The Iterative Solution to Paradoxes for Propositions

Bruno Whittle

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When propositions arrive on the scene, there would seem to be every reason to imbue them with structure. For example, if two sentences are made up of words with different meanings, or if these words are arranged in different ways, then it seems possible for someone to believe the proposition expressed by one of these sentences, without believing that expressed by the other. The natural way to ensure that such pairs of propositions can always be distinguished is to take propositions to have structures that mirror those of the sentences that express them. So that is how propositions are conceived of in seminal works in the philosophy of language.[1] Similarly, it seems very plausible that logically complex propositions hold in virtue of simpler ones. Thus, it would seem that conjunctions hold because their conjuncts do, or that disjunctions hold because their (true) disjuncts do. But then, since a proposition cannot, presumably, explain itself, we are led to distinguish a proposition p from the conjunction of p with itself (or indeed its conjunction with anything else, and similarly in the case of disjunction). Again, that is, we seem led to embrace a structured conception of propositions.[2]

It is with consternation, then, that we discover that this way of thinking about propositions is subject to paradox: in particular, a version of Russell's paradox known as the Russell-Myhill, in which there has been a revival of interest.[3] Indeed, to add to our dismay, it is sometimes claimed that this argument reveals there to be something fundamentally amiss in the structured approach to propositions. The

[1] For example, Kaplan [1977], Salmon [1986] and Soames [1987].

[2] See Rosen [2010] for an account of grounding (i.e. the in-virtue-of relation) in terms of structured propositions.

[3] Examples of recent works on this topic include Menzel [2012], Deutsch [2014], Uzquiano [2015], Walsh [2016], Goodman [2017], Fritz [2021] and Sbardolini [2021].

suggestion is that some new idea is needed to redeem this approach, and that until this is found, the ultimate value of work that relies on it hangs in the balance. Russell's ramified theory of types is acknowledged as a possible refuge. But due to its forbidding complexity, not to mention aspects that seem ad hoc, this is seen as a last resort.[4] A related idea, that seems to be increasingly influential, is that the Russell-Myhill paradox is a limitative result, showing that propositions cannot possibly be as fine-grained as standard structured accounts would have it.[5]

The aim of the present work is to do battle with this tide: to argue that the demise of structured propositions has been exaggerated. I will argue that an attractive solution to Russellian paradoxes for propositions is already effectively in hand. Specifically, we can solve these in essentially the same way that we solve such paradoxes for sets. The solution that I have in mind in the case of sets is of course the iterative conception of that notion, codified as Zermelo-Fraenkel set theory (or ZFC, the 'C' standing for choice). (Having given the Russell-Myhill, and related paradoxes for propositions, in §1, this solution to paradoxes for sets will be recapitulated in §2.)

We *might* try to construct propositions purely out of sets (or sets together with urelements, i.e. the objects that we start with, before any sets have been formed). However, it is more natural, and more in keeping with the way that philosophers tend to think about these, to instead construct them out of sets together with properties. Thus, in §3, I explain how to modify the standard account of sets to incorporate these. Then, in §4, I give the account of propositions, out of these sets and properties.

On the proposed account, propositions are just as fine-grained as is assumed in applications in the philosophy of language or metaphysics. That is, their structures mirror those of the sentences that express them, and propositions concerned with different things (e.g. distinct properties) are themselves always distinguished. More generally, adopting this solution to paradoxes for propositions would seem to involve only relatively minor revisions to approaches in philosophy that rely on them.

A virtue of the proposal is that it is based on essentially a single idea, applied to both sets and properties. The idea is that these are arranged into a hierarchy, in what would seem to be the simplest possible way. Thus, to begin with (i.e. at the bottom of our hierarchy), we have things that are neither sets nor properties.

[4] For such assessments of the state of play, see Deutsch [2008: final paragraph] and Klement [2010: 38].

[5] For this lesson from the paradox, see Uzquiano [2015] and Goodman [2017] (and perhaps also Fritz [2021]).

We then have a first layer of sets and properties: namely, sets that contain these initially given things, and properties that apply to them. Next, we have sets that also contain entities from this first layer of sets and properties, and properties that also apply to them. And so on. Adopting this, apparently quite natural, picture of sets and properties is sufficient to solve the whole family of Russellian paradoxes, for sets, properties and propositions. Further, the solutions themselves all have broadly the same shape.

As I explain in §5, this unity of motivation also seems to shed an interesting light on the much maligned ramified theory of types: specifically, on a criticism of this, offered by both Ramsey and Gödel.

In §6, confident that the reader is now firmly on side, I acknowledge certain limitations of the proposal. For example, according to it, no proposition can quantify over absolutely all objects, or even over absolutely all propositions. In mitigation, I make two points. Firstly, these limitations seem to be of a piece with those faced by the iterative conception of set (or ZFC)—limitations that have not (rightly in my view) prevented that approach from gaining very wide acceptance. Secondly, I argue that the restrictions of the proposed account are in significant respects less stringent than those faced by standard approaches to higher-order logic, i.e. the framework that important recent work on paradoxes for propositions has been conducted in.

In §7 I compare the proposed account of propositions to alternatives that have been offered—arguing, of course, that in certain respects the account proposed here is preferable.

The basic idea behind the proposal of this paper, i.e. to solve Russellian paradoxes for propositions (and along the way also those for properties) in essentially the same way that we solve those for sets, would seem to be a very natural one. However, while there are places where ideas similar to those to follow have been put forward, I do not know of any sustained development of this approach. Further, it has certainly not received the attention in treatments of paradoxes for propositions that, I will argue, it warrants.[6]

[6] Two notable antecedents are Charles Parsons' approach to the liar paradox [1974], and a suggestion of George Bealer's about how we should solve paradoxes for properties [1982]. Parsons' solution to the Liar is in terms of propositions, and he advocates seeing these as stratified in a way analogous to sets under ZFC. He does not however give a worked out account of propositions along these lines. (I believe that the account proposed here is the natural way of implementing his basic idea, even if the resulting picture seems to differ in certain respects from that which he paints; I will not discuss such differences because the paper is already long enough as it is.) Bealer, on the other hand, suggests a solution to paradoxes for properties that is based on ZFC. The basic scheme is thus

1 Paradoxes

I begin by reviewing the family of paradoxes for propositions that this paper will offer a solution to. The most famous of these, the Russell-Myhill, is as follows.[7] This paradox concerns propositions to the effect that any member of some collection of propositions A is true. That is, propositions of the form[8]

$$\langle \forall p(p \text{ in } A \to p \text{ is true})\rangle.$$

To derive a contradiction, consider the collection R of all such propositions that do not belong to the collection they are about. That is, q belongs to R iff, for some collection of propositions A,

$$q = \langle \forall p(p \text{ in } A \to p \text{ is true})\rangle, \text{ and } q \text{ is not in } A.$$

Now consider $r = \langle \forall p(p \text{ in } R \to p \text{ is true})\rangle$. Is r in R? If it is, then for some A, $r = \langle \forall p(p \text{ in } A \to p \text{ is true})\rangle$, and r is not in A. But it would appear that[9]

$$(*) \quad \text{if } B \neq C, \text{ then } \langle \forall p(p \text{ in } B \to p \text{ is true})\rangle \neq \langle \forall p(p \text{ in } C \to p \text{ is true})\rangle.$$

After all, these propositions are about different collections! But then $r = \langle \forall p(p \text{ in } A \to p \text{ is true})\rangle$ implies $A = R$. This means that, if r *is* in R, then it isn't. So it isn't. But then, of course, it is.

This paradox is similar in form to the more familiar Russellian paradox concerned exclusively with collections (or classes or sets). That results from considering the collection of all those collections that are not members of themselves. The Russell-Myhill is identical, save for the fact that it is concerned not with whether a collection contains *itself*, but rather with whether it contains a certain proposition that is about it.

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similar to that of the account proposed here. The details, however, seem to be rather different: see note 19. In addition, in Whittle [2017a], I proposed an account of properties and propositions that is in certain respects similar to that of this paper. However, the account proposed here is different in many points of detail, and seems more straightforward (at least, it fits more comfortably with standard classical logic).

[7] See Russell [1903: appendix B] and Myhill [1958].

[8] I use 'collection' rather than 'set' or 'class' to emphasize that in this informal presentation we are not relying on a technical notion to be understood in terms of some specific theory. Also, if S is a sentence, then by $\langle S\rangle$ I mean the proposition that S expresses.

[9] This principle should be understood as making a universal claim about any collections of propositions B and C.

The argument seems to pose a challenge specifically for structured, or fine-grained, propositions, because it is only under such a conception that $(*)$ is plausible. If propositions are identified instead with sets of possible worlds, for example, then there would seem to be little motivation for this principle.

I have stated the paradox in terms of collections. But one can give similar ones in terms of properties of propositions, or pluralities of them. Similarly, there is nothing special about universal quantification. An even simpler paradox would instead focus on propositions of the form $\langle Fa \rangle$, for a property F and an object a. One would then consider a property R that applies to such a proposition $\langle Fa \rangle$ iff F does not itself apply to this proposition. If we consider a proposition $\langle Rb \rangle$, for any object b, then (just as in other cases) we end up with R applying to this proposition iff it doesn't. So there is a whole family of paradoxes in this general vein.

Peter Fritz has recently given an ingenious version of the Russell-Myhill argument that relies only on widely accepted principles about grounding. That is, his argument shows that we do not need to take principles such as $(*)$ as basic. Rather, we can derive a principle along these lines, sufficient for a version of the paradox, solely from principles about grounding.[10] Fritz sees his argument as making trouble for the notion of grounding. But it also just shows how deeply embedded the structured conception of propositions is, not only in the philosophy of language, but in metaphysics as well. In any case, the account of propositions proposed here can be used by ground-theorists, and requires little revision to how they are already operating.

2 Sets

In this section I give the standard approach to sets, as a prelude to, in the next section, giving a version of this account that incorporates properties, and then in the section after that, constructing propositions out of these sets and properties. The

[10] The idea behind Fritz's argument can be illustrated (as in his [forthcoming]) using arbitrary conjunction. (The official argument uses only standard connectives and quantifiers, but it is more intricate.) Thus, for any plurality pp of propositions, we assume that there is a proposition $\langle \bigwedge pp \rangle$ that is the conjunction of these. The key assumption (which is a version of a standard principle about grounding) is that for any plurality ff of facts (i.e. true propositions), $\langle \bigwedge ff \rangle$ is immediately grounded precisely by the members of ff. It follows that for distinct pluralities of facts ff and gg, $\langle \bigwedge ff \rangle \neq \langle \bigwedge gg \rangle$. We can then give a Russellian paradox of a familiar shape: by considering the plurality rr of facts f such that, for some facts gg, $f = \langle \bigwedge gg \rangle$, but f is not one of the gg. We get that $\langle \bigwedge rr \rangle$ is in rr iff it isn't. For the official version of the argument, see Fritz [forthcoming] (and for further discussion, [2021]).

idea of this standard approach is that sets are arranged into a hierarchy of layers, or 'ranks'. This hierarchy can be motivated by the thought that the members of a set are 'prior' to it. Thus, we initially start with urelements: objects that are given before any sets have been formed, essentially. So our first rank, U_0, is simply the set of all of these urelements, U. This first rank contains Socrates, the Eiffel Tower, Sicily, and so on—but not yet any sets. The next rank, U_1, consists of the members of U, together with sets corresponding to every possible collection of members of U_0. For example, {Socrates} (the singleton of Socrates), {the Eiffel Tower, Sicily}, the empty set, $\varnothing$, etc. Similarly, U_2 consists of U together with sets corresponding to every possible collection of members of U_1. So U_2 contains (in addition to everything that U_1 contains) $\{\varnothing\}$, and {Socrates, {Sicily}}.

In general, for each natural number n, $U_{n+1} = U \cup \mathcal{P}U_n$.[11] Then, after all these ranks indexed by natural numbers, we have $U_\omega = \bigcup_{n<\omega} U_n$ (that is, U_ω is the union of all the natural number-indexed ranks). Next there is $U_{\omega+1} = U \cup \mathcal{P}U_\omega$, $U_{\omega+2}$, and so on. Our total universe of sets (and urelements) consists of the members of these ranks. Note that if U_α and U_β are ranks, and U_α comes before U_β, then $U_\alpha \subseteq U_\beta$. It is conventional to define the *rank* of a set x as the least α with $x \in U_{\alpha+1}$. For example, the rank of $\varnothing$ is 0, while that of $\{\varnothing\}$ is 1.[12] In general, U_α consists of the urelements, together with all sets that are of rank less than α.

Russell's paradox (i.e. for sets) is blocked by the fact that there is simply no 'Russell set' of all non-self-membered sets. For, on this approach, no set belongs to itself: rather, sets only contain sets that are of lower rank. The Russell set would thus have to contain absolutely every set (since every set is non-self-membered)—including itself, which we have just seen is impossible. The paradox cannot even get started.

This approach to sets has become dominant not simply because of how naturally it is motivated, but also because of how fruitful and convenient it has proved to work with. The universe of sets that it yields has a simple structure that can easily be visualized, and it is often straightforward to see what is true in this. A case in point concerns the axioms of the standard theory of sets, Zermelo-Fraenkel with Choice (or ZFC). It is in general easy to see that these hold on the picture sketched above. For example, the axiom of pairing states that for any objects x and y, there is a set that contains these. And it is clear that this holds on the above picture: the set $\{x, y\}$ will be found at the first rank after the earliest that contains both x and

[11] If A is a set, then $\mathcal{P}A$ is its powerset: the set of its subsets.

[12] So 'rank' is used in two ways: for these increasing sets U_0, U_1, etc., and also (when we talk about the rank *of* a set) for the indices of these sets.

y. It is similarly easy to see that whenever the picture contains a set x, it also contains its union and powerset, and thus that the axioms stating the existence of these hold.[13] Further, when one defines another type of object, in terms of the sets of this universe, then the resulting class is typically well suited for theoretical work. For example, it is standard to define the notion of a model (i.e. interpretation of a formal language) as a species of set. And, under this conception of set, one of afforded a clear understanding of which models exist, and of what their basic properties are.[14]

3 Properties

A common way of thinking of, for example, the proposition that Socrates is pious is as a complex involving Socrates and piety (i.e. the property). However, the notion of a property gives rise to paradoxes: we have already seen that there are paradoxes involving both the notion of a property and that of a proposition; but for one purely involving properties consider the property of being a non-self-applying property. This means that if we are going to construct propositions out of properties, in the manner envisaged, then we are going to need some way of solving the paradoxes they are afflicted by. An approach with the virtue of economy would be to identify properties with sets in some way. For example, we could identify piety with its extension (the set of pious things); or with its intension (i.e. the function from possible worlds to the property's extension in that world, where this function would itself be understood set-theoretically). Our solution to Russell's paradox for sets would then double up as a solution to the corresponding paradox for properties.

The downside of this approach, however, is that it would involve a significant revision to the way that philosophers have found it natural and fruitful to conceive of propositions. For this has been in terms of properties, which, metaphysically speaking, are different from sets in important respects. For example, a melon having the property of being round *causes* it to look a certain way; its looking this way is not caused by the melon belonging to this or that set. Similarly, the gods (famously) love certain acts *because* they have the property of being pious; they surely do not love these acts because of which sets they belong to. Or again: it would seem

[13]The union of a set is the set of its members' members. Indeed, I would argue that the only axiom of ZFC that cannot be read off the picture in this way is that of replacement: for discussion of this axiom, see Potter [2004: 296-98]. Boolos [1971] claims that Choice is also an exception. I disagree, but will not try to make that case here.

[14]For more on the iterative conception and ZFC, see Boolos [1971, 1989], Goldrei [1996] or Kunen [2011].

to be part of Socrates's nature that he has various properties, such as that of being human; but it is not part of this nature that he belongs to any sets (cf. Fine [1994]). Thus, if we can, it would seem preferable to incorporate genuine properties into our framework. In this section I explain how to do this, i.e. how to add properties to the hierarchy of sets of the previous section.

The idea is that properties are arranged into a hierarchy, precisely analogous to that of sets. In the case of sets, this hierarchy was motivated by the thought that the members of a set are prior to that set, which can be articulated as the claim that the existence of the members *grounds* the existence of the set. Stratification is then required to avoid, for example, the apparent impossibility of a fact grounding itself (since if x belonged to itself, we would be committed to x's existence being one of the things that grounds that very fact). Can our hierarchy of properties be motivated similarly? Possibly: one might here draw on Aristotle, who in the *Categories* refers to particular objects as *primary* substances, and writes that these 'underlie' everything else (i.e. including properties). I must confess, however, that this is a way of thinking that I struggle to make complete sense of.

How then to motivate stratification in the case of properties? In fact, I believe that the strongest rationale is the simplicity and elegance of the hierarchy itself. As with the corresponding conception of set, this has the upshot that one will have a clear understanding of which properties there are, and of which features they have. For example, we noted how, in the case of sets, one can easily observe that these satisfy certain foundational principles (such as the axioms of ZFC). Similarly, on the proposed conception of property, one will be able to recognize analogous principles for these (e.g. that for any property of properties F, there is a property that applies exactly to those things that some F-property applies to).

But I also believe that something else can be said here, which brings us closer to the 'priority' motivation in the case of sets. This something else is concerned specifically with properties that have definitions (i.e. what are known as 'real' definitions, because they concern worldly items, rather than linguistic or psychological ones). Thus, suppose that the property G is defined by the formula A:

$$Gx =_{\mathrm{df}} A.$$

In this case, it is plausible that when G applies to an object b, this is *because* A holds of b, i.e.[15]

$$Gb \rightarrow (A(b) < Gb). \tag{D}$$

[15] Here $<$ is the relation of (full) ground.

For example, given that being a bachelor is defined as being an unmarried man, if Frank is a bachelor, then this is because he is an unmarried man.[16] Similarly, it seems plausible that if G does *not* apply to an object, then this is because A does not hold of it.

But now we quickly find ourselves in choppy waters. In particular, we find ourselves committed to a fact explaining itself. For consider, e.g., the property of self-application, defined as follows:

$$Sx =_{\mathrm{df}} x \text{ applies to } x.$$

If S applies to itself, then (by (D)) this is because S applies to itself. But, similarly, if S doesn't apply to itself, then (by the corresponding principle for when a property does not apply) this holds because S does not apply to itself. Either way we have (as in the case of sets) the apparent impossibility of a fact grounding itself.

The natural way of avoiding such problems is to insist that properties are organized into a hierarchy, where a property can only apply to things that emerged at earlier levels. In that case, when we define a property of a particular level, that definition (and the corresponding principles) will hold only of things at lower levels. In particular, on this picture, any property of self-application S will reside at a certain level of the hierarchy, and its definition will apply only to things at lower levels: so $\neg SS$ will hold not because the negation of the definiens holds (i.e. because $\neg SS$ itself holds); rather, S will fail to apply to itself simply because it lives at too high a level of the hierarchy. In this section, then, I extend the standard approach to sets to incorporate properties, where these are stratified in this way.

We do though face a choice about how these stratified properties should be combined with our stratified sets. One option would be to start with a hierarchy of sets (just as in §2), and then to add a hierarchy of properties 'on top'. In that case properties could apply to sets, but could not belong to them. The disadvantage of this, however, is that sets seem to be just as valuable a tool for working with properties, as with any other sort of object. That is, just as we want to have at our disposal sets of people, or natural numbers, for example, so we would like there to be sets of properties. Indeed, for our purposes sets containing properties will be particularly useful, since they will provide a straightforward means of constructing propositions. An alternative option would be to start with the hierarchy of properties, and then to have one of sets after that: so properties could belong to, but could not be had by, sets. But this is surely unsatisfactory: sets have properties as much as any other type of object does! The better course would therefore seem to be the more

[16] For this principle relating real definitions to grounding, see Rosen [2015] and Correia [2017].

egalitarian one of having a single hierarchy of both sets and properties. This will allow sets to contain properties, and also to have them.

We again start with our set U of urelements: objects that are neither sets nor properties. I now use 'W' for our ranks. So the first rank W_0 is U.

The next rank (as before) adds sets corresponding to every possible collection of members of W_0. But it also adds n-ary properties (for n a positive natural number) that apply to the members of W_0. I use S_1 for these sets, and P_1^n for the n-ary properties. So

$$W_1 = U \cup S_1 \cup \bigcup_{0<n<\omega} P_1^n.$$

In general, I will assume that no set is a property, and that if $n \neq m$, then no n-ary property is also an m-ary one.

Exactly which properties does, for example, P_1^1 contain? Unlike sets, properties do not seem to be extensional, i.e. there can be distinct properties that apply to the same things (e.g. being an author of this paper with a heart, vs being an author of this paper with a kidney). However, I will not try to defend, or even state, a completely specific account of which sets correspond to more than one property, and if so how many they correspond to. Different, prima facie viable, accounts would seem to be compatible with the general framework on offer. On the other hand, I will assume that properties are *abundant* in the sense that every set corresponds to at least one property. More precisely, I assume that for every subset A of U (i.e. every $A \in S_1$), there is Q in P_1^1 that corresponds to A, in the sense that Q applies exactly to the members of A. Similarly, I assume that for every set A of ordered pairs of members of U, there is Q in P_1^2 such that: for any $x, y \in U$, Q applies to x, y iff $\langle x, y\rangle \in A$. And so on for ordered n-tuples for $n > 2$ (and also for subsequent ranks).[17]

Our next rank, W_2, then consists of U together with S_2 (sets whose members are all in W_1), and P_2^n (n-ary properties that apply to members of W_1). And so on,

[17] It will perhaps be helpful to have the definition of n-tuples before our minds in what follows, so I give this now. Thus, the ordered pair of x and y is defined:

$$\langle x, y\rangle =_{\mathrm{df}} \{\{x\}, \{x, y\}\}.$$

Then, for $n \geq 2$,

$$\langle x_1, \dots, x_{n+1}\rangle =_{\mathrm{df}} \langle\langle x_1, \dots, x_n\rangle, x_{n+1}\rangle.$$

(The ordered single of x, $\langle x\rangle$, is simply x itself.)

We are, then, using the same notation for propositions as for ordered tuples: something that, given how we will construct propositions, is natural, even if it is strictly speaking an abuse.

up to W_ω, then $W_{\omega+1}$ etc. As before, if x is a set or property, then by the *rank* of x I mean the least α with $x \in W_{\alpha+1}$.

One further piece of notation: we need an analogue of $\in$ but for properties. I will use η for this. So η_n is an $(n+1)$-ary predicate symbol, and $\eta_n Q a_1 \ldots a_n$ is true iff Q is an n-ary property that applies to $a_1, \ldots, a_n$. However, since it is more familiar, I will write $Q a_1 \ldots a_n$ as shorthand for $\eta_n Q a_1 \ldots a_n$.

Russell's paradox for properties is of course solved just as his paradox for sets was: no property (on this approach) applies to itself, and so there can be no property R that applies to every non-self-applying property.

In an appendix I explain how to modify the axioms of ZFC to include this parallel hierarchy of properties, and I sketch the straightforward argument to the effect that this modified theory is consistent (relative to the consistency of ZFC).[18,19]

[18] A natural question at this point: now that we have all these properties, do we still need the sets? One advantage of having sets in addition to properties is that, given that the former (but apparently not the latter) are extensional, we are freed from having to make certain arbitrary choices. For example, if we want to consider a model whose domain consists of Socrates, the Eiffel Tower and Sicily, we can simply take this to be *the* set of these things; we do not have to choose from among the (presumably) many properties that apply to them. Further, since so much work in both mathematics and philosophy has already been developed in terms of sets, it is convenient to be able to avail ourselves of this, without having to reformulate it. Having said that, however, it would be perfectly possible to give a version of the account of propositions to follow purely in terms of properties (and urelements), if one so desired. Thus, when our theory of properties is an addition to a pre-existing one of sets, our axioms for properties can essentially piggyback on those for sets, asserting (roughly) that for every set there is a corresponding property, and vice versa (see §B.2). In the absence of sets this strategy would of course no longer be available. One would instead give axioms directly for properties that are akin to those of ZFC. One would then substitute properties for sets in standard applications: e.g. one would define ordinals as properties, and one would then use those (rather than the set-theoretic ordinals) to describe the hierarchy of properties.

[19] Bealer [1982] suggests an approach to paradoxes for properties inspired by ZFC. The implementation of this basic idea is quite different from the one in this section, however: rather than using ZFC as a guide to which properties there are (as in this section), Bealer uses it as a guide to how the application relation behaves. Thus, Bealer does not solve Russell's paradox for properties by denying that there is a general property of not self-applying; he solves it by rejecting the intuitive account of which things this property applies to, i.e. he denies that it applies to exactly the properties that do not apply to themselves. An advantage of the account proposed here, I believe, is that our universe of properties (and sets and urelements) has a simpler and more straightforward shape.

4 Propositions

When it comes to constructing propositions, we face a choice (similar to that faced in the case of properties): do we identify these with sets of a certain sort, or do we take them to be sui generis entities? In this case, I think that each option has something to be said for it. Identifying propositions with sets has the advantage of economy, and of allowing us to make use of techniques and results that many readers will be familiar with. Further, this option does not seem to clash with standard ways of thinking about propositions (in the way that identifying properties with sets would). For it *is* standard to think of propositions as ordered n-tuples (at least to a first approximation), and the standard way of thinking of those is as sets. On the other hand, taking propositions to be sui generis has the advantage that it would avoid the element of arbitrariness involved in conjuring ordered tuples out of (unordered) sets (and the sense that there is a trick involved). For the purposes of this initial presentation, I believe that the advantages of identifying propositions with sets outweigh the disadvantages. However, the alternative course also seems viable, and much of what I will say would still hold under it.

To keep things manageable, I will focus exclusively on the propositions that are expressed by a standard first-order language (i.e. the sort of formal language that most of us will be most familiar with); the issue of what one should say about the propositions expressed by natural languages will be left for another time (and probably another person!). Given this focus, the natural strategy is to construct propositions that relatively precisely mirror the structure of the sentences of this type of formal language.[20] Of course, the notion of a sentence of such a language is defined via the more general one of a formula, and we will similarly define propositions via an analogous more general notion (that of a 'propositional function').

Thus, as a reminder, a standard way of defining formulas in a first-order language is as follows:

- If F is an n-ary predicate symbol, and $t_1, \ldots, t_n$ are variables or individual constants, then $Ft_1 \ldots t_n$ is a *formula*.

- If A and B are formulas, and x is a variable, then $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$, $\forall xA$ and $\exists xA$ are *formulas*.

[20] However, I will restrict attention to first-order languages that do not contain function symbols (other than individual constants). The proposal could certainly be extended to allow for functions as constituents (i.e. in analogy with function symbols). But this would require us to address an issue that I think would be a distraction here (see Kaplan [1977: 496] and Whittle [2017b: 5032-33]).

Sentences are then defined as formulas without free occurrences of variables.[21]

4.1 The Account Itself

To construct propositional functions (that is, proposition-like objects, but with free variables) I help myself to a disjoint pair of sets.[22] These are the set of two truth values, $\mathsf{Value} = \{t, f\}$, and a countably infinite set of variables, $\mathsf{Var} = \{v_1, v_2, \ldots\}$.[23] I will assume that these are both subsets of U (but nothing really turns on that).

How should we define our propositional connectives, i.e. the constituents of propositions (and propositional functions more generally) that are expressed by $\neg$, $\wedge$ etc.? Since a sentence of the form $\neg A$ is true iff A itself has a certain truth value (i.e. falsity), it is natural to think of the constituent expressed by $\neg$ as a property of truth values: namely, that which applies to a value iff, were A to have it, $\neg A$ would be true. Similarly, we can think of the constituent expressed by $\wedge$ as a relation (i.e. binary property) of truth values. So our propositional connectives are:

- $\neg$ is a unary property that applies exactly to f;
- $\wedge$ is a relation that applies exactly to t, t;
- $\vee$ is a relation that applies exactly to t, t; t, f; and f, t;
- and similarly for $\rightarrow$ and $\leftrightarrow$.

Thus a proposition p that results from combining $\neg$ with some other proposition q will be true if q has the truth value that $\neg$ applies to, i.e. if q is false; otherwise p will be false. Similarly a proposition that results from combining $\wedge$ with q and r will be true iff $\wedge$ applies to the truth values of q and r, that is, iff q and r are true.

What, next, about the constituents corresponding to $\forall$ and $\exists$? If Qx is a quantifier, then whether a sentence QxA is true depends on which set of things A is true of. For example, if Q is $\forall$, then for the sentence to be true, this set must be

[21] If Q is $\forall$ or $\exists$, then the *scope* of an occurrence of Qx in a formula is Qx together with the formula that immediately follows it. An occurrence of a variable x is *free* if it is not within the scope of an occurrence of $\forall x$ or $\exists x$.

[22] So, to be clear, propositional functions (as I use the term) are not in fact *functions* in the usual sense. This use of the term fits with at least some of the things that Russell says in [1908].

[23] On the approach proposed here, both propositional functions and propositions will contain variables. Since structured propositions are intended to mirror the sentences that express them, and the latter contain variables, this seems quite natural to me. Nevertheless, I explain how to give a version of the proposal that eliminates variables in an appendix.

the whole domain. On the other hand, if Q is $\exists$, then the set must merely be non-empty. We can thus take $\forall$ and $\exists$ to express properties of subsets of the domain D that we are quantifying over:[24]

- $\forall_D$ is a property that applies exactly to D
- $\exists_D$ is a property that applies exactly to the non-empty subsets of D.

A *propositional domain* is any set D with $D \cap \mathsf{Value} = D \cap \mathsf{Var} = \varnothing$. Intuitively, propositional domains are the sets that propositions can quantify over, or, more broadly, that they are about.[25] By an n-ary property *on* D I mean such a property that applies only to members of D. If D is a propositional domain, then propositional functions on D are defined as follows.

- If R is an n-ary property on D, and $a_1, \ldots, a_n \in D \cup \mathsf{Var}$, then

  $$\langle R, \langle a_1, \ldots, a_n \rangle \rangle$$

  is a *propositional function* on D.
- If p and q are *propositional functions* on D, and $x \in \mathsf{Var}$, then the following are also *propositional functions* on D:
  - $\langle \neg, p \rangle$
  - $\langle \wedge, \langle p, q \rangle \rangle$
  - similarly with $\vee$, $\rightarrow$ or $\leftrightarrow$ in place of $\wedge$
  - $\langle \langle \forall_D, x \rangle, p \rangle$
  - $\langle \langle \exists_D, x \rangle, p \rangle$

On this account, then, propositional functions are all ordered pairs, where the first member tells us which form of function we are dealing with: i.e. n-ary atomic, negation, conjunction, disjunction, universal etc. Given our assumptions (that no n-ary property is an m-ary one for $n \neq m$, that propositional domains are disjoint from Value, and that no property is a set), it is easy to see that each function is

[24] I discuss alternative possible choices for propositional connectives and quantifiers in note 47.

[25] I insist that these domains are disjoint from Var to avoid the difficulty of distinguishing propositional functions with free variables from propositions *about* variables. And the domains must be disjoint from Value to allow us to distinguish, for example, atomic propositions about propositions from negations.

of exactly one of these forms.[26] Further, we have the desired fine-grained identity conditions within each class. For example, if

$$\langle R, \langle a_1, \ldots, a_n \rangle \rangle = \langle T, \langle b_1, \ldots, b_n \rangle \rangle,$$

then $R = T$, and $a_i = b_i$ for each i (in virtue of the identity conditions for n-tuples). Similarly, if $\langle \wedge, \langle p, q \rangle \rangle = \langle \wedge, \langle p', q' \rangle \rangle$, then $p = p'$ and $q = q'$, and so on.

Further, propositions are defined as propositional functions without any free occurrences of variables.[27] Finally, truth for propositions (and satisfaction for propositional functions more broadly) is defined essentially just as for first-order languages.

4.2 Initial Discussion

On the iterative conception of set, which has served as our starting point and guide, no set belongs to itself. Rather, these are organized into ranks, and a set will only ever contain those of lower rank. And the situation is similar on the iterative conception of properties that we added to this in the previous section. Correspondingly, a distinctive feature of the proposed account of propositions is that no proposition is 'about' itself; rather, a proposition will only be about things of lower rank.[28] The basic idea is that, if D is a propositional domain, then no proposition on D belongs to D. But that is not *quite* right, because if p is a quantifier-free proposition on D, then it will also be a proposition on any domain E that includes D. However, any proposition p will have a *least* domain: i.e. a set D such that, if p is a proposition on E, then $D \subseteq E$. We can think of a proposition's least domain as comprising the things that it is about. And we have: a proposition is always of higher rank than the members of its least domain. This means, in particular, that a proposition that contains a quantifier will never be within the range of that quantifier.[29]

[26] Well, there is actually one possible exception: if the domain D is a singleton, then (for all that we have assumed) $\forall_D = \exists_D$, meaning that the universal functions are identical to the existential ones. This exception is natural: since if there is only one thing, to be true of everything comes to the same thing as merely being true of something. However, one could, if desired, insist that in this case $\forall_D$ and $\exists_D$ are coextensive but distinct properties, eliminating even this exception.

[27] These are defined as in the linguistic case. Thus, if $\langle \langle \forall_D, x \rangle, p \rangle$ occurs in a propositional function, then the *scope* of that occurrence of $\langle \forall_D, x \rangle$ is this whole pair $\langle \langle \forall_D, x \rangle, p \rangle$. (And similarly in the case of $\exists_D$.) An occurrence of a variable is *free* if it is not within the scope of an occurrence of $\langle \forall_D, x \rangle$ or $\langle \exists_D, x \rangle$.

[28] Note that since propositions are ordered pairs, i.e. sets, we have already defined their rank.

[29] If p is a quantified proposition, in virtue of containing either $\forall_D$ or $\exists_D$, then D is the unique (and so in particular the least) domain of p.

The paradoxes of §1 are solved in essentially the same way that Russell's paradox for sets was. Thus, consider the paradox involving propositions of the form $\langle Fa \rangle$ (= $\langle F, a \rangle$, on this account). This resulted from considering a property R that applies to every such proposition where F does not itself apply to $\langle Fa \rangle$. However, every proposition of the form $\langle F, a \rangle$ is such that F does not apply to it (for $\langle Fa \rangle$ is always of greater rank than F, which is in turn of greater rank than anything it applies to). R would thus have to be a property that applies to every unary proposition, including those of the form $\langle Rb \rangle$ for some object b, which is impossible.

The solution of the Russell-Myhill is similar. This, recall, involved propositions of the form

$$\langle \forall p(p \text{ in } A \to p \text{ is true}) \rangle \tag{†}$$

for some collection A of propositions. On our view, the notion of a set explicates that of a collection. Thus, the relevant propositions are those of form (†) but for some *set* A. The paradox resulted from considering the collection R of propositions of this form where the proposition does not itself belong to A. But, on the proposed account, no set will contain a proposition that itself contains that set as a constituent (since any such proposition will be of greater rank than the set, while the set's members will be of lower rank than the set). Our collection (i.e. set) R would thus have to contain *every* proposition of form (†): which is impossible, since this would mean in particular that it contained propositions of the form (†) about R itself.[30]

Fritz's argument, which is a version of the Russell-Myhill paradox, but which starts from principles about ground rather than grain, is blocked similarly. In particular, the Russellian property involved in that argument cannot exist on our ac-

[30] In the interest of full disclosure, I should confess that there is something that this talk of propositions of 'form (†)' is passing over: specifically, that on the proposed account there is no completely general membership relation, and similarly no completely general property of truth. (Just as, in ZFC, there is no set of all ordered pairs $\langle x, y \rangle$ with $x \in y$.) Rather, for any given rank α, there will be a membership relation $\in_\alpha$ that applies to x and y of rank less than α such that x belongs to y. And similarly there is a property true_α that applies to propositions of rank less than α that are true. Thus, a more careful account of the solution of the Russell-Myhill would have to incorporate such indices, but its essential shape would be as described.

A reader might at this point have the concern: if there is no overarching membership relation, how are we to make sense of the claims of set theory itself, for these are made using the single (unsubscripted) predicate symbol $\in$? The answer is that these can be understood not as quantifying over absolutely all sets, or as being about an overarching membership relation, but as being about some specific U_α and $\in_\alpha$: because we can choose α so that talking about U_α is for many purposes interchangeable with talking about the entire universe of sets (see §6).

count of properties.[31] More generally, the propositions of our account would seem to be just as fine-grained as is required by standard accounts of grounding. For example, we always distinguish p from $\langle p \wedge p \rangle$ (= $\langle \wedge, \langle p, p \rangle \rangle$) and $\langle p \vee p \rangle$ (both of these propositions will be of greater rank than p). This means that we can endorse the idea that conjunctions are grounded by their conjuncts, and disjunctions by their (true) disjuncts, without falling foul of the principle that ground is irreflexive.

In summary, on the approach that I have advocated, the whole family of Russellian paradoxes—for sets, properties, propositions or some combination of these—will be solved in essentially the same way, something that would seem fitting, given how similar the paradoxes would themselves seem to be.

Concerning the paradoxes for propositions in particular, we have seen how to give a natural, relatively simple solution to these by straightforwardly modifying the standard solution to Russell's paradox for sets. I have already noted that the propositions that result would seem to be well suited to applications in metaphysics (in particular, in connection with grounding). They would also seem to have just the fine-grained identity conditions sought after by philosophers of language who appeal to structured propositions.[32] The ultimate viability of this approach to propositions, at least as far as the threat of inconsistency is concerned, would thus seem to be secure.

What of the contention (of Uzquiano and Goodman) that the lesson of the Russell-Myhill paradox is that propositions are less fine-grained than structured accounts would have it? Our account, on which propositions are across the board

[31] To be more precise: in saying this I am thinking of the version of Fritz's argument that involves properties (see [2021]). The version of [forthcoming] instead involves pluralities (as in note 10), and is not yet solved on the proposed account simply because I haven't said anything about these. I would suggest that the natural thing to say is that talk of pluralities should, under the proposal, be explicated as talk of sets. This would mean that there is no plurality of all sets, or all propositions, for example. However, there is on the face of it something rather self-defeating, and inelegant, about adding propositions about such pluralities (e.g. about all propositions) 'on top' of those we have already introduced. For we will not then have succeeded in adding propositions about *all* propositions, but only all of those we had previously introduced. A similar point can be made about the possibility of admitting propositions about all sets (even if the issues in that case are a little more delicate). Of course, I do not imagine that these brief remarks will convert a devotee of plural quantification (understood along the lines of Boolos [1984], where there is, for example, a plurality of all sets), but it should at least convey the basic thought that a fuller treatment would develop.

[32] I should note however that when such writers specify the exact structure of propositions, they do not tend to allow these to contain variables. But in appendix A I explain how to give a version of the proposal that does away with variables. Thus, even if one wants to stick to that aspect of the literature, a version of the account is available.

just as fine-grained as is commonly assumed, would seem to challenge that. I would suggest the following analogy as a guide to how we should think about the situation. Consider the notion of a model, specifically for a first-order language, say. The intuitive version of this notion is subject to a Russell-style paradox. To give this, let us restrict attention to the language $\mathcal{L}$ whose only non-logical vocabulary is the unary predicate symbol F: in the following, by a 'model' I mean an interpretation of $\mathcal{L}$. Consider now the model R whose domain is the collection of all models, and in which F applies to precisely those models M such that, under M, F does not apply to M. In the by now familiar way, we get that F applies to R, under R, iff it doesn't.

Is the moral of this paradox that model-theorists are wrong to assume (or to work with a definition on which) models are fine-grained? Should they rather countenance the possibility that there are distinct collections A and B, such that some model that interprets F by A is identical to one in which it is interpreted by B? I would suggest not. A better description of the situation seems rather to be this. The probably most natural consistent notion of a model—i.e. that in terms of sets, which one finds in textbooks—*is* fine-grained. One need only open such a textbook in model theory to see how user-friendly this notion is, and how it has aided the development of the subject. Of course, the notion has its limitations. For example, there is no model in which the domain is the entire universe of sets, or even just the totality of all models. These limitations are significant, and certainly can't just be shrugged off. On the contrary, it is plausibly of interest to see if one can develop alternative notions that avoid them. And perhaps one thing to try is giving up on fineness of grain in certain problematic cases. Nevertheless, for the majority of applications, the standard, set-theoretic notion is what one should use.

I want to suggest that the situation with our proposed account of propositions is analogous. I hope that it will prove to be similarly user-friendly, and for similar reasons, i.e. the simplicity and naturalness of the iterative framework it is based on. In particular, as in previous cases, this framework seems to make it easy to tell what is true of propositions. For example, given any propositions p and q, it is clear that there will exist another that results from combining these with any given propositional connective. Or again, if p quantifies over domain D, and E is a superset of D, then we can see that there is a proposition that results from expanding the quantifiers in p to E. Further, we have seen that propositions have very straightforward identity conditions on this account (conditions which are easy to verify). I have no doubt that it is also of interest to develop alternative conceptions of proposition, including those that give up fineness of grain in certain cases. But I believe that for many applications, the account proposed here will best serve one's purposes.

I have emphasized that the account of properties and propositions offered here is inspired by, and analogous to, the iterative conception of set, and the theory that codifies it, ZFC. There is a point related to this that must be clarified, however. For in standard presentations of ZFC, there is assumed to be a single membership relation (referred to by $\in$), and the axioms are assumed to quantify over the entire universe of sets. On our approach, in contrast, properties always have a specific rank, and apply only to things of lower rank. This means that there cannot be properties that apply to sets of unbounded rank: such as a single membership relation, or quantifiers (which are a certain sort of property) that range over absolutely all sets. Does this mean that our approach isn't analogous to ZFC after all? Well, the point is that what is proposed here is an approach to properties that is analogous to the account *of sets* in ZFC (and the iterative picture it is based on). When one combines this picture of sets with properties that apply to sets of any rank (such as a general membership relation), one's picture of properties does not resemble that of sets (at least not in the most straightforward way). Consequently, one would seem to need some quite different solution to forms of Russell's paradox for properties (or propositions); whereas we have solved all versions of this paradox in really just the same way.[33]

5 A Single Idea

Thus, a distinctive feature of the overall view being proposed, and one that would seem to be a significant virtue, is that it is based on a single idea, applied to both sets and properties: i.e. the idea that these are stratified into the simple hierarchies described above, where sets can only contain things at lower levels, and properties can only apply to such.

As I noted in the introduction, people sometimes write as if Russell's ramified theory of types is the only consistent account of structured propositions currently available. However, this is seen as unattractive, because it is complicated, and perceived in some ways to be ad hoc. The picture proposed here is much simpler than Russell's, but I believe that there is a way in which it sheds light on that much maligned theory. For, as we have seen, the proposed account of propositions (and propositional functions more generally) is in fact 'ramified' in the sense that propositions are organized into ranks, and a proposition can only quantify over those of lower rank. (A departure from Russell's theory is the fact that our hierarchy is *cu-*

[33] See §6 for a discussion of how theorizing about sets can proceed even in the absence of a general membership relation or quantification over all sets.

mulative: later ranks include earlier ones. This is one reason that it is simpler.) However, the fact that our account is ramified in this sense does not require any special legislation: rather, it flows straightforwardly from the basic stratification, i.e. the idea that sets can only contain lower ranked things, and properties can only apply to such. For once quantifiers are treated not as sui generis constituents of propositions, but are instead identified with certain properties in a natural way, the ramified character of our hierarchy falls out of this basic restriction on properties.

This is striking, because an influential criticism of Russell's theory, offered by both Ramsey [1925: 187] and Gödel [1944: 455], sees it as embodying two quite independent sorts of restriction. Firstly, there is the restriction on application (which Ramsey and Gödel see as well motivated): i.e. propositional functions can only apply to things at lower levels of the hierarchy. And, secondly, there is the restriction on quantification (which Ramsey and Gödel are less sympathetic to): i.e. propositional functions can only quantify over things at lower levels of the hierarchy. On our approach, however, we reveal the (corresponding) restrictions on quantification to flow from the (corresponding) restriction on application. That is, we have a theory with the same broad features as Russell's, but where these are seen to be derived from a single idea.[34]

[34]There is an additional point that is worth making here. Probably the most criticized, even mocked, element of Russell's theory is the axiom of reducibility. This states, very roughly, that any propositional function is coextensive with one that is quantifier-free. This means, in effect, that the restrictions on quantification (in that theory) can in many cases be ignored: since even if a propositional function is at a high level of the hierarchy, in virtue of its quantifiers, there will be an equivalent at a lower level. This axiom was needed to enable various forms of mathematical reasoning, hampered by the restrictions on quantification, but it is seen by many as being ad hoc. It is thus worth noting that the analogue of the axiom of reducibility in our framework in fact holds, apparently as a result of entirely natural, and intrinsically motivated choices. The relevant principle says essentially that every propositional function is coextensive with a property. That is, since we can regard propositional functions as 'structured' properties, the principle says that every structured property is coextensive with an unstructured one. This follows (from standard principles about sets together with) our assumption that every set of n-tuples corresponds to an n-ary property (see §3). From this perspective, then, the axiom of reducibility can be seen as asserting simply that unstructured properties are abundant. I would suggest that one could mount a similar defence of the original axiom, in the context of Russell's theory, but that would require a much more in-depth discussion.

6 Limitations

I must now, alas, come to a proper discussion of the limitations of our proposal. The most serious of these stem from the fact that no property can apply to things of unbounded rank: rather, any property will itself have a certain definite rank, and is then only able to apply to things of rank lower than that. This means, for example, that there is no general property of truth that applies to true propositions of any rank. This is, without question, a real cost. It seems intuitively obvious that there is something—indeed, something natural and important—that all true propositions have in common. But—not so!—our account is forced to counter. Instead, on this account, for any rank α, there will be a property true$_\alpha$ that applies to all true propositions of rank less than α. This means that, given any set of propositions, there will be a property that applies exactly to the true members of the set. For example, as long as any language has only set-many sentences, then any (interpreted) language $\mathcal{L}$ will be such that there is a property that applies exactly to the true propositions expressed by sentences of $\mathcal{L}$. Indeed, given even any (possibly infinite) *set* X of languages, there will be a property that applies to every true proposition expressed by a sentence of one of the members of X. However, while we have a family of increasingly far-reaching properties of truth, we do not have one that 'reaches all the way around' and encompasses even propositions that contain that very property. For many purposes, e.g., when we are interested in a specific language, we can find a property that will serve our purposes. But the fact that we cannot have a single property of truth is a fundamental limitation.

And of course there is nothing special in this regard about truth: we similarly cannot have a property that applies to all propositions (propositions are not, on this view, a unified class!), nor can we have one that applies to all properties or all sets; nor can we respect the idea that there is a relation that the members of each set bear to that set (i.e. that there is such a thing as *the* membership relation!).

Since quantification is, for us, understood in terms of properties, this basic limitation also has implications for that. For example, just as there is no property that applies to all propositions, so there are no propositions that quantify over all propositions; just as there are none that quantify over all properties or all sets; or, of course, a fortiori, all objects of any stripe whatsoever. So we cannot, on this view, express claims such as:

(1) every proposition is true or false

(2) some truth is unknown

(3) every property has an extension

(4) every set has a powerset

In the case of some of these, a work around of sorts is available. Specifically, we can employ 'schematic generality' (see Russell [1908]). The idea is that we can assert *something like* (1) not by asserting a single proposition (that quantifies over all propositions), but rather by expressing a 'schema',

(1S) p is true or false.

So here we are committing ourselves to every proposition of the form $\langle p$ is true or false$\rangle$, for some proposition p. In (1S), 'p' is not a standard variable, but a 'schematic' one.[35] George Boolos [1998: 34] 'three quarters seriously' suggests that when someone writes such a schema we should take them to have produced infinitely many sentences, just with 'irredeemably bad handwriting'. There are though limits to how far this gambit will take us. For it seems that it will only allow us to produce surrogates of universal claims. Thus, even spotting ourselves the ability to write infinitely many sentences will not allow us to mimic the effect of (2), for example: since in this case we are not, essentially, saying that infinitely many claims of a certain form are true, but rather that *some* such claim is.[36]

The proposal of this paper, then, has real limitations. In defence of this approach, I will try to make two points. Firstly, these limitations are very similar to those faced by the standard approach to sets—limitations that have not stopped that from being very widely accepted! Secondly, I will compare the limitations of our proposal with those of higher-order logical frameworks (which have been the

[35] In fact, I am simplifying somewhat, since 'true' and 'false' must also be understood schematically here. A more careful schematic formulation of (1) would rather use a schematic variable α over ranks. Thus,

every$_\alpha$ proposition$_\alpha$ is true$_\alpha$ or false$_\alpha$.

Here 'every$_\alpha$' is a universal quantifier ranging over everything of rank less than α, while the other subscripted words denote the rank-restricted properties we have already met.

[36] Although (3) and (4) are universal-existential, and thus might seem to be similarly out of reach of this tactic, in virtue of their existential component, in these cases a purely universal surrogate would seem to be available (i.e. one whose only schematic variable is understood universally):

(3S) for every$_\alpha$ property$_\alpha$ x there is some$_\alpha$ y such that y is the extension$_\alpha$ of x

(4S) for every$_\alpha$ set$_\alpha$ x there is some$_{\alpha+1}$ y such that y is the powerset$_{\alpha+1}$ of x.

Still, this sort of trick won't work for every universal-existential, or of course more generally, by any means.

setting for interesting recent work on paradoxes for propositions). I will argue that in certain important respects, the restrictions imposed by that framework are more severe than those imposed by our account.

First, then: limitations of the iterative conception of set, or ZFC. The notion of a set is intended to be the formal analogue of the notion of a collection. And, intuitively, just as there is a collection of all prime numbers, and one of all planets, so there is one of all things whatsoever—but ZFC doesn't deliver such a set. For any rank α, there is a set of all the things that are urelements, or sets of rank less than α, i.e. U_α, but there is no set of everything. Similarly, ZFC allows a natural definition of a function, i.e. a set X of ordered pairs with the property that $y = z$ whenever $\langle x, y\rangle$ and $\langle x, z\rangle$ are both in X. This definition delivers the existence of many familiar functions, such as addition, multiplication or exponentiation on natural numbers: that is, the relevant sets of ordered pairs exist, according to ZFC. However, many of the functions that set theory is most concerned with, such as the generalizations of these arithmetical operations to transfinite ordinals or cardinals, do not exist according to this definition: the relevant sets of ordered pairs would have to contain members of unbounded rank, which is impossible. Again, increasingly extensive 'initial segments' of these functions exist, according to the set-theoretic definition, but the complete functions, whose domains are all of the ordinals, or all of the cardinals, do not. Finally, while, according to the set-theoretic definition of a model, for any rank α, there are models which have U_α as their domain, there is no model whose domain is the totality of all sets, or all models (as we noted above).

However, while these limitations are significant, they have not of course prevented this solution to Russell's paradox from gaining very wide acceptance. To describe it as the gold standard for theories of sets is probably an understatement. As I have emphasized (§2), this is in part because of the naturalness of the basic idea (the iterative conception of set), and in part because of how convenient the resulting universe has proved to work in. The virtues and vices of our proposed account of propositions would seem to be very similar—and I advocate for a similarly wide acceptance in this case (!).

I want now to consider higher-order logic. This framework is constituted by a hierarchy that is in certain respects similar to those of sets and properties that we have argued for. In this framework, one has, roughly, objects, then properties of objects, then properties of properties of objects, then properties of those, and so on.[37] But a key difference between this and the framework proposed here is that the higher-order approach is usually taken to be non-cumulative: thus, the second level of this hierarchy contains *only* properties of objects, it does not also contain the objects themselves; similarly, the third level of the hierarchy contains *only* new properties, it does not also contain those of the previous level. That is, the properties at the third level of this hierarchy (= second-level properties) only apply to first-level properties. There are no properties that apply both to objects and first-level properties, for example.

This feature is natural given standard ways of thinking about higher-order logic. On one such way of thinking, the motivation for strictly demarcating quantification into predicate position from quantification into name position is that if we substitute F for a in Fa (here a is a singular term, and F is predicate of objects), then the result is nonsense.[38] But then, when it comes to quantifying into the position of a predicate of object-predicates, we can hardly include in the range of this that of quantifiers into object-predicate position: for if we did that, instances of such generalizations would involve object-predicates applied to themselves. Thus, the different levels of our hierarchy must apparently be mutually exclusive. Another approach to higher-order logic sees its quantifiers as ranging over functions: thus, properties of objects are thought of as functions from objects to (unstructured) propositions of some sort.[39] On this approach, an expression of the form bc will make sense only if b denotes a function which includes the denotation of c within its domain. But then we cannot include objects within the range of quantifiers over properties of objects, on pain of such generalizations having instances that don't make sense. Again, then, it seems that the levels of our hierarchy must be disjoint.

Such non-cumulative approaches carry a cost, however. To try to bring this out, let us first return to the framework proposed in this paper, and consider sets: in particular, consider those sets that have exactly two members. Intuitively, it is obvious that all such sets have something in common. That is, it seems obvious that there is a property that applies to all such sets (and to nothing else). Unfortunately, the approach offered here fails to supply such a property. For every rank

[37] For higher-order logic in general, see Gamut [1991], Carpenter [1997] or Williamson [2003]. For discussions of the Russell-Myhill paradox within this framework, see Dorr [2016], Goodman [2017] or Fritz [2021]. The focus in this section is on the higher-order approach to properties: specifically, the way that these are stratified. In contrast, propositions are not usually stratified on this approach, with the consequence that they must be more coarse-grained than on structured accounts (see, e.g., Goodman [2017]).

[38] See, for example, Prior [1971: ch. 3].

[39] See Fritz [2021].

α, there is a property pair-set$_\alpha$ that applies to such sets of rank less than α; but there is no property that applies to *all* such pair-sets. Similarly, it seems obvious that every property that applies to exactly two things has something in common. Again, though, this judgement is not vindicated by our approach: there is a family of increasing properties of the form pair-property$_\alpha$, but none that apply to *all* pair-properties.

On a non-cumulative approach, however, there seems to be a significantly greater discrepancy between our intuitive judgements, and what the approach can vindicate. For on such an approach, even though it seems obvious that properties (i.e. of any level) that apply to exactly two things have something in common, there is not even a property that applies to all first- and second-level pair-properties, for example. But surely a first-level property that applies to exactly two objects has something in common with a second-level one that applies to exactly two first-level properties! Having to reject as illusory such intuitively correct judgements seems worse than simply failing to deliver an absolutely general property applying to all pair-properties, i.e. wherever they come in the hierarchy.

For another example, consider universal and existential quantifiers. On the higher-order approach (as in this paper), these are thought of as properties. For example, quantifiers over objects are second-level properties, quantifiers over first-level properties are third-level properties, and so on. Of course, the universal quantifier over objects $\forall_0$ is the dual of the existential one $\exists_0$, and vice versa: that is, each can be defined from the other, with the help of negation, in the familiar way. Similarly, the universal quantifier over first-level properties, $\forall_1$, is in the same way the dual of $\exists_1$. On the face of it, then, there is something that these pairs of quantifiers have in common! That is, there is a relation that each universal quantifier stands in to the existential one. After all, if one studies higher-order logic, one is surely missing something if one fails to appreciate that $\forall_1$ stands to $\exists_1$ as $\forall_0$ stands to $\exists_0$. However, any such relation is ruled out on a non-cumulative approach. Again, it is not that the proposal of this paper gives us everything that we might hope for: in particular, we do not have a relation that applies to every pair $\forall_D$, $\exists_D$, for any set D whatsoever. But we certainly do have a dual$_2$ relation, that applies to any property of rank 0 or 1 and its dual. More generally, for any set of properties of the form $\forall_D$, for example, we have a relation dual$_\alpha$ that will function in the desired way for all of these properties.[40]

[40] I have discussed non-cumulative versions of higher-order logic, but what about cumulative versions (as in Linnebo and Rayo [2012], for example)? The relation of such a theory to the framework proposed here would, to a great extent, depend on whether it is extensional. Suppose first that

Here is another way to think about the contrast between cumulative and non-cumulative approaches. It is an idea familiar from set theory that, even if one cannot quantify over absolutely all sets (as on the proposal of this paper), nevertheless, if one ascends far enough up the iterative hierarchy, one will reach a vantage point that enables something that is practically indistinguishable from this. That is, for a given application, one can usually choose α so that quantifying over U_α will allow one to theorize essentially just as if one was quantifying over absolutely all sets. For example, one can choose α so that, when one's quantifiers range over U_α, all of the axioms of ZFC will be true (i.e. if α is 'strongly inaccessible').[41] This means that, to a great extent, theorizing about the members of U_α resembles theorizing about absolutely all sets. Relatedly, while there is no model whose domain consists of the entire universe of sets, a model with this U_α as its domain is in many ways functionally equivalent to such a model. Thus, for many applications, the purpose that would be served by having a model whose domain contains all sets is served by the model based on U_α. And, again, while there is (for example) no addition function whose domain encompasses absolutely every ordinal, there *is* such a function whose domain encompasses all of the ordinals in U_α, and for many purposes this will be just as good.

Similarly, under the proposal of this paper, while quantification over absolutely all propositions is prohibited, theorizing *as if* one is doing this is often possible. For example, it is similarly plausible that one can find an α such that, when quantifying over W_α, all of the modified axioms of ZFC (i.e. which incorporate properties: see appendix B) will come out as true. The upshot is that for most linguistic or metaphysical applications, it is essentially business as usual. But nothing comparable can be said in mitigation of the red tape imposed by non-cumulative frameworks such as that of higher-order logic. For in such cases the issue is not merely that one cannot quantify over the whole hierarchy (or more generally that there are no

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the cumulative theory *is* extensional (as in Linnebo and Rayo [2012]). In this case, when it comes to giving an account of structured propositions, the theory would seem to be limited in essentially the same way that set theory is (see §3). For, in metaphysical terms, extensional higher-order entities would seem to be just as different from properties as sets are. On the other hand, if the higher-order cumulative theory is not extensional, then its entities could more plausibly be counted as genuine properties. In that case, the theory would be similar to the version of the present proposal with properties but no sets (see note 18). In particular, the absence of sets (or, at least, the absence of sets of higher-order entities) would lead to the same sort of inconveniences as in that case (note 18).

[41] There is also a general 'reflection' theorem to the effect that for any sentence S of the language of set theory, there is α such that S is true when interpreted as quantifying over absolutely all sets iff it is true when interpreted as quantifying over U_α. See theorem II.5.3 of Kunen [2011] for the result about set theory without urelements, but it also holds once urelements are permitted.

properties that stretch all the way up the hierarchy); it is that, even when one focuses merely on the first few levels, for example, the properties that one intuitively expects to have are nowhere to be found.

To summarize this section: yes, our approach has failings, but these are shared by something really popular; also, it could be worse.

7 Other Approaches

In this final section I compare the proposed account of propositions to two others that have been given.[42] We begin with the account of Fritz [2021]. The basic idea of this is to identify a structured proposition with a property that applies exactly to its constituents. For example, the proposition $\langle Fa \rangle$ would be identified with a relation that applies to F, a (and nothing else). This delivers propositions with fine-grained identity conditions. However, the non-cumulative character of the background hierarchy leads to what seem to me to be rather severe restrictions. For example, on this account, $\langle p \wedge q \rangle$ is (essentially) a 3-place property that applies exactly to $p, \wedge, q$.[43] Similarly, $\langle (p \wedge q) \wedge r \rangle$ is a 3-place property that holds of $\langle p \wedge q \rangle$, $\wedge$ and r—i.e. holds of the property just mentioned, and $\wedge$ and r. But this means that this latter proposition lives above $\langle p \wedge q \rangle$ in the hierarchy. And on a non-

[42] There are of course more approaches to the Russell-Myhill paradox than I can discuss in any depth here (see note 3). For example, in addition to the accounts considered in the text, there are proposals within the framework of Alonzo Church's Logic of Sense and Denotation, such as those of C. Anthony Anderson [1987] and Sean Walsh [2016]. On Anderson's approach, all talk of propositions must be relativized to a language: there is no general notion of a proposition, only that of a proposition$_{\mathcal{L}}$ (for a given language $\mathcal{L}$). However, at least from certain perspectives (e.g. those of metaphysical applications), one might hope for an account of propositions that is not tied to language in this way. Walsh's proposal is predicative in the sense that a formula defining a certain sort of higher-order entity cannot quantify over entities of that sort. The general project, which motivates this restriction, is a treatment of higher-order quantifiers that encompass only those that 'fall within our referential ken' [2016: 296]. Again, though, for at least some applications (especially in metaphysics) one might want an account of propositions that is not tied to our cognitive abilities. A very different approach, closer in spirit to that of the present work, has been put forward by Giorgio Sbardolini [2021]. This is based on the modal set theory of Linnebo [2013], and readers sympathetic to that may find the analogous treatment of propositions congenial. On the other hand, if one believes that the theoretical benefits of adding modal operators to our set theory can be attained more straightforwardly using only the resources of ZFC, then one might prefer the approach of this paper.

[43] I am ignoring Schönfinkeling (i.e. the practice of turning n-ary properties, for $n > 1$, which on this approach are thought of as functions, into *unary* functions).

cumulative approach, *that* means that no property or relation can apply to both of these propositions.

To see the cost of this, consider grounding (one of the applications that Fritz proposes). As he explains, no single grounding relation can apply to both $\langle p \wedge q \rangle$ and $\langle (p \wedge q) \wedge r \rangle$ (i.e. in the same argument position). This has the consequence that we cannot respect the judgement that $\langle p \wedge q \rangle$ holds in virtue of p and q, and $\langle (p \wedge q) \wedge r \rangle$ holds in virtue of its conjuncts *in the same sense*. That is, $\langle p \wedge q \rangle$ stands in one explanatory relation to its conjuncts, while $\langle (p \wedge q) \wedge r \rangle$ stands in another to its. From the perspective of the literature on grounding, and the intuitive judgements that lie behind it, this seems very revisionary. It is one of the main thrusts of this literature that there is a unified phenomenon (i.e. grounding!) which occurs in many different philosophical contexts. On this account, however, if one starts with p, q and r, and closes under conjunction—then there isn't even a single relation that captures the explanatory connections among these propositions.

Now, I certainly don't want to claim that the theory of this paper gives the proponent of grounding everything that they want. But it does seem to be significantly less revisionary. Thus, we cannot of course provide a single unified grounding relation, i.e. one that can apply to propositions of any rank whatsoever. Rather, for any grounding relation $<$, we will need a distinct relation to apply to propositions that themselves contain the original relation $<$. In general, as in other cases, we will have a grounding relation $<_\alpha$ that works as desired for propositions of rank less than α, for any rank α. This means that if we start with a set of propositions X, and then close under the truth-functional connectives, then there certainly will be a single grounding relation that will work as desired (i.e. as is assumed in the literature on grounding) for every member of the resulting set. Similarly, if we consider any interpreted language $\mathcal{L}$, or even any set of such languages, then there will again be a single relation that works for every proposition expressed by a sentence of a member of the set. This seems much closer to what the proponent of grounding hoped for.

A quite different approach is that of Deutsch [2014]. This solves the Russell-Myhill paradox by appealing to the distinction between sets and proper classes, or more precisely to the idea behind this: namely, that while sets can be members of either sets or proper classes, proper classes can be members of neither. So, similarly, the idea is that some propositions cannot be members of either sort of class (i.e. sets or proper classes). This certainly blocks the original version of paradox. I would argue, however, that the problem with this move is that it leads to unacceptable verdicts in the cases of other versions. For example, consider the simple Russellian paradox for propositions, involving those of the form $\langle Fa \rangle$ (§1). If we pursued Deutsch's strategy in this case, then we would end up with the result that some propositions are ineligible, not to belong to classes, but to have properties applying to them. But can we make sense of the idea that there are propositions that do not have any properties—propositions without qualities, as it were?[44] I should stress that Deutsch does not himself propose extending his solution from paradoxes for propositions involving classes to those involving properties. But if we don't so extend it, then we would seem to be solving very similar paradoxes in dissimilar ways. This would be in sharp contrast to the unified approach proposed here, one whose virtues I hope to have gone some way towards convincing the reader of.[45]

A Eliminating Variables

On the account proposed above, propositions contain variables. This seemed natural: we want propositions to mirror the structures of sentences. The latter contain variables. So shouldn't the former? I note however that when philosophers of language have expressed views on the nature of quantified propositions, they have conspicuously refrained from pursuing this option. Instead, they have done things in terms of proposition-valued functions, as follows.[46]

The basic idea is that $\langle \forall x F x \rangle$ is the pair of $\forall_D$ and the function that sends an object a to the proposition $\langle F a \rangle$. It is not clear to me that there are good reasons for wanting to avoid variables in this fashion. Nevertheless, in this appendix I explain how to eliminate variables from the proposed account, if desired.

We define a mapping from our propositional functions to variable-free propositions, relative to assignment, as follows. Thus, by an *assignment* on a propositional domain D I mean a function from Var into D. Given such an assignment σ on D, we essentially extend σ to a mapping from propositional functions (on D) into variable-free propositions, in the following way. (Here, if $a \in D$, then σa is simply a itself; and $\sigma(x/a)$ is the assignment that sends x to a and is otherwise just like σ.)

- $\langle R, \langle a_1, \ldots, a_n \rangle \rangle^\sigma = \langle R, \langle \sigma a_1, \ldots, \sigma a_n \rangle \rangle$

[44] Uzquiano [2015] makes a related point, but about pluralities rather than properties.

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[46] See Salmon [1986: appendix C], Soames [1987] and [2003: 101–6], and Pickel [2017].

- $\langle \neg, p\rangle^\sigma = \langle \neg, p^\sigma\rangle$
- $\langle \wedge, \langle p, q\rangle\rangle^\sigma = \langle \wedge, \langle p^\sigma, q^\sigma\rangle\rangle$ (similarly for $\vee$, $\rightarrow$ and $\leftrightarrow$)
- $\langle\langle \forall_D, x\rangle, p\rangle^\sigma = \langle \forall_D, g\rangle$, where g is the function that sends $a \in D$ to $p^{\sigma(x/a)}$
- $\langle\langle \exists_D, x\rangle, p\rangle^\sigma = \langle \exists_D, g\rangle$ (same g).

Of course, if p is a proposition, then $p^\sigma = p^\tau$ for any assignments σ and τ. Eliminating variables in this way leaves the main features of the account, for example, its solutions to paradoxes for propositions, essentially unaffected.[47]

B Axioms and Consistency

In this appendix I extend ZFC to incorporate properties, as in the account of §3, and then sketch an argument for the relative consistency of the resulting theory.

B.1 Language

We work in a first-order language with equality, whose non-logical predicate symbols are:

Unary: Set, Ind, Prop_n
Binary: $\in$
$(n+1)$-ary: η_n.

[47] I should note that sometimes writers make different choices about propositional connectives and quantifiers which, on the sort of iterative approach pursued here, would not work. For example, sometimes propositional connectives are thought of not as properties of truth values, but as properties of propositions. On an iterative approach, that would have the consequence that negation, for example, could not apply to propositions that contain it (since no property can apply to propositions that contain that property). Similarly, propositional quantifiers are sometimes thought of as properties, not of subsets of the domain, but of proposition-valued functions (such as g above). But this (on an iterative approach) would rule out propositions in which a single quantifier occurs twice, one occurrence within the scope of the other (since a property cannot apply to functions that have within their range propositions that contain that property). I do not believe that this is a great cost of our approach, however, since the choices of propositional connectives and quantifiers that *are* available to us seem perfectly natural. Of course, this treatment of the standard propositional connectives cannot to be extended to non-truth-functional ones, e.g. connectives for notions of grounding. On the proposed approach, these *would* be treated a properties of propositions, and so could not apply to propositions that themselves contain the notion of grounding in question. This certainly is a limitation, but as we have already seen, essentially, there is much to be said by way of mitigation (§§6 and 7).

The intended interpretation of Ind is to apply to urelements. As mentioned in §3, I abbreviate $\eta_n Q a_1 \dots a_n$ as $Q a_1 \dots a_n$. Further I will use s to range over sets. Thus, $\forall s F s$, for example, is an abbreviation of

$$\forall x(\text{Set } x \to Fx)$$

(for some variable x). Similarly, I use P_n to range over over n-place properties.[48]

B.2 Axioms

Basics

(i) $x \in y \to \text{Set } y$

(ii) $\eta_n x y_1 \dots y_n \to \text{Prop}_n x$

(iii) $\neg(Fx \wedge Gx)$, where F and G are distinct unary predicate symbols

Sets. Our axioms for these are simply those of ZFC. Or, more carefully, they are the standard axioms of ZFC, modified in the usual way to allow for urelements.[49] So, for example, the axiom of extensionality does not say that any objects with the same members are identical, but simply that any *sets* with the same members are.

Properties. Our axioms for properties assert a correspondence between these and sets. That is, every set (of the relevant sort) corresponds to a property, and vice versa. For ease of reading, I use abbreviations rather than writing these out in gory detail.

(i) x is a set of n-tuples $\to \exists P_n \forall y_1 \dots y_n (P_n y_1 \dots y_n \leftrightarrow \langle y_1, \dots, y_n \rangle \in x)$

(ii) $\exists x[x$ is a set of n-tuples $\wedge \forall y_1 \dots y_n (P_n y_1 \dots y_n \leftrightarrow \langle y_1, \dots, y_n \rangle \in x)]$

[48] In place of separate predicate symbols Prop_n for each n, we might use a binary predicate that applies to an n-ary property and the natural number n. This would be more expressive but less straightforward, it seems to me.

[49] Thus, in the main text, I used ZFC for the version of this theory that allows for urelements, but in fact the standard version restricts attention to pure sets. For the (standard) axioms of ZFC, see, e.g., Kunen [2011: 16–17].

B.3 Consistency

I now sketch an argument for the claim that if ZFC with urelements is consistent, then so is our extended theory. The natural way to do this is to work within ZFC with urelements, and construct a proper class model W of our theory. It is not that we are really assuming the existence of proper classes, however. Rather, talk of proper classes is, in the manner familiar from texts on set theory, officially to be understood as talk about formulas that correspond to these classes. Further, for simplicity I assume that there are infinitely many urelements (this assumption isn't essential). Thus, let u_1, u_2,...be a countable infinity of urelements. The domain of our model is the union of the following sets, which follow the notation of §3.[50]

$$\begin{aligned}
W_0 &= U - \{u_1, u_2, \dots\} \\
S_1 &= \mathcal{P} W_0 \\
P_1^n &= \mathcal{P}(W_0^n) \times \{u_n\} \\
W_1 &= W_0 \cup S_1 \cup \bigcup_{0<n<\omega} P_1^n \\
S_2 &= \mathcal{P} W_1 \\
&\vdots
\end{aligned}$$

Our logical vocabulary is then interpreted in the obvious way. Thus, $\text{Ind}^W = W_0$; Set^W is the union of the sets S_α for an ordinal α; and Prop_n^W is the union of the sets P_α^n. The membership symbol $\in$ holds of an object x and a member A of Set^W iff $x \in A$. Finally, η_n holds of a member Q of Prop_n^W and $a_1, \dots, a_n$ iff $\langle a_1, \dots, a_n \rangle$ belongs to the first member of Q. It is then relatively routine to verify that each of the axioms of our theory holds in this model.

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[50] Here W_0^n is the set of ordered n-tuples of members of W_0. And if A and B are sets, then $A \times B$ is the set of ordered pairs $\langle a, b \rangle$ with $a \in A$ and $b \in B$.

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