Modeling Term Structure of Cross-Currency Interest Rates

Tim Xiao

ABSTRACT

This article proposes a term structure model for dual-currency interest rate markets. The model assumes that volatility is a deterministic function of time alone. This volatility structure can reduce the dimension of the required state variables. An important special case is presented, which corresponds essentially to a Vasicek/Hull-White yield curve model in each currency. The model is very useful for pricing cross-currency derivatives.

Key Words: interest rate, cross currency, calibration, volatility structure.

An interest rate reflects the cost of borrowing or the reward of saving. Interest rate curve is the plot of maturities and associated interest rates that illustrates future interest rates in a clear and concise way. Interest rate curve is also called the term structure of interest rates.

Interest rate plays critical roles in finance and economy. It affects everyone and every business. One factor model is mathematically tractable, but may be insufficient to capture all dynamics of interest rate curve movements. As such, multi factor models of interest rates arise to have a better explanation of interest rate evolutions. A multi factor model assumes the movement of interest rates is determined by multiple state variables.

The most important contracts for calibrating interest rate term structure models are caplets and swaptions. To get a fast and stable model calibration, it is important to have closed from or semi-closed form approximation for their values.

A Cross Currency European Swaption is a European Swaption to enter into a swap to exchange cash flows in two different currencies. The underlying cross-currency swap can be fixed-to-fixed, fixed-to-floating and floating-to-floating types with possible floating spread and principal exchanges which may happen at the beginning of the swap or at the end of the swap or at both the beginning and the end.

There is a rich literature on interest rate modelling. The first analysis is one factor models, such as Vasicek (1977), and Cox, al et. (1985). These models assume that the movement of an interest rate curve is determined by a single state variable. This state variable is usually called the short rate that follows a stochastic diffusion process.

Medova al et. (2006) study interest rate data by using a three-factor interest rate curve model and the Kalman filter. The model captures the salient features of the whole term structure in forward simulation.

Yu and Ning (2019) propose an interest rate model by means of uncertain differential equations with jumps and derive a closed form price for zero-coupon bond. Verschuren (2019) develops a coherent framework on how to best incorporate negative interest rates in these studies through a single curve stochastic term structure model.

Kikuchi (2024) presents a new quadratic Gaussian short rate model with a stochastic lower bound to capture changes in the yield curve including negative interest rates. Akram (2020) presents a long-term interest rate model to reflect the central bank's actions influence the long-term interest rate primarily through the short-term interest rate.

Hansen (2023) presents a term structure model for no-arbitrage bond yields and realized bond market volatility and shows that conditional yield curve covariation is priced in long-term yields.

Levrero and Matteo [2019] study the relationship between short- and long-term interest rates and outline an asymmetry in the relationship. Bauer and Hamilton (2019) conclude that conventional tests of whether variables other than the level, slope and curvature can help predict bond returns have significant size distortions. Engel studies the relationship between interest rates, foreign exchanges, and risk premium.

This article presents a new interest rate model for cross-currency fixed income derivatives. The model has the property that all volatility is a deterministic function of time alone. In general, a deterministic volatility structure leads to a model such that if the underlying Brownian motion driving all uncertainty in both economies is of one dimension, then in general three-dimensional state variables are required to completely characterize the yield curve and exchange rate dynamics.

The analytic tractability of the constant correlation of the separable deterministic volatility (SDV) model provide closed form formulas for these values as functions of the yield curves and exchange rate.

By a further judicious choice of volatility structure, one can reduce the dimension of the required state variable. An important special case for applications is presented in which only three state variables are required. This case corresponds essentially to a Vasicek/Hull-White yield curve model in each currency. In this particular case, a general framework for European contingent claim valuation is also worked out.

The model concentrates on the evolution of the instantaneous forward rate. We will summarise some standard results and introduce a number of financial variables and concepts such as changes of numeraire.

The equations describing the dynamics involve a stochastic term which includes a multi-dimensional correlated Brownian motion. When calculating expected values of this Brownian motion it is necessary to specify a probability space and filtration to which the Brownian motion is adapted.

The rest of this paper is organized as follows: The model is presented in Section 1; Section 2 studies the relationship with other well-known models. Section 3 elaborates calibration. Numerical results are discussed in Section 4; the conclusions are given in Section 5.

1. Model

This section presents a general framework for multi-currency models of interest rate dynamics. Let:

$P^d_{t,\tau}$ = time t price of a domestic τ - maturity zero coupon bond of unit face value
$P^f_{t,\tau}$ = time t price of a foreign τ - maturity zero coupon bond of unit face value
S_t = exchange rate at time t; S_t is the price in units of domestic currency of one unit of foreign currency at time t

Fix a time horizon H, and choose as numeraire $N^d_t = P^d_{t,H} \big/ P^d_{0,H}$; we assume the existence of a probability measure Q^d (the domestic H-forward measure) with respect to which all cum-dividend N_t-relative security prices are martingales. It is also assumed that the information filtration $\mathcal{F}_t$ is generated by a D-dimensional Brownian motion Z^d_t.

Suppressing the superscript denoting the currency for simplicity, we suppose as given the following stochastic differential equation (SDE) for the instantaneous forward rate $f(u,w)$ of maturity w at time u:

$$df(u,w) = a(u,w)du + \sigma(u,w)dZ_u . \qquad (1)$$

The short rate may be shown to satisfy

$$dr_u = \left[\frac{\partial f}{\partial u}(t_0, u) + \sigma(u,u) \cdot v_{u,H} + \int_{t_0}^{u} \frac{\partial \sigma(s,u) \cdot \Sigma(s;u,H)}{\partial u} ds + \int_{t_0}^{u} \frac{\partial \sigma(s,u)}{\partial u} dZ_s \right] du + \sigma(u,u) dZ_u .$$

(2)

The domestic bond price is given by

$$P_{t,\tau}^{d} = \frac{P_{t_0,\tau}^{d}}{P_{t_0,t}^{d}} \exp\left(-\tfrac{1}{2} h^{d}\left(t_0, t; \tau, H\right) + \int_{t_0}^{t} \Sigma^{d}(u;t,\tau) dZ_u^{d} \right) \qquad (3)$$

The foreign bond price is given by

$$P_{t,\tau}^{f} = \frac{P_{t_0,\tau}^{f}}{P_{t_0,t}^{f}} \exp\left(q\left(t_0, t; \tau, H\right) - \tfrac{1}{2} h^{f}\left(t_0, t; \tau, H\right) + \int_{t_0}^{t} \Sigma^{f}(u;t,\tau) dZ_u^{d} \right) \qquad (4)$$

Finally, the spot exchange rate is

$$S_t = \frac{S_{t_0} P_{t_0,t}^{f}}{P_{t_0,t}^{d}} \exp\left(\begin{array}{l} \tfrac{1}{2}\left[-h^{d}(t_0,t;H,H) + h^{f}(t_0,t;H,H) + h^{S}(t_0,t;H) \right] \\ \qquad - \int_{t_0}^{t} v_{u,t}^{d} dZ_u^{d} + \int_{t_0}^{t} v_{u,t}^{f} dZ_u^{d} + \int_{t_0}^{t} v_S dZ_u^{d} \end{array} \right) \qquad (5)$$

The separable deterministic volatility (SDV) model, like all multi-currency interest rate models, is characterized by its volatility structure. The SDV model has a volatility structure of the following form:

$$
\begin{aligned}
&\sigma_i^d(u,t) = x_i^d(u) y_i^d(t), \qquad i = 1,\ldots,D \\
&\sigma_i^f(u,t) = x_i^f(u) y_i^f(t), \qquad i = 1,\ldots,D \qquad (6) \\
&v_{S,u} = \sigma_S(u)
\end{aligned}
$$

In this setting, the functions $h^*(t_0,t;\tau,H)$, $q(t_0,t;\tau,H)$. and $h^S(t_0,t;H)$ are deterministic, and one obtains the zero-coupon bond price and exchange rate processes:

The dynamics in this model are thus represented by the following three-dimensional stochastic processes:

$$
\begin{aligned}
&\tilde{A}_{i,t}^d = \int_{t_0}^{t} x_i^d(u) dZ_{i,u}^d \\
&\tilde{A}_{i,t}^f = \int_{t_0}^{t} x_i^f(u) dZ_{i,u}^d \qquad (7) \\
&\tilde{A}_{i,t}^S = \int_{t_0}^{t} \left(x_i^d(u) Y_i^d(u,t) - x_i^f(u) Y_i^f(u,t) + \sigma_{S,i}(u) \right) dZ_{i,u}^d
\end{aligned}
$$

These processes are jointly normally distributed with mean 0 and a covariance matrix that may be readily computed. This is done in detail in a special case of the model in the next section.

In this section, we present what is in some sense the minimal SDV dual-currency model. We will see that this model captures domestic curve risk, foreign curve risk, and FX risk while employing three factors, the minimal number required in order to do so. This particular parametrization of the SDV model is particularly convenient for applications.

Put $D=3$. Let

$$R=\begin{pmatrix} 1 & \rho_{df} & \rho_{dS} \\ \rho_{df} & 1 & \rho_{fS} \\ \rho_{dS} & \rho_{fS} & 1 \end{pmatrix} \tag{8}$$

be a positive definite correlation matrix with *constant* entries. Let $\Lambda=(\lambda_{ij})$ be any 3×3 matrix such that $\Lambda\Lambda^{\mathrm{tr}}=R$. The subscripts in Λ are from the ordered set $\{d,f,S\}$. Put $W_t=\Lambda Z_t^d$; W is a correlated Q^d-Brownian motion with quadratic variation $\langle W_i,W_j\rangle_t=\rho_{ij}t$.

The volatility structure is defined as follows: Let x_*, y^*, and σ_S be positive deterministic real-valued functions. In the notation of the previous section, put

$$\begin{aligned} x_i^d(u) &= x_d(u) \\ y_i^d(s) &= \lambda_{di}y^d(s) \end{aligned} \tag{9}$$

Let $Y^*(t_1,t_2)=\int_{t_1}^{t_2} y^*(s)ds$. In this situation, one can show that (9) may be written as follows:

$$\begin{aligned}
P_{t,\tau}^{d} &= \frac{P_{t_0,\tau}^{d}}{P_{t_0,t}^{d}} \exp\left(-\tfrac{1}{2} h^{d}\left(t_0,t;\tau,H\right) - Y^{d}\left(t,\tau\right)\int_{t_0}^{t} x_d(u)dW_{d,u} \right) \\
P_{t,\tau}^{f} &= \frac{P_{t_0,\tau}^{f}}{P_{t_0,t}^{f}} \exp\left(q\left(t_0,t;\tau,H\right) - \tfrac{1}{2} h^{f}\left(t_0,t;\tau,H\right) - Y^{f}\left(t,\tau\right)\int_{t_0}^{t} x_f(u)dW_{f,u} \right) \\
S_t &= \frac{S_{t_0} P_{t_0,t}^{f}}{P_{t_0,t}^{d}} \exp\left(\begin{array}{l} \tfrac{1}{2}\left[-h^{d}\left(t_0,t;H,H\right) + h^{f}\left(t_0,t;H,H\right) + h^{S}\left(t_0,t;H\right) \right] \\ \quad + \int_{t_0}^{t} x_d(u)Y^{d}\left(u,t\right)dW_{d,u} - \int_{t_0}^{t} x_f(u)Y^{f}\left(u,t\right)dW_{f,u} + \int_{t_0}^{t} \sigma_S(u)dW_{S,u} \end{array} \right)
\end{aligned} \tag{10}$$

These processes are jointly normally distributed with mean zero and covariance matrix $C(t_0,t)$ which will be explicitly determined below. For the moment, the introduction of these processes allows one to write more compactly as follows:

$$\begin{aligned}
P_{t,\tau}^{d} &= \frac{P_{t_0,\tau}^{d}}{P_{t_0,t}^{d}} \exp\left(-\tfrac{1}{2} h^{d}\left(t_0,t;\tau,H\right) - Y^{d}\left(t,\tau\right)\left(A_t^{d} - A_{t_0}^{d}\right)\right) \\
P_{t,\tau}^{f} &= \frac{P_{t_0,\tau}^{f}}{P_{t_0,t}^{f}} \exp\left(q\left(t_0,t;\tau,H\right) - \tfrac{1}{2} h^{f}\left(t_0,t;\tau,H\right) - Y^{f}\left(t,\tau\right)\left(A_t^{f} - A_{t_0}^{f}\right)\right) \\
S_t &= \frac{S_{t_0} P_{t_0,t}^{f}}{P_{t_0,t}^{d}} \exp\left(\tfrac{1}{2}\left[-h^{d}\left(t_0,t;H,H\right) + h^{f}\left(t_0,t;H,H\right) + h^{S}\left(t_0,t;H\right) \right] + \left(A_t^{S} - A_{t_0}^{S}\right)\right)
\end{aligned} \tag{11}$$

In the next section, this is rendered completely explicit, by building a formalism which allows for an efficient expression of $h^{*}\left(t_0,t;\tau,H\right)$, $q\left(t_0,t;\tau,H\right)$ and $h^{S}\left(t_0,t;H\right)$, as well as the covariance matrix of the state vector $A_t = \left(A_t^{d}, A_t^{f}, A_t^{S}\right)$. This will completely characterize the market dynamics in terms of the state variable A. The formalism introduced will also be of value in other contexts, e.g. parameter estimation for the model, and in the valuation of options on foreign exchange.

Valuation of the fixed coupon notes is obvious. To value the four floating notes, one requires the following:

$$\Pi^{\mathrm{DDF}}(t, t_i, t_i', s_i) = E_t^{Q_{s_i}^d}\left[1 / P_{t_i, t_i'}^d\right]$$

$$\Pi^{\mathrm{FDF}}(t, t_i, t_i', s_i) = E_t^{Q_{s_i}^d}\left[S_{s_i} / P_{t_i, t_i'}^d\right]$$

$$\Pi^{\mathrm{DFF}}(t, t_i, t_i', s_i) = E_t^{Q_{s_i}^d}\left[1 / P_{t_i, t_i'}^f\right]$$

$$\Pi^{\mathrm{FFF}}(t, t_i, t_i', s_i) = E_t^{Q_{s_i}^d}\left[S_{s_i} / P_{t_i, t_i'}^f\right]$$

The analytic tractability of the constant correlation SDV model provides closed-form formulas for these values as functions of the yield curves and exchange rate at time t.

To value a contingent claim on a portfolio of notes of the type defined in section 6.1, observe that, via the formulas above for the $\Pi^{\#}$, the value of such a portfolio at time $t = H$ is an explicit deterministic function of the two zero coupon curves and exchange rate at time H. In turn, this means that the portfolio value is an explicit deterministic function φ of the normally distributed state variable A_H. The option value is then of the form $P_{0,H}^d E^{Q_H^d}\left[\varphi(A_H)\right]$; this expectation may be evaluated numerically.

.

2. Connection to Short Rate Model

In this section, we show that the short rate model in each currency in the SDV constant correlation model is Vasicek/Hull-White. We suppress the currency superscript for ease of notation.

In this setting, the volatility σ of the instantaneous forward rate is given by

$$\sigma(u,t) = x(u)\,y(t) \qquad (13)$$

Recall that the short rate dynamics are given by the SDE

$$dr_u = \left[\frac{\partial f}{\partial u}(t_0,u) + \sigma(u,u)\cdot \nu_{u,H} + \int_{t_0}^{u} \frac{\partial \sigma(s,u)\cdot \Sigma(s;u,H)}{\partial u}\,ds + \int_{t_0}^{u} \frac{\partial \sigma(s,u)}{\partial u}\,dW_s\right]du + \sigma(u,u)\,dW_u,$$

(14)

in which $\nu_{u,H} = \Sigma(u,u,H) = -x(u)Y(u,H)$. We can rewrite this in the present context as follows: Put

$$\mu = \frac{\partial f}{\partial u}(t_0,u) - x(u)^2 y(u)Y(u,H) + y(u)^2 \int_{t_0}^{u} x(s)^2\,ds - y'(u)Y(u,H)\int_{t_0}^{u} x(s)^2\,ds + y'(u)\int_{t_0}^{u} x(s)\,dW_s$$

(15)

Then

$$dr_u = \mu_u du + x(u)y(u)dW_u \qquad (16)$$

We therefore can simplify the expression for μ:

$$\mu = \frac{\partial f}{\partial u}(t_0,u) - x(u)^2 y(u)Y\left(u,H\right) + y(u)^2 \int_{t_0}^{u} x(s)^2 ds - \frac{y'(u)}{y(u)} f(t_0,u) + \frac{y'(u)}{y(u)} r_u . \qquad (17)$$

Then the SDE for the short rate may be rewritten as

$$dr_u = \left[\theta(t_0,u,H) + \frac{y'(u)}{y(u)} r_u\right] du + x(u)y(u)dW_u . \qquad (18)$$

Thus one sees that this is Vasicek/Hull-White with mean reversion $\kappa(u) = -\frac{y'(u)}{y(u)}$ and short rate absolute volatility $\sigma(u) = x(u)y(u)$.

3. Calibration

In this particular model, parameter estimation consists in estimating the following quantities: 1) the interest rate volatility structure *functions* $x_*(s)$ and $y^*(u)$ for $* = d, f$; 2) the foreign exchange volatility *function* $\sigma_S(u)$; 3) the correlation coefficients ρ_{df} , ρ_{dS} , and ρ_{fS} .

The interest rate volatility functions may be estimated as in a single currency calibration problem. Typically, this is done by some sort of least squares error minimisation between model and market prices for liquid caps or swaptions, depending on the application to the derivative to be priced.

Estimating FX volatility and correlation is a bit more problematic. First, the FX volatility is *not* the Black-Scholes implied volatility. Indeed, the SDV model can be used to compute prices of FX options, and these model prices take into account interest rate risk (i.e., depend also upon $x_*(s)$ and $y^*(u)$), unlike Black-Scholes and its variants.

More problematic are the correlation coefficients. *A priori*, these are the correlations between the Brownian motions driving the evolution of the zero-coupon curves and exchange rate. Some method of relating these to observed quantities is required. Another problem is the assumption of constant correlation. However, correlation is estimated, it will almost surely be at least term-dependent, and so some sensible method of arriving at a single number must be devised.

In this section, we present an approach that proceeds using historical data to estimate correlation. Essentially, historical data is used to estimate the correlation coefficients, and then the FX volatility is adjusted to match FX option prices.

Fix a time horizon Δt and a maturity m, and put for convenience $t = t_0 + \Delta t$. Let $R^*(s, M)$ denote the continuously compounded zero coupon rate of maturity M observed at time s (i.e., maturity is at "calendar time" $s + M$).

We have explicitly included the dependence on maturity; this will be removed later.

On the other hand,

$$\mathrm{Cov}\left(\Delta\log P^d,\Delta\log P^f\right)=\tilde{\rho}_{df}(m)\mathrm{sd}\left(\Delta\log P^d\right)\mathrm{sd}\left(\Delta\log P^f\right), \qquad (19)$$

where $\tilde{\rho}_{df}(m)$ is the *sample* correlation between $\Delta\log P^d$ and $\Delta\log P^f$. Because $\Delta\log P^* = -m\Delta R^*$, $\tilde{\rho}_{df}(m)$ is also the sample correlation between the changes in zero coupon rates.

From the model, $\mathrm{sd}\left(\Delta\log P^*\right)= X_{**}\left(t_0,t\right)^{\frac{1}{2}}Y^*\left(t,t+m\right)$. Therefore

$$\rho_{df}(m)=\tilde{\rho}_{df}(m)\frac{X_{dd}\left(t_0,t\right)^{\frac{1}{2}}X_{ff}\left(t_0,t\right)^{\frac{1}{2}}}{X_{df}\left(t_0,t\right)}. \qquad (20)$$

In practise, Δt should be chosen to be small, e.g. 1 day. Since

$$\lim_{t\to t_0}\frac{X_{dd}\left(t_0,t\right)^{\frac{1}{2}}X_{ff}\left(t_0,t\right)^{\frac{1}{2}}}{X_{df}\left(t_0,t\right)}=1, \qquad (21)$$

it is convenient then to take the approximation $\rho_{df}(m)=\tilde{\rho}_{df}(m)$.

Put $\Delta \log S = \log S_{t_0+\Delta t} - \log S_{t_0}$. Let $\tilde{\rho}_{*S}(m)$ be the sample correlation coefficient between the change in zero coupon rates ΔR^* and $\Delta \log S$. From the model,

$$\mathrm{Cov}\left(\Delta \log P^*, \Delta \log S\right) = -C_{*S}\left(t_0, t\right) Y^*\left(t, t+m\right). \qquad (22)$$

On the other hand,

$$\begin{aligned}\mathrm{Cov}\left(\Delta \log P^*, \Delta \log S\right) &= -\tilde{\rho}_{*S}(m)\,\mathrm{sd}\left(\Delta \log P^*\right)\mathrm{sd}\left(\Delta \log S\right) \\ &= -\tilde{\rho}_{*S}(m) X_{**}\left(t_0, t\right)^{\frac{1}{2}} Y^d\left(t, t+m\right) C_{SS}\left(t_0, t\right)^{\frac{1}{2}}.\end{aligned} \qquad (23)$$

Using (22) or (23), one obtains

$$\rho_{*S}(m) = \frac{\tilde{\rho}_{*S}(m) X_{**}\left(t_0, t\right)^{\frac{1}{2}} C_{SS}\left(t_0, t\right)^{\frac{1}{2}} + \rho_{df}(m) X_{df}^{\hat{*}}\left(t_0, t, t\right) - X_{**}^{*}\left(t_0, t, t\right)}{X_{*S}\left(t_0, t\right)} \qquad (24)$$

where $\hat{d} = f$ and $\hat{f} = d$. One can show that

$$\lim_{t \to t_0} \frac{\tilde{\rho}_{*S}(m) X_{**}\left(t_0, t\right)^{\frac{1}{2}} C_{SS}\left(t_0, t\right)^{\frac{1}{2}} + \rho_{df}(m) X_{df}^{\hat{*}}\left(t_0, t, t\right) - X_{**}^{*}\left(t_0, t, t\right)}{X_{*S}\left(t_0, t\right)} = \tilde{\rho}_{*S}(m). \qquad (25)$$

It is therefore convenient to take the approximation $\rho_{*S}(m) = \tilde{\rho}_{*S}(m)$.

4. Numerical Results

The model volatility parameters are calibrated from returns of the zero rates. They obtain the variance of absolute returns on zero rates and try to match them by finding the best matching parameter values in the sense of least square. We use the parameters calibrated by the model to simulate the interest rate curves at given time buckets. The initial interest rate curve used for calibration is given below.

Table 1: Initial Interest Rate Curve

tenor(yr)	zero rate	tenor(yr)	zero rate	tenor(yr)	zero rate
0.002739726	0.000890098	1	0.002770497	8	0.026033924
0.019178082	0.001230174	1.5	0.003647494	9	0.02786755
0.038356164	0.001310163	2	0.005317523	10	0.029439501
0.083333333	0.001492994	3	0.009581143	12	0.032035763
0.166666667	0.001937198	4	0.013924717	15	0.034568351
0.25	0.002257539	5	0.017825326	20	0.036708282
0.5	0.002403464	6	0.021113788	25	0.037566367
0.75	0.0025523	7	0.023788876	30	0.03784863

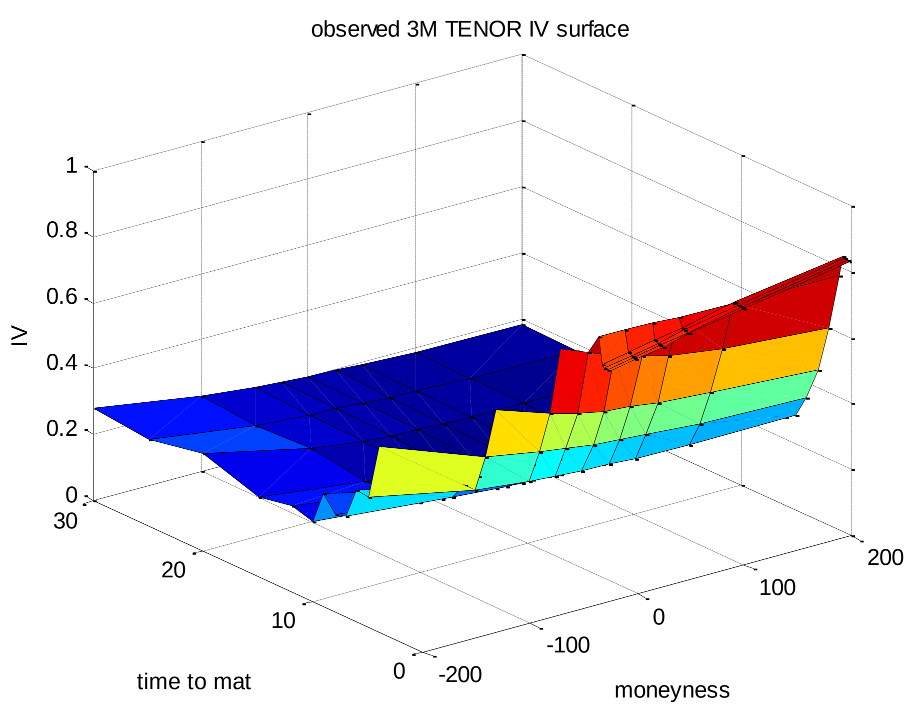
We first check the expected value of bond prices at each time bucket. Since drift parameters are piecewise constant and other parameters are constant, we can obtain analytic (closed) form of expected bond values at each time and at each maturity.

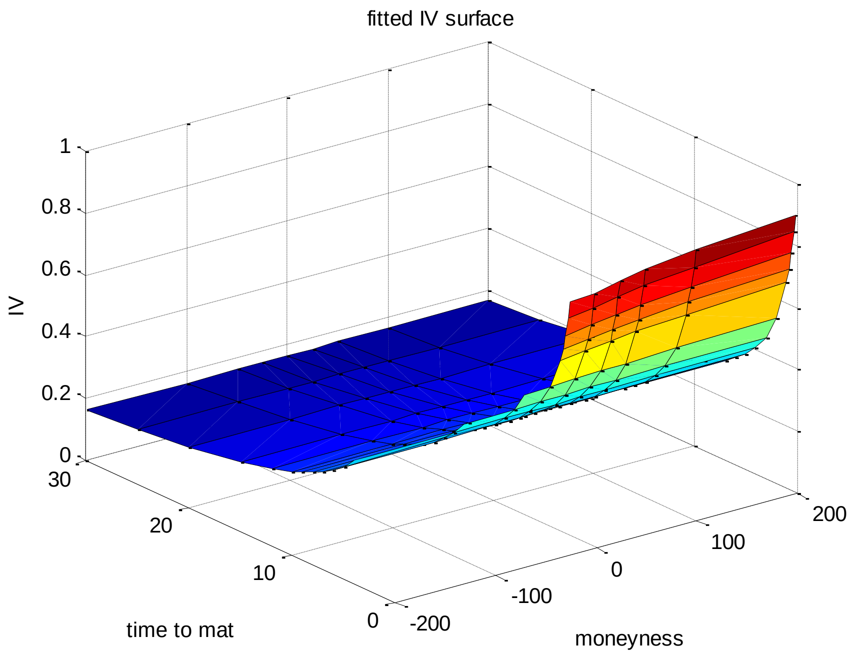
We want to maintain positive mean reversion speeds a and b while calibrating the parameters in order to keep the mean reversion feature in the model. We may tolerate one of mean reversion speeds to become negative. If this happens, we need to make sure that it remains small (-0.3 as a bound).

However, we need to be cautious if the mean reversion values are large. Any value bigger than four may not be desirable. We also want the correlation to be negative, but not at negative one. If the “naive" calibration results yield the correlation of negative one, we may manually set a parameter to another value so that we can insist on the flexibility of the two-factor model.

One characteristic of interest rate implied volatility surface is that the implied vol usually peaks at time to maturity about 18 months. The model will not be able to accommodate this feature since it’s convex. However, the value reconciliation will scale it up. The following graphs are based on observed market implied volatility.

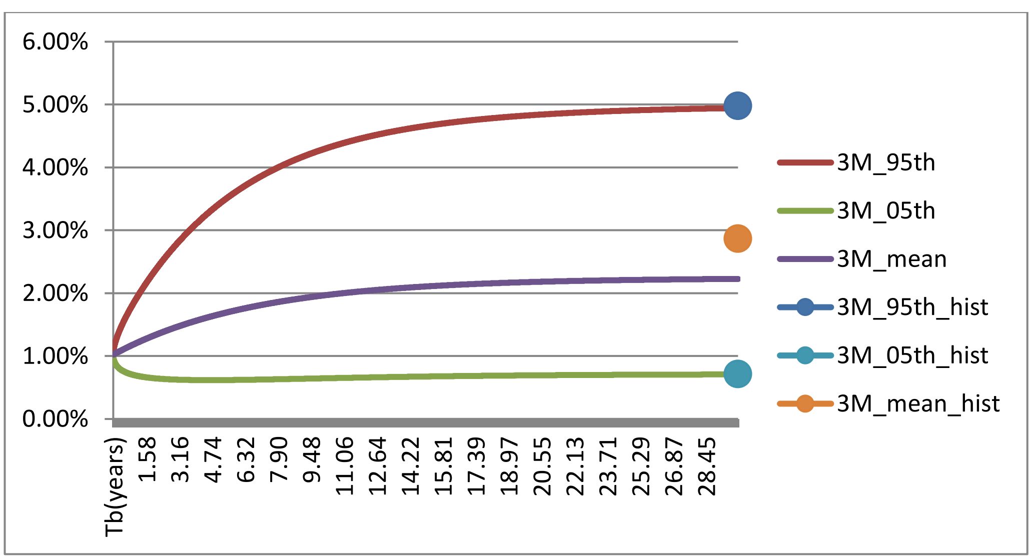
Figure 1: Market Implied Volatility





The plots to follow, beyond displaying the resulting simulated shape produced by the model, stress the flexibility that the model possesses due to the various calibrations proposed.

Figure 2: Simulation Results



5. Conclusion

This article presents an interest rate model for cross currency markets. All equations use the cash account as the numeraire. There are situations when another choice of numeraire can simplify calculations or add stability to numerical methods. We document some common alternatives and will note the choice of numeraire and induced measure explicitly.

All the results have been for a general forward rate volatility function. We also choose a specific form of the volatility that leads to a Markov, affine interest rate model. The model establishes a general result for the drift of the forward rate required to ensure an interest rate model is arbitrage free.

Furthermore, we formulate the multi-factor gaussian interest rate model as a multi dimensional partial differential equation. We also examine valuation of European style (see https://finpricing.com/lib/EqWarrant.html) contingent claims. The claims which shall be the focus of our attention have the property that they are options to exchange a portfolio of domestic notes for a portfolio of foreign notes or vice-versa, in which each note payment is either fixed or related to a par floating rate, and such that payments occur at known times.

Intuitively, multi-factor instantaneous forward rate term structure is unlikely to guarantee positive forward rate on the curve, since the two points on the curve are driven by factors not necessarily meeting that condition.

References:

Akram, T., and Li, H, 2020, An inquiry concerning long-term U.S. interest rates using monthly data, Applied Economics, 52 (24), 2594-2621.

Bauer, M.D. and Hamilton, J.D., 2018, Robust bond risk premia, Review of Financial Studies, 31, 399-448.

Cox, J., Ingersoll, J., and Ross, S. (1985), A theory of the term structure of interest rates, Econometrica, 53 (2), 385-407.

Engel, C., 2016, Exchange rates, interest rates, and risk premium, The American Economic Review, 106 (2), 436-474.

Hansen, A., 2023, A joint model for the term structure of interest rates and realized volatility, Journal of Financial Econometrics, 21 (4), 1196-1227.

Kikuchi, K. (2024), A term structure interest rate model with the Brownian bridge lower bound, Ann Finance.

Medova, E.A., Rietbergen, M.I., Villaverde, M. and Yong, Y.S., 2006, Modelling the long-term dynamics of yield curves, working paper, 2006.

Levrero, E.S., and Deleidi, M., 2019 The causal relationship between short- and long-term interest rates: an empirical assessment of the United States," MPRA paper No. 93608.

Vasicek, O, (1977), An equilibrium characterization of the term structure, Journal of Financial Economics, 5 (2), 177-188.

Verschuren, R. (2020), Stochastic interest rate modelling using a single or multiple curves: an empirical performance analysis of the Levy forward price model, Quantitative Finance, 20 (7), 1123-1148.

Yu, S. and Ning, Y., (2019), An interest rate model with jumps for uncertain financial markets, Physica A: Statistical Mechanics and its Applications, 527 (1).