet b) \sqcup (a \bullet c)$;

5. $a \bullet b \sqsubseteq c$ if and only if $a \sqsubseteq b \Rightarrow c$.

It is clear that a *Heyting algebra* (consult, e.g., Rasiowa and Sikorski [32]) is the special case of a Dunn semilattice in which $\bullet$ and $\sqcap$ are the same operation; for this reason, where $\mathbf{D} = \langle D, \mathbf{1}, \mathbf{0}, \bullet, \Rightarrow, \sqcup, \sqcap \rangle$ is a Heyting algebra, I will often omit $\bullet$. (Not every Dunn semilattice is a Heyting algebra; consult Example 4 below.)

A few elementary results concerning Dunn semilattices, some of which I will have occasion to appeal to in the sequel, are summarized without proof in Fact 1:

Fact 1. In any Dunn semilattice $\mathbf{D} = \langle D, \mathbf{1}, \mathbf{0}, \bullet, \Rightarrow, \sqcup, \sqcap \rangle$, the following obtain:

1. $a \sqsubseteq b$ implies $a \bullet c \sqsubseteq b \bullet c$;

2. $a \sqcap b \sqsubseteq a \bullet b \sqsubseteq a \sqcup b$;

[19] Of course, much of the *mathematics* behind Dunn monoids is older; see, e.g., Ward and Dilworth [37].

[20] In the interest of completeness, I should note that Meyer and Routley [26, pp. 419–420] discuss algebraic models for mingle-extended relevance logics en passant.

[21] Any Dunn semilattice will also have a greatest element (with respect to $\sqsubseteq$), viz., $\mathbf{0} \Rightarrow \mathbf{0}$, which will not in general be identical to $\mathbf{1}$.

3. $a \bullet (b \sqcap c) \sqsubseteq (a \bullet b) \sqcap (a \bullet c)$;

4. $(a \sqcap b) \bullet (c \sqcap d) \sqsubseteq (a \bullet c) \sqcap (b \bullet d)$.

There is an obvious way to generate a Dunn semilattice or Heyting algebra from a given mingle frame (Definition 5) or intuitionistic frame (Definition 6):

DEFINITION 12 (Filter Algebra)**.** Given a mingle frame $\mathfrak{F} = \langle S, 0, 1, \vee, \wedge \rangle$, the *filter algebra* over $\mathfrak{F}$, $\mathbb{A}(\mathfrak{F}) = \langle D, \mathbf{1}, \mathbf{0}, \bullet, \Rightarrow, \sqcup, \sqcap \rangle$, is defined as follows:

1. $D = \mathcal{F}(\mathfrak{F})$;

2. $\mathbf{1} = \uparrow 0$;

3. $\mathbf{0} = \uparrow 1$;

4. $I \bullet J = \{k \in S : \exists i \in I, \exists j \in J (i \vee j \leq_\wedge k)\}$;

5. $I \Rightarrow J = \bigcup \{K \in \mathcal{F}(\mathfrak{F}) : K \bullet I \subseteq J\}$;

6. $I \sqcup J = \{i \wedge j : i \in I, j \in J\}$;

7. $I \sqcap J = I \cap J$.

LEMMA 10. *Any filter algebra $\mathbb{A}(\mathfrak{F}) = \langle D, \mathbf{1}, \mathbf{0}, \bullet, \Rightarrow, \sqcup, \sqcap \rangle$ over a mingle frame $\mathfrak{F} = \langle S, 0, 1, \vee, \wedge \rangle$ is a Dunn semilattice.*

Proof. First, it must be verified that the operations, so defined, actually are operations on $\mathcal{F}(\mathfrak{F})$, i.e., that given filters, they yield filters. I examine just the cases of $\bullet$ and $\Rightarrow$.

It is clear that $I \bullet J$ is nonempty if I and J are. So suppose that $x, y \in I \bullet J$; then $\exists i, i' \in I$ and $\exists j, j' \in J$ such that $i \vee j \leq_\wedge x$ and $i' \vee j' \leq_\wedge y$. By join-distributivity and the facts that $(i \vee j) \wedge (i' \vee j) \leq_\wedge i \vee j \leq_\wedge x$ and $(i \vee j') \wedge (i' \vee j') \leq_\wedge i' \vee j' \leq_\wedge y$, $(i \wedge i') \vee (j \wedge j') = ((i \vee j) \wedge (i' \vee j)) \wedge ((i \vee j') \wedge (i' \vee j')) \leq_\wedge x \wedge y$, where $i \wedge i' \in I$ and $j \wedge j' \in J$. Thus, $x \wedge y \in I \bullet J$, as desired. Conversely, if $x \wedge y \in I \bullet J$, $\exists i \in I$ and $\exists j \in J$ such that $i \vee j \leq_\wedge x \wedge y$. The result then follows immediately from the facts that $x \wedge y \leq_\wedge x$ and $x \wedge y \leq_\wedge y$.

For any filters I and J, since $I \bullet \uparrow 1 \subseteq J$, clearly $I \Rightarrow J \neq \emptyset$. Suppose that $x, y \in I \Rightarrow J$; then $\exists X, Y \in \mathcal{F}(\mathfrak{F})$ such that $x \in X$ and $y \in Y$ with $X \bullet I \subseteq J$ and $Y \bullet I \subseteq J$. Consider the filter $X \sqcup Y$; we wish to show $(X \sqcup Y) \bullet I \subseteq J$. Suppose $z \in (X \sqcup Y) \bullet I$. Then $\exists i \in I$, $x' \in X$, and $y' \in Y$ such that $(x' \wedge y') \vee i = (x' \vee i) \wedge (y' \vee i) \leq_\wedge z$. But $X \bullet I \subseteq J$ implies that $x' \vee i \in J$ and $Y \bullet I \subseteq J$ implies that $y' \vee i \in J$, so $(x' \vee i) \wedge (y' \vee i) \in J$ (as J is meet-closed) and $z \in J$ (as J is upwards closed). This suffices to show $x \wedge y \in I \Rightarrow J$, since $x \wedge y \in X \sqcup Y$. Conversely, suppose $x \wedge y \in I \Rightarrow J$;

then $\exists K \in \mathcal{F}(\mathfrak{F})$ such that $x \wedge y \in K$ and $K \bullet I \subseteq J$. By upwards closure, $x, y \in K$, which suffices.

I omit the arguments that $\langle D, \mathbf{0}, \sqcup, \sqcap\rangle$ is a distributive lattice with bottom $\mathbf{0}$, that $\langle D, \mathbf{1}, \bullet\rangle$ is a meet-semilattice with top $\mathbf{1}$, and that $I \bullet \uparrow 1 = \uparrow 1$; these are fairly routine. It remains to verify the last two requirements from Definition 11. To show that $I \bullet (J \sqcup K) = (I \bullet J) \sqcup (I \bullet K)$, suppose that $x \in I \bullet (J \sqcup K)$; then for some $i \in I$, $j \in J$, and $k \in K$, $i \vee (j \wedge k) = (i \vee j) \wedge (i \vee k) \leq_\wedge x$. Clearly, $i \vee j \in I \bullet J$ and $i \vee k \in I \bullet K$, so $(i \vee j) \wedge (i \vee k) \in (I \bullet J) \sqcup (I \bullet K)$, from which the result follows by upwards closure. Conversely, suppose $x \in (I \bullet J) \sqcup (I \bullet K)$. Then $x = y \wedge z$ for some $i, i' \in I$, $j \in J$, and $k \in K$ such that $i \vee j \leq_\wedge y$ and $i' \vee k \leq_\wedge z$, and therefore, $(i \vee j) \wedge (i' \vee k) \leq_\wedge y \wedge z$. Then $j \wedge k \in J \sqcup K$ and $i \wedge i' \in I$, so $(i \wedge i') \vee (j \wedge k) \in I \bullet (J \sqcup K)$; but $(i \wedge i') \vee (j \wedge k) = ((i \vee j) \wedge (i' \vee j)) \wedge ((i \vee k) \wedge (i' \vee k)) \leq_\wedge (i \vee j) \wedge (i' \vee k) \leq_\wedge y \wedge z = x$, so $x \in I \bullet (J \sqcup K)$. Finally, it has to be verified that $I \bullet J \subseteq K$ if and only if $I \subseteq J \Rightarrow K$. From left to right, this is essentially immediate from the definition of $J \Rightarrow K$. Conversely, it suffices to show that $(J \Rightarrow K) \bullet J \subseteq K$.[22] Suppose $x \in (J \Rightarrow K) \bullet J$; then there is some y in some filter Y such that $Y \bullet J \subseteq K$ and some $z \in J$ such that $y \vee z \leq_\wedge x$. But then $y \vee z \in K$, so $x \in K$ by upwards closure, as desired. □

LEMMA 11. *Any filter algebra $\mathbb{A}(\mathfrak{F}) = \langle D, \mathbf{1}, \mathbf{0}, \bullet, \Rightarrow, \sqcup, \sqcap\rangle$ over an intuitionistic frame $\mathfrak{F} = \langle S, 0, 1, \vee, \wedge\rangle$ is a Heyting algebra.*

Proof. The argument is the same as that for Lemma 10, except we have to check that $I \bullet J = I \sqcap J$ for all filters I, J. From right to left, if $x \in I \sqcap J = I \cap J$, then $x \in I, J$, so $x \in I \bullet J$ as $x \vee x \leq x$. Conversely, if $x \in I \bullet J$, then there are $i \in I$ and $j \in J$ such that $i \vee j \leq x$; but $i \leq i \vee j \leq x$ and $j \leq i \vee j \leq x$ imply that $x \in I \cap J$, as required. (Obviously this argument depends on the fact that $\leq$ is unambiguous in an intuitionistic frame.) □

Example 4 (RM3). Recall the moderate Kleene bisemilattice from Example 3. I will presently show that the filter algebra over this frame is a reduct of the characteristic algebra for the logic **RM3**.[23] In particular, our algebra is $\mathbb{A} = \langle\{\mathbf{-1}, \mathbf{0}, \mathbf{1}\}, \mathbf{0}, \mathbf{-1}, \bullet, \Rightarrow, \sqcup, \sqcap\rangle$ where $\mathbf{-1} = \{1\}$, $\mathbf{0} = \{0, 1\}$, and $\mathbf{1} = \{0, .5, 1\}$—these are all the filters in this bisemilattice—and the connectives, defined by Definition 12, are displayed table-wise for convenience:[24]

[22] This follows from the general fact that $I \subseteq J$ and $J \bullet K \subseteq L$ imply $I \bullet K \subseteq L$. For if $x \in I \bullet K$, $i \vee k \leq_\wedge x$ for some $i \in I$ and $k \in K$. But $i \in I \subseteq J$, so $x \in L$ as $J \bullet K \subseteq L$.

[23] Consult, for example, Anderson and Belnap, Jr. [1, §29.12, p. 470], Brady [6, p. 9], or Priest [29, §7.4, pp. 124–125]. Note that I am omitting the negation table for **RM3**.

[24] I have named the values of the algebra specifically to call to mind the fact that **RM3** is one member of the infinite class of so-called *Sugihara matrices* (named after Sugihara

$\bullet$	$\{1\}$	$\{0,1\}$	$\{0,.5,1\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{0,1\}$	$\{1\}$	$\{0,1\}$	$\{0,.5,1\}$
$\{0,.5,1\}$	$\{1\}$	$\{0,.5,1\}$	$\{0,.5,1\}$

$\sqcap$	$\{1\}$	$\{0,1\}$	$\{0,.5,1\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$\{0,1\}$	$\{1\}$	$\{0,1\}$	$\{0,1\}$
$\{0,.5,1\}$	$\{1\}$	$\{0,1\}$	$\{0,.5,1\}$

$\Rightarrow$	$\{1\}$	$\{0,1\}$	$\{0,.5,1\}$
$\{1\}$	$\{0,.5,1\}$	$\{0,.5,1\}$	$\{0,.5,1\}$
$\{0,1\}$	$\{1\}$	$\{0,1\}$	$\{0,.5,1\}$
$\{0,.5,1\}$	$\{1\}$	$\{1\}$	$\{0,.5,1\}$

$\sqcup$	$\{1\}$	$\{0,1\}$	$\{0,.5,1\}$
$\{1\}$	$\{1\}$	$\{0,1\}$	$\{0,.5,1\}$
$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,.5,1\}$
$\{0,.5,1\}$	$\{0,.5,1\}$	$\{0,.5,1\}$	$\{0,.5,1\}$

Observe that **RM3** is not a Heyting algebra as, for example, $\mathbf{0}\bullet\mathbf{1} \neq \mathbf{0}\sqcap\mathbf{1}$. On the other hand, the filter algebra over strong Kleene (which is of course an intuitionistic frame, per Definition 6) does yield a Heyting algebra—indeed, the smallest Heyting algebra which is not a Boolean algebra.

I have examined how to obtain an algebraic structure from an operational frame; it is time to examine the converse. While there are several ways to get a mingle frame from a Dunn semilattice (cf. Punčochář [30, §5]), I will just consider the one which I find most natural. The reader will observe that the construction mirrors, algebraically, the canonical model construction in Definition 9 from Subsection 3.3.[25]

DEFINITION 13 (Filter Frame)**.** Given a Dunn semilattice $\mathbf{D} = \langle D, \mathbf{1}, \mathbf{0}, \bullet, \Rightarrow, \sqcup, \sqcap\rangle$, the *filter frame* over $\mathbf{D}$, $\mathfrak{F}(\mathbf{D}) = \langle S, 0, 1, \vee, \wedge\rangle$, is defined as follows:

1. $S = \mathcal{F}(\mathbf{D})$;[26]
2. $0 = \uparrow\mathbf{1}$;
3. $1 = \uparrow\mathbf{0} = D$;

[35]); these play an important role in the algebraic theory of **RM** [9]. Here I should also note an interesting anticipation of my work by Meyer, who in [1, §29.3.2, p. 400] very nearly presents Sugihara matrices as bisemilattices, discussing extensional and intensional orders of the pertinent sets of integers. Of course, an important difference is that neither $\sqsubseteq$ nor $\leq_\bullet$ in a Dunn semilattice need be a chain.

[25] Here I should note that in the canonical model construction, where $\circ$ is included in the language, $\Gamma \cdot \Delta$ could have been equivalently defined as $\{\theta : \exists\varphi \in \Gamma, \exists\psi \in \Delta(\vdash_{\mathbf{L}} (\varphi \circ \psi) \to \theta)\}$, which makes the connection even sharper.

[26] Just to be clear, $\mathcal{F}(\mathbf{D})$ is taken to be the set of $\sqcap$-filters in $\mathbf{D}$.

4. $I \vee J = \{k \in S : \exists i \in I, \exists j \in J(i \bullet j \sqsubseteq k)\}$;

5. $I \wedge J = I \cap J$.

LEMMA 12. *Any filter frame* $\mathfrak{F}(\mathbf{D}) = \langle S, 0, 1, \vee, \wedge\rangle$ *over a Dunn semilattice* $\mathbf{D} = \langle D, \mathbf{1}, \mathbf{0}, \bullet, \Rightarrow, \sqcup, \sqcap\rangle$ *is a mingle frame.*

Proof. The argument mirrors the proof of Lemma 5, so I will not belabor it for too long. It should, however, briefly be verified that when I and J are filters, $I \vee J$ is as well, since this is not entirely obvious. Suppose $a, b \in I \vee J$, so as to show that $a \sqcap b \in I \vee J$. Then $\exists i, i' \in I$ and $j, j' \in J$ such that $i \bullet j \sqsubseteq a$ and $i' \bullet j' \sqsubseteq b$; clearly, $(i \bullet j) \sqcap (i' \bullet j') \sqsubseteq a \sqcap b$. I and J are filters, so $i \sqcap i' \in I$ and $j \sqcap j' \in J$, whence $a \sqcap b \in I \vee J$ since $(i \sqcap i') \bullet (j \sqcap j') \sqsubseteq (i \bullet j) \sqcap (i' \bullet j') \sqsubseteq a \sqcap b$ by the assumptions, definition of $\vee$, and Fact 1. Conversely, if $a \sqcap b \in I \vee J$, that $a, b \in I \vee J$ is immediate from the facts that $a \sqcap b \sqsubseteq a$ and $a \sqcap b \sqsubseteq b$. Finally, it is obvious that $I \vee J$ is nonempty, since (ex hypothesi) I and J are. □

LEMMA 13. *Any filter frame* $\mathfrak{F}(\mathbf{D}) = \langle S, 0, 1, \vee, \wedge\rangle$ *over a Heyting algebra* $\mathbf{D} = \langle D, \mathbf{1}, \mathbf{0}, \Rightarrow, \sqcup, \sqcap\rangle$ *is an intuitionistic frame.*

Proof. The result follows from Lemma 12 and the observation that $0 = \uparrow \mathbf{1} \subseteq I$ for any filter I because in a Heyting algebra, $\mathbf{1}$ is the top element in the $\sqsubseteq$ order and therefore is contained in any filter. □

Given a Dunn semilattice, an algebraic model is obtained by assigning elements of the algebra to propositional variables:[27]

DEFINITION 14 (Model). A *Dunn semilattice model* is a structure $\mathfrak{M}^a = \langle \mathbf{D}, \nu\rangle$ where $\mathbf{D} = \langle D, \mathbf{1}, \mathbf{0}, \bullet, \Rightarrow, \sqcup, \sqcap\rangle$ is a Dunn semilattice and $\nu : \Pi \to D$ is extended to the full language in the obvious way:

1. $\nu(\bot) = \mathbf{0}$;

2. $\nu(t) = \mathbf{1}$;

3. $\nu(\varphi \wedge \psi) = \nu(\varphi) \sqcap \nu(\psi)$;

4. $\nu(\varphi \vee \psi) = \nu(\varphi) \sqcup \nu(\psi)$;

5. $\nu(\varphi \circ \psi) = \nu(\varphi) \bullet \nu(\psi)$;

6. $\nu(\varphi \to \psi) = \nu(\varphi) \Rightarrow \nu(\psi)$.

[27] For the purposes of algebraic semantics, it is natural to assume **RM0** is formulated in the full language.

A *Heyting algebraic model* is defined in essentially the same way, with Heyting algebras playing the role of Dunn semilattices and the irrelevant connectives and clauses being omitted.

DEFINITION 15 (Validity)**.** Where $\mathfrak{M}^a = \langle \mathbf{D}, \nu \rangle$ is a Dunn semilattice model, φ is valid in $\mathfrak{M}^a$ ($\models^{\mathfrak{M}^a} \varphi$) if $\mathbf{1} \sqsubseteq \nu(\varphi)$. φ is Dunn semilattice *valid* ($\models^a_{\mathbf{RM0}} \varphi$) if $\models^{\mathfrak{M}^a} \varphi$ for every Dunn semilattice model $\mathfrak{M}^a = \langle \mathbf{D}, \nu \rangle$. Heyting validity ($\models^a_{\mathbf{J}} \varphi$) is defined analogously.

LEMMA 14. *If* $\models^a_{\boldsymbol{RM0}} \varphi$ *(*$\models^a_{\boldsymbol{J}} \varphi$*), then* $\models_{\boldsymbol{RM0}} \varphi$ *(*$\models_{\boldsymbol{J}} \varphi$*).*

Proof. For the case of **RM0**, suppose $\not\models_{\mathbf{RM0}} \varphi$; then there is some mingle model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ such that $\not\models^{\mathfrak{M}}_0 \varphi$. Let $\mathbb{A}(\mathfrak{F}) = \langle D, \mathbf{1}, \mathbf{0}, \bullet, \Rightarrow, \sqcup, \sqcap \rangle$ be the filter algebra over $\mathfrak{F}$; by Lemma 10, this is a Dunn semilattice. The Dunn semilattice countermodel is defined to be $\mathfrak{M}^a = \langle \mathbb{A}(\mathfrak{F}), \nu \rangle$ where $\nu(p) = V(p)$. By an induction that is essentially trivial in virtue of Lemmata 4 and 10, $\nu(\psi) = [\psi]^{\mathfrak{M}}$ for all ψ. But then, clearly, $\uparrow 0 = \mathbf{1} \not\sqsubseteq \nu(\varphi) = [\varphi]^{\mathfrak{M}}$, as $0 \notin [\varphi]^{\mathfrak{M}}$ ex hypothesi. So, $\not\models^{\mathfrak{M}^a} \varphi$, which suffices. The case of **J** is essentially the same, but Lemma 11 fulfills the role of Lemma 10. □

LEMMA 15. *If* $\models_{\boldsymbol{RM0}} \varphi$ *(*$\models_{\boldsymbol{J}} \varphi$*), then* $\models^a_{\boldsymbol{RM0}} \varphi$ *(*$\models^a_{\boldsymbol{J}} \varphi$*).*

Proof. For the case of **RM0**, suppose $\not\models^a_{\mathbf{RM0}} \varphi$. Then there is a Dunn semilattice model $\mathfrak{M}^a = \langle \mathbf{D}, \nu \rangle$ where $\mathbf{D} = \langle D, \mathbf{1}, \mathbf{0}, \bullet, \Rightarrow, \sqcup, \sqcap \rangle$ is a Dunn semilattice and $\mathbf{1} \not\sqsubseteq \nu(\varphi)$. Let $\mathfrak{F}(\mathbf{D}) = \langle S, 0, 1, \vee, \wedge \rangle$ be the filter frame over $\mathbf{D}$; by Lemma 12, this is a mingle frame. The mingle countermodel is defined to be $\mathfrak{M} = \langle \mathfrak{F}(\mathbf{D}), V \rangle$ where, for all p, $V(p) = \{I \in S : \nu(p) \in I\}$. Clearly, each $V(p)$ is a filter in $\mathfrak{F}(\mathbf{D})$ since $I, J \in V(p)$ if and only if $\nu(p) \in I, J$ if and only if $\nu(p) \in I \cap J$ if and only if $I \cap J \in V(p)$ and every $V(p)$ is nonempty (containing, e.g., 1). Thus, $\mathfrak{M}$ is a mingle model.

It must be shown that for all ψ and all filters I, $\models^{\mathfrak{M}}_I \psi$ if and only if $\nu(\psi) \in I$. The argument for this result is entirely analogous to that for Lemma 7, so I will just briefly examine the case of $\to$. Suppose $\models^{\mathfrak{M}}_J \theta$ and $\nu(\theta \to \chi) = \nu(\theta) \Rightarrow \nu(\chi) \in I$. By the induction hypothesis, $\nu(\theta) \in J$, so as $(\nu(\theta) \Rightarrow \nu(\chi)) \bullet \nu(\theta) \sqsubseteq \nu(\chi)$, $\nu(\chi) \in I \vee J$, which suffices by the induction hypothesis. Conversely, suppose that $\nu(\theta \to \chi) = \nu(\theta) \Rightarrow \nu(\chi) \notin I$ and consider $I \vee \uparrow \nu(\theta)$. If it were the case that $\nu(\chi) \in I \vee \uparrow \nu(\theta)$, then $i \bullet k \sqsubseteq \nu(\chi)$ for some $i \in I$ and $\nu(\theta) \sqsubseteq k$. By Fact 1, $\nu(\theta) \sqsubseteq k$ implies $i \bullet \nu(\theta) \sqsubseteq i \bullet k \sqsubseteq \nu(\chi)$, whence $i \sqsubseteq \nu(\theta) \Rightarrow \nu(\chi)$ and $\nu(\theta) \Rightarrow \nu(\chi) \in I$, which is impossible. So $\nu(\theta) \in \uparrow \nu(\theta)$ and $\nu(\chi) \notin I \vee \uparrow \nu(\theta)$ imply $\models^{\mathfrak{M}}_{\uparrow \nu(\theta)} \theta$ and $\not\models^{\mathfrak{M}}_{I \vee \uparrow \nu(\theta)} \chi$ by the induction hypothesis, which yields the result.

Now, since $\mathbf{1} \not\sqsubseteq \nu(\varphi)$, $\nu(\varphi) \notin 0 = \uparrow \mathbf{1}$, whence $\not\models^{\mathfrak{M}}_0 \varphi$ by the immediately preceding induction. Therefore, $\not\models_{\mathbf{RM0}} \varphi$, as desired. The case involving **J** is essentially the same, but Lemma 13 plays the role of Lemma 12. □

THEOREM 6 (Algebraic Soundness and Completeness). $\vdash_{\boldsymbol{RM0}} \varphi$ *(*$\vdash_{\boldsymbol{J}} \varphi$*) if and only if* $\models^a_{\boldsymbol{RM0}} \varphi$ *(*$\models^a_{\boldsymbol{J}} \varphi$*).*

Proof. Immediate from Theorems 1–4 and Lemmata 14 and 15. □

Theorem 6 could of course have been proved much more directly, using a routine Lindenbaum construction for the algebraic completeness component; but the proof I have given sheds considerably more light on the relationship between the algebraic and operational semantics presented in this chapter.

5 Concluding Remarks

In this chapter, I examined operational and algebraic semantics for **RM0** and **J**. Adapting work of Humberstone [18], I showed that **RM0** is determined by a certain class of bisemilattices, taken as frames, whereas **J** is determined by the subclass of those frames which are lattices. I also examined algebraic semantics for both **RM0** and **J** and showed how to transform operational models into equivalent algebraic models and vice-versa.

One clear takeaway from this chapter is that **RM0** and **J** are very closely related. This is not only apparent semantically, in the fact that intuitionistic frames and Heyting algebras are natural special cases of mingle frames and Dunn semilattices respectively, but in the fact that **J** can be straightforwardly exactly translated into $\mathbf{RM0}^t$ per Theorem 5. In [38], I presented extensions of Urquhart's semilattice relevance logic **S** which might be thought of as (quasi-)relevant companions of **J** and **KC** (Jankov's logic). Such logics, in my view, could hold appeal to relevantists of a constructivist bent (or constructivists of a relevantist bent). In view of the results of this chapter, I think that **RM0** is another system that could hold appeal to such logicians.

Another clear takeaway is that the operational semantics of Humberstone [18] deserves more attention than it has received. As I showed, Humberstone's semantics importantly anticipated more recent developments in inquisitive semantics (as illustrated in the work of, for example, Punčochář [30]). In fact, though, this chapter only scratches the surface of what can be done by extending or modifying the Humberstone framework. In unpublished work, I have shown how the operational semantics of this chapter can be used to characterize a variety of intuitionistic and relevant modal logics, with embedding results forthcoming for intuitionistic modal systems and their relevant companions; without doubt, the algebra of such logics will also prove a rich vein for future study.

This chapter leaves open a number of interesting problems, both philosophical and technical. I have not attempted to articulate a philosophical account of the operational semantics developed here for either **RM0** or **J** (in this respect, **RM0** would appear to be on worse footing than the systems surveyed in [38], which have clear philosophical motivation). This is emphatically *not* because I do not think the semantics can be well-motivated, but rather because this is, by design, a technical piece. I leave to future work, my own or others', the project of interpreting this semantics.[28]

On the technical side, much more work could still be done even just on the model theory of **RM0** and **J**. One example: while I have examined operational and algebraic models for both of these systems and shown how to move between them, both of these logics already have relational modelings (ternary in the case of **RM0** [23], binary in the case of **J** [21]) which I have not discussed. It would be valuable to examine the relation of those semantics to the semantics presented here.

Dedication

I dedicate this chapter to the memory of J. Michael Dunn, a great logician and generous human being.

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[28] It could just as well have been left to *past* and future work, in view of the fact that Humberstone [18] (not to mention the inquisitive semanticists) has some informal things to say about how to interpret his semantics. But I confess that my own interpretive views, germinal though they are, do not entirely align with his.

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