Llull, Leibniz, and the Logic of Discovery

This version of the paper corrects typographical errors in "Llull and Leibniz: The Logic of Discovery," *Catalan Review* IV, Nos. 1–2 (July–December 1990), 75–83.

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Abstract: Llull and Leibniz both subscribed to conceptual atomism: the belief that the majority of concepts are compounds constructed from a relatively small number of primitive concepts. Llull worked out techniques for finding the logically possible combinations of his primitives, but Leibniz criticized Llull's execution of these techniques. This paper argues that Leibniz was right about things being more complicated than Llull thought but that he was wrong about the details. The paper attempts to correct these details.

Keywords: Llull, Ars magna, conceptual atomism, logic of discovery, Leibniz

A logic conceived as a formal calculus is one thing; the same logic applied is another. When logic is applied, *why* is it applied? To what use is it put? We are perhaps most accustomed to answer in terms of justification; given a truth, logic may show us why it is true. But there is another answer, prominent from the early modern period down to the present day, which is that logic can also be used to discover.

One of the pioneers of this point of view was Leibniz. His efforts to forge a logic of discovery were informed by a kind of conceptual atomism, a belief that the majority of concepts are compounds constructed from a relatively small number of primitives. He claimed that "a kind of alphabet of human thoughts can be worked out and that *everything can be discovered and judged by a comparison of the letters of the alphabet and an analysis of the words made from them*."[1] Such an alphabet, as Leibniz says, could be used to judge *and* discover. Not only would it serve to demonstrate propositions already held to be true—a logic of justification—it could also be used to invent or discover new truths—a logic of discovery. What would a logic of discovery look like? Leibniz's answer, his *ars inveniendo*, was to have two parts: one combinatorial, to generate questions, and one analytic, to answer them.[2]

How closely Leibniz's thinking here parallels the *Ars magna* (hereafter the Art) of Ramón Llull is not sufficiently known. The parallel is not accidental, though the historical links between Llull, Lullism, and Leibniz are by no means simple.[3] As a young man of twenty, Leibniz was both fascinated and repelled by the Art. The fascination came from Llull's having anticipated some of his leading ideas. The repulsion came from Llull's mathematical naiveté, a consequence of having lived some four centuries before the developments in combinatorial mathematics upon which Leibniz hoped to base his own logic of discovery. (Leibniz, in fact, was the first to use the term 'combinatorial' in its modern sense.) Both of Leibniz's reactions linger in the objectives of this note: to outline what it was about the Art that fascinated Leibniz, first of all; and, after correcting a mistake in Leibniz's critique of Llull, to extend it in a new direction.

Llull anticipated Leibniz in the belief that human reason was a matter of combining a few primitive notions. To specify these notions, Llull devised a conceptual alphabet which, he believed, limned the basic structure of the universe. In the later, ternary phase of the Art (ca. 1290–1308), the alphabet takes the following form.[4]

	Fig. A	Fig. T	Questions and Rules	Subjects	Virtues	Vices
B	goodness	difference	whether?	God	justice	avarice
C	greatness	concordance	what?	angel	prudence	gluttony
D	eternity*	contrariety	of what?	heaven	fortitude	lust
E	power	beginning	why?	man	temperance	pride
F	wisdom	middle	how much?	imaginative	faith	accidie
G	will	end	of what kind?	sensitive	hope	envy
H	virtue	majority	when?	vegetative	charity	ire
I	truth	equality	where?	elementative	patience	lying
K	glory	minority	how? and with what?	instrumentative	pity	inconstancy

* or duration

Each of the alphabet's six columns is meant to depict one of the universe's fundamental structural features. The first column (under 'Fig. A') lists the Llullian dignities, externalizations of the divine personality from which the world's goodness, greatness, duration, and so forth emanate in Neoplatonic fashion. The second column is composed of what Llull takes to be the primary logical categories. The third column details the kinds of questions that can be asked; the fourth, the medieval ontological hierarchy; and the fifth and sixth, the essential moral categories.

Llull also anticipated Leibniz in recognizing that such an alphabet was the key to a logic of discovery. Moreover, the combinatory and analytic parts of Leibniz's *ars inveniendi* are clearly prefigured in the alphabet's function. Combining the "letters" of the alphabet, which were in fact words, produced "words", which were (roughly) sentences. Once the harvest of all logically possible combinations of the alphabet's letters was in (corresponding to the combinatory part of Leibniz's *ars inveniendi*), the Art was to be used to winnow the false combinations from the true (the analytic part). The result, the wheat, would be the sum total of the most general truths about the world—the definitive philosophy.

But in *De arte combinatoria*, Leibniz faults Llull's execution of the combinatorial part of this task.[5] Llull considered only binary and ternary combinations of the letters of the alphabet, but unary all the way up through nonary combinations are possible.[6] Therefore, Leibniz argued, the 9 letters of each column can be combined in $2^9 - 1 = 511$ possible ways. And, since there are 6 columns, Llull's simple alphabet yields the astounding number of $511^6 = 17{,}804{,}320{,}388{,}674{,}561$ possible combinations.

Actually, Leibniz's figure is either too low or too high. The total number k of unary through n-ary combinations that can be obtained from n things without repetition is given by $k = 2^n - 1$. But Leibniz proceeds, in effect, by applying this formula to only the first column of the alphabet, obtaining 511, and raising that result to the sixth power. To see that this skews the results, the reader might try following Leibniz's procedure to answer two questions about the model M:

$$\begin{matrix} a & d \\ b & e \\ c & f \end{matrix} .$$

i) How many unary through n-ary combinations without repetitions are there where $n = 6$ (the number of letters in M)? Applying (1) in Leibniz's fashion to the first column of M yields 7, and squaring it gives 49. But the correct procedure of applying (1) to the entire matrix gives $2^6 - 1 = 63$ combinations, making 49 too low. ii) Where $n = 6$, how many unary through ternary combinations without repetitions are there? The number of combinations without repetitions of n things taken r at a time is given by

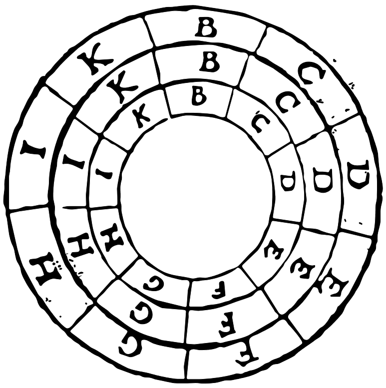
$$C_{n,r} = \frac{n!}{r!(n-r)!}.$$

Hence there are 6 unary + 15 binary + 20 ternary = 41 combinations, making 49 too high.

The same thing happens with Llull's alphabet. If Leibniz wanted the total number of unary through n-ary combinations without repetitions where n = 55 (the number of letters in the alphabet), that number is $2^{55} - 1$, all of 39 orders of magnitude larger than the figure in *De arte combinatoria*. On the other hand, if he wanted the total number of unary through nonary combinations without repetitions for the same number of letters, that is a number on the order of 10^9, which is 7 orders of magnitude smaller than Leibniz's figure.

Nevertheless, Leibniz was right about things being much more complicated than Llull thought. In the remainder of this note, I offer a very modest second to Leibniz's critique. Instead of the alphabet, however, I will focus on the table that appears for the first time in the *Taula general* (1293) and remains intact down to the *Ars generalis ultima* and the *Ars brevis* (both 1308), the final versions of the Art. The table was designed with two very different functions in mind: to automatically provide a middle term for a sound categorical syllogism on any subject whatsoever, and to exhaustively tabulate the ternary combinations of the first two columns of the alphabet. The remarks that follow concern only the second of these functions.

Llull's table was generated from the Fourth Figure of his Art, which is reproduced below.



As one can see, the Fourth Figure is composed of three concentric circles, each compartmentalized by the variables B through K from the extreme left of the alphabet. The outer circle is to be thought of as fixed and the two inner circles as movable so as to produce the various ternary combinations of variables. In the manuscripts and some of the earliest printed editions, the inner circles really did move; they were cut out and joined to the center of the outer circle by a thread knotted at both ends. The Fourth Figure was thus a primitive logical machine.

Here is how it generates the table. Given the 9 variables B through K, there are

$$\frac{9!}{3!(9-3)!} = 84$$

ternary combinations without repetitions of variables. They are as follows.

1 BCD	2 BCE	3 BCF	4 BCG	5 BCH	6 BCI	7 BCK	8 BDE	9 BDF	10 BDG	11 BDH	12 BDI
13 BDK	14 BEF	15 BEG	16 BEH	17 BEI	18 BEK	19 BFG	20 BFH	21 BFI	22 BFK	23 BGH	24 BGI
25 BGK	26 BHI	27 BHK	28 BIK	29 CDE	30 CDF	31 CDG	32 CDH	33 CDI	34 CDK	35 CEF	36 CEG

37 CEH	38 CEI	39 CEK	40 CFG	41 CFH	42 CFI	43 CFK	44 CGH	45 CGI	46 CGK	47 CHI	48 CHK
49 CIK	50 DEF	51 DEG	52 DEH	53 DEI	54 DEK	55 DFG	56 DFH	57 DFI	58 DFK	59 DGH	60 DGI
61 DGK	62 DHI	63 DHK	64 DIK	65 EFG	66 EFH	67 EFI	68 EFK	69 EGH	70 EGI	71 EGK	72 EHI
73 EHK	74 EIK	75 FGH	76 FGI	77 FGK	78 FHI	79 FHK	80 FIK	81 GHI	82 GHK	83 GIK	84 HIK

Each of these combinations is incorporated in the table at the head of one of 84 columns. The remainder of each column is composed of 19 variations on the combination at its head and the letter *T*. The complete table has 84 × 20 = 1680 compartments, therefore. For our limited purposes, however, the abbreviated table from the *Ars brevis* will suffice.[7] It lists only 7 of the 84 columns.

BCD	CDE	DEF	EFG	FGH	GHI	HIK
BCTB	CDTC	DETD	EFTE	FGTF	GHTG	HITH
BCTC	CDTD	DETE	EFTF	FGTG	GHTH	HITI
BCTD	CDTE	DETF	EFTG	FGTH	GHTI	HITK
BDTB	CETC	DFTD	EGTE	FHTF	GITG	HKTH
BDTC	CETD	DFTE	EGTF	FHTG	GITH	HKTI
BDTD	CETE	DFTF	EGTG	FHTH	GITI	HKTK
BTBC	CTCD	DTDE	ETEF	FTFG	GTGH	HTHI
BTBD	CTCE	DTDF	ETEG	FTFH	GTGI	HTHK
BTCD	CTDE	DTEF	ETFG	FTGH	GTHI	HTIK
CDTB	DETC	EFTD	FGTE	GHTF	HITG	IKTH
CDTC	DETD	EFTE	FGTF	GHTG	HITH	IKTI
CDTD	DETE	EFTF	FGTG	GHTH	HITI	IKTK
CTBC	DTCD	ETDE	FTEF	GTFG	HTGH	ITHI
CTBD	DTCE	ETDF	FTEG	GTFH	HTGI	ITHK
CTCD	DTDE	ETEF	FTFG	GTGH	HTHI	ITIK
DTBC	ETCD	FTDE	GTEF	HTFG	ITGH	KTHI
DTBD	ETCE	FTDF	GTEG	HTFH	ITGI	KTHK
DTCD	ETDE	FTEF	GTFG	HTGH	ITHI	KTIK
TBCD	TCDE	TDEF	TEFG	TFGH	TGHI	THIK

Llull uses T as an interpretive device: all variables appearing before it are to be interpreted by reading across the alphabet to its first column; all variables coming after it are interpreted by reading across to the second. Hence *BCTB* stands for 'goodness', 'greatness', and 'difference', while *TBCD* stands for 'difference', 'concordance', and 'contrariety'. What Llull has done, in effect, is to construct each column from the possible ternary combinations of the 6 "letters" that are the values of the variables at the head of the column. The column *BCD*, for example, is composed of the 20 combinations of the values of the variables *B*, *C*, and *D*.

There are two critical points to be made about this table. The first is that Llull restricts himself unduly to combinations that, when interpreted, have no repetitions. All ternary combinations from the table are considered meaningful, with *BCDT*, for example, being interpreted as 'Goodness is as great as eternity'.[8] But if that makes sense, so does *BCCT*, 'Goodness is as great as greatness'. Yet *BCCT*—and all the other combinations with repetitive values—are excluded from the table. If we include them, the total number of triples is not 84 but $9^3 = 729$.[9]

The second point is that even if we assume only Llull's 84 combinations without repetitive values, the table still turns out to be more complicated than it appears. When Llull interprets *BCDT*, for example, as 'Goodness is as great as eternity', he ignores the fact that there are 5 other ways of ordering the variables and 5 other equally legitimate interpretations.

BDCT	Goodness is as eternal as greatness.
CBDT	Greatness is as good as eternity.
CDBT	Greatness is as eternal as goodness.
DBCT	Eternity is as good as greatness.
DCBT	Eternity is as great as goodness.

He does not register the difference between the variables, where order does not matter (*BCD* = *DCB*), and the variables' interpretations, where order matters indeed ('Goodness is as great as eternity' ≠ 'Eternity is as great as goodness'). Thus one might expect some sort of mix-up about combinations, which are not ordered, and permutations, which are. That is in fact what happens. Llull calculates the number of *combinations* of the 6 values taken 3 at a time:

$$\frac{6!}{3!(6-3)!} = 20.$$

What he should have done, however, is to calculate the number of *permutations*. The number of permutations without repetitions for n things taken r at a time is given by

$$P_{n,r} = \frac{n!}{(n-r)!}.$$

Hence the number of permutations of the 6 values taken 3 at a time is:

$$\frac{6!}{(6-3)!} = 120.$$

What would a corrected table, one with unique entries for all and only the 3-place permutations without repetitions of values, look like? It would be larger than Llull's original, of course. Since the problem is not the ternary combinations of 9 variables but the ternary permutations of their 18 values, the table would have

$$\frac{18!}{(18-3)!} = 4896$$

different compartments. The procedure I am recommending here is none other than Llull's in an analogous situation. To evacuate the *binary* combinations of variables from the Third Figure, he specifies the possible permutations of their values.[10]

The foregoing critique of both Llull and Leibniz admittedly concerns matters of detail. But it is submitted in the belief that one of the sources of philosophical progress is a clearer understanding of its past, and that the larger, still tantalizing project of a logic of discovery can be carried forward only through a mastery of such details, past as well as present.

[1] Leibniz, *Mathematische Schriften*, ed. C.I. Gerhardt (Berlin and Halle, 1849–63; rpt. Darmstadt: Hildesheim, 1962), vol. 7, p. 185. Emphasis in original.

[2] *Opuscles et fragments inédits de Leibniz*, ed. Louis Couturat (Paris, 1903; rpt. Darmstadt: Hildesheim, 1961), p. 167.

[3] Catherine Wilson, *Leibniz's Metaphysics: A Historical and Comparative Study* (Princeton and Manchester: Princeton and Manchester Univ. Presses, 1989), ch. 1.

[4] The alphabet is taken from the *Ars brevis*. See Anthony Bonner's *Selected Works of Ramon Llull (1232–1316)* (Princeton: Princeton Univ. Press, 1985), vol. I, p. 581. On the various phases of the Art, see ibid., I, pp. 56–57.

[5] *Dissertatio de arte combinatoria* in *Philosophische Schriften*, ed. Paul Ritter and Willy Kabitz (Berlin: Akademie, 1971), vol. I, p. 193.

[6] The 9 unary combinations would have been useless to Llull, who was interested only in combination that, when interpreted, bear truth values.

[7] Bonner, p. 597.

[8] When interpreting the compartments at the head of the column, *T* is understood as the last letter. *BCD*, then, is interpreted as *BCDT*.

[9] Prantl argued that this was the correct number in *Geschichte der Logik im Abendlande* (Leipzig, 1867; rpt. Graz: Akademische Druck-u. Verlagsanstalt, 1955), vol. III, p. 162 n 90.

[10] Bonner, p. 598.