Truth and Generalized Quantification[*]

Bruno Whittle

April 6, 2018

Abstract

Kripke [1975] gives a formal theory of truth based on Kleene's strong evaluation scheme. It is probably the most important and influential that has yet been given—at least since Tarski. However, it has been argued that this theory has a problem with generalized quantifiers such as $\text{All}(\varphi, \psi)$, i.e. all φs are ψ, or $\text{Most}(\varphi, \psi)$. Specifically, it has been argued that such quantifiers preclude the existence of just the sort of language that Kripke aims to deliver, that is, one that contains its own truth predicate. In this paper I solve the problem by showing how Kleene's strong scheme, and Kripke's theory that is based on it, can in a natural way be extended to accommodate the full range of generalized quantifiers.

Kripke [1975] famously presented a semantic framework for languages that contain their own truth predicates, a framework designed to handle the Liar and related paradoxes. Kripke's central theory, based on Kleene's strong evaluation scheme, is probably the most important and influential formal account of truth that has yet been given—at least since that of Tarski [1935]. It is a straightforward implementation of the idea that one can assert (or deny) that a sentence is true precisely if one can assert (deny) the sentence itself. According to the theory, the sentences that are true are those that can be seen to be such by repeated application of this idea. Further, many subsequent proposals take this theory as their starting point. To name just a few examples: the proposals of Skyrms [1984], Gaifman [1992, 2000], Cobreros *et. al* [2013], as well as that of Maudlin [2004]. However, Maudlin also raises a problem for the theory (one that, as he is well aware, applies equally to his

---

own proposal). The problem is this: it seems that the theory cannot accommodate the generalized quantifier $All(\varphi, \psi)$, i.e. all φs are ψ. In fact, however, if Maudlin is right, then the problem is far more widespread than he seems to recognize. For his line of thought would extend to a whole range of generalized quantifiers, such as Most, The, Both and so on. But this would then be a serious limitation of Kripke's theory. For, as is now widely accepted, quantification in natural language is via just such generalized quantifiers.[1] Further, the problem would seem to affect not only Kripke's theory, but also many that are based on it, i.e. a significant portion of the recent literature on truth. Indeed, the problem would threaten to affect even the basic—and apparently very natural—idea that Kripke's theory codifies.

However, in this paper I solve the problem by showing how Kleene's strong scheme, and Kripke's theory that uses it, can in a natural way accommodate the full range of generalized quantifiers. I present the problem in §1, and the solution in §2.[2]

1 The Problem

Maudlin [2004: 59–64] considers the possibility of augmenting the first-order languages, in terms of which Kripke's theory is presented, with the generalized quantifier All.[3] There is of course a tradition of formalizing sentences of the form 'all φs are ψ' in first-order logic: as $\forall x(\varphi \to \psi)$. But there are a number of reasons to be dissatisfied with this strategy. One (which Maudlin gives) is that there does not seem to be anything in the English sentence corresponding to $\to$. Another (which

---

[1]Shaw [2016] presses this expanded version of Maudlin's argument, concluding that Kripke's theory is indeed limited in this way.

[2]There has in fact been significant discussion of the generalized quantifier All in connection with theories of truth. However, these treatments have been within approaches importantly different from the strong Kleene version of Kripke's theory, and they do not address Maudlin's problem. For example, Priest [2006] and Beall [2009] consider how to handle this quantifier within the dialetheist framework. (For the relation of the proposal of this paper to dialetheism, see note 23.) Beall *et al.* [2006] is concerned with relevant logic. Field [2014, 2016] focuses on accounts of All that preserve certain classical laws, i.e. logical truths: again an approach very different from the strong Kleene version of Kripke's theory, since under that scheme there are no logical truths. (I address the issue of laws governing All in §2.2.)

[3]Unless otherwise stated, by 'Kripke's theory' I mean the strong Kleene proposal of [1975]. In fact, this is not really a single proposal but a family of them, where different members correspond to different fixed points. For simplicity, I focus on the least fixed point proposal, but everything that I say carries straightforwardly over to the other members of the family.

Maudlin doesn't give but could have) is that no comparable trick is available in the case of many other generalized quantifiers, such as Most or Finite.[4,5]

However, Maudlin claims that it is impossible to add All to our language without completely undermining Kripke's theory. For such an addition, he claims, would preclude the existence of just the sort of language that Kripke aims to deliver: namely, one that contains its own truth predicate.

Before giving Maudlin's argument, I introduce the concepts that Kripke's theory involves. The theory makes use of partially interpreted languages, i.e. those whose sentences can take the value true, false *or* undefined. More carefully, let L be a standard formal language: either a first-order language, or such a language augmented with generalized quantifiers such as All, Most etc.[6] A *total* (or *classical*) *L-interpretation* is a pair $\mathfrak{A}$ of a non-empty set A, the *domain* of $\mathfrak{A}$; together with a function that assigns to each individual constant c of L a member $c^{\mathfrak{A}}$ of A, and to each n-ary predicate symbol P of L a subset $P^{\mathfrak{A}}$ of A^n (the *extension* of P under $\mathfrak{A}$). A *partial L-interpretation* is just like a total one, except that an n-ary predicate symbol P is assigned a pair $\langle P_+^{\mathfrak{A}}, P_-^{\mathfrak{A}} \rangle$ of disjoint subsets of A^n; $P_+^{\mathfrak{A}}$ is the *extension* of P under $\mathfrak{A}$, while $P_-^{\mathfrak{A}}$ is the *anti-extension*.[7] P is true of the members of $P_+^{\mathfrak{A}}$ under $\mathfrak{A}$, false of those of $P_-^{\mathfrak{A}}$, and undefined of the remaining members of A^n. More generally, which formulas are true or false under which assignments under $\mathfrak{A}$ depends on which evaluation scheme is employed. Unless otherwise stated, we assume that the standard connectives and quantifiers are handled using Kleene's strong scheme.[8] If $\mathfrak{A}$ is a total (partial) L-interpretation, then $\langle L, \mathfrak{A} \rangle$ is a *totally* (*partially*) *interpreted language*.

Maudlin's argument is then as follows. Let $\mathcal{L}$ be the partially interpreted language under consideration. Maudlin suggests that if $\mathcal{L}$ contains All, then formulas containing this should be given truth conditions as in (A). Here $\varphi(x)$ and $\psi(x)$ are

---

[4] That is, there is no way to define these in first-order logic: see Barwise and Cooper [1981].

[5] In fact, Maudlin also gives another reason: he claims that in the context of Kripke's theory, i.e. partially interpreted languages understood via Kleene's strong scheme, $\forall x(\varphi \rightarrow \psi)$ has the wrong truth condition. On the contrary, he claims that any adequate formalization of 'all φs are ψ' should have truth condition as in (A) below. As will become clear, I do not think that *this* is a good reason to reject the first-order formalization. Nevertheless, the reasons just given in the text seem quite sufficient to motivate augmenting first-order languages with All.

[6] For simplicity, I inessentially assume that L does not contain function symbols other than individual constants.

[7] Thus, a total L-interpretation $\mathfrak{A}$ is not in general a partial one. However, $\mathfrak{A}$ may be identified with the partial L-interpretation $\mathfrak{A}'$ that is just like $\mathfrak{A}$ except that if P is an n-ary predicate symbol, then $P^{\mathfrak{A}'} = \langle P^{\mathfrak{A}}, A^n - P^{\mathfrak{A}} \rangle$.

[8] For details see §2.

formulas of $\mathcal{L}$,[9] and $\varphi_+^{\mathcal{L}}$ is the extension of φ in $\mathcal{L}$—i.e. the set of things φ is true of. In contrast, $\varphi_-^{\mathcal{L}}$ is the anti-extension: the set of things it is false of.[10]

(A) $\text{All}_x(\varphi, \psi)$ is true iff $\varphi_+^{\mathcal{L}} \subseteq \psi_+^{\mathcal{L}}$.

Why think that this is the right truth condition? Well, when we say 'all φs are ψ' we seem to be saying something about the φs—and *only* about these. Thus, the truth condition for $\text{All}_x(\varphi, \psi)$ should apparently be a condition on the φs, i.e. the members of $\varphi_+^{\mathcal{L}}$, and nothing else.[11] But once we have decided that the truth condition for $\text{All}_x(\varphi, \psi)$ should be concerned exclusively with the members of $\varphi_+^{\mathcal{L}}$, it seems obvious that it should be that to the effect that these are also all members of $\psi_+^{\mathcal{L}}$—which is precisely what (A) says.

  If we give $\text{All}_x(\varphi, \psi)$ the truth condition as in (A), then it will not be equivalent to the traditional first-order formalization of 'all φs are ψ', i.e. $\forall x(\varphi \rightarrow \psi)$. For example, if some $a \notin \varphi_+^{\mathcal{L}} \cup \varphi_-^{\mathcal{L}}$, then $\text{All}_x(\varphi, \varphi)$ will of course be true (given (A)), but $\forall x(\varphi \rightarrow \varphi)$ will not be.[12]

  It turns out, however, that given standard assumptions about resources for self-reference, (A) precludes $\mathcal{L}$ from containing its own truth predicate. That is, if T is a unary predicate symbol of $\mathcal{L}$, then (A) entails that $T_+^{\mathcal{L}}$ is not the set of true sentences of $\mathcal{L}$.[13] For let λ be[14]

$$\text{All}_x(T(x) \wedge x = \lambda, x \neq x).$$

---

[9] By writing $\varphi(x)$ I mean that the formula has no free variables except possibly x.

[10] Thus, in (A) and the discussion to follow, attention is for simplicity restricted to formulas with at most x free. However, extension to the more general case is straightforward. I write 'All_x' rather than '$\text{All}x$' for ease of reading.

[11] The intuitive idea that $\text{All}_x(\varphi, \psi)$ is about the extension of φ (and nothing else) is intimately related to the formal claim that All is conservative. Thus, in the context of a totally interpreted language $\mathcal{K}$, the semantic value of All can be represented as a relation R between sets, i.e. $\text{All}_x(\varphi, \psi)$ is true iff $R(\varphi^{\mathcal{K}}, \psi^{\mathcal{K}})$ (where $\varphi^{\mathcal{K}}$ and $\psi^{\mathcal{K}}$ are the extensions of φ and ψ, respectively). All is then said to be *conservative* if, for any sets A and B, $R(A, B)$ iff $R(A, A \cap B)$. See Barwise and Cooper [1981] (where they use the terminology of 'living on' for this property) or Westerståhl [1989].

[12] Under Kleene's strong scheme, a universal quantification is true only if all of its instances are, and $\chi \rightarrow \zeta$ is undefined if both χ and ζ are. But then $\forall x(\varphi \rightarrow \varphi)$ is not true, in virtue of the undefined instance $\varphi(x/a) \rightarrow \varphi(x/a)$. Here and throughout I assume for simplicity that every object a in the domain of $\mathcal{L}$ is denoted by an individual constant of $\mathcal{L}$, and, in our metalanguage, I use the same terms for object and constant (e.g. I use $\varphi(x/a)$ for a sentence of $\mathcal{L}$).

[13] The assumption that T is a predicate symbol is just for simplicity, the point applies equally well to compound formulas.

[14] I am of course speaking slightly loosely in using 'λ' to characterize λ. More carefully, λ is a sentence that is equivalent to $\text{All}_x(T(x) \wedge x = \ulcorner\lambda\urcorner, x \neq x)$, where $\ulcorner\lambda\urcorner$ is a term denoting λ (or a code for λ).

Thus (reading T as truth) λ says: all truths identical to me are self-distinct. By (A), λ is true iff $\left[\,T(x)\wedge x = \lambda\,\right]^{\mathcal{L}}_{+} \subseteq (x \neq x)^{\mathcal{L}}_{+} = \varnothing$, i.e. iff $\lambda \notin T^{\mathcal{L}}_{+}$. That is, $T^{\mathcal{L}}_{+}$ is *not* the set of true sentences.

The standard argument for the existence of an interpretation of T in which it applies exactly to the truths is blocked, because this argument relies on monotonicity, which (A) violates. An evaluation scheme is monotonic if, whenever we sharpen the interpretation of our intended truth predicate T, nothing that was true (or false) ceases to be true (or false, respectively).[15] Thus, if $\langle S_{+}, S_{-}\rangle$ is a potential interpretation of T (i.e. S_{+} and S_{-} are disjoint subsets of the domain), then I say that a sentence is true (false) in $\langle S_{+}, S_{-}\rangle$ if it is true (false) when T is interpreted by $\langle S_{+}, S_{-}\rangle$ (i.e. holding the interpretation of other expressions fixed). Further, let $j(S_{+}, S_{-})$ be $\langle R_{+}, R_{-}\rangle$, where R_{+} (R_{-}) is the set of sentences that are true (false) in $\langle S_{+}, S_{-}\rangle$. The condition of monotonicity is then

$$\langle U_{+}, U_{-}\rangle \leq \langle V_{+}, V_{-}\rangle \Rightarrow j(U_{+}, U_{-}) \leq j(V_{+}, V_{-}).$$

If $\langle S_{+}, S_{-}\rangle$ is a fixed point for j, i.e. $j(S_{+}, S_{-}) = \langle S_{+}, S_{-}\rangle$, then the sentences that are true (false) in $\langle S_{+}, S_{-}\rangle$ are precisely those that are in S_{+} (S_{-}). Thus, when T is interpreted by $\langle S_{+}, S_{-}\rangle$, our language contains its own truth predicate in a very natural sense. Further, given monotonicity, it is easy to see that j will have a fixed point.[16] However, when All is interpreted by (A), λ produces a violation of monotonicity, because $\langle\varnothing, \varnothing\rangle \leq \langle\{\lambda\}, \varnothing\rangle$, yet $j(\varnothing, \varnothing) \not\leq j(\{\lambda\}, \varnothing)$ (for λ is true in $\langle\varnothing, \varnothing\rangle$ but not in $\langle\{\lambda\}, \varnothing\rangle$). So the monotonicity-based argument for the existence of a fixed point no longer goes through.

It might seem, therefore, that Kripke's theory cannot handle All. Maudlin puts the point even more starkly. The generalized quantifier All, he writes, 'suffers a deadly defect: it is not free from paradox' [2004: 64].

As I noted in the introduction, however, the problem that Maudlin raises is in fact far more widespread than he recognizes: for it seems to apply to a whole range of other generalized quantifiers.[17] To illustrate, consider Most. The line of thought behind (A) leads to the following truth conditions.[18]

---

[15]$\langle S_{+}, S_{-}\rangle$ sharpens $\langle R_{+}, R_{-}\rangle$ (written $\langle R_{+}, R_{-}\rangle \leq \langle S_{+}, S_{-}\rangle$) if $R_{+} \subseteq S_{+}$ and $R_{-} \subseteq S_{-}$.

[16]Consider the sequence of interpretations $\langle R^{\mu}_{+}, R^{\mu}_{-}\rangle$ for ordinals μ as follows: $\langle R^{0}_{+}, R^{0}_{-}\rangle = \langle\varnothing, \varnothing\rangle$; $\langle R^{\eta+1}_{+}, R^{\eta+1}_{-}\rangle = j(R^{\eta}_{+}, R^{\eta}_{-})$; and, if θ is a limit ordinal, then $\langle R^{\theta}_{+}, R^{\theta}_{-}\rangle = \langle\bigcup_{\eta<\theta} R^{\eta}_{+}, \bigcup_{\eta<\theta} R^{\eta}_{-}\rangle$. Monotonicity ensures that this sequence is increasing (i.e. $\langle R^{\mu}_{+}, R^{\mu}_{-}\rangle \leq \langle R^{\eta}_{+}, R^{\eta}_{-}\rangle$ whenever $\mu \leq \eta$). But then, given that there are more ordinals than sentences of $\mathcal{L}$, we eventually reach μ with $\langle R^{\mu+1}_{+}, R^{\mu+1}_{-}\rangle = \langle R^{\mu}_{+}, R^{\mu}_{-}\rangle$.

[17]The extension of Maudlin's argument to other generalized quantifiers is given in Shaw [2016].

[18]$|A|$ is the cardinality of A.

(M) $\mathrm{Most}_x(\varphi, \psi)$ is true iff $|\varphi_+^{\mathcal{L}} \cap \psi_+^{\mathcal{L}}| > |\varphi_+^{\mathcal{L}} - \psi_+^{\mathcal{L}}|$.

That is, $\mathrm{Most}_x(\varphi, \psi)$ is true iff the things that φ and ψ are *both* true of outnumber the things that only φ is true of. Again, (M) seems inevitable given the initial thought that 'most φs are ψ' is about the φs, i.e. the members of $\varphi_+^{\mathcal{L}}$, and *only* those. For it follows that the truth condition for Most must be concerned exclusively with the members of $\varphi_+^{\mathcal{L}}$. But then it seems obvious that this condition must be: more of these members are ψ (i.e. in $\psi_+^{\mathcal{L}}$) than are not; which is exactly what (M) says.

Again, however, (M) spells disaster for Kripke's theory. For example,[19] let α be $0 = 0$; let W be a unary predicate symbol that is true of the sentences written on the wall (and false of everything else); and let ρ be

$$\mathrm{Most}_x(T(x) \wedge W(x), x \neq \rho).$$

Thus ρ says: most truths on the wall are not me. Suppose further that the only things written on the wall are α and ρ. (M), together with $\alpha \in T_+^{\mathcal{L}}$, gives: ρ is true iff $|[T(x) \wedge W(x)]_+^{\mathcal{L}} \cap (x \neq \rho)_+^{\mathcal{L}}| = |\{\alpha\}| = 1 > |[T(x) \wedge W(x)]_+^{\mathcal{L}} - (x \neq \rho)_+^{\mathcal{L}}|$; i.e. iff $\rho \notin T_+^{\mathcal{L}}$. That is, $T_+^{\mathcal{L}}$ is not the set of truths. As with (A), (M) violates monotonicity, and so blocks the standard argument for the existence of a fixed point.

But this is just one more example: it is straightforward to extend Maudlin's argument to cover a whole range of generalized quantifiers. For example, The, Both, No, Finite and so on. Kripke's theory thus seems unable to handle any of these.

This would be very significant, for the following two reasons. Firstly, it seems clear that such generalized quantifiers are the best way of formalizing quantification in natural language.[20] But, secondly, Kripke's theory, i.e. the strong Kleene proposal of [1975], plays an absolutely central role in the subsequent literature on truth. It is not simply that many writers take it to be the best theory that we have—it is also that many alternative proposals take it as their starting point. These include the proposals of Skyrms [1984], Gaifman [1992, 2000], Maudlin [2004] and Cobreros *et. al* [2013]. Indeed, Kripke's theory seems to be a straightforward implementation of the basic idea behind it: i.e. that we can assert (or deny) that a sentence is true precisely when we can assert (deny) the sentence itself. These arguments thus threaten to show even that there is a flaw with this idea. Our predicament appears dire.

I should note that there are theories to which the above arguments do *not* extend: for example, the supervaluationist proposals of Kripke [1975] or the revi-

---

[19] This example is similar to one in Shaw [2016].

[20] For arguments to this effect, see Barwise and Cooper [1981], Higginbotham and May [1981] and Westerståhl [1989].

sion theory of truth.[21,22] To illustrate, consider the basic supervaluationist theory of Kripke [1975], and the example of All. As we saw, if we accept that the truth condition of $\text{All}_x(\varphi, \psi)$ must concern only the members of $\varphi^{\mathcal{L}}_+$, then Maudlin's (A) seems inevitable. However, a supervaluationist approach would resist this initial move. Rather, $\text{All}_x(\varphi, \psi)$ will be true in $\mathcal{L}$ iff it is true in every classical sharpening of $\mathcal{L}$. This means that the truth of $\text{All}_x(\varphi, \psi)$ depends not just on $\varphi^{\mathcal{L}}_+$ but also on possible sharpenings of $\varphi^{\mathcal{L}}_+$. Further, it is easy to see that monotonicity is retained on this approach, and thus that the argument for the existence of a fixed point goes through as before. In the resulting language, λ is treated like a standard Liar sentence: it is neither true nor false. Similar remarks apply to Most and other generalized quantifiers.

Nevertheless, it would be a major blow if a theory as central to our theorizing about truth as Kripke's strong Kleene proposal was unable to incorporate such quantifiers.[23]

2 The Solution

Fortunately, there is a way of resisting Maudlin's argument (and its extensions). In this section, I show how Kleene's strong scheme, and Kripke's theory that uses it, can in fact be naturally extended to incorporate the full range of generalized quantifiers.[24]

---

[21]For the latter, see Gupta [1982], Herzberger [1982] and Gupta and Belnap [1993].

[22]In contrast, although I will not discuss the case in any detail, these arguments *do* seem to extend to the weak Kleene proposal of Kripke [1975].

[23]Another family of proposals that it is natural to ask about here are those in the dialetheist tradition. For these also employ Kripke's strong Kleene fixed point construction—but taking the third value to represent being both true and false. See, e.g., Dowden [1984] and Beall [2009: 18–24]. Does Maudlin's argument extend to these proposals? It seems not. For in this case (A) is unmotivated: since on such an approach the third value represents true and false, objects a such that $\varphi(x/a)$ receives the third value should clearly not be ignored in the truth condition of $\text{All}_x(\varphi, \psi)$ (as they are in (A)). And similarly for (M). Nevertheless, even within the dialetheist framework, one faces the question of how, in general, to incorporate generalized quantifiers—and the solution proposed in §2 would seem to apply just as well there as it does in more orthodox contexts. However I do not elaborate on this point here.

[24]The line of thought of this section can similarly be used to extend Kleene's weak scheme, and that version of Kripke's theory. But I do not do this here.

2.1 The Idea Behind the Scheme

We should start by getting clear on the idea that lies behind Kleene's strong scheme. Its treatment of connectives is given by the following truth tables.[25]

α	β	$\alpha \wedge \beta$	$\alpha \vee \beta$
t	t	t	t
t	f	f	t
t	u	u	t
f	t	f	t
f	f	f	f
f	u	f	u
u	t	u	t
u	f	f	u
u	u	u	u

α	$\neg\alpha$
t	f
f	t
u	u

Similarly, $\forall x\varphi$ is true according to this scheme if every instance is, false if some instance is false, and undefined otherwise. $\exists x\varphi$ is true if some instance is, false if every instance is false, and undefined otherwise.

Where do these treatments of these logical operators come from? They are not of course pulled out of thin air. Rather, there is a clear guiding idea behind them. I am going to argue in this section that this idea can in fact be naturally extended to generalized quantifiers, and that, when we do this, the resulting treatment is immune to the problems of §1.

So what is the idea? To illustrate, consider $\wedge$. The basic idea is this: $\alpha \wedge \beta$ gets a standard value s (i.e. t or f) iff one has already done enough to determine that $\alpha \wedge \beta$ will get this value—however one turns the us into standard values. That is, $\alpha \wedge \beta$ gets s iff however one turns us into standard values, $\alpha \wedge \beta$ gets s. For example, we have f on the sixth row, because however one turns the u under β into a standard value, $\alpha \wedge \beta$ will get f. But we have u on the third row, because turning the u into t gives $\alpha \wedge \beta$ the value t, while turning it into f gives $\alpha \wedge \beta$ the value f.

This idea can easily be made precise: here's the simplest way. To a first approximation, the semantic value of a connective can be thought of as a function from assignments of values to the letters 'α' and 'β'. This is essentially what truth tables do. Of course, we do not really want to claim that the semantic value of a connective involves these greek letters—our official such values are given later in

---

[25]Although I focus on the Boolean connectives everything that I say applies equally well to others, such as $\rightarrow$ and $\leftrightarrow$.

this section. However, these toy models allow a nice illustration of the idea behind Kleene's strong scheme.

Thus, in these terms, the classical semantic value of $\wedge$ can be thought of as a function whose domains consists of all (total) functions from $\{\alpha, \beta\}$ to $\{t, f\}$, and whose range is included in (indeed, in this case is) $\{t, f\}$. In contrast, the strong Kleene semantic value can be thought of as a *partial* function, whose domain consists of all *partial* functions from $\{\alpha, \beta\}$ into $\{t, f\}$, and whose range is again included in $\{t, f\}$. Thus, for example, the partial function that sends α to t and is undefined for β corresponds to the row of the truth table with t under α and u under β. The strong Kleene semantic value is then determined by the following condition, where $\mathsf{Partial}_o$ is the set of partial functions from $\{\alpha, \beta\}$ into $\{t, f\}$, and Total_o is the set of total ones; and $\wedge_o^C$ and $\wedge_o^{SK}$ are the classical and strong Kleene semantic values of $\wedge$, respectively.[26]

($*$) For any $g \in \mathsf{Partial}_o$ and standard value s, $\wedge_o^{SK}(g) = s$ iff for any $h \in \mathsf{Total}_o$ sharpening g, $\wedge_o^C(h) = s$.

Similarly in the case of all the other connectives.

Similarly, too, in the cases of $\forall$ and $\exists$. In the context of a partially interpreted language, the semantic value of a formula $\varphi(x)$ with exactly one free variable can be thought of as a pair of disjoint sets (i.e. subsets of the domain); and the semantic value of a quantifier can be thought of as a function from such pairs to standard values.[27] The strong Kleene semantic values of $\forall$ and $\exists$ are then determined by the following condition.

($\dagger$) $\forall_o^{SK}$ ($\exists_o^{SK}$) sends $\langle S_+, S_- \rangle$ to standard value s iff $\forall_o^C$ ($\exists_o^C$) sends every total sharpening of $\langle S_+, S_- \rangle$ to s.

Our official semantic values unify these conditions as follows. Let Assign be the set of assignments, that is, total functions from the set of variables to the domain. Then, in the classical context, we can take the semantic value of a formula ψ to be a total function $H : \mathsf{Assign} \to \{t, f\}$ as follows:

$$H(g) = \begin{cases} t & \text{if } \psi \text{ is true under } g \\ f & \text{if } \psi \text{ is false under } g. \end{cases}$$

---

[26] The subscript 'o' indicates that these are merely first approximations.

[27] Again, these are first approximations. Our official semantic values are given below.

Note that H is a total function in the classical context, because ψ is always either true or false under g. In the context of a partially interpreted language, by contrast, the semantic value of a formula becomes a partial function $I : \text{Assign} \to \{t, f\}$, with the same definition as H, except that the function is only partial since ψ might be neither true nor false under g:[28]

$$I(g) = \begin{cases} t & \text{if } \psi \text{ is true under } g \\ f & \text{if } \psi \text{ is false under } g. \end{cases}$$

Let Total (Partial) be the set of total (partial) functions from Assign to $\{t, f\}$. Note that Total $\subseteq$ Partial. In the classical context, the semantic value of an n-ary logical operator is a total n-ary function on Total. Thus, if $H, H' \in$ Total, we have[29]

$$\neg^C(H)(g) = \begin{cases} t & \text{if } H(g) = f \\ f & \text{if } H(g) = t \end{cases}$$

$$\wedge^C(H, H')(g) = \begin{cases} t & \text{if } H(g) = H'(g) = t \\ f & \text{if } H(g) = f \text{ or } H'(g) = f \end{cases}$$

$$\forall_x^C(H)(g) = \begin{cases} t & \text{if for each } a \text{ in the domain, } H(g(x/a)) = t \\ f & \text{if for some } a \text{ in the domain, } H(g(x/a)) = f. \end{cases}$$

We can extend the functions $\neg^C$, $\wedge^C$ and $\forall_x^C$, currently defined only on Total, to functions $\neg^{SK}$, $\wedge^{SK}$ and $\forall_x^{SK}$ defined on all of Partial with exactly the same definition, this time applied to partial functions H and H'.[30,31]

---

[28]The relationship between these official semantic values and our previous way of thinking about the semantic value of $\varphi(x)$ is as follows. If, to our first approximation, the semantic value of φ was $\langle S_+, S_- \rangle$, then its official semantic value is I such that

$$I(g) = \begin{cases} t & \text{if } g(x) \in S_+ \\ f & \text{if } g(x) \in S_-. \end{cases}$$

[29]I write $\forall_x^C$ rather than $(\forall x)^C$ for ease of reading. Here $g(x/a)$ is the assignment that sends x to a but is in other respects the same as g.

[30]Strictly speaking, we should in this case write $\forall_x^{SK, \mathfrak{A}}$ rather than $\forall_x^{SK}$, where $\mathfrak{A}$ is the partial interpretation at issue, since the value of this function depends on the domain. But since the domain will be clear from the context, for readability I omit the extra superscript.

[31]The relation of the official semantic values of logical operators to our first approximations of these is as follows. If g and g' are partial functions, then $g(x) = g'(y)$ means that either both are defined and they are equal, or both are undefined. If $h(\alpha) = H(g)$, and $h(\beta) = H'(g)$, then

On this approach, the semantic value of a formula is defined by induction as follows. Let K be a language, and let $\mathfrak{A}$ be a total K-interpretation. The semantic value of a formula φ of K under $\mathfrak{A}$ is written $\varphi^{\mathfrak{A}}$. If P is an n-ary predicate symbol and $u_1, \ldots, u_n$ are terms, then[32]

$$[P(u_1, \ldots, u_n)]^{\mathfrak{A}}(g) = t \text{ iff } \langle u_1^{\mathfrak{A},g}, \ldots, u_n^{\mathfrak{A},g} \rangle \in P^{\mathfrak{A}}.$$

If σ is an n-ary logical operator (i.e. a connective or a quantifier together with a variable), then

$$[\sigma(\varphi_1, \ldots, \varphi_n)]^{\mathfrak{A}}(g) = \sigma^C(\varphi_1^{\mathfrak{A}}, \ldots, \varphi_n^{\mathfrak{A}})(g).$$

Similarly, if $\mathfrak{B}$ is a partial K-interpretation, then

$$[P(u_1, \ldots, u_n)]^{\mathfrak{B}}(g) = t \text{ iff } \langle u_1^{\mathfrak{B},g}, \ldots, u_n^{\mathfrak{B},g} \rangle \in P_+^{\mathfrak{B}},$$

$$[P(u_1, \ldots, u_n)]^{\mathfrak{B}}(g) = f \text{ iff } \langle u_1^{\mathfrak{B},g}, \ldots, u_n^{\mathfrak{B},g} \rangle \in P_-^{\mathfrak{B}};$$

$$[\sigma(\varphi_1, \ldots, \varphi_n)]^{\mathfrak{B}}(g) = \sigma^{SK}(\varphi_1^{\mathfrak{B}}, \ldots, \varphi_n^{\mathfrak{B}})(g).$$

The relationship between the classical and strong Kleene semantic values of the logical operators can be stated as follows. Here σ is an n-ary logical operation.

(‡) For any $H_1, \ldots, H_n \in \text{Partial}$, $g \in \text{Assign}$ and standard value s,

$\sigma^{SK}(H_1, \ldots, H_n)(g) = s$ iff for any $I_1, \ldots, I_n \in \text{Total}$ with $H_i \subseteq I_i$,
for $i = 1, \ldots, n$, $\sigma^C(I_1, \ldots, I_n)(g) = s$.

As this account of the strong Kleene scheme brings out, it is in a certain way similar to the supervaluationist scheme. Specifically, both crucially involve quantification over classical sharpenings of partial semantic values. But the key difference is this. In the strong Kleene case, this quantification plays a 'local' role: it gives the semantic values of specific lexical items, i.e. the logical operators. In contrast, in the supervaluationist case, the quantification plays a 'global' role: it occurs in the account of what it means for whole formulas to be true or false. It is because of this difference that the supervaluationist scheme, unlike the strong Kleene one, is sensitive to when a single predicate symbol occurs more than once in a formula.

$\wedge^{SK}(H, H')(g) = \wedge_o^{SK}(h)$. Similarly, $\forall_x^{SK}(H)(g) = \forall_o^{SK}(\langle S_+, S_- \rangle)$, if $S_+ = \{a \in A : H(g(x/a)) = t\}$ and $S_- = \{a \in A : H(g(x/a)) = f\}$ (where A is the domain). Similarly for other standard logical operators, and for the classical case.

[32] Here $u_i^{\mathfrak{A},g}$ is the denotation of u_i under $\mathfrak{A}$ and g, defined in the usual way.

Hence the contrasting treatments of $\varphi \vee \neg\varphi$, $\neg\neg\varphi \to \varphi$, etc. As one might put it in a slogan: supervaluationism = strong Kleene + coordination.[33]

More carefully, the relationship is as follows. Let $\mathcal{K}$ be a partially interpreted language, and let $\mathcal{K}'$ be another such language, such that each predicate symbol P of $\mathcal{K}$ has been replaced by countably many 'copies' P_1, P_2, …; where $P_i = Q_j$ only if $P = Q$ and $i = j$; and where each P_i is interpreted in $\mathcal{K}'$ as P is interpreted in $\mathcal{K}$. Now let φ be a formula of $\mathcal{K}$, and let φ' be the result of replacing each occurrence of a predicate symbol P in φ with a distinct 'copy' P_i (i.e. different copies for distinct occurrences of a single symbol). We then have: φ is true (false) in $\mathcal{K}$ under the strong Kleene scheme (and assignment g) iff φ' is true (false) in $\mathcal{K}'$ under the supervaluationist scheme (and g).[34]

2.2 Applying the Idea

With that on the table, what to say about generalized quantification? Since we have arrived at a completely general principle relating the strong Kleene and classical semantic values of logical operators, i.e. (‡), we can simply apply this to generate the strong Kleene values of generalized quantifiers. And we are going to arrive at the values for these quantifiers that are determined by (‡). But it will be more instructive to proceed somewhat less abstractly.

Thus, consider first All. The classical treatment is as follows. Here $\mathcal{K}$ is a totally interpreted language, and $\varphi^{\mathcal{K}}$ is the extension of φ in $\mathcal{K}$.

$$\text{All}_x(\varphi, \psi) \text{ is true iff } \varphi^{\mathcal{K}} \subseteq \psi^{\mathcal{K}}.$$

As before, let us to a first approximation think of the semantic value of All as a function from extensions (in the classical case) or pairs of extensions and anti-extensions (in the partial one) to standard values. Then if we follow the guiding idea behind the strong Kleene scheme, we will give All the semantic value determined by the following condition.[35] (Here $\text{All}^{SK}(B, C, D, E)$ is shorthand for $\text{All}(\langle B, C\rangle, \langle D, E\rangle)$; the official semantic value of All_x, generated by (‡), is related to this first approximation just as in the case of $\forall$.)

(A1) For any standard value s,

$$\text{All}^{SK}(S_+^P, S_-^P, R_+^P, R_-^P) = s \text{ iff for any total } \langle S_+^T, S_-^T\rangle \geq \langle S_+^P, S_-^P\rangle \text{ and}$$
$$\langle R_+^T, R_-^T\rangle \geq \langle R_+^P, R_-^P\rangle, \text{All}^C(S_+^T, R_+^T) = s.$$

[33] I.e. coordination in the sense of Fine [2007].

[34] This is proved by a routine induction on the degree of complexity of φ.

[35] In this section I omit the subscript 'o' for ease of reading.

This then reduces to (here $B^c = A - B$, where A is the domain)

(A2) $\text{All}^{SK}(S_+^P, S_-^P, R_+^P, R_-^P) = \begin{cases} t & \text{iff } (S_-^P)^c \subseteq R_+^P \\ f & \text{iff } S_+^P \cap R_-^P \neq \varnothing. \end{cases}$

Clearly, (A1) is quite different from (A): although the truth condition of (A1) entails that of (A), the reverse is certainly not the case. For example, if the interpretation of T is $\langle U_+, U_- \rangle$, with $U_+ \cup U_- \neq A$, then $\text{All}_x(T(x), T(x))$ is neither true nor false, according to (A1); but it is of course true, according to (A). Indeed, it is easy to see that, understood via (A1), $\text{All}_x(\varphi, \psi)$ is equivalent to $\forall x(\varphi \rightarrow \psi)$—just as in the classical case. It follows that the sentence λ that caused difficulty for (A) is handled like a familiar sort of Liar sentence under (A1); and, further, that when All is understood via (A1), monotonicity is restored. Thus, the argument for the existence of a fixed point goes through just as before. It seems, therefore, that far from suffering from a 'deadly defect', All is no more paradoxical than the familiar first-order logical operators. Rather, we have found a natural way of extending Kripke's theory to accommodate this.

The fact that $\text{All}_x(\varphi, \psi)$ ends up being equivalent to $\forall x(\varphi \rightarrow \psi)$ does mean that (just as in the classical case) adding it to our language does not allow the expression of any new truth conditions. What we have seen, however, is that this treatment of All emerges naturally from the same basic idea that generated the treatments of the more familiar connectives and quantifiers. Further, as we will see, our approach to All naturally applies to other generalized quantifiers, such as Most: and these certainly do allow the expression of new truth conditions.

Before considering these additional quantifiers, however, there are some further issues to take up in connection with All. First, we should return to the argument of §1 that I presented in favour of (A). This started with the idea that 'all φs are ψ' is about the φs, and only these. This was then taken to mean that the truth conditions for $\text{All}_x(\varphi, \psi)$ should concern only the members of $\varphi_+^{\mathcal{L}}$. In contrast, the truth condition given by (A1) concerns the members of the (possibly larger) set $(\varphi_-^{\mathcal{L}})^c$. On reflection, however, this more liberal construal of the original idea is surely very natural. For on perhaps the most natural understanding of partially interpreted languages—and certainly that which seems natural in connection with truth—the members of $(\varphi_+^{\mathcal{L}} \cup \varphi_-^{\mathcal{L}})^c$ are such that it is *indeterminate* whether they are φs. But then of course one cannot ignore these when it comes to the question of whether all φs are ψ: since it is not determinate that these are not in fact φs; that is, they may be φs. The correct understanding of the original idea, then, should insist only that the truth condition of $\text{All}_x(\varphi, \psi)$ is a condition on $(\varphi_-^{\mathcal{L}})^c$—which is of course compatible with (A1).

A final point about All is this. On the proposed treatment, there are no logical truths. For example, $\text{All}_x(\varphi, \varphi)$ does not hold with full generality. This might be thought to constitute an objection: mustn't any adequate treatment yield this principle (perhaps among others)? No: our aim was to extend Kleene's strong scheme (and Kripke's theory based on it) to generalized quantifiers. If one has chosen to use this scheme for the standard connectives and quantifiers, then one has already made one's peace with the fact that none of the classical laws for these come out as logical truths. There would thus seem to be little motivation for insisting that certain classical laws for All (such as $\text{All}_x(\varphi, \varphi)$) must be maintained. Of course, things would be different if any adequate treatment of All had to include (A)—then $\text{All}_x(\varphi, \varphi)$ would indeed be logically true. We have seen, however, that contrary to what has been supposed, the natural extension of Kleene's strong scheme to All does not include (A).[36] So much, then, for the supposed objection.

Consider now Most. The classical treatment is

$$\text{Most}_x(\varphi, \psi) \text{ is true iff } |\varphi^{\mathcal{K}} \cap \psi^{\mathcal{K}}| > |\varphi^{\mathcal{K}} - \psi^{\mathcal{K}}|.$$

As before, the guiding idea of the scheme (or (‡)) yields

(M1) For any standard value s,

$$\text{Most}^{SK}(S_+^P, S_-^P, R_+^P, R_-^P) = s \text{ iff for any total } \langle S_+^T, S_-^T \rangle \geq \langle S_+^P, S_-^P \rangle \text{ and } \langle R_+^T, R_-^T \rangle \geq \langle R_+^P, R_-^P \rangle, \text{Most}^C(S_+^T, R_+^T) = s.$$

This then reduces to

(M2) $\text{Most}^{SK}(S_+^P, S_-^P, R_+^P, R_-^P) = \begin{cases} t & \text{iff } |S_+^P \cap R_+^P| > |(S_-^P)^c - R_+^P| \\ f & \text{iff } |(S_-^P)^c \cap (R_-^P)^c| \leq |S_+^P - (R_-^P)^c|. \end{cases}$

This treatment of Most again satisfies the more liberal—and plausible—construal of the idea that $\text{Most}_x(\varphi, \psi)$ is about the φs (and nothing else). For the truth

_____

[36] One might insist that there is a sense in which, on the strong Kleene version of Kripke's theory, there *are* in fact logical truths (involving the standard connectives and quantifiers). Firstly, one might note that if the base language is classical, then any instance of a classical law that does not contain the truth predicate will come out as true on this theory. Secondly—an alternative strategy— one might point out that the classical laws, although sometimes untrue, are never false. I am grateful to a referee for this journal for suggesting these strategies. However, each strategy applies equally well to the proposed treatment of All: if the base language is classical, then it is similarly the case that instances of classical laws (e.g. $\text{All}_x(\varphi, \varphi)$) that do not contain T are true; further, even when these do contain T, they will never be false. Again, then, the proposed extension of Kleene's strong scheme would seem to be on a par with the original as far as logical truth is concerned.

condition is again in terms of the members of $(\varphi^{\mathcal{L}}_-)^c$. Further, it is easy to show that this restores monotonicity, and thus that the argument for the existence of a fixed point goes through just as in the first-order case.[37]

The sentence that caused problems for (M), ρ, is handled as follows. If $\langle U_+, U_- \rangle$ is a potential interpretation of T, with $\alpha \in U_+$, let $\mathcal{L}_U$ be the result of interpreting T by $\langle U_+, U_- \rangle$. With (M), we got the disastrous result that ρ is true in $\mathcal{L}_U$ iff $\rho \notin U_+$. With (M1), we get that ρ is true in $\mathcal{L}_U$ iff $\left| [T(x) \wedge W(x)]^{\mathcal{L}_U}_+ \cap (x \neq \rho)^{\mathcal{L}_U}_+ \right| = \left| \{\alpha\} \right| = 1 > \left| [[T(x) \wedge W(x)]^{\mathcal{L}_U}_-]^c - (x \neq \rho)^{\mathcal{L}_U}_+ \right|$; i.e. iff $\rho \in U_-$. This is just what we get in the case of a standard Liar sentence[38]—and far less disastrous. Similarly, ρ is false in $\mathcal{L}_U$ iff $\rho \in U_+$. Again, just as with a standard Liar. Consequently, in any fixed point $\langle V_+, V_- \rangle$, ρ will be neither true nor false in $\langle V_+, V_- \rangle$.

Further, it is easy to see that other generalized quantifiers, such as Finite, Both, etc. can be accommodated in just the same way.

To drive home how natural this treatment of generalized quantifiers is, I consider one final example. For one of the first examples of Kripke [1975] in fact involves 'most' (even though Kripke does not himself consider how to give a formal treatment of this). Thus, Kripke supposes that Jones asserts

(1) Most of Nixon's assertions about Watergate are false.

As Kripke points out, although there are many situations in which (1) seems straightforwardly true or false (e.g. if Nixon's utterances about Watergate are themselves all so), there are other situations in which it seems paradoxical. For example, if (1) is Jones's sole utterance about Watergate, while Nixon's is

(2) Everything Jones says about Watergate is true.

In this case, it seems that (1) is true iff (2) is false iff (1) is not true.

The proposed treatment of generalized quantifiers, however, seems to give exactly the right results. Thus, suppose that we formalize these as

---

[37] Indeed, one can show that as long as the logical operators of our language satisfy (‡), monotonicity obtains. Thus, suppose that $\langle U_+, U_- \rangle \leq \langle V_+, V_- \rangle$ are potential interpretations of T. We show by induction on the degree of φ that if φ is true (false) in $\langle U_+, U_- \rangle$, under an assignment g, then it is also true (false) in $\langle V_+, V_- \rangle$ under g. There are two cases: φ is atomic, or φ is $\sigma(\psi_1, \ldots, \psi_n)$ for some n-ary logical operator σ. But the atomic case is obvious, so suppose φ is $\sigma(\psi_1, \ldots, \psi_n)$. For each ψ_i, let H_i^U be the partial function that sends an assignment to t (f) iff ψ_i is true (false) in $\langle U_+, U_- \rangle$; and similarly for H_i^V. By the inductive hypothesis, $H_i^U \subseteq H_i^V$, for each i. But then it follows from (‡) that, if $\sigma^{SK}(H_1^U, \ldots, H_n^U) = s \ (\in \{t, f\})$, then $\sigma^{SK}(H_1^V, \ldots, H_n^V) = s$, which establishes our claim.

[38] I.e. β with $\beta = \neg T(\beta)$.

(1*) $\text{Most}_x(N(x), \neg T(x))$

(2*) $\text{All}_x(J(x), T(x))$

where N and J are unary predicate symbols of $\mathcal{L}$. If N is true only of sentences that are straightforwardly true or false, e.g. do not contain T (and false of everything else), then (1*) will similarly be straightforwardly true or false. That is, it will be true in the fixed point constructed iff more of the sentences that N applies to are true in this fixed point than are false in it; otherwise it will be false. In contrast, if $N_+^{\mathcal{L}} = \{(2^*)\}$, and $N_-^{\mathcal{L}} = A - \{(2^*)\}$; and similarly $J_+^{\mathcal{L}} = \{(1^*)\}$ while $J_-^{\mathcal{L}} = A - \{(1^*)\}$; then it is easy to see that, on the proposed treatment of the quantifiers, neither (1*) nor (2*) will be true or false in the fixed point. For if $\langle U_+, U_- \rangle$ is a potential interpretation of T, then (1*) will be true in $\langle U_+, U_- \rangle$ iff (2*) $\in U_-$; and false in $\langle U_+, U_- \rangle$ iff (2*) $\in U_+$; and similarly for (2*). Thus, neither (1*) nor (2*) will be added to the extension or anti-extension of T in the construction of the fixed point.

It seems, then, that Kleene's strong scheme, and Kripke's theory that is based on it, can in a natural way be extended to cover the full range of generalized quantifiers. These quantifiers thus pose no threat to Kripke's theory or the many proposals that are based on it—or, indeed, to the basic idea that lies behind it.[39]

References

Barwise, J. and R. Cooper. 1981. Generalized Quantifiers in Natural Language. *Linguistics and Philosophy* 4: 159–219.

Beall, J. 2009. *Spandrels of Truth*. Oxford: Clarendon Press.

Beall, J., R. T. Brady, A. P. Hazen, G. Priest and G. Restall. 2006. Relevant Restricted Quantification. *Journal of Philosophical Logic* 35: 587–98.

Cobreros, J., P. Égré, D. Ripley and R. van Rooij. 2013. Reaching Transparent Truth. *Mind* 122: 841–66.

Dowden, B. H. 1984. Accepting Inconsistencies from the Paradoxes. *Journal of Philosophical Logic* 13: 125–30.

Field, H. 2014. Naive Truth and Restricted Quantification: Saving Truth a Whole Lot Better. *Review of Symbolic Logic* 7: 147–91.

— 2016. Indicative Conditionals, Restricted Quantification, and Naive Truth. *Review of Symbolic Logic* 9: 181–208.

Fine, K. 2007. *Semantic Relationism*. Malden: Blackwell.

Gaifman, H. 1992. Pointers to Truth. *Journal of Philosophy* 89: 223–61.

— 2000. Pointers to Propositions. In A. Chapuis and A. Gupta (eds), *Circularity, Definition, and Truth*: 79–121. New Delhi: Indian Council of Philosophical Research.

Gupta, A. 1982. Truth and Paradox. *Journal of Philosophical Logic* 11: 1–60.

Gupta, A. and N. Belnap. 1993. *The Revision Theory of Truth*. Cambridge, MA: MIT Press.

Herzberger, H. G. 1982. Notes on Naive Semantics. *Journal of Philosophical Logic* 11: 61–102.

Higginbotham, J. and R. May. 1981. Questions, Quantifiers and Crossing. *Linguistic Review* 1: 41–80.

Kripke, S. 1975. Outline of a Theory of Truth. *Journal of Philosophy* 72: 690–716.

Maudlin, T. 2004. *Truth and Paradox: Solving the Riddles*. Oxford: Clarendon Press.

Priest, G. 2006. *In Contradiction*, second edition, Oxford: Clarendon Press.

Shaw, J. R. 2016. Theories of Truth and Generalized Quantification. Paper presented at the annual meeting of the Pacific Division of the American Philosophical Association, San Francisco.

Skyrms, B. 1984. Intensional Aspects of Self-Reference. In R. L. Martin (ed.), *Recent Essays on Truth and the Liar Paradox*: 119–31. Oxford: Clarendon Press.

Tarski, A. 1935. The Concept of Truth in Formalized Languages. In his *Logic, Semantics, Metamathematics: Papers from 1923 to 1938*, translated by J. H. Woodger, second edition, J. Corcoran (ed.), 1983: 152–267. Indianapolis: Hackett.

Westerståhl, D. 1989. Quantifiers in Formal and Natural Languages. In D. Gabbay and F. Guenthner (eds), *Handbook of Philosophical Logic, Volume IV*: 1–131. Dordrecht: D. Reidel.