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On the Mutual Definability of the Notions of Entailment, Rejection, and Inconsistency

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Abstract: In this paper, two axiomatic theories $\boldsymbol{T}^{-}$ and $\boldsymbol{T}'$ are constructed, which are dual to Tarski's theory $\boldsymbol{T}^{+}$ (1930) of deductive systems based on classical propositional calculus. While in Tarski's theory $\boldsymbol{T}^{+}$ the primitive notion is the classical consequence function (entailment) Cn^{+}, in the dual theory $\boldsymbol{T}^{-}$ it is replaced by the notion of Słupecki's rejection consequence Cn^{-} and in the dual theory $\boldsymbol{T}'$ it is replaced by the notion of the family *Incons* of inconsistent sets. The author has proved that the theories $\boldsymbol{T}^{+}$, $\boldsymbol{T}^{-}$, and $\boldsymbol{T}'$ are equivalent.

Keywords: deductive system; entailment; rejection; inconsistency; Tarski's consequence theories; rejection theory; inconsistency theory; equivalence of theories

1. Introduction

The results of metamathematical research by Tarski from the 1930s have determined the main development line of modern logic. Thanks to Tarski's papers dated from the 1930s, we have a contemporary understanding of metamathematics and metalogic as a separate and formalizable deductive science. The axioms formulated by Tarski began the axiomatizaton of part of the theory of deductive systems. We mean here the following axioms:

a. the general consequence theory (see Tarski [1]),
b. the so-called richer theory of deductive systems (see Tarski [2]).

The notions of consequence (inferential entailment), rejection, and inconsistency (resp. consistency) can be included into important syntactic concepts of metamathematics, *i.e.*, the theory of deductive, logical, and mathematical systems, in particular axiomatic systems. Basic syntactic concepts are: the notion of a *meaningful sentence*, a sentence (a *well-formed expression*, in short: a *wfe*) and the concept of *consequence* (*entailment*) for *wfes*, and, thus, the concept of a *proof* and a *thesis*, so every deductive science is a set of meaningful sentences (*wfes*), theses that are the consequences of a set of sentences.

The *set S of all wfes* is precisely defined in every specific deductive system. Then it is conceived as the smallest set including a set of simple expressions (of the vocabulary of its language) and closed under the syntactic rules (operations) of constructing composed expressions from simpler ones.

If A is any set of *wfes* of S of the system, then by means of its postulated *inference rules* of the set R (operations) we can derive (deduce) new sentences called *consequences* of the set A. The precise definition of set $Cn(A)$ of *all consequences of the set A* should be given in every specific deductive system; the schema of a definition of $Cn(A)$ can be formulated as follows (see Tarski [3], chapters III and V, and *cf.* Wójcicki [4], Czelakowski [5]): $Cn(A)$ is the intersection of all sets that contain the set A and are closed under the given inference rules of R, *i.e.*, the set is the smallest set including the set A and closed with respect to inference rules in R.

If A^+ is the set of axioms of a deductive system and R is the set of its inference rules, then it is useful to accept the relativized notion of the consequence with respect to A^+ and R:

$$Cn_{A^+,R}(A) = Cn(A^+ \cup A).$$

If $A = \varnothing$ then the set of consequences $Cn(A^+)$ is called the *set of theses of the system*. An expression e of this system is its *thesis* if and only if e has a proof on the basis of the axioms of A^+ and inference rules of R (*cf.* [4–7]).

One of the most important requirements that we put on deductive systems is their consistency (non-contradiction). The notion of consistency and, in opposition – the notion of inconsistency (contradiction) – may be defined in two ways:

(*) A deductive system is *consistent* if and only if not every its sentential expression is its thesis (*i.e.*, there are sentential expressions of this system that are not its theses).

A deductive system is *inconsistent* if and only if its every *wfe* is its thesis.

(**) A deductive system is *consistent* if and only if no two contradictory sentential expressions are at the same time its theses (*i.e.*, no sentence such that itself and its negation are its theses).

A deductive system is *inconsistent* if and only if a sentential expression of the system and its negation are theses of the system (*i.e.*, some two contradictory sentences of the system belong to its theses).

The notions defined in (**) can be introduced only in deductive systems with a negation connective that is at least intuitionistic (see: Tarski [1,3]).

The notion of *rejection* was introduced into metalogic by Łukasiewicz [8–11], in relation to the axiomatic method for characterizing some deductive systems, both as asserted systems (by acceptation, systems in the usual sense) and as refutation, rejection systems (where some sentential expressions are rejected on the basis of sentences previously rejected, false, or not accepted in these systems).

The research into refutation systems for well-known asserted logic systems, introduced by Łukasiewicz [9–11] and his pupil Jerzy Słupecki, was continued in Słupecki's research circle and beyond. Significant results were achieved by Tomasz Skura. His very important results in this direction are given in his book [12]. The notion of *rejection* can be understood as a notion complementary to the *consequence* notion.

If A^- is the set of *rejection axioms* of a deductive system and R^- is the set of its *rejection rules*, then the set $Cn^-(A^-)$ is called the *set of rejected sentences of the system*. An expression e of the set is a rejected sentence of the systems if and only if e has a rejection proof on the basis of rejection axioms of A^-, theses, and rejection rules of R^-.

Łukasiewicz used the rejection rule: if an implication is a thesis of the system and its antecedent is rejected then its consequent is rejected too.

For any set A of *wfes* of a deductive systems formalized on two levels, we intuitively assume that the *consequence operation Cn* leads from true, asserted *wfes* of the set A to true, accepted expressions of $Cn(A)$ in the systems, while *rejection operation* Cn^- leads from false, non-accepted *wefs* of the set A^- to false, non-accepted expressions of the set $Cn^-(A^-)$.

Łukasiewicz's research, in which he used the notion of rejection, was aimed to define a system that would be *decidable* in the sense that "we could decide whether any particular expression of this system has to be asserted or rejected" (*cf.* [10], chapter 4.). (Let us note that Rudolf Carnap pursued a similar goal in relation to classical propositional calculus; *cf.* Carnap [13], §28 and Carnap [14], §20; discussion of Łukasiewicz's and Carnap's approach can be found in Citkin [15]). The aim was connected with proving the theorem that rejected sentential expressions of a deductive system are only its expressions that are not its theses. He used the notion of a *consistent system* differently from its use above. The system is *consistent* (or *inconsistent*) if none of its theses is rejected (or its every expression is rejected).

General notions of consequence, rejection, and consistency (inconsistency) can be formalized in the general theory $\boldsymbol{T}$ of deductive systems built by Alfred Tarski [1], containing an axiomatic characterization of its two primitive notions: the set S of all *wfes* of language of an arbitrary but fixed deductive system and the consequence operation Cn^+ ($Cn = Cn^+$) for such a system, by enriching $\boldsymbol{T}$ by Słupecki's definition of the general notions of the rejection operation Cn^- [16], and the definition of the family *Incons* of all *inconsistent* sets. In this theory, some basic, general properties of the consequence operation Cn^+, the rejection operation Cn^-, and the notion of *Incons* (*Consist*) are described. The notions Cn^- and *Incons* are defined in the theory $\boldsymbol{T}$ and in its extension $\boldsymbol{T}^+$ presented by Tarski [2,3,17,18], by means of the consequence operation Cn^+. In $\boldsymbol{T}^+$, next to primitive terms S and Cn^+, there are two new primitive terms: the symbols c and n, which are metalogical names of functors of implication and negation, which in the intended interpretation are the connectives of the classical sentential calculus; the theory $\boldsymbol{T}^+$ contains an axiomatic characterization of the notion of consequence today known as the classic consequence. It is the theory of deductive systems based on the classical sentential calculus.

Definability of the notions of consequences (entailment and rejection), *i.e.*, operations Cn^+ and Cn^- by means of the notion of inconsistence (the family *Incons* of inconsistent sets) can be shown in the theory $\boldsymbol{T}'$ of inconsistence, in which primitive notions are the set S and the family *Incons* and the defined notions: consequences Cn^+ and Cn^-.

Definability of the notions of consequence (entailment) and inconsistence, *i.e.*, the operation Cn^+ and the family *Incons* by means of the notion of rejection (the operation Cn^-), can be shown in theory $\boldsymbol{T}^-$ of rejection, in which primitive notions are set S and the operation Cn^- and the defined notions are the notions of consequence Cn^+ and inconsistency *Incons*.

In Sections 2–4, we present the axiom systems of these three theories: $\boldsymbol{T}^+$, $\boldsymbol{T}^-$ and $\boldsymbol{T}'$ (The theories $\boldsymbol{T}^+$ and $\boldsymbol{T}^-$ were presented or sketched earlier in [19–21]; see also axiom systems in [7,22]. The axioms of the theory $\boldsymbol{T}'$ were discussed in [23].). In Section 5, we prove that they are equivalent, leading to a finding that notions consequence operation Cn^+, rejection operation Cn^-, and the family *Incons* are mutually definable, which is the purpose of this paper. The mutual definability of those notions is essential to the methodology of deductive sciences, since it proves that the properties of some main syntactic notions of deductive systems based on the classical logic can be described on the basis of other assumptions, without diminishing the range of those properties.

Let us recall that two theories *T1* and *T2* are equivalent if all theorems and definitions of *T1* are theorems or definitions of *T2* and conversely. For the proof of equivalencies of axiomatic theories *T1* and *T2*, it is sufficient to prove that all axioms and definitions of *T1* are theorems or definitions of *T2*, and conversely all axioms and definitions of *T2* are theorems or definitions of *T1*.

The theoretical research described in this paper is the continuation and development of the metamathematical research, initiated by renowned representatives of the Lvov–Warsaw School: Łukasiewicz, Tarski, and Słupecki. One can attempt to apply them to the research on deductive systems based on non-classical logic; however, that is not the purpose of this paper.

2. Tarski's Theories of Deductive Systems; Consequence Theories $\boldsymbol{T}$ and $\boldsymbol{T}^+$

Tarski's so-called *general theory of deductive systems*, or *general consequence theory*, denoted by $\boldsymbol{T}$, is based on the following primitive notions (see [1,17,18]):

* the set S *of all sentential expressions* (*sentences*) of an arbitrary, but fixed language of a deductive system, and
* the *consequence operation* Cn^+ ($Cn^+ = Cn$) defined on the family $P(S)$ of all subsets of the set S, *i.e.*, the operation $Cn^+: P(S) \to P(S)$, which to any set X of sentences assigns the set Cn^+X of all its *consequences*, *i.e.*, all sentences that sentences of X *entail*, that are sentences deducible from sentences of X.

In the axioms of Tarski's theories, the variables $x, y, z, \ldots$ are assumed to run over sentences of set S, while the variables $X, Y, Z, \ldots$ represent a subset of set S.

2.1. Axioms, Definitions, and Theorems within Theory T

The original axiom system for the general consequence theory $\boldsymbol{T}$ (see Tarski [1]) is the following:

A1. $0 < \mathrm{card}(S) \leqslant \aleph_0$ — denumerability of S,
A2. $X \subseteq Cn^+X \subseteq S$ — reflexivity of Cn^+,
A3. $Cn^+Cn^+X = Cn^+X$ — the consequence Cn^+ is idempotent,
A4#. $Cn^+X = \bigcup \{Cn^+Y \mid Y \subseteq X \;\&\; \mathrm{card}(Y) < \aleph_0\}$ — the consequence Cn^+ is finitistic.

The axiom A4# can be replaced by the following pair of formulas that are equivalent to it:

A4. $X \subseteq Y \Rightarrow Cn^+X \subseteq Cn^+Y$ — the consequence Cn^+ is monotonic,
A5. $Cn^+X \subseteq \bigcup \{Cn^+Y \mid Y \subseteq X \;\&\; \mathrm{card}(Y) < \aleph_0\}$.

For the theory $\boldsymbol{T}$ we most frequently use the axiom system consisting of axioms A1–A5.

We can enrich theory $\boldsymbol{T}$ by definitions of basic syntactic notions, in particular the notions of interest to us: *inconsistency* (the family *Incons* of inconsistent sets) and *rejection* (the rejection operation Cn^- corresponding to the notion of rejection introduced by Łukasiewicz).

D1a. $X \in Cons \Leftrightarrow Cn^+X \neq S$,
D1. $X \in Incons \Leftrightarrow Cn^+X = S$,
D2. $Cn^-X = \{y \mid \exists x \in X\, (x \in Cn^+\{y\})\}$.

The formulas on the left-hand side of the above definitions may be read as follows: X is a *consistent set*, X is an *inconsistent set*, and Cn^-X is the *set of all sentences rejected* on the basis of the sentences of set X.

In accordance with D1, a set (a system) is *inconsistent* if and only if all sentences of S are its consequences. It follows from D2 that a sentence y is *rejected* on the basis of set X if and only if there exists at least one sentence in X that is a consequence of y (that sentence y entails).

The *rejection function* Cn^- was defined by Słupecki in [16] within the general consequence theory $\boldsymbol{T}$. Słupecki proved that it satisfies axioms A2–A5 of the general consequence theory $\boldsymbol{T}$, *i.e.*, the following theorems in $\boldsymbol{T}$ enriched by D2 hold:

A2⁻. $X \subseteq Cn^-X \subseteq S$ — reflexivity of Cn^-,
A3⁻. $Cn^-Cn^-X = Cn^-X$ — the operation Cn^- is idempotent,
A4⁻. $X \subseteq Y \Rightarrow Cn^-X \subseteq Cn^-Y$ — the operation Cn^- is monotonic,
A5⁻. $Cn^-X \subseteq \bigcup\{Cn^-Y \mid Y \subseteq X \;\&\; \mathrm{card}(Y) < \aleph_0\}$ — the operation Cn^- is finitistic.

So, we can call the operation Cn^- the *rejection consequence.* It is a generalization of the notion of rejection used by Łukasiewicz. The intuitive meaning of this consequence can be described by the following theorem:

T1. $Cn^+X \subseteq X \Rightarrow Cn^-\,(S \backslash X) \subseteq S \backslash X$.

We omit here an easy proof of T1. It is given, e.g., in [7], below Theorem 3.3.

If we accept that the consequence operation Cn^+ is *infallible*: from true (or accepted as true) sentences of the set X it leads to true (or accepted as true) consequences of the set Cn^+X, then in accordance with T1 the operation Cn^- is an *anti-infallible* consequence operation: from false (or not accepted as true, rejected) sentences of set $S \backslash X$ it leads to false (or not accepted as true, rejected) sentences of set $Cn^-\,(S \backslash X)$.

Słupecki in [16] proved that the rejection consequence Cn^- satisfies the conditions:

T2⁻. $Cn^-(\varnothing) = \varnothing$ — Cn^- is normal,
T3⁻. $Cn^-(X \cup Y) = Cn^-X \cup Cn^-Y$ — Cn^- is additive.

From A1, A2^{-}–A5^{-}, T2, and T3 there follows the condition:

C^{-1}. $Cn^{-}X = \{y \mid \exists x \in X\ (Cn^{-}\{y\} \subseteq Cn^{-}\{x\})\}$,

which is the so-called *unit consequence condition* for the rejection consequence Cn^{-}. The notion of the unit consequence was introduced and analyzed in this author's previous paper [21]. Every unit consequence satisfies axioms A1–A5 of general consequence theory $\boldsymbol{T}$ and it is an additive and normal function. Reversely, every additive and normal function satisfying Tarski's axioms A1–A5 is a unit consequence operation.

The condition C^{-1} will be the fundamental axiom of the theory $\boldsymbol{T}^{-}$.

2.2. Axioms, Definitions, and Theorems within Theory $\boldsymbol{T}^{+}$

2.2.1. Deductive Systems Theories Based on Classical Sentential Calculi

Theory $\boldsymbol{T}$ is the fundamental theoretical basis of the formalization of the theory of deductive science. Richer theories based on theory $\boldsymbol{T}$ concern only deductive systems based on a propositional logic.

The symbol "$Cn^{+}X$" is then understood as the set $Cn^{+}{}_{A}X$, *i.e.*, the set of consequences of the set X with respect to a set A (which usually is the set of logical axioms) and inference rules. The set $Cn^{+}X$ can be intuitively understood as the smallest set containing the set $A \cup X$ and closed with respect to inference rules. The set $Cn^{+}\varnothing$ is then conceived of as the set of all logical theses of the system. When the deductive system theory is based on the classical sentential calculus, then it contains, besides primitive notions S and Cn^{+} of the theory $\boldsymbol{T}$, as many primitive notions as there are in the sentential calculus on which the deductive system is based. Thus, if connectives implication → and negation ~ are primitive notions of classical calculus, then their metalogical counterparts c and n, characterized by *specific axioms* of the theory, belong to the primitive notions of the richer theory of deductive systems based on this calculus.

In addition, if the connectives conjunction ∧ and disjunction ∨ belong to the primitive notions of the calculus, then their metalogical counterparts & and d, characterized by some additional *specific axioms* of the theory, belong to the primitive notions of the richer deductive systems theory based on this calculus.

Theories of deductive systems based on classical sentential calculi are called *theories of classical consequence.* We will present the axiom system of the original so-called *Tarski's enriched deductive system theory* denoted by $\boldsymbol{T}^{+}$ and the axioms of its equivalent modification $\boldsymbol{T}^{++}$. Tarski's ideas were elaborated by Pogorzelski and Słupecki [24], Słupecki *et al.* [19,20], Wójcicki [4], Pogorzelski and Wojtylak [6], and the current author [22,23], among others.

2.2.2. Tarski's Theory $\boldsymbol{T}^{+}$ of Classical Consequence

The original theory $\boldsymbol{T}^{+}$ (see Tarski [2]) is the theory of deductive systems based on the classical implicational-negational sentential calculus (e.g., Łukasiewicz calculus). Its adequate axiom system includes axioms A1–A5 of $\boldsymbol{T}$ and the following specific axioms characterizing the classical consequence by means of counterparts of the classical connectives of implication → and negation ~, *i.e.*, metalanguage connectives c and n, which are primitive notions of $\boldsymbol{T}^{+}$:

A6^{+}. $\mathrm{c}xy, \mathrm{n}x \in S$,
A7^{+}. $\mathrm{c}xy \in Cn^{+}X \Leftrightarrow y \in Cn^{+}(X \cup \{x\})$,
A8^{+}. $Cn^{+}\{x, \mathrm{n}x\} = S$,
A9^{+}. $Cn^{+}\{x\} \cap Cn^{+}\{\mathrm{n}x\} = Cn^{+}\varnothing$.

The direct implication of the axiom A7^{+} corresponds to the *modus ponens* rule because from it and axioms of $\boldsymbol{T}$ follows the implication:

(l1) $x, \mathrm{c}xy \in Cn^{+}X \Rightarrow y \in Cn^{+}X$

The proof of lemma (l1) is given in Section 5.1.2.

The reverse implication of A7^{+} is a form of the deduction theorem.

MT1. If α is a sentential expression, in which, apart from variables, there occur at most symbols c and n and α is a substitution of a thesis of the classical implicational-negational sentential calculus, then the expression

$$\alpha \in Cn^{+}\varnothing$$

is a theorem of the theory $\boldsymbol{T}^{+}$.

The meta-theorem is based on a result of Tarski (see [3], chapter III, p. 33, Theorem 3*). A proof of MT1 we may find in [25], chapter IV, § 8. It is obvious that α as a substitution of a thesis of the classical sentential calculus is here a formula of the language of the theory $\boldsymbol{T}^{+}$; it is not a formula of substitutional calculus.

MT1 follows from the fact that the theorems of $\boldsymbol{T}^{+}$ are all expressions stating that the substitution of any axiom of Łukasiewicz's implicational-negational calculus belongs to the set $Cn^{+}\varnothing$ and by (l1) (for $X = \varnothing$).

In $\boldsymbol{T}^{+}$, we can define metalogical counterparts & and d of the classical connectives: conjunction $\wedge$ and disjunction $\vee$, respectively. Thus,

D&. $\&(x, y) \underset{df}{=} \mathrm{nc}x\mathrm{n}y,$

Dd. $\mathrm{d}(x, y) \underset{df}{=} \mathrm{cn}xy.$

Metalogical connectives & and d are assumed as two new primitive notions in a modification $\boldsymbol{T}^{++}$ of the theory $\boldsymbol{T}^{+}$.

2.2.3. The Theory of Classical Consequence $\boldsymbol{T}^{++}$

Primitive notions of the theory $\boldsymbol{T}^{++}$ are notions: S, Cn^{+}, c, n, &, and d. The axioms of the theory $\boldsymbol{T}^{++}$ are A1–A5, the axiom

A6^{++}. $\mathrm{c}xy, \mathrm{n}x, \&(x,y), \mathrm{d}(x,y) \in S,$

axioms A7^{+}–A9^{+} of the theory $\boldsymbol{T}^{+}$ and two more axioms on conjunction and disjunction:

A10^{+}. $Cn^{+}(X \cup \{\&(x, y)\}) = Cn^{+}(X \cup \{x, y\}),$

A11^{+}. $Cn^{+}(X \cup \{\mathrm{d}(x, y)\}) = Cn^{+}(X \cup \{x\}) \cap Cn^{+}(X \cup \{y\}).$

Let us note that the axiom A11^{+} can be replaced by the equivalent axiom:

A$'$11^{+}. $Cn^{+}\{\mathrm{d}(x, y)\} = Cn^{+}\{x\} \cap Cn^{+}\{y\}.$

We may prove the following meta-theorem analogous to MT1 and derivable from it and from definitions D& and Dd:

MT2. If α is a sentential expression in which apart from variables there occur at most the symbols c, n, &, and d; and α is a substitution of a thesis of the classical sentential calculus with the primitive notions implication, negation, conjunction, and disjunction, then the expression

$$\alpha \in Cn^{+}\varnothing$$

is a theorem of the theory $\boldsymbol{T}^{++}$.

For example, a theorem of $\boldsymbol{T}^{+}$ is the expression:

$$\mathrm{c}y\mathrm{d}(x,\ y)\ \in Cn^{+}\varnothing$$

because the expression $\text{c}y\text{cn}xy$ is a substitution of a thesis of Łukasiewicz's implicational-negational sentential calculus and from MT1 it follows that the expression $\text{c}y\text{cn}xy \in Cn^{+}\varnothing$, is a theorem of $\boldsymbol{T}^{+}$; thus, on the basis of Dd, a theorem of $\boldsymbol{T}^{+}$ is: $\text{c}y\text{d}(x, y) \in Cn^{+}\varnothing$. The expression: $\text{c}y\text{d}(x, y) \in Cn^{+}\varnothing$, is also, of course, a theorem of the theory $\boldsymbol{T}^{++}$, because on the basis of axioms A2 and A′11^{+} we get in $\boldsymbol{T}^{++}$ the expression: $\text{d}(x, y) \in Cn^{+}\{y\}$, and using the reverse implication of A7^{+} (putting $X/\varnothing$ and x/y) we have: $\text{c}y\text{d}(x, y) \in Cn^{+}\varnothing$.

The theories $\boldsymbol{T}^{+}$ and $\boldsymbol{T}^{++}$ are equivalent: each of them contains axioms and definitions of the other among its theorems, only when we add definitions D& and Dd to the theory $\boldsymbol{T}^{+}$.

3. The Theory T^{-} of Rejection Consequence

Primitive notions of the theory $\boldsymbol{T}^{-}$ are: S, Cn^{-}, c, n. The notions & and d in $\boldsymbol{T}^{-}$ are defined. We assume that the variables $x, y, z, \ldots, x_1, x_2, \ldots$ run over sentences of set S of all sentential expressions, while the variables $X, Y, Z, \ldots$ run over family $P(S)$.

The operation Cn^{-}: $P(S) \to P(S)$ to any set X of sentences of S assigns the set $Cn^{-}X$ of all its sentences rejected on the basis of set X.

The axioms of theory $\boldsymbol{T}^{-}$ are noted by means of the terms & of k-ary conjunction ($k \geqslant 1$) and $\sim$ which have the following definitions:

D&a. $\&(x_1) = x_1$,

b. $\&(x_1, x_2) = \text{nc}x_1\text{n}x_2$.

c. $\&(x_1, x_2, \ldots, x_{n+1}) = \&(x_1, \&(x_2, \ldots, x_{n+1}))$.

D^{-}2. $x \sim y \Leftrightarrow \forall z\, (x \in Cn^{-}\{z\} \Leftrightarrow y \in Cn^{-}\{z\})$.

The inductive definition D&a.–c. is a generalization of the definition D& in $\boldsymbol{T}^{+}$ and it is adapted in $\boldsymbol{T}^{+}$. The definition D^{-}2 says that two sentences x and y are equivalent (symbolically: $x \sim y$) if and only if they are rejected on the basis of the same sentences. It is easy to see that the relation $\sim$ is the equivalence relation.

The original axiom system for the theory $\boldsymbol{T}^{-}$ was given in my dissertation [21,26]. Below, we give its modification (*cf.* [19]):

A^{-}1. $0 < \text{card}(S) \leqslant \aleph_0$,

A^{-}2. $\text{c}xy, \text{n}x \in S$,

A^{-}3. $Cn^{-}X = \{y \mid \exists x \in X(Cn^{-}\{y\} \subseteq Cn^{-}\{x\})\}$,

A^{-}4. $y \in Cn^{-}\{\text{c}xy\}$,

A^{-}5. $x \in Cn^{-}\{y\} \Leftrightarrow Cn^{-}\{\text{c}xy\} = S$,

A^{-}6. $x_1 \in Cn^{-}\{y_1\} \wedge x_2 \in Cn^{-}\{y_2\} \Rightarrow \&(x_1, x_2) \in Cn^{-}\{\&(y_1, y_2)\}$,

A^{-}7. $Cn^{-}\{\text{c}\&(x, y)z\} = Cn^{-}\{\text{c}x\text{c}yz\}$,

A^{-}8. $\&(x, \text{n}x) \in Cn^{-}\{\text{y}\}$,

A^{-}9. $x, \text{n}x \in Cn^{-}\{y\} \Rightarrow Cn^{-}\{y\} = S$,

A^{-}10a. $\&(x, y) \sim \&(y, x)$,

b. $\&(\&(x, y), z) \sim \&(x, \&(y, z))$,

c. $y \sim z \Rightarrow \&(x, y) \sim \&(x, z)$,

d. $\&(x, x) \sim x$.

The axioms A^{-}1 and A1 in $\boldsymbol{T}$ are identical. The axioms A^{-}2 and A6^{+} in $\boldsymbol{T}^{+}$ are also identical. The important axiom A^{-}3 is the condition C^{-1} of unit consequence for the operation Cn^{-} (which in the theory $\boldsymbol{T}$, enriched with its definition D2, is a theorem).

In [21] it was proved that every operation satisfying the condition C^{-1} (*i.e.*, the axiom A^{-}3) also satisfies axioms of Tarski's theory $\boldsymbol{T}$ and it is an additive and normal operation. So, the following expressions are theorems of the theory $\boldsymbol{T}^{-}$:

A2$^-$. $X \subseteq Cn^-X \subseteq S$ — reflexivity of Cn^-,
A3$^-$. $Cn^-Cn^-X = Cn^-X$ — the operation Cn^- is idempotent,
A4$^-$. $X \subseteq Y \Rightarrow Cn^-X \subseteq Cn^-Y$ — the operation Cn^- is monotonic,
A5$^-$. $Cn^-X \subseteq \bigcup \{Cn^-Y \mid Y \subseteq X \,\&\, \mathrm{card}(Y) < \aleph_0\}$ — the operation Cn^- is finitistic.
T2$^-$. $Cn^-(\varnothing) = \varnothing$ — Cn^- is normal,
T3$^-$. $Cn^-(X \cup Y) = Cn^-X \cup Cn^-Y$ — Cn^- is additive.

The operation Cn^- is called the *rejection consequence* and the theory $\boldsymbol{T}^-$ describing its properties is called the theory of rejection consequence. Some specific properties of the rejection consequence are established by the axioms A$^-$4–A$^-$10.

For example, the axiom A$^-$4 states that any sentence y is rejected on the basis of any implication with the consequent y; the axiom A$^-$8 states that every conjunction of two contradictory sentences is rejected on the basis of any sentence, and according to axiom A$^-$9, if two contradictory sentences are rejected on the basis of a sentence, then each sentence of S is rejected on basis of the sentence.

In the theory $\boldsymbol{T}^-$, we can define the operation of classical consequence Cn^+ and the notion of inconsistency (exactly the family *Incons*). The operation Cn^+ is defined in $\boldsymbol{T}^-$ by means of the notion of the set $\boldsymbol{\&}X$.

D**&**. If $X = \varnothing$, then $\boldsymbol{\&}X = \varnothing$, and if $X \neq \varnothing$, then $\boldsymbol{\&}X$ is the set of all conjunctions (in the sense of the definition D&a.–c.) built from different sentences of set X.

The operation Cn^+ is defined in the following way:

DCn^+. $y \in Cn^+X \Leftrightarrow (\exists x \in \boldsymbol{\&}X(x \in Cn^-\{y\} \wedge X \neq \varnothing)) \vee (Cn^-\{y\}= S \wedge X = \varnothing)$.

In accordance with the above definition, if set X is the empty set, then a sentence y is its consequence when each sentence of S is rejected on the basis of y, and if X is a nonempty set, the sentence y is its consequence when on the basis of y some conjunction of sentences of X is rejected.

Of course, in the theory $\boldsymbol{T}^-$, the definition of the family *Incons* of inconsistent, contradictory sets is the same as in the theory $\boldsymbol{T}$, *i.e.*,

D1. $X \in Incons \Leftrightarrow Cn^+X = S$.

4. The Theory $\boldsymbol{T}'$ of Inconsistency

The notion of the family *Incons* of inconsistent, contradictory sets, besides the notions of set S, negation n, and implication c, is the primitive notion of the theory $\boldsymbol{T}'$ formalizing the notion. The axioms of the theory are the following expressions:

A1$'$. $0 < \mathrm{card}(S) \leqslant \aleph_0$,
A2$'$. $Incons \subseteq P(S)$,
A3$'$. $X \subseteq Y \wedge X \in Incons \Rightarrow Y \in Incons$,
A4$'$. $X \in Incons \Rightarrow \exists Y(Y \subseteq X \wedge \mathrm{card}(Y) < \aleph_0 \wedge Y \in Incons)$,
A5$'$. $\mathrm{c}xy, \mathrm{n}x \in S$,
A6$'$. $X \cup \{x\}, X \cup \{\mathrm{n}x\} \in Incons \Rightarrow X \in Incons$,
A7$'$. $\{x, \mathrm{n}x\} \in Incons$,
A8$'$. $X \cup \{x, \mathrm{n}y\} \in Incons \Leftrightarrow X \cup \{\mathrm{nc}xy\} \in Incons$.

The specific axioms characterizing the family *Incons* say successively: this is the subfamily of all subsets of sentences of S (A2$'$); a superset of an inconsistent set is an inconsistent set itself (A3$'$); every inconsistent set includes a finite inconsistent set (A4$'$); if two sets: the set X with an added sentence and the set X with the added negation of this sentence are inconsistent, then the set X is itself inconsistent (A6$'$); any set containing two contradictory sentences is inconsistent (A7$'$); the set we get from set X by adding to it a sentence x and the negation of a sentence y is inconsistent if and only if set X with the added negation of implication with the antecedent x and the consequent y is inconsistent (A8$'$).

In the theory $\boldsymbol{T}'$ of inconsistency, we define the notion of classical consequence (inferential entailment), properly the operation Cn^+, in the following way:

D$'Cn^+$. $x \in Cn^+X \Leftrightarrow X \cup \{\mathrm{n}x\} \in Incons$.

It states that a sentence is a consequence of set X (some sentences of X entail the sentence) if and only if by adding the negation of the sentence to set X we get the inconsistent set.

In $\boldsymbol{T}'$, the notion of rejection, properly the notion of rejection consequence Cn^-, is defined in the same way as in $\boldsymbol{T}$ and $\boldsymbol{T}^+$, *i.e.*,

D$'Cn^-$. $Cn^-X = \{y \mid \exists x \in X(x \in Cn^+\{y\})\}$.

5. Equivalence of the Theories $\boldsymbol{T}^+$, $\boldsymbol{T}^-$, and $\boldsymbol{T}'$

In Sections 2–4, we showed that specific notions of each of the theories $\boldsymbol{T}^+$, $\boldsymbol{T}^-$, and $\boldsymbol{T}'$ may be defined in the other ones. In order to show that the notions of entailment (consequence), rejection, and inconsistency are mutually definable, we have to justify that the three theories are equivalent. We recall that the equivalence of two axiomatic theories will be proved if all axioms and definitions of one of these theories are theorems or definitions of the second one and the reverse: all axioms and definitions of the second theory are theorems or definitions of the first.

5.1. Equivalence of Theories $\boldsymbol{T}^+$ and $\boldsymbol{T}^-$

5.1.1. Proofs of Axioms and Definitions of $\boldsymbol{T}^+$ in $\boldsymbol{T}^-$

Let us observe again that the first axioms of the theories $\boldsymbol{T}^+$ and $\boldsymbol{T}^-$ are identical. Also, the axioms A6$^+$ and A$^-$2 are identical, and the definition of the family *Incons* (D1) is the same in both theories. So, it will suffice to prove that axioms A2–A5 and A7$^+$–A9$^+$ of the theory $\boldsymbol{T}^+$ for the classical consequence Cn^+ as well as the definition D2 of the rejection operation Cn^- are proved in the theory $\boldsymbol{T}^-$. (Some proofs in this section are modeled after those given in the paper [19] (in its Appendix).)

To prove the axioms of $\boldsymbol{T}^+$ in the theory $\boldsymbol{T}^-$ we will specifically use the definition DCn^+ and the following properties of the set **&**X, the conjunction &, and the relation ~ that follow from D**&**, D&a.–c., D2 and the axiom A$^-$10, respectively:

(i) $\&\{x\} = \{x\}$,
(ii) $X \subseteq \boldsymbol{\&}X$,
(iii) $X \subseteq Y \Rightarrow \boldsymbol{\&}X \subseteq \boldsymbol{\&}Y$,
(iv) If $i1, i2, \ldots, in$ is a permutation of numbers $1, 2, \ldots, n$, then $\&(x_1, x_2, \ldots, x_n) \sim \&(x_{i1}, x_{i2}, \ldots, x_{in})$,
(v) $y, z \in \boldsymbol{\&}X \Rightarrow \exists x \in \boldsymbol{\&}X\ (x \sim \&(y, z))$,
(vi) $x_1, x_2, \ldots, x_n \in \boldsymbol{\&}X \Rightarrow \exists x \in \boldsymbol{\&}X\ (x \sim \&(x_{i1}, x_{i2}, \ldots, x_{in})$.

We will use, of course, the basic properties of the rejection operation Cn^-. In accordance with A$^-$3, the operation is a unit consequence operation and satisfies the expressions: A2$^-$–A5$^-$ and thus, in particular, we have:

(vii) $x \in Cn^-\{x\}$,
(viii) $x \in Cn^-\{y\} \Leftrightarrow Cn^-\{x\} \subseteq Cn^-\{y\}$,
(ix) $y \in Cn^-X \Leftrightarrow \exists x \in X\ (y \in Cn^-\{x\})$.

From (ix) it follows that a sentence rejected on the basis of a set of sentences is rejected on the basis of only one sentence of the set.

From axioms of $\boldsymbol{T}^-$ we obtain also some further lemmas.

From A$^-$6 of $\boldsymbol{T}^-$ we have

(x) $x_1 \in Cn^-\{y_1\} \wedge \ldots \wedge x_n \in Cn^-\{y_n\} \Rightarrow \&(x_1, \ldots, x_n) \in Cn^-\ \&\{y_1, \ldots, y_n\}$.

From $A^{-}6$, $A^{-}10d$ and $D^{-}2$ we get

(xi) $Cn^{-}\{x\} = Cn^{-}\{y\} = S \Rightarrow Cn^{-}\{\&(x, y)\} = S$.

From $A^{-}10a$, $D^{-}2$, $A^{-}5$ and $A^{-}7$ it follows:

(xii) $Cn^{-}\{\mathrm{c}\&(x, y)z\} = S \Leftrightarrow Cn^{-}\{\mathrm{c}y\mathrm{c}xz\} = S$.

From $A^{-}10d$,c,a and $D^{-}2$, $A^{-}7$ we get

(xiii) $Cn^{-}\{\mathrm{c}\&(x, z)\mathrm{c}xy\} = S \Leftrightarrow Cn^{-}\{\mathrm{c}\&(x, z)y\} = S$.

From $A^{-}5$, the lemma (ii), and again $A^{-}5$ it follows immediately:

(xiv) $\&(x, z) \in Cn^{-}\{y\} \Rightarrow z \in Cn^{-}\{\mathrm{c}xy\}$.

Making use of $\mathrm{D}Cn^{+}$, D**&**, and (i) we have the next lemma:

(xv) $x \in Cn^{+}\{y\} \Leftrightarrow y \in Cn^{-}\{x\}$.

Now, we may prove axioms A2–A5 and $A7^{+}$–$A9^{+}$ of $\boldsymbol{T}^{+}$ in theory $\boldsymbol{T}^{-}$. First, we will prove the axioms of Tarski's theory $\boldsymbol{T}$ in $\boldsymbol{T}^{-}$.

A2. $X \subseteq Cn^{+}X \subseteq S$.

Proof. For $X = \varnothing$, A2 is true. If $X \neq \varnothing$, then every sentence y in X is in $\boldsymbol{\&}X$ on the basis of (ii), and on the basis of (vii): $y \in Cn^{-}\{y\}$, thus we get: $\exists x \in \boldsymbol{\&}X$ $(x \in Cn^{-}\{y\})$ and using the definition $\mathrm{D}Cn^{+}$ we have: $y \in Cn^{+}X$; y also belongs to S. □

A4. $X \subseteq Y \Rightarrow Cn^{+}X \subseteq Cn^{+}Y$.

Proof. Let us assume that $X \subseteq Y$. If $X = Y = \varnothing$, then A4 is true. Let $X = \varnothing$, $Y \neq \varnothing$ and $y \in Cn^{+}X$. Then, in accordance with $\mathrm{D}Cn^{+}$, we state that $Cn^{-}\{y\} = S$ and because there exist $y_1 \in Y$, $y_1 \in \boldsymbol{\&}Y$ on the basis of (i), and $y_1 \in Cn^{-}\{y\}$. Thus, using $\mathrm{D}Cn^{+}$, we get: $y \in Cn^{+}Y$. So $Cn^{+}X \subseteq Cn^{+}Y$ and A4 holds.

If $X \neq \varnothing$ and $y \in Cn^{+}X$, from $\mathrm{D}Cn^{+}$ we get that there is $x_1 \in \boldsymbol{\&}X$ such that $x_1 \in Cn^{-}\{y\}$. Then, because $X \subseteq Y$, on the basis of (iii) $x_1 \in \&Y$ and using again $\mathrm{D}Cn^{+}$, we have: $y \in Cn^{+}Y$. So $Cn^{+}X \subseteq Cn^{+}Y$ also in the last case, which proves that A4 holds. □

A3. $Cn^{+}Cn^{+}X = Cn^{+}X$.

Proof. First, we note that $Cn^{+}X \neq \varnothing$ because from (vii), $A^{-}5$ and $\mathrm{D}Cn^{+}$: $cxx \in Cn^{+}\varnothing$ and from A4 it follows that $\mathrm{c}xx \in Cn^{+}X$. Let us assume that $z \in Cn^{+}Cn^{+}X$. We will prove that $z \in Cn^{+}X$. Since $Cn^{+}X \neq \varnothing$, from $\mathrm{D}Cn^{+}$ and (ii) we have: $y_1 \in \boldsymbol{\&}Cn^{+}X$ and $y_1 \in Cn^{-}\{z\}$. On the basis of D**&** we can accept that $y_1 = \&(t_1, \ldots, t_n)$ and $t_1, \ldots, t_n \in Cn^{+}X$. Let us consider two cases:

$$\text{(c1) } X = \varnothing \text{ and (c2) } X \neq \varnothing.$$

In case (c1), on the basis $\mathrm{D}Cn^{+}$ we have: $Cn^{-}\{t_1\} = \ldots = Cn^{-}\{t_n\} = S$, and $Cn^{-}\{\&(t_1, \ldots, t_n)\} = S$ in accordance with (xi). Thus, $Cn^{-}\{y_1\} = S$ and for any sentence s of S we get: $s \in Cn^{-}\{y_1\}$. Hence and (viii): for any s of S, $s \in Cn^{-}\{z\}$ and $Cn^{-}\{z\} = S$, and on the basis of $\mathrm{D}Cn^{+}$ we get: $z \in Cn^{+}X$. So

(1) $X = \varnothing \Rightarrow Cn^{+}Cn^{+}X \subseteq Cn^{+}X$.

Let us consider now the case (c2). Since $t_1, \ldots, t_n \in Cn^{+}X$ from $\mathrm{D}Cn^{+}$ it follows that $x_1, \ldots, x_n \in \boldsymbol{\&}X$ and $x_1 \in Cn^{-}\{t_1\} \wedge \ldots \wedge x_n \in Cn^{-}\{t_n\}$. Making use the lemma (x), we get: $\&(x_1, \ldots, x_n) \in Cn^{-}\{\&(t_1, \ldots, t_n)\}$. From (vi) we conclude that there is $x_0 \in \boldsymbol{\&}X \wedge x_0 \sim \&(x_1, \ldots, x_n)$. Then, because $y_1 = \&(t_1, \ldots, t_n)$, from $D^{-}2$ we obtain: $x_0 \in Cn^{-}\{y_1\}$, and because $y_1 \in Cn^{-}\{z\}$, from (viii) we have: $x_0 \in Cn^{-}\{z\}$, and $x_0 \in \boldsymbol{\&}X$, hence using $\mathrm{D}Cn^{+}$, we have: $z \in Cn^{+}X$. So

(2) $X \neq \varnothing \Rightarrow Cn^{+}Cn^{+}X \subseteq Cn^{+}X$.

From (1) and (2) we have

(3) $Cn^{+}Cn^{+}X \subseteq Cn^{+}X$.

The reverse inclusion to (3) follows immediately from A2. □

A7^{+}. $cxy \in Cn^{+}X \Leftrightarrow y \in Cn^{+}(X \cup \{x\})$

Proof (⇒). Let $cxy \in Cn^{+}X$. In the case when $X = \varnothing$ from DCn^{+} we have: $Cn^{-}\{cxy\} = S$ and from A^{-}5 we state that $x \in Cn^{-}\{y\}$, hence from (i) and DCn^{+} we get: $y \in Cn^{+}\{x\}$, and by A4, we obtain: $y \in Cn^{+}(X \cup \{x\})$.

Let us now assume that $X \neq \varnothing$. Since $cxy \in Cn^{+}X$, from DCn^{+} there exists $z_1 \in \&X$ such that $z_1 \in Cn^{-}\{cxy\}$. From D**&** for z_1 there are such sentences $x_1, \ldots, x_n \in X$ that $z_1 = \&(x_1, \ldots, x_n)$.

Now, we will consider two cases:

(a) when sentence x is one of the elements of the sequence $(x_1, \ldots, x_n)$ and
(b) when x is different from each element of the sequence $(x_1, \ldots, x_n)$.

Let us first consider the case:

(a) $x = x_{i1} \wedge 1 \leqslant i1 \leqslant n$.

From (iv) we have

(a1) $\&(x_1, x_2, \ldots, x_n) \sim \&(x_{i1}, x_{i2}, \ldots, x_{in})$, where $i1, i2, \ldots, in$ is a permutation of numbers $1, 2, \ldots, n$.

Let us assume that

(a2) $z = \&(x_{i2}, \ldots, x_{in})$.

Then from (a), (a1) and D^{-}2 we get

(a3) $z_1 = \&(x, z) \in Cn^{-}\{cxy\}$.

From (a3) and A^{-}5 we have: $Cn^{-}\{c\&(x, z)cxy\} = S$, hence from the lemma (xiii) we get

(a4) $Cn^{-}\{c\&(x, z)y\} = S$.

Using again the axiom A^{-}5 but to (a4), we get

(a5) $\&(x, z) \in Cn^{-}\{y\}$

and because $z_1 = \&(x, z) \in \&X$, on the basis of (iii): $\&(x, z) \in \&(X \cup \{x\})$, and from (a5) and D$Cn^{+}$ we get: $y \in Cn^{+}(X \cup \{x\})$.

Let us consider now the other case:

(b) $x \neq x_1 \wedge \ldots \wedge x \neq x_n$.

We know that $z_1 \in \&X$ and $z_1 \in Cn^{-}\{cxy\}$, thus it follows from D**&** that

(b1) $\&(x, z_1) \in \&(X \cup \{x\})$

and from A^{-}5 that

(b2) $Cn^{-}\{cz_1cx\,y\} = S$.

The lemma (xii) and (b2) imply the formula: $Cn^{-}\{c\&(x, z_1)y\} = S$, from which, by means of $A^{-}5$, we obtain

(b3) $\&(x, z_1) \in Cn^{-}\{y\}$.

So, (b1), (b3), and the assumption: $X \neq \varnothing$, on the basis of DCn^{+} provide: $y \in Cn^{+}(X \cup \{x\})$. □

Proof (⇐). Let us suppose that $y \in Cn^{+}(X \cup \{x\})$. If $X = \varnothing$, then $y \in Cn^{+}\{x\}$ and by (i) and DCn^{+} we get: $x \in Cn^{-}\{y\}$, and in accordance with $A^{-}5$ we have: $Cn^{-}\{cxy\} = S$, hence by DCn^{+} we get: $cxy \in Cn^{+}X$.

Let us now assume that $X \neq \varnothing$. Using DCn^{+} to: $y \in Cn^{+}(X \cup \{x\})$, we state that: there is a sentence: $z_1 \in \mathbf{\&}(X \cup \{x\})$ such that $z_1 \in Cn^{-}\{y\}$, hence on the basis of $A^{-}4$ and (viii) $z_1 \in Cn^{-}\{cxy\}$. Let us observe that from $z_1 \in \mathbf{\&}(X \cup \{x\})$ it follows that there are $x_1, x_2, \ldots, x_n \in X \cup \{x\}$ such that $z_1 = \&(x_1, x_2, \ldots, x_n)$ and we have to consider two cases:

(a) when x is an element of the sequence $(x_1, x_2, \ldots, x_n)$ and
(b) when x is different from each element of the sequence.

Let us first consider the case:

(a) $x = x_{i1} \wedge 1 \leqslant i1 \leqslant n$.

From (iv) we have

(a1) $z_1 = \&(x_1, x_2, \ldots, x_n) \sim \&(x_{i1}, x_{i2}, \ldots, x_{in})$, where $i1, i2, \ldots, in$ is a permutation of numbers $1, 2, \ldots, n$.

From D&c. and (a) we have

(a2) $\&(x_{i1}, x_{i2}, \ldots, x_{in}) = \&(x_{i1}, \&(x_{i2}, \ldots, x_{in})) = \&(x, \&(x_{i2}, \ldots, x_{in}))$.

Thus, from (a1) and (a2) we get

(a3) $z_1 \sim \&(x, \&(x_{i2}, \ldots, x_{in}))$.

Since $z_1 \in Cn^{-}\{y\}$, from $D^{-}2$ and (a3) it follows that $\&(x, \&(x_{i2}, \ldots, x_{in})) \in Cn^{-}\{y\}$ and by (xiv)

(a4) $\&(x_{i2}, \ldots, x_{in}) \in Cn^{-}\{cxy\}$.

It is easy to see that on the basis of (a) and: $x_1, x_2, \ldots, x_n \in X \cup \{x\}$, we obtain

(a5) $\&(x_{i2}, \ldots, x_{in}) \in \mathbf{\&}X$.

However, (a5) and (a4) are true, and $X \neq \varnothing$. So according to DCn^{+}: $cxy \in Cn^{+}X$. □

A5. $Cn^{+}X \subseteq \bigcup \{Cn^{+}Y \mid Y \subseteq X \,\&\, \mathrm{card}(Y) < \aleph_0\}$.

It is sufficient to prove the following implication:

A5*a*. $y \in Cn^{+}X \Rightarrow \exists Y\, (Y \subseteq X \,\&\, \mathrm{card}(Y) < \aleph_0 \wedge y \in Cn^{+}Y)$.

Proof. Let us assume that $y \in Cn^{+}X$. If $X = \varnothing$, then the consequent of the above implication is true. So we will consider the case when $X \neq \varnothing$. Then by DCn^{+} there is $z_1 \in \mathbf{\&}X$ such that $z_1 \in Cn^{-}\{y\}$, hence making use D**&**, we obtain: $z_1 = \&(x_1, x_2, \ldots, x_n)$ and $x_1, x_2, \ldots, x_n \in X$. Hence, $\&(x_1, x_2, \ldots, x_n) \in Cn^{-}\{y\}$ and by $A^{-}5$ we have: $Cn^{-}\{c\&(x_1, x_2, \ldots, x_n)y\} = S$. Making use of $A^{-}7$, we state that: $Cn^{-}\{cx_1c\&(x_2, \ldots, x_n)y\} = S$. Then, by induction we get: $Cn^{-}\{cx_1cx_2 \ldots cx_ny\} = S$ and making use of DCn^{+}, we obtain: $cx_1cx_2 \ldots cx_ny \in Cn^{+}\varnothing$, hence by $A^{+}7$: $cx_2 \ldots cx_ny \in Cn^{+}\{x_1\}$ and by induction we have: $y \in Cn^{+}\{x_1, x_2, \ldots, x_n\}$, and because of $\{x_1, x_2, \ldots, x_n\} \subseteq X \wedge \mathrm{card}\{x_1, x_2, \ldots, x_n\} < \aleph_0$, the consequent of A5*a* is true. □

A8$^+$. $Cn^+\{x, \mathrm{n}x\} = S$.

Proof. Making use of the axioms A$^-$8 and A$^-$5, we have: $Cn^-\{\mathrm{c\&}(x, \mathrm{n}x)y\} = S$, hence by A$^-$7: $Cn^-\{\mathrm{c}x\mathrm{cn}xy\} = S$. Thus, from D$Cn^+$ we obtain: $\mathrm{c}x\mathrm{cn}xy \in Cn^+\varnothing$ and twice making use of A7$^+$, we have: $\mathrm{cn}xy \in Cn^+\{x\}$ and $y \in Cn^+\{x, \mathrm{n}x\}$, hence: $Cn^+\{x, \mathrm{n}x\} = S$. □

A9$^+$. $Cn^+\{x\} \cap Cn^+\{\mathrm{n}x\} = Cn^+ \varnothing$.

Proof. Let $y \in Cn^+\{x\}$ and $y \in Cn^+\{\mathrm{n}x\}$. It is easy to show that $x \in Cn^-\{y\}$ and $\mathrm{n}x \in Cn^-\{y\}$. Then, in accordance with A$^-$9: $Cn^-\{y\} = S$, and making use of DCn^+ we get: $y \in Cn^+\varnothing$. So, the inclusion: $Cn^+\{x\} \cap Cn^+\{\mathrm{n}x\} \subseteq Cn^+\varnothing$ is valid. The reverse inclusion is also valid because if $y \in Cn^+\varnothing$, then $Cn^-\{y\} = S$ on the basis of DCn^+, and for any x: $x \in Cn^-\{y\}$ and $\mathrm{n}x \in Cn^-\{y\}$, so by the lemma (xv): $y \in Cn^+\{x\}$ and $y \in Cn^+\{\mathrm{n}x\}$. □

It remains to prove that the definition D2 of the rejection operation Cn^- (Słupecki's definition) added to the theory $\boldsymbol{T}$ (and to the theory $\boldsymbol{T}^+$) is a theorem in the theory $\boldsymbol{T}^-$.

D2. $Cn^-X = \{y \mid \exists x \in X\ (x \in Cn^+\{y\}\}$.

Proof. According to the lemma (ix): $y \in Cn^-X \Leftrightarrow \exists x \in X\ (y \in Cn^-\{x\})$. The formula: $x \in Cn^+\{y\} \Leftrightarrow y \in Cn^-\{x\}$ is the lemma (xv) in $\boldsymbol{T}^-$. So, $y \in Cn^-X \Leftrightarrow \exists x \in X\ (x \in Cn^+\{y\})$. □

We have justified that theory $\boldsymbol{T}^+$ of the classical consequence (entailment), together with the definition of rejection consequence, is included in theory $\boldsymbol{T}^-$ of rejection consequence, So $\boldsymbol{T}^+ \subseteq \boldsymbol{T}^-$ and the notion of the consequence operation Cn^+ is definable in the theory $\boldsymbol{T}^-$ by means of the rejection operation Cn^-.

Conclusion 1. The notion of consequence (entailment) is definable by rejection.

In the next subsection, we will justify the reverse statement.

5.1.2. Proofs of Axioms and Definitions of $\boldsymbol{T}^-$ in $\boldsymbol{T}^+$

It will suffice to prove that axioms A$^-$3–A$^-$10 and definition DCn^+ of the operation Cn^+ in theory $\boldsymbol{T}^-$ are theorems of theory $\boldsymbol{T}^+$ with the added definition D2 of the rejection operation Cn^-; definitions D&a.–c., D$^-$2, and D& in $\boldsymbol{T}^-$ are the same in $\boldsymbol{T}^+$.

First, we note that axiom A$^-$3 of $\boldsymbol{T}^-$ is the condition C^{-1} of a unit consequence operation that follows from theorems A2$^-$–A5$^-$ of $\boldsymbol{T}^+$ with D2. The theorems were proved by Słupecki (see Section 2.1).

We will omit easy proofs of axioms A$^-$4–A$^-$10 in $\boldsymbol{T}^+$. However, we have to prove in $\boldsymbol{T}^+$ enriched by D2 the definition DCn^+. For this purpose we will use the following lemma of $\boldsymbol{T}^+$ in its proof:

(l1) $\mathrm{c}xy \in Cn^+X \wedge x \in Cn^+X \Rightarrow y \in Cn^+X$,
(l2) $\mathrm{c}xx \in Cn^+\varnothing \wedge \mathrm{c}y\mathrm{c}xy \in Cn^+\varnothing$ (see MT1),
(l3) $x \in Cn^-\{y\} \Leftrightarrow y \in Cn^+\{x\} \Leftrightarrow \mathrm{c}xy\ Cn^+\varnothing$,
(l4) $Cn^+\{x_1, x_2, \ldots, x_n\} = Cn^+\{\&(x_1, x_2, \ldots, x_n)\}$,
(l5) $y \in Cn^+\varnothing \Leftrightarrow Cn^-\{y\} = S$.

The lemma (l1) is a counterpart of the *modus ponens* rule.

Proof (l1)**.** Let us assume that

(1) $\mathrm{c}xy \in Cn^+X$,
(2) $x \in Cn^+X$.

It follows from A2 and (2) that $X \cup \{x\} \subseteq Cn^{+}X$, hence by A4 and A3 we get

(3) $Cn^{+}(X \cup \{x\}) \subseteq Cn^{+}X$.

Making use of (1) and A^{+}7, we obtain

(4) $y \in Cn^{+}(X \cup \{x\})$. Thus, by (4) and (3) we have: $y \in Cn^{+}(X)$. □

Proof (l2)**.** By A2: $x \in Cn^{+}\{x\}$, hence, by A7^{+}: c$xx \in Cn^{+}\varnothing$. And, on the basis of A2 we state that $y \in Cn^{+}\{x, y\}$. So, twice making use of A7^{+}, we have: c$xy \in Cn^{+}\{$y$\}$ and cyc$xy \in Cn^{+}\varnothing$. □

The lemma (l3) follows immediately from D2 and A7^{+}.
The lemma (l4) follows by induction from A10^{+} and definition D&c.

Proof (l5) ($\Rightarrow$)**.** Let $y \in Cn^{+}\varnothing$. In accordance with (l2), we have: cyc$xy \in Cn^{+}\varnothing$, hence from (l1) we get: c$xy \in Cn^{+}\varnothing$; thus by (l3) we get: $x \in Cn^{-}\{y\}$. However, x is any sentence of S, so $S \subseteq Cn^{-}\{y\}$ and on the basis of A2^{-} we get: $Cn^{-}\{y\} = S$. □

Proof (l5) ($\Leftarrow$)**.** Let $Cn^{-}\{y\}= S$. Thus, c$xx \in Cn^{-}\{y\}$ and by (l3) cc$xxy \in Cn^{+}\varnothing$. Hence, making use of (l1) and (l2), we obtain: $y \in Cn^{+}\varnothing$. □

Now, we may prove DCn^{+}.

DCn^{+}. $y \in Cn^{+}X \Leftrightarrow (\exists x \in \&X\ (x \in Cn^{-}\{y\} \wedge X \neq \varnothing)) \vee (Cn^{-}\{y\} = S \wedge X = \varnothing)$.

Proof ($\Rightarrow$)**.** Let $y \in Cn^{+}$X. If X $= \varnothing$, then $y \in Cn^{+}\varnothing$ and by (l5) $Cn^{-}\{y\} = S$. Thus, the consequent of the implication is true. If $X \neq \varnothing$, then on the basis of axiom A5 by assumption there is a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that $y \in Cn^{+}\{x_1, x_2, \ldots, x_n\}$. Thus, by (l4) $y \in Cn^{+}\{\&(x_1, x_2, \ldots, x_n)\}$ and according to D**&**: $\&(x_1, x_2, \ldots, x_n) \in \&X$. So, making use of (l3): $\exists x \in \&X\ (x \in Cn^{-}\{y\} \wedge X \neq \varnothing)$ and the first constituent of the proving disjunction is valid. □

Proof ($\Leftarrow$)**.** Let us assume that: $(\exists x \in \&X\ (x \in Cn^{-}\{y\} \wedge X \neq \varnothing) \vee (Cn^{-}\{y\}= S \wedge X = \varnothing)$. We will consider two cases. In the first case, when the second constituent of the above disjunction is valid, $y \in Cn^{+}X$ by (l5).

In the second case, when the first constituent of the disjunction is valid, there exists such $z_1 \in \&X$ that $z_1 \in Cn^{-}\{y\} \wedge X \neq \varnothing$. Then it follows from D**&** that $z_1 = \&(x_1, x_2, \ldots, x_n)$ and $x_1, x_2, \ldots, x_n \in X$. Making use of the lemma (l3), we get: $y \in Cn^{+}\{\&(x_1, x_2, \ldots, x_n)\}$ and by lemma (l4) $y \in Cn^{+}\{x_1, x_2, \ldots, x_n\}$. Since $\{x_1, x_2, \ldots, x_n\} \subseteq X$, from A4 we have: $y \in Cn^{+}$X. □

We have justified that theory $\boldsymbol{T}^{-}$ of the rejection consequence, together with the definition of classical consequence, is included in theory $\boldsymbol{T}^{+}$ of the classical consequence: $\boldsymbol{T}^{-} \subseteq \boldsymbol{T}^{+}$. So, the notion of the rejection operation Cn^{-} is definable in theory $\boldsymbol{T}^{+}$ by means of the consequence operation Cn^{+}.

Conclusion 2. The notion of rejection is definable by the notion of entailment (consequence).

5.1.3. Mutual Definability of the Notions of Entailment and Rejection.

In Section 5.1.1, we proved that $\boldsymbol{T}^{+} \subseteq \boldsymbol{T}^{-}$, and in Section 5.1.2, we proved that $\boldsymbol{T}^{-} \subseteq \boldsymbol{T}^{+}$. Thus, we can state the equivalence the theories $\boldsymbol{T}^{+}$ and $\boldsymbol{T}^{-}$. From Conclusions 1 and 2 we make

Conclusion 3. The notions of entailment and rejection are mutually definable.

5.2. Equivalence of Theories T^+ and T'

5.2.1. Proofs of Axioms and Definitions of $\boldsymbol{T}^+$ in $\boldsymbol{T}'$

In this part, we will show that the axioms A1′–A8′ and the definition D′Cn^+ of the theory $\boldsymbol{T}'$ imply axioms A1–A5, A$^+$6–A$^+$9, and definition D1 of theory $\boldsymbol{T}^+$.

At the beginning, let us note that axioms A1 and A1′, and also A$^+$6 and A5′ are identical.

In proofs of the remaining axioms of $\boldsymbol{T}^+$ on the ground of $\boldsymbol{T}'$ we will make use of the following lemmas of $\boldsymbol{T}'$:

L1. For every natural number n:

$$X \notin Incons \wedge \forall i \in \{1, 2, \ldots, n\}(X \cup \{\mathrm{n}x_i\} \in Incons) \Rightarrow X \cup \{x_1, x_2, \ldots, x_n\} \notin Incons.$$

L2. $Cn^+X \in Incons \Rightarrow X \in Incons$.

The proof of L1 is by induction with respect to n. Let

(1) $X \notin Incons$,
(2) $\forall i \in \{1, 2, \ldots, n\}(X \cup \{\mathrm{n}x_i\} \in Incons)$.

If $n = 1$, then the lemma follows immediately from the axiom A6′.
Let us assume now that for $k < n$

(3) $X \cup \{x_1, x_2, \ldots, x_k\} \notin Incons$.

From (2) it follows that $X \cup \{\mathrm{n}x_{k+1}\} \in Incons$, hence by the axiom A3′ we have

(4) $X \cup \{x_1, x_2, \ldots, x_k\} \cup \{\mathrm{n}x_{k+1}\} \in Incons$.

Making use of (4), (3) and the axiom A6′, we obtain

(5) $X \cup \{x_1, x_2, \ldots, x_k\} \cup \{x_{k+1}\} = X \cup \{x_1, x_2, \ldots, x_k, x_{k+1}\} \notin Incons$.

So, if our lemma is true for $k < n$, then it is also true for k+1, which was to be shown. □

The proof of the lemma (L2) is by contradiction. Let us assume that

(1) $Cn^+X \in Incons$ and
(2) $X \notin Incons$.

In accordance with assumption (1) and axiom A4′, it follows that there exists a finite, inconsistent subset Y_1 of the set Cn^+X. Let $Y_1 = \{x_1, x_2, \ldots, x_k\}$ be that set. Then $Y_1 \subseteq Cn^+X$ and $Y_1 \in Incons$. Thus, by D′Cn^+ we get: $\forall i \in \{1, 2, \ldots, n\}(X \cup \{\mathrm{n}x_i\} \in Incons)$. Thus, by lemma (L1) and assumption (2) we have: $X \cup Y_1 \notin Incons$, but it, in accordance with A3′, leads to the contradictory statement: $Y_1 \notin Incons$. □

Now, we proceed to proofs of the axioms of $\boldsymbol{T}^+$ on the ground of the theory $\boldsymbol{T}'$. At the beginning, let us note that the axioms A1 and A1′, and also A$^+$6 and A5′ are identical.

A2. $X \subseteq Cn^+X \subseteq S$.

Proof. Let $x \in X$. Hence, $\{x, \mathrm{n}x\} \subseteq X \cup \{\mathrm{n}x\}$ and by axioms A7′and A3′: $X \cup \{\mathrm{n}x\} \in Incons$. Then, according to D′$Cn^+$: $x \in Cn^+X$, and on the basis of our conventions about variables: $Cn^+X \subseteq S$. □

A4. $X \subseteq Y \Rightarrow Cn^+X \subseteq Cn^+Y$.

Proof. Let $X \subseteq Y$ and $x \in Cn^+X$. Thus, from D′Cn^+ it follows that: $X \cup \{\mathrm{n}x\} \in Incons$, and making use of our assumption and the axiom A3′: $Y \cup \{\mathrm{n}x\} \in Incons$. Thus, by D′$Cn^+$ we get: $x \in Cn^+Y$. Hence, $Cn^+X \subseteq Cn^+Y$. □

A3. $Cn^+Cn^+X \subseteq Cn^+X$.

Proof. Let $x \in Cn^+Cn^+X$. Then, by $\mathrm{D}'Cn^+$ we have: $Cn^+X \cup \{\mathrm{n}x\} \in Incons$. On the basis of A4, A2 we state that: $Cn^+X \subseteq Cn^+(X \cup \{\mathrm{n}x\})$ and $\{\mathrm{n}x\} \subseteq Cn^+(X \cup \{\mathrm{n}x\})$, thus $Cn^+X \cup \{\mathrm{n}x\} \subseteq Cn^+(X \cup \{\mathrm{n}x\})$ and by A3′ $Cn^+(X \cup \{\mathrm{n}x\}) \in Incons$. Thus, by the lemma L2 we have: $X \cup \{\mathrm{n}x\} \in Incons$ and in accordance with $\mathrm{D}'Cn^+$: $x \in Cn^+X$. □

A5. $Cn^+X \subseteq \bigcup \{Cn^+Y \mid Y \subseteq X \ \& \ \mathrm{card}(Y) < \aleph_0\}$.

Proof. Let $x \in Cn^+X$. Thus, it follows from $\mathrm{D}'Cn^+$ that: $X \cup \{\mathrm{n}x\} \in Incons$, and on the basis of A4′ there is such a set Y_1 that

(1) $Y_1 \subseteq X \cup \{\mathrm{n}x\}$ and card $(Y_1) < \aleph_0$ and
(2) $Y_1 \in Incons.$

From (1) we can also state that

(3) $Y_1 \backslash \{\mathrm{n}x\} \subseteq X$ and card $(Y_1 \backslash \{\mathrm{n}x\}) < \aleph_0$.

Let us also observe that $Y_1 = Y_1 \backslash \{\mathrm{n}x\} \cup \{\mathrm{n}x\}$ or $Y_1 \cup \{\mathrm{n}x\} = Y_1 \backslash \{\mathrm{n}x\} \cup \{\mathrm{n}x\}$. Thus, making use of (2) and A3′, we obtain: $Y_1 \backslash \{\mathrm{n}x\} \cup \{\mathrm{n}x\} \in Incons.$ Thus, on the basis of $\mathrm{D}'Cn^+$ we get: $x \in Cn^+(Y_1 \backslash \{\mathrm{n}x\})$. Hence, on the basis of (3) we see that the proving inclusion is valid. □

A7⁺. $\mathrm{c}xy \in Cn^+X \Leftrightarrow y \in Cn^+(X \cup \{x\})$.

Proof. The following equivalences are satisfied in accordance with $\mathrm{D}'Cn^+$ and A8′:

$$y \in Cn^+(X \cup \{x\}) \Leftrightarrow X \cup \{x, \mathrm{n}y\} \in Incons \Leftrightarrow X \cup \{\mathrm{nc}xy\} \in Incons \Leftrightarrow \mathrm{c}xy \in Cn^+X.$$

Thus, A7⁺ is proved. □

A8⁺. $Cn^+\{x, \mathrm{n}x\} = S$.

Proof. It follows from the axioms A7′ and A3′ that $\{x, \mathrm{n}x, \mathrm{n}y\} \in Incons$, thus by means of $\mathrm{D}'Cn^+$ we have: $y \in Cn^+\{x, \mathrm{n}x\}$. Thus, $S \subseteq Cn^+\{x, \mathrm{n}x\}$ is true and by A2: $Cn^+\{x, \mathrm{n}x\} = S$. □

A9⁺. $Cn^+\{x\} \cap Cn^+\{\mathrm{n}x\} = Cn^+\varnothing$.

Proof. We will prove that the inclusion: $Cn^+\{x\} \cap Cn^+\{\mathrm{n}x\} \subseteq Cn^+\varnothing$ is true; the reverse inclusion follows immediately from A4.

Let $y \in Cn^+\{x\}$ and $y \in Cn^+\{\mathrm{n}x\}$. Thus, by $\mathrm{D}'Cn^+$ we have:

(1) $\{\mathrm{n}y\} \cup \{x\} \in Incons$ and
(2) $\{\mathrm{n}y\} \cup \{\mathrm{n}x\} \in Incons.$

Let us assume that $y \notin Cn^+\varnothing$. Then by $\mathrm{D}'Cn^+$: $\{\mathrm{n}y\} \notin Incons$, and from (2) and A6′: $\{\mathrm{n}y\} \cup \{x\} \notin Incons.$ It would be contrary to (1). Hence: $y \in Cn^+\varnothing$, and the proving inclusion is valid. □

It remains to prove that definition D1 accepted in theories $\boldsymbol{T}$ and $\boldsymbol{T}^+$ is a theorem in theory $\boldsymbol{T}'$.

D1. $X \in Incons \Leftrightarrow Cn^+X = S$.

Proof (⇒). If $X \in Incons$ then by A3′ also $\forall x\,(X \cup \{\mathrm{n}x\}) \in Incons$, and according to $\mathrm{D}'Cn^+$: $\forall x\,(x \in Cn^+X)$. Thus and A2: $Cn^+X = S$.□

Proof (⇐). If $Cn^+X = S$, then for each x of S $\{x, \mathrm{n}x\} \subseteq Cn^+X$. But according to A7′: $\{x, \mathrm{n}x\} \in Incons$. Thus, by A3′: $Cn^+X \in Incons$. Making use of the lemma (L2), we conclude: $X \in Incons.$ □

We have justified that theory $\boldsymbol{T}^{+}$ of the classical consequence (inferential entailment), together with the definition of the family of inconsistent sets, is included in theory $\boldsymbol{T}'$ of inconsistency. So $\boldsymbol{T}^{+} \subseteq \boldsymbol{T}'$ and the notion of consequence operation Cn^{+} is definable in theory $\boldsymbol{T}'$ by means of family *Incons* of inconsistent sets.

Conclusion 4. The notion of consequence (entailment) is definable by the notion of inconsistency.

5.2.2. Proofs of Axioms and Definitions of $\mathbf{T}'$ in $\boldsymbol{T}^{+}$

In this part, we will show that axioms A1–A5, $A^{+}6$–$A^{+}9$ and definition D1 of theory $\boldsymbol{T}^{+}$ imply axioms A1′–A8′ and definition $D'Cn^{+}$ of theory $\boldsymbol{T}'$.

At the beginning, let us note that axioms A1 and A1′, and also $A^{+}6$ and A5′ are identical, whereas A2′ is a consequence of the accepted convention about variables.

Let us also notice that in $\boldsymbol{T}^{++}$ all the theorems in the form $\alpha \in Cn\varnothing$ are valid, where α is a metalogical substitution of any thesis of the classical sentential calculus with connectives: $\rightarrow, \sim, \wedge, \vee$ (see Section 2.2.3: MT2).

A3′. $X \subseteq Y \wedge X \in Incons \Rightarrow Y \in Incons.$

Proof. Let $X \subseteq Y$ and $X \in Incons$. Then, by A4, D1 and A2: $S = Cn^{+}X \subseteq Cn^{+}Y \subseteq S$. Thus: $Y \in Incons.$ □

A4′. $X \in Incons \Rightarrow \exists Y\, (Y \subseteq X \wedge \mathrm{card}(Y) < \aleph_0 \wedge Y \in Incons).$

Proof. Let $X \in Incons$. Thus, $Cn^{+}X = S$. So, by $A6^{+}$: $\{x, \mathrm{n}x\} \subseteq Cn^{+}X$. Making use now of A4, A3 and $A10^{+}$, we state that: $Cn^{+}\{\&(x, \mathrm{n}x)\} \subseteq Cn^{+}X$, hence by A2: $\&(x, \mathrm{n}x) \in Cn^{+}X$. Then, on the basis of A5 there is a finite subset Z_1 of X such that $\&(x, \mathrm{n}x) \in Cn^{+}Z_1$, thus by A4, A3 and A2: $Cn^{+}\{\&(x, \mathrm{n}x)\} \subseteq Cn^{+}Z_1 \subseteq S$. Hence, by $A10^{+}$ and $A8^{+}$: $Cn^{+}Z_1 = S$, and the consequent of the proving implication is true. □

$D'Cn^{+}$. $x \in Cn^{+}X \Leftrightarrow X \cup \{nx\} \in Incons.$

Proof (⇒). Let $x \in Cn^{+}X$. Then, on the basis of A5: $x \in Cn^{+}\{x_1, \ldots, x_n\}$ and $\{x_1, \ldots, x_n\} \subseteq X$. Thus, by the lemma (l4) we get: $x \in Cn^{+}\{\&(x_1, \ldots, x_n)\}$, and by $A7^{+}$ we have: $\mathrm{c}\&(x_1, \ldots, x_n)x \in Cn^{+}\varnothing$. A thesis of the classical sentential calculus is: $(p \rightarrow q) \rightarrow (p \wedge \sim q \rightarrow r)$, so its substitution by MT2 leads to: $\mathrm{cc}\&(x_1, \ldots, x_n)x\mathrm{c}\&((\&(x_1, \ldots, x_n),\mathrm{n}x)y \in Cn^{+}\varnothing$, and making use of the lemma (l1) of $\boldsymbol{T}^{+}$, we get: $\mathrm{c}\&(\&(x_1, \ldots, x_n),\mathrm{n}x)y \in Cn^{+}\varnothing$; thus, by $A7^{+}$: $y \in Cn^{+}\{\&(\&(x_1, \ldots, x_n),\mathrm{n}x)\}$; hence, on the basis of $A10^{+}$, the axioms of $\boldsymbol{T}$ and the lemma (l4):

$$y \in Cn^{+}(\{\&(x_1, \ldots, x_n\} \cup \{\mathrm{n}x\}) = Cn^{+}(Cn^{+}\{\&(x_1, \ldots, x_n\}\} \cup \{\mathrm{n}x\}) =$$
$$Cn^{+}(Cn^{+}\{x_1, \ldots, x_n\} \cup \{\mathrm{n}x\}) = Cn^{+}(\{x_1, \ldots, x_n\} \cup \{\mathrm{n}x\}) \subseteq Cn^{+}(X \cup \{\mathrm{n}x\}).$$

So, $y \in Cn^{+}(X \cup \{\mathrm{n}x\})$. However, y is any sentence of S, so by $A2^{+}$ we get: $S \subseteq Cn^{+}(X \cup \{\mathrm{n}x\}) \subseteq S$. Thus, in accordance with D1: $X \cup \{\mathrm{n}x\} \in Incons.$ □

Proof (⇐). Let us assume that $X \cup \{\mathrm{n}x\} \in Incons$. Then, it follows from D1 that: $Cn^{+}(X \cup \{\mathrm{n}x\}) = S$. Thus, any sentence of S, in particular $\&(x,\mathrm{n}x)$, belongs to $Cn^{+}(X \cup \{\mathrm{n}x\})$. Hence and $A7^{+}$: $\mathrm{cn}x\&(x,\mathrm{n}x) \in Cn^{+}X$. However, by MT2 $\mathrm{ccn}x\&(x,\mathrm{n}x)\mathrm{cn}\&(x,\mathrm{n}x)x \in Cn^{+}\varnothing \subseteq Cn^{+}X$. Thus, making use of the lemma (l1), we have: $\mathrm{cn}\&(x,\mathrm{n}x)x \in Cn^{+}X$. Hence, by (l1) and the fact from MT2: $\mathrm{n}\&(x,\mathrm{n}x) \in Cn^{+}\varnothing \subseteq Cn^{+}X$, we obtain: $x \in Cn^{+}X.$ □

A6′. $X \cup \{x\}, X \cup \{\mathrm{n}x\} \in Incons \Rightarrow X \in Incons.$

Proof. Let us note that in $\boldsymbol{T}^{+}$ a theorem is the equality: $Cn^{+}(X \cup \{x\}) = Cn^{+}(X \cup \{\mathrm{nn}x\})$. Thus, if the antecedent of the proving implication is valid, then by $D'Cn^{+}$: $x, \mathrm{n}x \in Cn^{+}X$. Thus, making use of the axioms A4, A3, $A8^{+}$ and A2, we have: $S \subseteq Cn^{+}X \subseteq S$. Thus, by D1 we get: $X \in Incons.$ □

A7′. $\{x, \mathrm{n}x\} \in Incons$.

A7′ follows immediately from A8$^+$ and D1. □

A8′. $X \cup \{x, \mathrm{n}y\} \in Incons \Leftrightarrow X \cup \{\mathrm{nc}xy\} \in Incons$.

Proof. The following equivalences follow from D′Cn^+ and A7$^+$:
$X \cup \{x, \mathrm{n}y\} \in Incons \Leftrightarrow y \in Cn^+(X \cup \{x\}) \Leftrightarrow \mathrm{c}xy \in Cn^+X \Leftrightarrow X \cup \{\mathrm{nc}xy\} \in Incons$.
Thus, $X \cup \{x, \mathrm{n}y\} \in Incons \Leftrightarrow X \cup \{\mathrm{nc}xy\} \in Incons$. □

We have justified that theory $\boldsymbol{T}'$ of inconsistency, together with definition of the consequence operation Cn^+, is included in theory $\boldsymbol{T}^+$ of classical consequence. So $\boldsymbol{T}' \subseteq \boldsymbol{T}^+$ and the notion of inconsistency is definable in theory $\boldsymbol{T}^+$ by means of consequence operation Cn^+.

Conclusion 5. The notion of inconsistency is definable by the notion of consequence (entailment).

5.2.3. Mutual Definability of the Notions of Entailment and Inconsistency

In Section 5.2.1 we proved that $\boldsymbol{T}^+ \subseteq \boldsymbol{T}'$, and in Section 5.2.2 we proved that $\boldsymbol{T}' \subseteq \boldsymbol{T}^+$. Thus, we can state the equivalence of theories $\boldsymbol{T}^+$ and $\boldsymbol{T}'$.

From Conclusions 4 and 5 we obtain

Conclusion 6. The notions of entailment and inconsistency are mutually definable.

5.3. Results

In Section 5.1 it was shown that

(1) theory $\boldsymbol{T}^+$ and theory $\boldsymbol{T}^-$ are equivalent; thus,
(2) theory $\boldsymbol{T}^-$ and theory $\boldsymbol{T}^+$ are equivalent.

In Section 5.2, it was proved that

(3) theory $\boldsymbol{T}^+$ and theory $\boldsymbol{T}'$ are equivalent.

Thus, it follows from statements (2) and (3) that

(4) theory $\boldsymbol{T}^-$ and theory $\boldsymbol{T}'$ are equivalent.

Thus, in view of statements (1), (3), and (4) and the assumption that definitions of conjunctions D&a.–c., the set of conjunctions D& and the disjunction Dd are the same in all of the theories under consideration, we conclude:

(5) Theories $\boldsymbol{T}^+$, $\boldsymbol{T}^-$, and $\boldsymbol{T}'$ are mutually equivalent.

Hence, the notion of the consequence operation (entailment) Cn^+ can be replaced by the notion of the rejection operation (rejection) Cn^- or by the notion of family *Incons* of inconsistent sets (inconsistency) and the latter notion can be replaced by the former ones.

In this way, on the basis of Conclusion 3 (Section 5.1) and Conclusion 6 (Section 5.2) we may say that

(6) The notions of entailment, rejection, and inconsistency are mutually definable.

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References

1. Tarski, A. Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften, I. *Monatshefte Math. Phys.* **1930**, *37*, 361–404. [CrossRef]
2. Tarski, A. Über einige fudamentale Begriffe der Metamathematik. *Comptes Rendus Séances Socété Sci. Lett. Vars.* **1930**, *23*, 22–29.
3. Tarski, A. *Logic, Semantics, Metamathematics: Papers from 1928 to 1938*; Hackett Publishing Company: Indianapolis, IN, USA, 1983.
4. Wójcicki, R. *Theory of Logical Calculi*; Kluwer Academic Publishers Group: Dordrecht, The Netherlands, 1988.
5. Czelakowski, J. *Protoalgebraic Logics*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2001.
6. Pogorzelski, W.A.; Wojtylak, P. *Completeness Theory for Propositional Logics*; Birkhäuser: Basel, Switzerland, 2008.
7. Wybraniec-Skardowska, U.; Waldmajer, J. On pairs of dual consequence operations. *Log. Univ.* **2011**, *5*, 177–203. [CrossRef]
8. Łukasiewicz, J. Logika dwuwartościowa. *Prz. Filoz.* **1921**, *23*, 189–205.
9. Łukasiewicz, J. O sylogistyce Arystotelesa (On Aristotle's syllogistic). Sprawozdania z czynności i posiedzeń Polskiej Akademii Umiejętności (PAU). In Proceedings of the Activities and Meetings of Polish Skills Academy, Warszawa, Poland, 9 June 1939.
10. Łukasiewicz, J. *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, 2nd ed.; Oxford University Press: Oxford, UK, 1951.
11. Łukasiewicz, J. A system of modal logic. *J. Comput. Syst.* **1953**, *1*, 111–149.
12. Skura, T. *Refutation Methods in Modal Propositional Logic*; Semper: Warszawa, Poland, 2013.
13. Carnap, R. *Introduction to Semantics*; Harvard University Press: Cambridge, MA, USA, 1942.
14. Carnap, R. *Formalization of Logic*; Harvard University Press: Cambridge, MA, USA, 1943.
15. Citkin, A. A meta-logic of inference rules. *Syntax Log. Log. Philos.* **2015**, *24*, 313–337.
16. Słupecki, J. *Funkcja Łukasiewicza (Łukasiewicz's function), Zeszyty Naukowe Uniwersytetu Wrocławskiego*; Seria B; University of Wrocław: Wrocław, Poland, 1953; Volume 3, pp. 33–40.
17. Tarski, A. Grundzüge des Systemenkalküls, part I. *Fundam. Math.* **1935**, *25*, 503–526.
18. Tarski, A. Grundzüge des Systemenkalküls, part II. *Fundam. Math.* **1936**, *26*, 283–301.
19. Słupecki, J.; Bryll, G.; Wybraniec-Skardowska, U. Theory of rejected propositions, part I. *Stud. Log.* **1971**, *29*, 76–123. [CrossRef]
20. Słupecki, J.; Bryll, G.; Wybraniec-Skardowska, U. Theory of rejected propositions, part II. *Stud. Log.* **1972**, *30*, 97–107. [CrossRef]
21. Wybraniec-Skardowska, U. *Teoria Zdań Odrzuconych (Theory of Rejected Sentences); Z Badań nad Teorią Zdań Odrzuconych (Studies in the Theory of Rejected Sentences)*; Seria B; Teacher's Training College Press: Opole, Poland, 1969; Volume 2, pp. 5–131.
22. Wybraniec-Skardowska, U. Foundations for the formalization of metamathematics and axiomatizations of consequence theories. *Ann. Pure Appl. Log.* **2004**, *127*, 243–266. [CrossRef]
23. Wybraniec-Skardowska, U. *Wzajemna definiowalność pojęć wynikania i sprzeczności (On mutual definability of the notions of entailment and inconsistency)*; Zeszyty Naukowe Wyższej Szkoły Pedagogicznej w Opolu; Mathematics Series; Teacher's Training Opole College: Opole, Poland, 1974; Volume 15, pp. 77–86.
24. Pogorzelski, W.A.; Słupecki, J. Podstawowe własności systemów dedukcyjnych opartych na nieklasycznych logikach (Basic properties of deductive systems based on non-classical logics), part I. *Stud. Log.* **1960**, *9*, 163–176. [CrossRef]
25. Borkowski, L. *Logika Formalna. Systemy Logiczne, Wstęp do Metalogiki*, 2nd ed.; PWN: Warszawa, Poland, 1977.
26. Wybraniec-Skardowska, U.; Bryll, G. *Z Badań nad Teorią Zdań Odrzuconych (Studies in the Theory of Rejected Sentences)*; Seria B; Teacher's Training College Press: Opole, Poland, 1969; Volume 2.

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