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# Strong normalization of a symmetric lambda calculus for second order classical logic

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**Abstract** We extend Barbanera and Berardi's symmetric lambda calculus [2] to second order classical propositional logic and prove its strong normalization.

## 1 Introduction

In late 1980's, T. Griffin's observation [5] on relation between the law of excluded middle and control operators in programming languages stimulates general interest in reduction rules for classical logic. In such studies, one often encounters non-determinacy, in the sense that the same deduction has different normal forms. The most well known example arises when we consider cut-elimination of sequent calculi. Another example occurs in a $\lambda\mu$-calculus with symmetric structural reduction rules, which Parigot suggests in order to ensure that normal forms of the natural number type are Church numerals [6].

In spite of these examples, non-deterministic reduction for classical logic do not seem well studied except some systems for propositional or first order logic [1], [2], [3]. One of the reasons of this situation is that, to the author's knowledge, strong normalization of such calculi for higher order logic is not known. In this paper, we prove strong normalization of an extension of Barbanera and Berardi's symmetric lambda calculus to second order classical propositional logic.

Our method of proving strong normalization is, as one may expect, Tait-Girard's method of reducibility candidates. Parigot has already used such a

*Mathematics Subject Classification:* 03F05, 03B40 method in his proof of strong normalization for a second order $\lambda\mu$-calculus [7]. Girard also gives a proof of strong normalization of classical linear logic using an adaptation of Tait-Girard's method to Tait calculi [4]. Unfortunately, their methods, which one could consider a negative translation, do not seem to work on a non-deterministic calculus like a symmetric lambda calculus. Barbanera and Berardi discover a suitable definition of reducibility for such a calculus. But since their notion of reducibility of a formula $A$ mutually depends on the notion of reducibility of its negation $A^{\perp}$, characterization of reducibility candidates is not obvious. We will overcome this difficulty by extending Barbanera and Berardi's construction of reducibility candidates to infinitary logical connectives and defining reducibility candidates as the smallest set closed under such construction.

The organization of this paper is the following. In Section 2, we introduce $\lambda^2_{sym}$, an extension of Barbanera and Berardi's symmetric lambda calculus. Section 3 is devoted to prove its strong normalization.

## 2 Description of $\lambda^2_{sym}$

In this section, we present $\lambda^2_{sym}$, a symmetric lambda calculus for second order classical propositional logic.

**Definition 1 (Proper types)** Type variables *are symbols* $X_1, X_2, \cdots$. *We use* $X, Y, \cdots$ *as metasymbols of them.* Proper types *(denoted* $A, B, A_i, \cdots$*) are defined inductively as follows.*

1. *If $X$ is type variable, $X$ and $X^{\perp}$ are proper types.*
2. *If $A_1$ and $A_2$ are proper types, $A_1 \wedge A_2$ and $A_2 \vee A_2$ are proper types.*
3. *If $A$ is a proper type and $X$ is a type variable, $\forall XA$ and $\exists XA$ are proper types. These constructs bind $X$ in $A$.*

Negation $A^{\perp}$ *for each proper type $A$ is defined by De Morgan's law and double negation elimination. The* substitution $A[B/X]$ *is defined as the usual manner.*

**Definition 2 (Types)** Types *(denoted* $C, D, C_i, \cdots$*) are proper types and the symbol* $\perp$.

**Definition 3 (Terms)** Variables of a proper type $A$ *are symbols* $x^A_1, x^A_2, \cdots$. *We use* $x, y, \cdots$ *as metasymbols.* Terms of type $C$ *(denoted* $t, u, t_i, u_i \cdots$*) is defined inductively as follow.*

1. *If $x$ is a variable of a proper type $A$, $x$ is a term of type $A$.*
2. *If $t_i$ is a term of a proper type $A_i$ for $i = 1$ and $2$, $\langle t_1, t_2\rangle$ is a term of type $A_1 \wedge A_2$*

3. *If $t$ is a term of a proper type $A_i$ for $i = 1$ or $2$, $\sigma_i(t)$ is a term of type $A_1 \vee A_2$.*
4. *If $t$ is a term of a proper type $A$ and does not contain a free variable whose type has $X$ as a free type variable, $\varPi X.t$ is a term of type $\forall X A$. This construct binds $X$ in $t$.*
5. *If $t$ is a term of a proper type $A[B/X]$, $(B)t$ is a term of type $\exists X A$.*
6. *If $t_1$ is a term of a proper type $A$ and $t_2$ is of $A^{\perp}$, $t_1 * t_2$ is a term of type $\perp$.*
7. *If $t$ is a term of type $\perp$ and $x$ is a variable of a proper type $A$, $\lambda x.t$ is a term of type $A^{\perp}$. This construct binds $x$ in $t$.*

*The* substitution $t[B_1/X_1, \cdots, u_1/x_1, \cdots]$ *are defined as a term obtained from $t$ by replacing each free occurrence of $X_i$ and $x_i$ by $B_i$ and $u_i$.*

**Definition 4 (Reduction rules)** *The* basic reduction rules of $\lambda^2_{sym}$ *are the following.*

$$\begin{array}{lcc}
(\beta) & (\lambda x.t) * u \Rightarrow t[u/x] & t * (\lambda x.u) \Rightarrow u[t/x] \\
(\pi) & \langle t_1, t_2\rangle * \sigma_i(u) \Rightarrow t_i * u & \sigma_i(t) * \langle u_1, u_2\rangle \Rightarrow t * u_i \\
(\varPi) & \varPi X.t * (A)u \Rightarrow t[A/X] * u & (A)t * \varPi X.u \Rightarrow t * u[A/X] \\
(\eta) & \lambda x.t * x \Rightarrow t & \lambda x.x * t \Rightarrow t \\
(Triv) & E[t] \Rightarrow t &
\end{array}$$

*where in $\eta$-rules, $x$ is not a free variable of $t$ and in $Triv$ -rules, the types of $t$, $E[-]$ are $\perp$ and $E[-]$ does not bind any free variables of $t$ .*

*The* one-step reduction relation *(denoted $\Rightarrow_1$) is defined as the compatible closure of the basic reduction rules. The* reduction relation *(denoted $\Rightarrow$) is defined as the reflexive and transitive closure of the one-step reduction relation.*

## 3 Strong normalization of $\lambda^2_{sym}$

**Definition 5 (Reduction sequence, strong normalizability)** *For a term $t$, a* reduction sequence of $t$ *is defined as a sequence of terms $t_1, t_2, \cdots$ where $t_1 = t$ and $t_i \Rightarrow_1 t_{i+1}$ for all $i = 1, 2, \cdots$. If all reduction sequences of $t$ are finite, $t$ is* strongly normalizable.

**Theorem 1** *All terms of $\lambda^2_{sym}$ are strongly normalizable.*

The rest of the paper is devoted to prove Theorem 1. In Subsection 3.1, we define operators on sets of terms which correspond logical connectives. The set of reducibility candidates is defined as the smallest set closed under these operators in Subsection 3.2. Finally, we prove Theorem 1 in Subsection 3.3.

*3.1 Interpretations of logical connectives*

$\mathcal{V}_A, \mathcal{T}_A, \mathcal{N}_A$ are defined as the sets of variables, terms, strongly normalizable terms of type $A$ respectively. $\mathcal{T}, \mathcal{N}$ are the unions of all $\mathcal{T}_A, \mathcal{N}_A$. $I$ denotes non-empty sets of indices. If a term $t$ is strongly normalizable, $w(t)$ is the maximal length of reduction sequences of $t$.

**Definition 6** *1. For $\beta \subset \mathcal{T}_A$, $\beta^\perp$ is the set $\{t \in \mathcal{T}_{A^\perp} | \forall u \in \beta, t * u \in \mathcal{N}\}$.*
*2. For $\beta \subset \mathcal{T}_A$, $\mathcal{L}(\beta)$ is the set $\{\lambda x.t | x \in \mathcal{V}_A, \forall u \in \beta, t[u/x] \in \mathcal{N}\}$.*
*3. $\mathrm{Neg}_A(\beta)$ for $\beta \subset \mathcal{T}_{A^\perp}$ denotes the set $\mathcal{V}_A \cup \beta^\perp \cup \mathcal{L}(\beta)$*
*4. $\mathrm{Pair}(\beta_1, \beta_2)$ denotes the set $\{\langle t_1, t_2\rangle | t_1 \in \beta_1, t_2 \in \beta_2\}$.*
*5. Let $(B_i)_{i\in I}$ be a family of proper types and $(\beta_i)_{i\in I}$ be a family of $\beta_i \subset \mathcal{T}_{A[B_i/X]}$. We define as follows.*

$$\prod_{i\in I} \beta_i := \{\Pi X.t \in \mathcal{T}_{\forall X A} | \forall i \in I, t[B_i/X] \in \beta_i\}$$

**Definition 7** *Let $\beta_i \subset \mathcal{T}_{A_i}$ for $i = 1, 2$ and $\mathrm{Neg}_{\beta_1 \wedge \beta_2}(\beta)$ be $\mathcal{V}_{A_1 \wedge A_2} \cup \mathrm{Pair}(\beta_1, \beta_2) \cup \mathcal{L}(\beta)$ for $\beta \subset \mathcal{T}_{{A_1}^\perp \vee {A_2}^\perp}$. $\beta_1 \wedge \beta_2$ and $\beta_1 \vee \beta_2$ are defined as follows.*

$$\begin{aligned} \beta_1 \wedge \beta_2 &:= \text{the least fixed point of } \mathrm{Neg}_{\beta_1 \wedge \beta_2} \circ \mathrm{Neg}_{{A_1}^\perp \vee {A_2}^\perp} \\ \beta_2 \vee \beta_2 &:= \qquad\qquad \mathrm{Neg}_{A_1 \vee A_2}({\beta_1}^\perp \wedge {\beta_2}^\perp) \end{aligned}$$

*Remark.* Since $\mathrm{Neg}_{\beta_1 \wedge \beta_2}$ and $\mathrm{Neg}_{{A_1}^\perp \vee {A_2}^\perp}$ are decreasing operators on countable sets, $\mathrm{Neg}_{\beta_1 \wedge \beta_2} \circ \mathrm{Neg}_{{A_1}^\perp \vee {A_2}^\perp}$ is the increasing operator on countable sets. Let

$$X_\nu := \mathrm{Neg}_{\beta_1 \wedge \beta_2} \circ \mathrm{Neg}_{{A_1}^\perp \vee {A_2}^\perp}(\bigcup_{\mu<\nu} X_\mu)$$

for an ordinal $\nu$. Then $X_{\omega_1}$ is the least fixed point of the operator, where $\omega_1$ is the first uncountable ordinal.

**Definition 8** *Let $(B_i)_{i\in I}$ be a family of proper types and $(\beta_i)_{i\in I}$ be a family of $\beta_i \subset \mathcal{T}_{A[B_i/X]}$. $\mathrm{Neg}_{\bigwedge \beta_i}(\beta)$ is defined as $\mathcal{V}_{\forall X A} \cup \prod_{i\in I} \beta_i \cup \mathcal{L}(\beta)$ for $\beta \subset \mathcal{T}_{\exists X A^\perp}$. Then $\bigwedge_{i\in I} \beta_i$ and $\bigvee_{i\in I} \beta_i$ are defined as follows.*

$$\begin{aligned} \bigwedge_{i\in I} \beta_i &:= \text{the least fixed point of } \mathrm{Neg}_{\bigwedge \beta_i} \circ \mathrm{Neg}_{\exists X A^\perp} \\ \bigvee_{i\in I} \beta_i &:= \qquad\qquad \mathrm{Neg}_{\exists X A}(\textstyle\bigwedge_{i\in I} {\beta_i}^\perp) \end{aligned}$$

*Remark.* Similarly to the above remark, $X_{\omega_1}$ is the least fixed point of $\mathrm{Neg}_{\bigwedge \beta_i} \circ \mathrm{Neg}_{\exists X A^\perp}$ if we define

$$X_\nu := \mathrm{Neg}_{\bigwedge \beta_i} \circ \mathrm{Neg}_{\exists X A^\perp}(\bigcup_{\mu<\nu} X_\mu)$$

for an ordinal $\nu$.

**Lemma 1** *The following hold.*

1. *Let $\emptyset \neq \beta_i \subset \mathcal{N}_{A_i}$ for $i = 1, 2$. Suppose $\beta_1, \beta_2$ are closed under the reduction relation. Then if $t \in \beta_i{}^\perp$ for some $i \in I$, $\sigma_i(t) \in (\beta_1 \wedge \beta_2)^\perp$.*
2. *Let $(B_i)_{i\in I}$ be a family of types and $(\beta_i)_{i\in I}$ be a family of $\emptyset \neq \beta_i \subset \mathcal{N}_{A[B_i/X]}$. Suppose $\beta_i$ is closed under the reduction relation for each $i \in I$. Then if $t \in \beta_i{}^\perp$ for some $i \in I$, $(B_i)t \in (\bigwedge_{i\in I} \beta_i)^\perp$.*

*Proof* We only prove 2. We prove $(B_i)t \in X_{\omega_1}{}^\perp$ by induction on $\omega_1$. ($X_\nu$ is defined as in Remark of Definition 8.) It suffices to prove that for each $u \in X_\nu$, $(B_i)t * u \in \mathcal{N}$. Let us examine the different possibilities for $u$. Note that $u \in \mathcal{N}$ from the fact $\beta_i \subset \mathcal{N}$.

First, we consider the case where $u \in \mathcal{V}_{\forall X A}$. The thesis holds because $t \in \mathcal{N}$.

The case where $u \in \prod_{i\in I} \beta_i$. Then we have $u \equiv \mathit{\Pi} X.u_1$ and $\forall i \in I$, $u_1[B_i/X] \in \beta_i$. We examine the different possibilities for one-step reduction of $(B_i)t * u$.

1. The case where $(B_i)t * u \Rightarrow_1 (B_i)t' * u'$ for $t \Rightarrow t'$ and $u \Rightarrow u'$. Since $t' \in \beta_i{}^\perp$ and $u' \in \prod_{i\in I} \beta_i$, we have the thesis by induction hypothesis on $w(t) + w(u)$.
2. The case where $(B_i)t * u \Rightarrow_1 t * u_1[B_i/X]$. Since $u_1[B_i/X] \in \beta_i$, $t * u_1[B_i/X] \in \mathcal{N}$.
3. $(B_i)t * u \Rightarrow_1 s$ and $s$ is a subterm of $t$ or $u$. Since $t$ and $u$ are strongly normalizable, their subterm $s$ is also strongly normalizable.

The case where $u \in \mathcal{L}(\mathrm{Neg}_{\exists X A^\perp}(\bigcup_{\mu<\nu} X_\mu))$. Let $u \equiv \lambda x.u_1$. We examine the different possibilities for one-step reduction of $(B_i)t * u$.

1. The case where $(B_i)t * \lambda x.u_1 \Rightarrow_1 (B_i)t' * \lambda x.u_1'$. Since $t' \in \beta_i{}^\perp$ and $\lambda x.u_1' \in \mathcal{L}(\mathrm{Neg}_{\exists X A^\perp}(\bigcup_{\mu<\nu} X_\mu))$ hold, the thesis follows from induction hypothesis on $w(t) + w(u_1)$.
2. The case where $(B_i)t * u \Rightarrow_1 u_1[(B_i)t/x]$ by $(\beta)$ or $(\eta)$. By induction hypothesis,
$$(B_i)t \in \bigcap_{\mu<\nu} X_\mu{}^\perp \subset \mathrm{Neg}_{\exists X A^\perp}(\bigcup_{\mu<\nu} X_\mu).$$
From the hypothesis of $u$, we have the thesis.
3. $(B_i)t * u \Rightarrow_1 s$ and $s$ is a subterm of $t$ or $u$. Since $t$ and $u$ are strongly normalizable, their subterm $s$ is also strongly normalizable.

*3.2 Reducibility candidates*

**Definition 9** *For a proper type $A$, we define $\alpha_A \subset \mathcal{T}_A$ and $\overline{\alpha_A} \subset \mathcal{T}_{A^\perp}$ as follows.*

$$\begin{aligned} \alpha_A &:= \text{the least fixed point of } \mathrm{Neg}_A \circ \mathrm{Neg}_{A^\perp} \\ \overline{\alpha_A} &:= \qquad\qquad \mathrm{Neg}_{A^\perp}(\alpha_A) \end{aligned}$$

*Remark.* We have the fact $\alpha_A = \mathrm{Neg}_A(\overline{\alpha_A})$ and $\overline{\alpha_A} = \mathrm{Neg}_{A^\perp}(\alpha_A)$ from the definition above.

**Definition 10** *For each proper type $A$, the set $\mathcal{R}_A$ of reducibility candidates is defined by mutual induction as follows. $\mathcal{R}$ denotes the union of all $\mathcal{R}_A$.*

1. *$\alpha_A \in \mathcal{R}_A$ and $\overline{\alpha_A} \in \mathcal{R}_{A^\perp}$.*
2. *If $\beta_i \in \mathcal{R}_{A_i}$ for $i = 1, 2$, $\beta_1 \wedge \beta_2 \in \mathcal{R}_{A_1 \wedge A_2}$ and $\beta_1 \vee \beta_2 \in \mathcal{R}_{A_1 \vee A_2}$.*
3. *Let $(B_i)_{i \in I}$ be a family of proper types. If $\beta_i \in \mathcal{R}_{A[B_i/X]}$ for each $i \in I$, $\bigwedge_{i \in I} \beta_i \in \mathcal{R}_{\forall X A}$ and $\bigvee_{i \in I} \beta_i \in \mathcal{R}_{\exists X A}$.*

**Proposition 1** *If $\beta \in \mathcal{R}_A$, then $\mathcal{V}_A \subset \beta \subset \mathcal{N}_A$.*

*Proof* $\beta$ can be written $\mathrm{Neg}_1(\mathrm{Neg}_2(\gamma))$ where for $i = 1, 2$, $\mathrm{Neg}_i$ is one of $\mathrm{Neg}_B$, $\mathrm{Neg}_{\beta_1 \wedge \beta_2}$, $\mathrm{Neg}_{\beta_1 \vee \beta_2}$, $\mathrm{Neg}_{\bigwedge \beta_i}$, $\mathrm{Neg}_{\bigvee \beta_i}$. the fact $\mathcal{V}_A \subset \beta$ and $\mathrm{Neg}_2(\gamma) \neq \emptyset$ follow. From induction hypothesis on the construction of $\beta$ and the fact that $\mathrm{Neg}_2(\gamma)$ is non-empty, $\mathcal{V}_A$, $\mathrm{Neg}_2(\gamma)^\perp$, $\mathcal{L}(\mathrm{Neg}_2(\gamma))$, $\mathrm{Pair}(\beta_1, \beta_2)$ in the case of $\beta = \beta_1 \wedge \beta_2$ and $\Pi \beta_i$ in the case of $\beta = \bigwedge_{i\ I} \beta_i$ are subsets of $\mathcal{N}$. We have $\mathrm{Neg}_1(\mathrm{Neg}_2(\gamma)) \subset \mathcal{N}$. □

**Proposition 2** *For $\beta \in \mathcal{R}_A$, the following hold.*

1. *$\beta$ is closed under the reduction relation.*
2. *$\beta^\perp \in \mathcal{R}_{A^\perp}$ and $\beta^{\perp\perp} = \beta$.*

The proof of Proposition 2 is induction on the construction of $\beta$. On each induction step, first we establish the clause 1 of Proposition 2 and next prove the clause 2 of the proposition. Our proof is divided to Lemmata 2, 3, 4.

**Lemma 2** *$\alpha_A$ and $\overline{\alpha_A}$ satisfy the clauses 1, 2 of Proposition 2.*

*Proof* We have the equation

$$\alpha_A = \mathrm{Neg}_A(\overline{\alpha_A}) = \mathcal{V}_A \cup \overline{\alpha_A}^\perp \cup \mathcal{L}(\overline{\alpha_A}).$$

$\mathcal{V}_A$, $\overline{\alpha_A}^\perp$ are closed under the reduction relation. Moreover, if $t \in \mathcal{L}(\overline{\alpha_A})$ and $t \Rightarrow_1 t'$, then $t \in \overline{\alpha_A}^\perp$ in the case where a $\eta$-rule is applied to the outermost $\lambda$ of $t$, or $t \in \mathcal{L}(\overline{\alpha_A})$. The first clause of Proposition 2 for $\alpha_A$ follows.

Next, we prove ${\alpha_A}^\perp \in \mathcal{R}_{A^\perp}$. Eventually, we prove that ${\alpha_A}^\perp$ is equal to $\overline{\alpha_A}$. Since $\overline{\alpha_A} = \mathrm{Neg}_{A^\perp}(\alpha_A)$, ${\alpha_A}^\perp \subset \overline{\alpha_A}$ immediately follows. We prove the converse.

Let $t \in \overline{\alpha_A}$, $u \in \alpha_A$. We prove that $t * u \in \mathcal{N}$. We consider the different possibilities for $t$.

The case where $t \in \mathcal{V}_{A^\perp}$. Since $u \in \mathcal{N}$, we have $t * u \in \mathcal{N}$.

The case where $t \in {\alpha_A}^\perp$. This implies the thesis since $u \in \alpha_A$.

The case where $t \in \mathcal{L}(\alpha_A)$. Let $t \equiv \lambda x.t_1$. We consider the possibilities for one-step reduction of $t * u$.

1. $t * u \Rightarrow_1 t' * u'$. By the similar discussion of the proof of the clause 1, $t' \in \mathcal{L}(\alpha_A)$ or $t' \in {\alpha_A}^\perp$ and $u' \in \alpha_A$. From induction hypothesis on $w(t) + w(u)$, the thesis follows.
2. $t * u \Rightarrow_1 t_1[u/x]$. From hypothesis on $t$.
3. $u \equiv \lambda x.u_1$ and $t * u \Rightarrow_1 u_1[t/x]$. Since $u \in \overline{\alpha_A}^\perp$ or $u \in \mathcal{L}(\overline{\alpha_A})$, we have $u_1[t/x] \in \mathcal{N}$.
4. $t * u \Rightarrow_1 s$ and $s$ is a subterm of $t$ or $u$. Since $t$ and $u$ are strongly normalizable, their subterm $s$ is also strongly normalizable.

For $\overline{\alpha_A}$, the proof is similar. $\square$

**Lemma 3** *If $\beta \subset \mathcal{T}_A$ is equal to one of $\beta_1 \wedge \beta_2$, $\beta_1 \vee \beta_2$, $\bigwedge \beta_i$, $\bigvee \beta_i$ and $\beta_i$ satisfies clauses 1, 2 of Proposition 2, then $\beta$ is closed under the reduction relation.*

*Proof* First, we consider the case where $\beta$ is equal to $\beta_1 \vee \beta_2$ or $\bigvee \beta_i$. For some $\gamma \in \mathcal{R}$, $\beta = \mathrm{Neg}_A(\gamma)$. Since $\mathcal{V}_A$ and $\gamma^\perp$ are closed under the reduction relation, we consider only the case where $t \in \mathcal{L}(\gamma)$. Let $t \equiv \lambda x.t_1$. We examine the different possibilities for one-step reduction of $t$.

1. $\lambda x.t_1 \Rightarrow_1 \lambda x.t_1'$ and $t_1 \Rightarrow_1 t_1'$. Then $\lambda x.t_1' \in \mathcal{L}(\gamma)$.
2. $t \equiv \lambda x.u * x$ and $t \Rightarrow_1 u$. Since $t \in \mathcal{L}(\gamma)$, $u \in \gamma^\perp$. Hence, $t' \in \gamma^\perp \subset \mathrm{Neg}_A(\gamma) = \beta$.

Next, we consider the case where $\beta = \bigwedge \beta_i$. Since $\mathcal{V}_A$ and $\beta_i$ are closed under the reduction relation, and from a discussion on $\mathcal{L}(\mathrm{Neg}_{A^\perp}(\beta))$ similar to the above, it suffices to prove that if $\lambda x.t * x \in \mathcal{L}(\mathrm{Neg}_{A^\perp}(\beta))$ and $\lambda x.t * x \Rightarrow_1 t$ by $(\eta)$, we have $t \in \beta$. Note that $\lambda x.t * x \in \mathcal{L}(\mathrm{Neg}_{A^\perp}(\beta))$ implies $t \in \mathrm{Neg}_{A^\perp}(\beta)^\perp$. Since $t$ has a type $A$ whose outermost connective is universal, we have the following possibilities.

1. $t \in \mathcal{V}_A$. By Proposition 1, $t \in \beta$.
2. $t \equiv \mathit{\Pi} X.t_1$. We prove $t_1[B_i/X] \in \beta_i$. Let $u \in {\beta_i}^\perp$. From Lemma 1, we have $(B_i)u \in (\bigwedge \beta_i)^\perp \subset \mathrm{Neg}_{A^\perp}(\beta)$. By hypothesis on $t$ , $t * (B_i)u \in \mathcal{N}$. Hence, $t_1[B_i/X] * u \in \mathcal{N}$. This means $t_1[B_i/X] \in \beta_i^{\perp\perp}$. $\beta_i^{\perp\perp}$ is equal to $\beta_i$ from hypothesis on $\beta_i$.

3. $t \equiv \lambda x.t_1$. This implies $t \in \mathcal{L}(\mathrm{Neg}_{A^\perp}(\beta))$.

The case where $\beta = \beta_1 \wedge \beta_2$ is treated similarly. □

**Lemma 4** *Let $\beta_k \in \mathcal{R}_{A_k}$ for $k = 1, 2$, $(B_i)_{i \in I}$ be a family of proper types and $\beta_i \in \mathcal{R}_{A[B_i/X]}$ for $i \in I$. Assume that $\beta_k, \beta_i$ satisfy the clauses 1,2 of Proposition 2. Then we have the following equations.*

$$(\beta_1 \wedge \beta_2)^\perp = {\beta_1}^\perp \vee {\beta_2}^\perp \tag{1}$$

$$(\beta_1 \vee \beta_2)^\perp = {\beta_1}^\perp \wedge {\beta_2}^\perp \tag{2}$$

$$(\bigwedge_{i \in I} \beta_i)^\perp = \bigvee_{i \in I} {\beta_i}^\perp \tag{3}$$

$$(\bigvee_{i \in I} \beta_i)^\perp = \bigwedge_{i \in I} {\beta_i}^\perp \tag{4}$$

*Proof* We will prove (3) and (4). The proofs of (1) and (2) are similar.

The proof of (3). Since $\bigvee {\beta_i}^\perp = \mathrm{Neg}_{\exists X A^\perp}(\bigwedge \beta_i^{\perp\perp})$ and $\beta_i^{\perp\perp} = \beta_i$, $(\bigwedge \beta_i)^\perp \subset \bigvee {\beta_i}^\perp$. Hence, it suffices to prove that if $t \in \bigvee {\beta_i}^\perp$ then for all $u \in \bigwedge \beta_i$, $t * u \in \mathcal{N}$. We consider only the case where $t \in \mathcal{L}(\bigwedge \beta_i)$. Let $t \equiv \lambda x.t_1$. Note that if $\beta \in \mathcal{R}$ satisfies the clauses of Proposition 2, the same hold for $\beta^\perp$.

1. $t * u \Rightarrow_1 \lambda x.t_1' * u'$ and $t_1 \Rightarrow t_1'$, $u \Rightarrow u'$. From Lemma 3, we have $\lambda x.t_1' \in \bigvee {\beta_i}^\perp$ and $u \in \bigwedge \beta_i$. By induction hypothesis on $w(t) + w(u)$.
2. $t * u \Rightarrow_1 t_1[u/x]$. From hypothesis on $t$.
3. $u \equiv \lambda y.u_1$ and $t * u \Rightarrow_1 u_1[t/x]$. From the fact $u \in \mathcal{L}(\bigvee {\beta_i}^\perp)$.
4. $t * u \Rightarrow_1 s$ and $s$ is a subterm of $t$ or $u$. Since $t$ and $u$ are strongly normalizable, their subterm $s$ is also strongly normalizable.

The proof of (4). First we prove $\bigwedge {\beta_i}^\perp \subset (\bigvee \beta_i)^\perp$. For this purpose, it suffices to prove that if $t \in \bigwedge {\beta_i}^\perp$ and $u \in \bigvee \beta_i$ then $t * u \in \mathcal{N}$. We consider only the case where $u \in \mathcal{L}(\bigwedge {\beta_i}^\perp)$. Let $u \equiv \lambda x.u_1$. We prove that if $t * u \Rightarrow_1 v$, $v \in \mathcal{N}$ as follows.

1. $v \equiv \lambda x.u_1' * t'$ and $u_1 \Rightarrow u_1'$, $t \Rightarrow t'$. By Lemma 3, we have $t' \in \bigwedge {\beta_i}^\perp$. From induction hypothesis on $w(u) + w(t)$.
2. $v \equiv u_1[t/x]$. From hypothesis of $u$.
3. $t \equiv \lambda y.t_1$ and $v \equiv t_1[u/y]$. We have $\bigwedge {\beta_i}^\perp = \mathrm{Neg}_{\bigwedge {\beta_i}^\perp}(\bigvee \beta_i)$, since $\bigwedge {\beta_i}^\perp = \mathrm{Neg}_{\bigwedge {\beta_i}^\perp} \circ \mathrm{Neg}_{\exists X A^\perp}(\bigwedge {\beta_i}^\perp)$ and $\bigvee \beta_i = \mathrm{Neg}_{\exists X A^\perp}(\bigwedge {\beta_i}^\perp)$. From the shape of $t$, $t \in \mathcal{L}(\bigvee \beta_i)$. Hence we have $t_1[u/y] \in \mathcal{N}$.
4. The case where $v$ is a subterm of $t$ or $u$. From the fact $t, u \in \mathcal{N}$.

Next, we prove $(\bigvee \beta_i)^\perp \subset \bigwedge {\beta_i}^\perp$. Let $t \in (\bigvee \beta_i)^\perp$. We will prove $t \in \bigwedge {\beta_i}^\perp$ by consideration of the different possibilities for the shape of $t$.

1. $t \in \mathcal{V}_{\forall X A^\perp}$. Since $\mathcal{V}_{\forall X A^\perp} \subset \bigwedge {\beta_i}^\perp$, $t \in \bigwedge {\beta_i}^\perp$.
2. $t \equiv \mathit{\Pi} X.t_1$. Assume $u \in \beta_i$. From Lemma 1, $(B_i)u \in (\bigwedge {\beta_i}^\perp)^\perp$. We have $t * (B_i)u \in \mathcal{N}$ from the fact $(\bigwedge {\beta_i}^\perp)^\perp \subset \bigvee \beta_i$. Since $t * (B_i)u \Rightarrow t_1[B_i/X] * u$, we have $t_1[B_i/X] * u \in \mathcal{N}$. Hence we can see $t_1[B_i/X] \in {\beta_i}^\perp$ and therefore, $t \in \mathit{\Pi} {\beta_i}^\perp$.
3. $t \equiv \lambda x.t_1$. This implies $t \in \mathcal{L}(\bigvee \beta_i)$. Since $\mathcal{L}(\bigvee \beta_i) \subset \bigwedge {\beta_i}^\perp$, we have $t \in \bigwedge {\beta_i}^\perp$. $\square$

*Proof (Proof of Proposition 2)* By induction on the construction of $\beta$, using Lemmata 2, 3, 4. $\square$

**Lemma 5** *Let $\beta \in \mathcal{R}_A$ and $t \in \mathcal{L}(\beta)$. Then $t \in \beta^\perp$.*

*Proof* Since $\beta^\perp \in \mathcal{R}$ can be written $\mathrm{Neg}_*(\beta)$, where $\mathrm{Neg}_*$ is one of $\mathrm{Neg}_A$, $\mathrm{Neg}_{\beta_1 \wedge \beta_2}$, $\mathrm{Neg}_{\bigwedge \beta_i}$, we have the thesis. $\square$

### *3.3 Proof of Theorem 1*

**Definition 11** *An* interpretation *$\xi$ is a map from the set of type variables to $\mathcal{R}$. We define $\xi[\beta/X]$ as an interpretation which satisfies $\xi[\beta/X](X) = \beta$ and $\xi[\beta/X](Y) = \xi(Y)$ for $Y \neq X$. $\xi$ is extended to arbitrary types using the following clauses.*

$$\begin{aligned}
\xi(\perp) &= \mathcal{N}_\perp \\
\xi(X^\perp) &= \xi(X)^\perp \\
\xi(A_1 \wedge A_2) &= \xi(A_1) \wedge \xi(A_2) \\
\xi(A_1 \vee A_2) &= \xi(A_1) \vee \xi(A_2) \\
\xi(\forall X A) &= \textstyle\bigwedge_{\beta \in \mathcal{R}} \xi[\beta/X](A) \\
\xi(\exists X A) &= \textstyle\bigvee_{\beta \in \mathcal{R}} \xi[\beta/X](A)
\end{aligned}$$

**Lemma 6** *Let $A, B$ be proper types and $\xi$ be an interpretation. Then, we have*

$$\xi[\xi(B)/X](A) = \xi(A[B/X]).$$

*Especially, $\xi(B)^\perp = \xi(B^\perp)$.*

*Proof* By induction on the construction of $A$. Only the case where $A \equiv X^\perp$ is non-trivial. In this case, we have the thesis using Lemma 4 repeatedly. $\square$

**Proposition 3** *Let $t$ be a term of type $A$, $x_1^{A_1}, \cdots, x_n^{A_n}$ be the free variables of $t$, $X_1, \cdots, X_m$ be the free type variables of $t$ and $\xi$ be an interpretation. Assume that for each $X_j$, $B_j$ is a proper type which satisfies $\xi(X_j) \subset \mathcal{T}_{B_j}$ and a term $t_i \in \xi(A_i)$ is given for each $x_i^{A_i}$. Then*

$$t[B_1/X_1, \cdots, B_m/X_m, t_1/x_1^{A_1}, \cdots, t_n/x_n^{A_n}] \in \xi(A).$$

*Proof* Induction on the construction of $t$. In the following proof, $\tilde{B}, \tilde{u}$ denote $B[B_1/X_1, \cdots], u[B_1/X_1, \cdots, t_1/x_1^{A_1}, \cdots]$ for each type $B$ and term $u$.

$t$ is the variable $x_i^{A_i}$. The thesis follows from $t_i \in \xi(A_i)$.

$A \equiv A'_1 \vee A'_2$ and $t \equiv \sigma_i(t'_i)$. By induction hypothesis, we have $\tilde{t'_i} \in \xi(A'_i)$. By Lemma 1, $\sigma_i(\tilde{t'_i}) \in (\xi(A'_1)^\perp \wedge \xi(A'_2)^\perp)^\perp$. By Lemma 4, we have $\sigma_i(\tilde{t'_i}) \in \xi(A'_1) \vee \xi(A'_2)$.

$A \equiv A'_1 \wedge A'_2$ and $t \equiv \langle t'_1, t'_2 \rangle$. By induction hypothesis, for each $t'_i$ we have $\tilde{t'_i} \in \xi(A_i)$. The thesis follows from the definition of $\xi(A'_1) \wedge \xi(A'_2)$.

$A \equiv \exists X A'_1$ and $t \equiv (B)t'_1$. We have $\tilde{t'_1} \in \xi(A'_1[B/X])$ by induction hypothesis. By Lemma 6, we have $\tilde{t'_1} \in \xi[\xi(B)/X](A'_1)$. By Lemma 1,

$$(\tilde{B})\tilde{t'_1} \in (\bigwedge_{\beta \in \mathcal{R}} \xi[\beta/X](A'_1)^\perp)^\perp.$$

From Lemma 4,

$$(\tilde{B})\tilde{t'_1} \in \bigvee_{\beta \in \mathcal{R}} \xi[\beta/X](A'_1).$$

$A \equiv \forall X A'_1$ and $t \equiv \mathit{\Pi} X.t'_1$. We can safely assume that $X$ is not contained in $B_1, \cdots, B_m$ and $t_1, \ldots, t_n$ as a free type variable. By induction hypothesis and the fact that $t_i \in \xi[\beta/X]$ for each $1 \leq i \leq n$, we have $\tilde{t'_1}[B/X] \in \xi[\beta/X](A'_1)$ for each proper type $B$ and $\beta \in \mathcal{R}_B$. This implies

$$\mathit{\Pi} X.\tilde{t'_1} \in \bigwedge_{\beta \in \mathcal{R}} \xi[\beta/X](A'_1).$$

$t \equiv \lambda x.t'_1$. We can safely assume that $x$ is not contained in $t_1, \cdots, t_n$ as a free variable. By induction hypothesis, $\tilde{t'_1}[u/x] \in \mathcal{N}$ for each $u \in \xi(A^\perp)$. This implies $\lambda x.\tilde{t'_1} \in \mathcal{L}(\xi(A)^\perp)$. By Lemma 5, we have $t \in \xi(A)$.

$t \equiv t'_1 * t'_2$. By induction hypothesis, we have $\tilde{t'_1} \in \xi(A_1)$ and $\tilde{t'_2} \in \xi({A_1}^\perp)$. The thesis follows from the fact $\xi({A_1}^\perp) = \xi(A_1)^\perp$. $\square$

*Proof (Proof of Theorem 1)* In the previous proposition, let $\xi(X)$ be $\alpha_X$ for each type variable $X$ and $t_i \equiv x_i^{A_i}$ for each free variable $x_i^{A_i}$. Then we have $t \in \xi(A)$. From Proposition 1, it follows $t \in \mathcal{N}$. $\square$

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