Mathematics, Isomorphism, and the Identities of Objects

Graham WHITE*

Former lecturer at Queen Mary, University of London

Journal of Knowledge Structures & Systems

December 2021

Vol.: 2 Issue: 2 Pages: 56-58

Abstract

We compare the medieval projects of commentaries and disputations with the modern projects of formal ontology and of mathematics.

Key words: Identity; Isomorphism; Mathematics; Formal ontology; Elliptic curves

Augusto (2021) uses the contemporary language of formal ontology to examine the work of the medieval Dominican philosopher and theologian Dietrich of Freiberg. In doing so, he brings two worlds into contact: one is the medieval enterprise of writing commentaries, and conducting disputations, on authoritative texts, and the other is the modern world of formal ontology. In an enterprise like this, issues of translation are naturally important, although they may go unnoticed. Besides the perennial issues of translation, there are other sensitive issues with this paper. One in particular is this: the contemporary project of formal ontology is institutionally at home in computer science, where it is used to bridge between descriptions of phenomena (often formulated in natural language) and formalisations of these descriptions in some appropriate language or data processing formalism, aided by modern mathematics, and with the semantics, either of modern mathematics, or of formal ontology. So the following questions arise: firstly, what are the limitations of the medieval practices of writing commentaries and conducting disputations? Secondly, what are the limitations of the modern practice of using formal ontology to construct computer languages and data processing formalisms? And, thirdly: where are the discontinuities? That is, what parts of medieval intellectual practice cannot be expressed in modern terms, and vice versa? One part of modern mathematical practice which cannot be expressed in medieval terms is the following: mathematical objects are related by two,

*✉ g.graham.white@gmail.com

rather than just one, relationship. One is equality (as with ordinary objects) and one is isomorphism, which is weaker than equality. If two objects are isomorphic, then they share all of their mathematical properties, but two objects can be isomorphic without being equal. For an example, consider elliptic curves:[1] these are curves given by equations of the form

$$y^2 = ax^3 + bx^2 + cx + d,$$

where x and y are variables, and a, b, c and d are constants. There are many isomorphisms between these curves: we can change the constants so that there is only a one-parameter family of curves, for example

$$y^2 = x\left(x - 1\right)\left(x - \lambda\right),$$

where λ is a parameter. But does this family of curves contain each isomorphism class only once? No: if we define

$$j\left(\lambda\right) = 1728\frac{g_2^3}{g_2^3 - 27g_3^2},$$

where $g_2 = -\frac{1}{3}\lambda^2 + \frac{1}{3}\lambda - \frac{1}{3}$, and $g_3 = -\frac{1}{27}\left(\lambda^3 - 6\lambda^2 - 3\lambda + 9\right)$, then two elliptic curves are isomorphic iff their two corresponding values of j are equal. We might like to think that there would be a one-parameter family of elliptic curves whose parameter is j, but there is not. We can see this from the formula above for g_2: this has zero of order 2 at $\lambda = 0$, so, if we let x be a value near 0, then there will be two values of λ near 0 for which $j\left(\lambda\right) = x$, and so, correspondingly, two elliptic curves with that value of λ. Thus, pairs of distinct but isomorphic elliptic curves will be dense near $\lambda = 0$: consequently, there can be no neighbourhood of $\lambda = 0$ which does not contain isomorphic but distinct elliptic curves. This insight dates back to before the nineteenth century, but was studied extensively then. Section 2.2 of Augusto (2021) runs into trouble here, because it asks, of mathematical objects,

> are sets universals or particulars? And are "three-angled polygon" and "three-sided polygon" the same or different particulars?

Now mathematical reality, as we have seen in the above example, has a distinction between equality and isomorphism, so that one must be careful when talking about sameness and difference of mathematical objects without being careful about the distinction between identity and isomorphism. Furthermore, as we have seen from the example of elliptic curves, we cannot somehow evade the issue by talking of isomorphism classes instead of objects: because of the density of isomorphism classes of elliptic curves, we cannot just look at the isomorphism classes and put a non-trivial topology on them. And, because these phenomena were discovered in the nineteenth century, we cannot assume that philosophy has had a pioneering role with these issues.

[1]See Mumford et al. (2002); see also the article "Elliptic curve" in [1].

References

Augusto, L. M. (2021). Bridging mainstream and formal ontology: A causality-based upper ontology in Dietrich of Freiberg. *Journal of Knowledge Structures & Systems*, *2*(2), 1-35.

Mumford, D., Series, C., & Wright, D. (2002). *Indra's pearls.* Cambridge: Cambridge University Press.

Online Resources

[1] https://en.wikipedia.org/w/index.php?title=Elliptic_curve&oldid=951213572. [Accessed 17-April-2020]

Cite this article as:

White, G. (2021). Mathematics, isomorphism, and the identities of objects. *Journal of Knowledge Structures & Systems*, *2*(2), 56-58.

EDITORIAL INFORMATION

Editor-in-chief:[a] Luis M. Augusto

[a]Commentary articles are reviewed only by the editor-in-chief for appropriateness of content and style.