# Epistemic Infinite Regress and the Limits of Metaphysical Knowledge

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**Abstract**

I will explore the paradoxical nature of epistemic access. By critiquing the traditional conception of mental states that are labelled as 'knowledge', I demonstrate the susceptibility of these states to an infinite regress, thus, challenging their existence and validity. I scrutinise the assumption that an epistemic agent can have complete epistemic access to all facts about a given object while simultaneously being ignorant of certain truths that impact the very knowledge claims about the object. I further analyse the implications of this paradox when attempting to fine-grain the parameters of epistemic access, which happens to reveal the vacuity of such definitions and the inevitability of the regress.

## 1 Introduction

I will address a specific paradox that arises when considering the scope of an individual's knowledge concerning a particular object or set of objects. I posit that the traditional understanding of knowledge as a mental state is flawed due to its vulnerability to an infinite regress of justification. This paradox will also be explored across various epistemological scenarios, including multi-agent systems, to demonstrate the fragility of our knowledge constructs. My analysis is to be in favour of the thesis of epistemic eliminativism.

## 2 The Paradox of Epistemic Access

### 2.1 The Initial Argument

Let $j$ be an epistemic agent with access to a set of facts $\{P_1, P_2, \ldots, P_n\}$ about a corresponding set of objects $\{\Omega_1, \Omega_2, \ldots, \Omega_n\}$, with the exception of facts $\{P_{n+1}, P_{n+2}, \ldots, P_{n+n}\}$ that are beyond $j$'s epistemic reach.

### 2.2 The Epistemic Challenge

Assume an object $\Omega$, such that $j$ has epistemic access to all facts about $\Omega$. However, consider a fact $P_{n+n}$ not within $j$'s knowledge base but is relevant to $\Omega$, such that:

$$(a.)\;\; P_{n+n} \text{ is true and } j \text{ knows some fact } P \text{ about } \Omega \tag{1}$$

and $(a.)$ is outside of $j$'s knowledge. Therefore, $j$ does not have epistemic access to every fact about $\Omega$.

### 2.3 Refinement of Parameters

An attempt to refine the parameters leads to the definition:

$$(a'.) \quad j \text{ has epistemic access to every truth known to be true about } \Omega \tag{2}$$

This definition sidesteps the paradox but is vacuous. It will be demonstrated that if we attempt to reduce the statement down to the facts that render said statement, the paradox will ensue. A reduction of $(a'.)$, denoted as $(a')_2$, posits that $j$ knows every fact about a 'conceptual portion' of $\Omega$, which is labelled as $\Omega_{\frac{n}{m}}$.

### 2.4 The Paradox Revisited

However, we encounter:

$$(b.) \quad P_{n+n} \text{ is true and } j \text{ knows some fact } P' \text{ about } \Omega_{\frac{n}{m}} \tag{3}$$

$$(c.) \quad P_{n+n} \text{ is true and } j \text{ knows some fact } P'' \text{ about } \Omega_{\frac{n\pm1}{m\pm1}} \tag{4}$$

$$(...) \tag{5}$$

If a true statement that is unknown to $j$ can undermine $j$'s epistemic access to every fact about $\Omega$, it can similarly undermine access to every fact about any incomplete part $(\mu)$ of $\Omega$, and an incomplete part $(\mu')$ of $\mu$, and then $\mu''$ of $\mu'$ of $\mu'''$ of $\mu''$ of $\mu'$ of $\mu$ of $\Omega$; $\mu''''$ of $\mu'''$ of $\mu''$ of $\mu'$ of $\Omega$ ... $\mu_n$ of $\mu_{n-1}$ ... of $\mu''''$ of $\mu'''$ of $\mu''$ of $\mu'$ of $\Omega$. So, it is perpetually ad infinitum.

## 3 The Multi-Agent Epistemic Paradox

### 3.1 Theorem: Persistence of the Epistemic Paradox in Multi-Agent Frameworks

Let $\mathscr{A} = \{a_1, a_2, \ldots, a_k\}$ be a set of epistemic agents, and let $\mathscr{K}$ represent the collective knowledge base of the system. For any given object $\Omega$, and a set of facts $\mathscr{F} = \{P_1, P_2, \ldots, P_n\}$ about $\Omega$, there exists a fact $P_{n+1}$ not within $\mathscr{K}$, but relevant to $\Omega$, such that the collective epistemic access to every fact about $\Omega$ is incomplete.

### 3.2 Proof

Construct a Kripke model $M = (W, R, V)$ where:

- $W$ is a set of possible worlds representing all possible states of knowledge.
- $R$ is a binary relation on $W$ representing the accessibility relation, which is reflexive and transitive.
- $V$ is a valuation function assigning truth values to each proposition in each world.

Let $\Box$ represent the modal operator for 'necessarily' and $\Diamond$ for 'possibly'. We define $\Box_a P$ to mean 'agent $a$ knows $P$', and $\Diamond_a P$ to mean 'it is possible for agent $a$ to know $P$'.

Assume that:

$$\forall a \in \mathscr{A}, \forall P \in \mathscr{F}, \Box_a P \tag{6}$$

meaning that every agent knows all the facts in $\mathscr{F}$.

However, for $P_{n+1}$:

$$\exists w \in W, \neg V(w, P_{n+1}) \tag{7}$$

meaning there is some world where $P_{n+1}$ is not known.

Since $R$ is reflexive and transitive, we have:

$$\forall a \in \mathscr{A}, \Diamond_a \neg P_{n+1} \tag{8}$$

indicating that it is possible for each agent to not know $P_{n+1}$.

Given the common knowledge operator $C_{\mathscr{A}}$, which represents what is common knowledge among all agents in $\mathscr{A}$, we can express:

$$C_{\mathscr{A}}(\bigwedge_{P \in \mathscr{F}} P) \wedge \Diamond_{\mathscr{A}} \neg P_{n+1} \tag{9}$$

This states that while all facts in $\mathscr{F}$ are common knowledge, there is a possibility that $P_{n+1}$ is not known by the collective.

To demonstrate the paradox, we show that for any subset $\mathscr{F}' \subseteq \mathscr{F}$ and any agent $a$ in $\mathscr{A}$:

$$\Box_a (\bigwedge_{P \in \mathscr{F}'} P) \to \Diamond_a \neg P_{n+1} \tag{10}$$

This implies that no matter how comprehensive the subset of known facts $\mathscr{F}'$ is, there is always the possibility that an agent $a$ does not know $P_{n+1}$, which is relevant to $\Omega$. This potential ignorance persists even if $\mathscr{F}'$ is equivalent to $\mathscr{F}$, the full set of known facts about $\Omega$.

The paradox arises because the existence of $P_{n+1}$, unknown to $a$, undermines the claim that $a$ has complete epistemic access to $\Omega$. This is true for each agent in $\mathscr{A}$, and by extension, for the collective knowledge base $\mathscr{K}$.

Furthermore, if we consider the possibility of new facts emerging over time, the paradox deepens. Let $\mathscr{F}_t$ represent the set of all known facts at time $t$. Then:

$$\forall t, \exists P_{t+1} \notin \mathscr{F}_t \text{ and relevant to } \Omega \tag{11}$$

This suggests that at any given moment, there could be a fact not yet discovered or known that is pertinent to $\Omega$, thus perpetuating the cycle of incomplete epistemic access.

# 4 Model-Theoretic Implications of the Epistemic Paradox

## 4.1 Theorem: Incompleteness of Epistemic Models

Given any epistemic model $M$ and a set of agents $\mathscr{A}$, there exists at least one fact $P$ about an object $\Omega$ such that $P$ is not contained within the collective knowledge base $\mathscr{K}$ of $\mathscr{A}$, regardless of the completeness of $M$.

## 4.2 Proof

Let $M = (W, R, V)$ be an epistemic model where:

- $W$ is a non-empty set of possible worlds.

- $R$ is an accessibility relation on $W$ that is Euclidean, serial, and transitive.
- $V$ is a valuation function mapping each world $w \in W$ to a set of propositions.

The collective knowledge operator $C_{\mathscr{A}}$ is to be defined as follows:

$$C_{\mathscr{A}} P \Leftrightarrow \forall a \in \mathscr{A}, \Box_a P \tag{12}$$

where $\Box_a P$ means that agent $a$ knows $P$.

Assume for contradiction that:

$$\forall P, C_{\mathscr{A}} P \text{ or } C_{\mathscr{A}} \neg P \tag{13}$$

This would imply that for every proposition $P$, it is either common knowledge among $\mathscr{A}$ that $P$ is true, or it is common knowledge that $P$ is false.

However, consider the proposition $P_\Omega$ which states 'There exists a fact about $\Omega$ that is not known by any agent in $\mathscr{A}$'. If $C_{\mathscr{A}} P_\Omega$ were true, then it would be common knowledge that there is an unknown fact about $\Omega$, which contradicts the assumption of completeness. On the other hand, if $C_{\mathscr{A}} \neg P_\Omega$ were true, then there would be no unknown facts about $\Omega$, which is the very fact that is supposedly unknown.

Therefore, we must conclude that:

$$\exists P_\Omega, \neg C_{\mathscr{A}} P_\Omega \wedge \neg C_{\mathscr{A}} \neg P_\Omega \tag{14}$$

So, it has thus been demonstrated that there is at least one fact about $\Omega$ that is neither common knowledge to be true nor common knowledge to be false among the agents in $\mathscr{A}$, thus proving the incompleteness of the epistemic model $M$.

# 5 Dynamic Epistemic Logic and the Epistemic Paradox

Dynamic epistemic logic extends classical epistemic logic by introducing actions and events that can change the state of knowledge. One of the fundamental actions in DEL is the public announcement, which can be used to model the dissemination of information to all agents in a system.

## 5.1 Public Announcements and Knowledge Update

Consider a public announcement action $\alpha$ that announces a fact $P$. The effect of $\alpha$ on the knowledge state of an agent $a$ is represented by the update operator $[!\alpha]$, such that:

$$[!\alpha] \Box_a P \text{ iff } \Box_a P \text{ after the announcement of } \alpha \tag{15}$$

## 5.2 Modeling the Paradox with Public Announcements

Let's construct a scenario where a fact $P_{n+1}$ is publicly announced to the set of agents $\mathscr{A}$. Before the announcement, the fact is not part of the collective knowledge base $\mathscr{K}$, but after the announcement, it becomes common knowledge.

### 5.3 The Paradox in the Context of DEL

The paradox manifests in DEL as follows:

1. Before the announcement of $P_{n+1}$, it is not known by any agent:

$$\forall a \in \mathscr{A}, \neg \Box_a P_{n+1} \tag{16}$$

2. After the announcement, $P_{n+1}$ becomes common knowledge:

$$[!\alpha] C_{\mathscr{A}} P_{n+1} \tag{17}$$

3. However, the announcement of $P_{n+1}$ reveals the existence of a new fact $P_{n+2}$, previously unknown and relevant to $\Omega$:

$$[!\alpha] \Diamond_{\mathscr{A}} \neg P_{n+2} \tag{18}$$

### 5.4 The Infinite Regress in DEL

The infinite regress arises because each public announcement of a new fact $P_{n+k}$ leads to the acknowledgment of yet another unknown fact $P_{n+k+1}$:

$$\forall k \geq 1, [!\alpha_k] \Diamond_{\mathscr{A}} \neg P_{n+k+1} \tag{19}$$

where $\alpha_k$ is the announcement of $P_{n+k}$.

## 6 Semantic Ramifications of the Epistemic Paradox

### 6.1 Theorem: Semantic Instability in Epistemic Modal Logic

Given an epistemic modal logic system $\mathscr{L}$ with a set of agents $\mathscr{A}$, the semantics of $\mathscr{L}$ are inherently unstable due to the epistemic paradox, leading to a perpetual state of semantic indeterminacy.

### 6.2 Proof

Consider the standard semantic apparatus of epistemic modal logic, where:

- A model $M = (W, R, V)$ is defined, with $W$ as a set of possible worlds, $R$ as an accessibility relation, and $V$ as a valuation function.
- The truth of a proposition $P$ at a world $w$ is given by $V(w, P)$.
- The knowledge of an agent $a$ is represented by the modal operator $\Box_a$, with $\Box_a P$ meaning 'agent $a$ knows $P$'.

The epistemic paradox introduces a proposition $P_\Omega$ such that:

$$\forall w \in W, \exists P_\Omega \text{ that is neither true nor false in } V(w, P_\Omega) \tag{20}$$

This implies that for any world $w$, there is a fact about $\Omega$ that is semantically indeterminate.

As a result, the knowledge operator $\Box_a$ cannot be applied to $P_\Omega$ in a meaningful way, since:

$$\neg(\Box_a P_\Omega \vee \Box_a \neg P_\Omega) \tag{21}$$

This indicates that agent $a$ cannot know $P_\Omega$ nor its negation, leading to semantic instability.

### 6.3 Semantic Indeterminacy and Infinite Regress

The semantic indeterminacy of $P_\Omega$ leads to an infinite regress of semantic instability. For every proposition $P_{n+k}$ known by $a$, there exists a $P_{n+k+1}$ that is semantically indeterminate:

$$\forall k \geq 1, \exists P_{n+k+1}, \neg V(w, P_{n+k+1}) \text{ for some } w \in W \tag{22}$$

This perpetuates the cycle of incomplete knowledge and semantic instability within the system $\mathscr{L}$.

## 7 The Epistemic Paradox and the Knowledge Generalisation Rule

The knowledge generalisation rule in epistemic logic, typically denoted as $\Box$, states that if an agent knows a proposition, then the agent knows that they know this proposition. Formally, if $\Box_a P$ is true, then $\Box_a \Box_a P$ should also be true. However, the epistemic paradox challenges this rule by introducing scenarios where an agent's knowledge may be incomplete or uncertain.

### 7.1 Formalisation of the Knowledge Generalisation Rule

The knowledge generalisation rule can be formalised as follows:

$$\forall a \in \mathscr{A}, \forall P \in \mathscr{P}, (\Box_a P \to \Box_a \Box_a P) \tag{23}$$

where $\mathscr{A}$ is the set of all agents and $\mathscr{P}$ is the set of all propositions.

### 7.2 Challenges Posed by the Epistemic Paradox

The epistemic paradox suggests that for any known proposition $P$, there may exist another proposition $Q$ that is relevant to $P$ but is not known by the agent. This undermines the certainty required by the knowledge generalisation rule, as the agent cannot be said to know that they know $P$ if $Q$ is unknown,

$$\exists a \in \mathscr{A}, \exists P \in \mathscr{P}, \exists Q \in \mathscr{P}, (\Box_a P \wedge \neg \Box_a Q \wedge Rel(P, Q)) \tag{24}$$

where $Rel(P, Q)$ denotes a relevance relation between propositions $P$ and $Q$.

### 7.3 Implications for Modal Possible World Kripke Frames

In the context of Kripke frames for modal logic, the paradox implies that an agent's knowledge in a world $w$ may be incomplete due to the existence of inaccessible worlds containing relevant truths. This challenges the reflexivity of the accessibility relation required by the knowledge generalisation rule,

$$\exists w \in W, \exists u \in W, (w, u) \notin R \wedge (\exists P, V(u, P) \wedge Rel(V(w), P)) \tag{25}$$

where $R$ is the accessibility relation and $V$ is the valuation function.

### 7.4 Revisiting the Knowledge Generalisation Rule

The knowledge generalisation rule may need to be reconsidered. One approach is to introduce a level of uncertainty into the rule, allowing for the possibility that an agent may not know all the implications of their knowledge,

$$\forall a \in \mathscr{A}, \forall P \in \mathscr{P}, (\Box_a P \to \Diamond_a \Box_a P) \tag{26}$$

where $\Diamond_a$ denotes the possibility operator, indicating that it is possible for the agent to know that they know $P$.

## 8 The Paradox and Common Knowledge Logic

In this section, we explore the epistemic paradox within the framework of common knowledge logic. We introduce a formal system $\mathscr{L}_{CK}$ to analyze the structure of common knowledge in the presence of the paradox.

### 8.1 Formal System $\mathscr{L}_{CK}$

Let $\mathscr{L}_{CK}$ be a formal language of common knowledge logic defined over a set of propositions $\mathscr{P}$, a set of agents $\mathscr{A}$, and the following symbols:

- Logical connectives: $\wedge$, $\vee$, $\neg$, $\to$
- Modal operators: $\Box_a$ for each $a \in \mathscr{A}$, and $C_{\mathscr{A}}$
- A countably infinite set of variables: $x_1, x_2, \ldots$
- Quantifiers: $\forall$, $\exists$
- Auxiliary symbols: (, ), ,

### 8.2 Axioms of $\mathscr{L}_{CK}$

The system $\mathscr{L}_{CK}$ includes axioms for propositional logic, axioms for modal logic $K$ for each agent, and the following axioms specific to common knowledge:

$$\begin{aligned}
CK1: &\quad C_{\mathscr{A}} P \to \Box_a P, \forall a \in \mathscr{A} \\
CK2: &\quad C_{\mathscr{A}} P \to C_{\mathscr{A}} C_{\mathscr{A}} P \\
CK3: &\quad \Box_a (P \to Q) \to (\Box_a P \to \Box_a Q) \forall a \in \mathscr{A} \\
CK4: &\quad \neg C_{\mathscr{A}} \bot \\
CK5: &\quad (C_{\mathscr{A}} P \wedge C_{\mathscr{A}} (P \to Q)) \to C_{\mathscr{A}} Q
\end{aligned}$$

### 8.3 Semantics of $\mathscr{L}_{CK}$

The semantics of $\mathscr{L}_{CK}$ are given by a Kripke structure $M = (W, R, V)$ where:

- $W$ is a non-empty set of possible worlds.
- $R$ is a set of accessibility relations $R_a$ for each $a \in \mathscr{A}$, where each $R_a$ is an equivalence relation.
- $V$ is a valuation function mapping each world $w \in W$ to a set of propositions.

## 8.4 The Paradox in $\mathscr{L}_{CK}$

We introduce a proposition $P_\Omega$ representing an unknown fact about an object $\Omega$. The paradox arises when considering the common knowledge of $P_\Omega$:

$$C_{\mathscr{A}} P_\Omega \to \bigwedge_{a \in \mathscr{A}} \Box_a P_\Omega \tag{27}$$

However, if $P_\Omega$ is truly unknown, then it cannot be the case that:

$$\bigwedge_{a \in \mathscr{A}} \Box_a P_\Omega \tag{28}$$

This leads to a contradiction with the axiom $CK1$.

## 8.5 Resolving the Paradox

To resolve the paradox, we may need to revise the axioms or the semantics of $\mathscr{L}_{CK}$. One approach is to introduce a new operator $\Box^*_{\mathscr{A}}$ that captures the collective knowledge with the possibility of unknown facts:

$$\Box^*_{\mathscr{A}} P \Leftrightarrow \bigwedge_{a \in \mathscr{A}} \Box_a P \wedge \exists Q (Q \text{ is unknown and } Rel(P, Q)) \tag{29}$$

This operator allows for the acknowledgment of collective knowledge while admitting the possibility of unknown relevant propositions.

## 8.6 The Paradox Revisited

The introduction of the operator $\Box^*_{\mathscr{A}}$ was an attempt to reconcile the paradox within the framework of $\mathscr{L}_{CK}$. However, this operator inadvertently leads to a new paradoxical situation. The operator suggests that for every known fact $P$, there exists a potentially unknown fact $Q$ that is relevant to $P$. This undermines the very foundation of common knowledge, as the collective knowledge base $\mathscr{K}$ can never be complete.

## 8.7 The Failure of $\Box^*_{\mathscr{A}}$

The operator $\Box^*_{\mathscr{A}}$ fails to resolve the paradox due to the following reasons:

- It introduces a new level of uncertainty that is not accounted for in the original axioms of $\mathscr{L}_{CK}$.
- It violates the closure properties of knowledge, as it allows for the existence of unknown facts that are relevant to known facts.
- It leads to a recursive problem where for every $\Box^*_{\mathscr{A}} P$, there must exist a $\Box^*_{\mathscr{A}} Q$ that is unknown, ad infinitum.

## 8.8 The Inherent Limitations of $\mathscr{L}_{CK}$

The system $\mathscr{L}_{CK}$ is inherently limited in its ability to encapsulate the epistemic paradox. The axioms and semantics of $\mathscr{L}_{CK}$ presuppose a level of completeness and certainty that is contradicted by the paradox. The paradox demonstrates that there will always be unknown facts that escape the grasp of common knowledge, rendering the system incomplete.

### 8.9 The Unresolvable Nature of the Paradox

The epistemic paradox remains unresolvable within the confines of $\mathscr{L}_{CK}$ due to the following:

$$\forall P \in \mathscr{P}, \exists Q \in \mathscr{P}, (\Box^*_{\mathscr{A}} P \wedge \neg \Box^*_{\mathscr{A}} Q \wedge Rel(P, Q)) \tag{30}$$

This formula states that for every proposition $P$ that is collectively known, there exists a related proposition $Q$ that is not collectively known. This relation $Rel(P, Q)$ ensures the perpetuation of the paradox.

## 9 Epistemic Closure and the Paradox

The principle of epistemic closure holds that if a subject $S$ knows $p$, and $p$ entails $q$, then $S$ should also know $q$, provided that $S$ recognizes the entailment. Formally, this can be expressed as:

$$K_S(p) \wedge (p \to q) \to K_S(q) \tag{31}$$

where $K_S(p)$ denotes that subject $S$ knows proposition $p$.

### 9.1 Formalisation of the Paradox

The paradox arises when we consider a proposition $p$ that entails its own non-knowledge. Let $p$ be the proposition "Proposition $q$ is not known by $S$", which can be formalised as:

$$p \equiv \neg K_S(q) \tag{32}$$

If $S$ knows $p$, then by epistemic closure, $S$ should also know $\neg K_S(q)$, which leads to a contradiction since $S$ cannot know the proposition that is defined as not being known by $S$.

### 9.2 Challenges to Epistemic Closure

The paradox challenges the principle of epistemic closure by demonstrating that there exist propositions that, if known, invalidate the closure principle. This can be represented as:

$$K_S(p) \wedge (p \equiv \neg K_S(q)) \nrightarrow K_S(\neg K_S(q)) \tag{33}$$

### 9.3 Implications for Epistemic Logic

The implications of this paradox for epistemic logic are significant. It suggests that the principle of epistemic closure may not hold universally, especially in cases where self-referential propositions are involved. To address this, we might consider revising the standard models of epistemic logic to accommodate exceptions to closure.

### 9.4 Revised Principle of Epistemic Closure

A possible revision to the principle of epistemic closure to account for the paradox could be to introduce a conditional that excludes self-referential knowledge claims:

$$K_S(p) \wedge (p \to q) \wedge \neg(p \equiv \neg K_S(q)) \to K_S(q) \tag{34}$$

This revised principle maintains closure for propositions that do not entail their own non-knowledge, thus avoiding the paradoxical situation.

### 9.5 Continuation of the Paradox Despite the Revised Principle

The revised principle of epistemic closure attempts to circumvent the paradox by excluding self-referential knowledge claims. However, the paradox persists due to the nature of epistemic conditions that involve potential self-reference or circularity. The revised principle states:

$$K_S(p) \wedge (p \to q) \wedge \neg(p \equiv \neg K_S(q)) \to K_S(q) \tag{35}$$

Yet, consider a proposition $r$ that is equivalent to $p$ but is not explicitly self-referential. Let $r$ be such that:

$$r \equiv (p \wedge \neg K_S(q)) \tag{36}$$

If $S$ knows $r$, then by the revised principle, $S$ should also know $q$, provided $r$ does not explicitly entail its own non-knowledge. However, since $r$ is equivalent to $p$, knowing $r$ implies knowing $p$, which leads to the same contradiction as before:

$$K_S(r) \wedge (r \equiv (p \wedge \neg K_S(q))) \nrightarrow K_S(q) \tag{37}$$

Thus, even with the revised principle, the paradox continues to manifest through propositions that are equivalent to self-referential claims without being explicitly self-referential.

## Conclusion

It has become evident that the traditional conceptualisation of knowledge, characterised by the possibility of complete epistemic access, is fundamentally flawed. The paradox of epistemic access, as demonstrated through various epistemological frameworks, reveals an inherent vulnerability to an infinite regress of unknown truths. This regress not only challenges the possibility of attaining complete knowledge but also calls into question the validity of what we term as 'knowledge'.

The attempts to refine the parameters of epistemic access, whether through semantic adjustments in epistemic modal logic or through the introduction of multi-agent systems, have been shown to be futile. Each strategy, rather than resolving the paradox, only serves to further illuminate the depth of the problem, leading to an endless cycle of semantic indeterminacy and epistemic uncertainty.

Therefore, we must conclude that the conventional understanding of knowledge acquisition is in error.

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