논리연구 25-3(2022) pp. 313-332

Fuzzy R Systems and Algebraic Routley-Meyer Semantics*

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【Abstract】 Here algebraic Routley-Meyer semantics is addressed for two fuzzy versions of the logic of relevant implication **R**. To this end, two versions $\mathbf{R}^{\mathbf{t}}$ and $\mathbf{R}^{\mathbf{T}}$ of **R** and their fuzzy extensions $\mathbf{FR}^{\mathbf{t}}$ and $\mathbf{FR}^{\mathbf{T}}$, respectively, are first discussed together with their algebraic semantics. Next algebraic Routley-Meyer semantics for these two fuzzy extensions is introduced. Finally, it is verified that these logics are sound and complete over the semantics.

【Key Words】 Routley-Meyer Semantics, Relevance Logic, Fuzzy Logic, **FR**, $\mathbf{FR}^{\mathbf{t}}$, $\mathbf{FR}^{\mathbf{T}}$

Received: Aug. 5th, 2022, Revised: Sep. 14th, 2022, Accepted: Sep. 30th, 2022.

* This research was supported by “Research Base Construction Fund Support Program” funded by Jeonbuk National University in 2022. I have to thank the referees for their helpful comments.

1. Introduction

As is well known, fuzzy logic deals with the vagueness of our natural language and relevance logic the relevance in our arguments or implications. As a common area of these two logics, Yang (2008; 2009; 2015b) introduced fuzzy-relevance logic. Especially, he dealt with fuzzy-relevance logic systems related to the well-known relevant logic system **R** and its neighbors in his (2015b).

However, the completeness results for the logic systems were just provided algebraically, although the completeness results for **R** were established both algebraically and relationally. One interesting fact is that binary Kripke-style semantics for fuzzy **R** have been provided by Yang (2012; 2019). This ensures that one can provide a relational semantics for fuzzy **R**. But he did not introduced Routley-Meyer semantics for fuzzy **R**.

Routley-Meyer semantics was first introduced as a ternary relational semantics for relevance logics (see Routley & Routley (1972a; 1972b; 1973)). In particular, Dunn (1986) dealt with this semantics for **R**. In the early 2010s, Yang (2013) noted that there exist at least three versions of **R**, i.e., $\mathbf{R}^0$ (the **R** without propositional constants), $\mathbf{R}^\mathbf{t}$ (the **R** with propositional constants **t**, **f**), and $\mathbf{R}^\mathbf{T}$ (the **R** with propositional constants **t**, **f**, **T**, **F**).

Note that the Routley-Meyer semantics introduced by Dunn (1986) is just for $\mathbf{R}^0$. Each Routley-Meyer semantics for $\mathbf{R}^\mathbf{t}$ and $\mathbf{R}^\mathbf{T}$, respectively, was instead introduced by Yang (2015a). Especially, Yang (2012; 2019) extended $\mathbf{R}^\mathbf{t}$ to $\mathbf{FR}^\mathbf{t}$, the least fuzzy extension of $\mathbf{R}^t$, and provided Kripke-style semantics for it. Then, since Routley-Meyer semantics is just a ternary generalization of the so-called Kripke semantics, these series of facts give rise to the following question:

● Can we introduce Routley-Meyer semantics for **FR**, in particular for $\mathbf{FR}^t$?

As a positive answer to this question, we provide such semantics for two fuzzy versions of **R**, i.e., the fuzzy extensions of $\mathbf{R}^t$ and $\mathbf{R}^T$. To this end, in Section 2, we first introduce the systems $\mathbf{FR}^t$ and $\mathbf{FR}^T$ as fuzzy versions of $\mathbf{R}^t$ and $\mathbf{R}^T$, respectively, define the corresponding algebraic structures, and establish algebraic completeness for them. In Section 3, we first introduce Routley-Meyer semantics for these systems and then prove that these logics are complete with respect to the Routley-Meyer semantics. More precisely, we provide algebraic Routley-Meyer semantics for the logics in the sense that completeness results are indirectly provided using algebraic completeness of the logics.

2. Preliminaries: logics and algebraic semantics

In this section, we introduce $\mathbf{FR}^t$ and $\mathbf{FR}^T$ as fuzzy extensions of $\mathbf{R}^t$ and $\mathbf{R}^T$, respectively. First, the language for $\mathbf{FR}^t$ is a countable sentential language with *FOR* (the set formulas) inductively constituted from *AS* (a set of atomic sentences), constant **f**, connectives $\vee$, $\wedge$, $\rightarrow$, and the defined connectives as follows: $\mathbf{t} := \mathbf{f} \rightarrow \mathbf{f}$; $P_t := P \wedge \mathbf{t}$; $\sim P := P \rightarrow \mathbf{f}$; $P \leftrightarrow Q := (P \rightarrow Q) \wedge (Q \rightarrow P)$; $P \& Q := \sim(P \rightarrow \sim Q)$. The language for $\mathbf{FR^T}$ is obtained from the language for $\mathbf{FR^t}$ by adding constant **F** together with the defined connective **T** as $\mathbf{F} \rightarrow \mathbf{F}$.

The other notations and terminology for $R^l \in \{\mathbf{FR^t}, \mathbf{FR^T}\}$ are as usual. We introduce R^l as a consequence relation $\vdash$ in Hilbert style.

Definition 2.1 (i) (Yang (2012)) $\mathbf{FR^t}$ is axiomatized by the axioms and rules below:[1)]

A1. $P \rightarrow P$ (SI, self-implication)
A2. $(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$ (SF, suffixing)
A3. $(P \rightarrow (P \rightarrow Q)) \rightarrow (P \rightarrow Q)$ (CR, contraction)
A4. $(P \rightarrow (Q \rightarrow R)) \leftrightarrow (Q \rightarrow (P \rightarrow R))$ (PM, permutation)
A5. $(P \wedge Q) \rightarrow P$, $(P \wedge Q) \rightarrow Q$ ($\wedge$-E, $\wedge$-elimination)
A6. $((P\rightarrow Q)\wedge(P\rightarrow R)) \rightarrow (P\rightarrow(Q\wedge R))$ ($\wedge$-I, $\wedge$-introduction)
A7. $P \rightarrow (P \vee Q)$, $Q \rightarrow (P \vee Q)$ ($\vee$-I, $\vee$-introduction)
A8. $((P\rightarrow R)\wedge (Q\rightarrow R)) \rightarrow ((P\vee Q)\rightarrow R)$ ($\vee$-E, $\vee$-elimination)
A9. $(P\wedge(Q\vee R))\rightarrow((P\wedge Q)\vee(P\wedge R))$ (D, distributivity)
A10. $P \leftrightarrow (\mathbf{t} \rightarrow P)$ (PP, push and pop)
A11. $\sim\sim P \rightarrow P$ (DNE, double negation elimination)
A12. $(P \rightarrow Q)_t \vee (Q \rightarrow P)_t$ (PL_t, **t**-prelinearity)
$P \rightarrow Q, P \vdash Q$ (mp, modus ponens)

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1) A6, indeed, is redundant in $\mathbf{FR^t}$. However, we introduce it so as to verify that $\mathbf{R^t}$ is the $\mathbf{FR^t}$ omitting A12. Notice that the system deleting A6 and A12 is not $\mathbf{R^t}$ (cf see Anderson & Belnap (1975), Anderson, Belnap, & Dunn (1992), Dunn (1976)).

P, Q ⊢ P ∧ Q (adj, adjunction).

(ii) $\mathbf{FR}^{\mathrm{T}}$ is an axiomatic expansion of $\mathbf{FR}^{\mathrm{t}}$ with the constant **F** and its corresponding axiom:

A13. **F** → P.

The axiom A12 is needed for linearity. Notice that in mathematical fuzzy logic a logic is in general called *fuzzy* in case it is complete on linearly ordered models (see e.g. Cintula (2006)). Notice further that the two versions $\mathbf{R}^{\mathrm{t}}$ and $\mathbf{R}^{\mathrm{T}}$ of **R** are the $\mathbf{FR}^{\mathrm{t}}$ deleting A12 and the $\mathbf{FR}^{\mathrm{T}}$ omitting A12, respectively.

Proposition 2.2 (Yang (2012; 2015a)) $\mathbf{FR}^{\mathrm{t}}$ proves:

(1) (P & (Q & R)) ↔ ((P & Q) & R) (&-ASS, &-associativity)

(2) (P ∧ Q) → (P & Q)

(3) (P & (Q ∧ R)) ↔ ((P & Q) ∧ (P & R))

(4) (P → (Q ∨ R)) ↔ ((P → Q) ∨ (P → R))

(5) ((P → (Q ∨ R)) ∧ (Q → R)) → (P → R)

(6) (P & Q) → (Q & P) (&-C, &-commutativity)

(7) (P → (Q → R)) ↔ ((P & Q) → R) (RE, residuation)

(8) P → (P & P) (&-CTR. &-contraction)

(9) ∼∼P ↔ P (DN, double negation)

(10) (P → Q) → (∼Q → ∼P) (CP, contraposition)

Note that Proposition 2.2 (5) is not a theorem in $\mathbf{R}^{\mathrm{t}}$ (see Dunn (1986)).

A *theory* over R^{l} ∈ {$\mathbf{FR}^{\mathrm{t}}$, $\mathbf{FR}^{\mathrm{T}}$} is a set T of formulas. We define a *proof* in a theory T over R^l as a sequence of formulas, every member of which is either a member of T, an axiom of R^l, or derived by its preceding members using the rules in Definition 2.1. T $\vdash$ P, more exactly T $\vdash_{R^l}$ P, means that P can be proved in T with respect to R^l, i.e., there is an R^l-proof of P in T. The following is the relevant deduction theorem (RDT_t):

Proposition 2.3 (Meyer, Dunn, & Leblanc (1976)) Let T be a theory, and P, Q formulas.

(RDT_t) T $\cup$ {P} $\vdash$ Q if and only if T $\vdash$ $P_t \rightarrow$ Q.

We henceforth use the notations "$\sim$", "$\rightarrow$", "$\vee$", and "$\wedge$" both as unary and binary connectives and as unary and binary operators.

Let x_1 := x $\wedge$ 1. We define the algebraic counterpart of R^l as follows.

Definition 2.4 (i) A *commutative distributive pointed residuated lattice* is a structure **A** = (A, 1, 0, *, $\vee$, $\wedge$, $\rightarrow$) such that:

(1) (A, *, 1) is a commutative monoid.

(2) (A, $\vee$, $\wedge$) is a distributive lattice.

(3) a $\leq$ (b $\rightarrow$ c) if and only if (a * b) $\leq$ c, for each a, b, c $\in$ A (residuation).

(4) 0 is an element in A.

(ii) A *bounded commutative distributive pointed residuated lattice* is a commutative distributive pointed residuated lattice satisfying:

(1′) (A, ⊥, ⊤, ∨, ∧) is a bounded distributive lattice, where ⊥ and ⊤ are bottom and top elements, respectively.

(iii) (Dunn-algebras, Anderson-Belnap (1975), Anderson, Belnap, & Dunn (1992)) A *Dunn-algebra* is a commutative distributive pointed residuated lattice satisfying:

(4) $a \leq (a * a)$ for all $a \in A$ (contraction).

(5) $((a \rightarrow 0) \rightarrow 0) \leq a$ for all $a \in A$ (double negation elimination).

(iv) (FR^1-algebras) An *FR^l-algebra* is a Dunn-algebra satisfying:

(6) $1 \leq (a \rightarrow b)_1 \vee (b \rightarrow a)_1$ (pl_t).

(v) ($FR^\top$-algebras) An *$FR^\top$-algebra* is an FR^1-algebra satisfying (1′).

All the FR^1- and $FR^\top$-algebras are henceforth called *R^l-algebras*. We further define negation and equivalence operations as follows: $\sim a := a \rightarrow 0$ and $a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a)$. Using $\sim$ and $\rightarrow$, one might define * as follows: $a * b := \sim(a \rightarrow \sim b)$ and similarly, using $\sim$ and *, $\rightarrow$ as follows: $a \rightarrow b := \sim(a * \sim b)$. The class of all R^l-algebras is a variety denoted by $\mathcal{R}^l$.

We say that an R^l-algebra is *linearly ordered* in case the ordering of its algebra is connected, i.e., $a \leq b$ or $b \leq a$ for each $a, b \in A$. For an R^l-algebra $\mathcal{A}$, an *$\mathcal{A}$-evaluation* (shortly evaluation) is a map $v : FOR \rightarrow \mathcal{A}$ such that $v(\mathbf{f}) = 0$, $v(P \rightarrow Q) = v(P) \rightarrow v(Q)$, $v(P \vee Q) = v(P) \vee v(Q)$, $v(P \wedge Q) = v(P) \wedge v(Q)$, (and hence $v(\sim P) = \sim v(P)$, $v(P \& Q) = v(P) * v(Q)$, and $v(\mathbf{t}) = 1$).

Let $\mathcal{A}$ be an R^l-algebra, T a theory, P a formula, and K a class of R^l-algebras. P is said to be an *1-tautology* in $\mathcal{A}$, shortly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), in case v(P) $\geq$ 1 for each evaluation v; an evaluation v is said to be an *$\mathcal{A}$-model* of T if 1 $\leq$ v(P) for each P $\in$ T. By *Mod(T, $\mathcal{A}$)*, we denote the class of $\mathcal{A}$-models of T; P is a *semantic consequence* of T with respect to K, denoted by T $\vDash_{\mathsf{K}}$ P, if Mod(T, $\mathcal{A}$) = Mod(T $\cup$ {P}, $\mathcal{A}$) for each $\mathcal{A} \in \mathsf{K}$; $\mathcal{A}$ is said to be an $\mathbf{R}^l$*-algebra* if and only if P is R^l-provable in T (i.e. T $\vdash_{R^l}$ P) implies P is also a semantic consequence of T with respect to the class {$\mathcal{A}$} (i.e. T $\vDash_{\{A\}}$ P), $\mathcal{A}$ an R^l-algebra. By *MOD(*R^l*)* (*MOD*^{l}*(*R^l*)* respectively), we denote the class of $\mathbf{R}^l$-algebras (linearly ordered $\mathbf{R}^l$-algebras respectively). Finally, we write T $\vDash^{(l)}_{R^l}$ P in place of T $\vDash_{MOD^{(l)}R^l}$ P.

First, it is verified that classes of provably equivalent formulas are an R^l-algebra. For a fixed theory T on $R^l \in$ {$\mathbf{FR^t}$, $\mathbf{FR^T}$} and a formula P, define $[P]_T$ as the set of all formulas Q such that T $\vdash_{R^l}$ P $\leftrightarrow$ Q. By A_T, we denote the set of the classes $[P]_T$. Moreover, define: 1 = $[\mathbf{t}]_T$, 0 = $[\mathbf{f}]_T$, ($\top$ = $[\mathbf{T}]_T$, $\bot$ = $[\mathbf{F}]_T$,) $[P]_T \to [Q]_T = [P \to Q]_T$, $[P]_T \vee [Q]_T = [P \vee Q]_T$, $[P]_T \wedge [Q]_T = [P \wedge Q]_T$, and $[P]_T * [Q]_T = [P \& Q]_T$. We denote the algebra formed from these definitions by $\boldsymbol{A_T}$.

Proposition 2.5 (Yang (2012; 2015a)) Let T be a theory on $R^l \in$ {$\mathbf{FR^t}$, $\mathbf{FR^T}$}. Then $\mathbf{A}_T$ is an $\mathbf{R}^l$-algebra.

Proof: We just consider the **t**-prelinearity condition (6). Let T $\vdash_{R^l}$ $(P \to Q)_\mathbf{t} \vee (Q \to P)_\mathbf{t}$. Then, since $[\mathbf{t}]_T \leq (([P]_T \to [Q]_T)$

$\wedge$ $[\mathbf{t}]_T$) $\vee$ ((($[Q]_T \rightarrow [P]_T$) $\wedge$ $[\mathbf{t}]_T$), one can ensure that (6) holds. For other ones, see Proposition 2.8 in Yang (2012) and Proposition 2.8 in Yang (2015a). □

Theorem 2.6 (Completeness) Let T be a theory over $R^l \in \{\mathbf{FR^t}, \mathbf{FR^T}\}$ and P be a formula. T $\vdash_{R^l}$ P if and only if T $\vDash_{R^l}$ P if and only if T $\vDash^l_{R^l}$ P.

Proof: T $\vdash_{R^l}$ P if and only if T $\vDash_{R^l}$ P: ($\Rightarrow$) This direction is obvious. ($\Leftarrow$) Proposition 2.5 ensures that $\mathbf{A}_T \in$ MOD(R^l) and that v $\in$ Mod(T, $\mathbf{A}_T$) for $\mathbf{A}_T$-evaluation v defined as v(Q) = $[Q]_T$. Then, T $\vdash_{R^l}$ $\mathbf{t} \rightarrow$ P because $1 \leq$ v(P) = $[P]_T$ follows from T $\vDash_{R^l}$ P. Hence, by (mp), one obtains that T $\vdash_{R^l}$ P since T $\vdash_{R^l}$ $\mathbf{t}$.

T $\vDash_{R^l}$ P if and only if T $\vDash^l_{R^l}$ P: The claim follows from the fact that every R^l-algebra is a subdirect product of linearly ordered R^l-algebras, see Lemma 3.7 in Cintula (2006) for the subdirect representation. □

3. Algebraic Routley-Meyer semantics for Rl

Here we consider algebraic Routley-Meyer semantics for R^l, i.e., $R^l \in \{\mathbf{FR^t}, \mathbf{FR^T}\}$.

3.1 Semantics

We first introduce *Routley-Meyer (RM) frames* for R^l,

Definition 3.1 (i) (RM frames, Yang (2020)) An *RM frame* is a structure **RF** = (RF, 1, R), where 1 is a special element in RF and R $\subseteq$ RF^3. The elements of **RF** are called *nodes*.

(ii) (Linear RM frames, Yang (2021)) *Linear RM* (simply, RM^l) *frame* is an *RM frame* **RF** = (RF, 1, R) equipped with a relation $\leq$, where (RF, $\leq$) forms a linearly ordered set.

(iii) (Operational RM frames, Yang (2020)) An *operational RM frame* is an RM frame **RF** = (RF, 1, $\leq$, $*$, R), where (RF, 1, $*$) is a groupoid with identity and R satisfies the below postulates:

p_s. R1ab and R1ba imply a = b for each a, b $\in$ RF;

p_t. R1ab and R1bc imply R1ac for each a, b, c $\in$ RF;

$p_{\leq}$. a $\leq$ b if and only if R1ba for each a, b $\in$ RF.

(iv) ((Pointed, residuated) Fine operational RM^l frames, Yang (2021)) A *Fine operational* RM^l (simply, F-RM^l) *frame* is an operational RM frame, where * has the definition (df_F) c $\leq$ (a * b) := Rabc[2)] and R satisfies the following postulates: for each a, b $\in$ RF,

p^l. R1ab or R1ba.

An F-RM^l frame is said to be *pointed* if it also has an arbitrary element 0; a (pointed) F-RM^l frame is called *residuated* in case it has a residuum $\rightarrow$ defined as a $\rightarrow$ b := sup{c: (a $*$ c) $\leq$ b} for each a, b $\in$ RF.

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2) The reason to call this a Fine operational frame is that (df_F) is the order reversely considered definition of Fine's one, i.e., he defined R as follows: c $\geq$ a * b := Rabc (see Fine (1974).)

(v) (FR^1 frames) Let ~a := a → 0 for all a ∈ RF. A pointed, residuated F-RM^l frame is said to be an *FR^l frame* if it further satisfies the following definitions and postulates:

df1. R^2abcd := (∃x)(Rabx ∧ Rxcd) for each a, b, c, d ∈ RF;

df2. $R^2a(bc)d$:= (∃x)(Raxd ∧ Rbcx) for each a, b, c, d ∈ RF;

df3. a → b := ~(a * ~b) for each a, b ∈ RF;

p_e. Rabc implies Rbac for each a, b, c ∈ RF;

p_a. R^2abcd if and only if $R^2a(bc)d$ for each a, b, c, d ∈ RF;

p_c. Raaa for each a ∈ RF;

p_{inv}. ~~a = a for each a ∈ RF.

(vi) ($FR^⊤$ frames) An FR^1 frame is said to be *bounded* if it has the bottom and top elements ⊥, ⊤ with respect to the linear order ≤. A bounded FR^1 frame is said to be an *$FR^⊤$ frame*.

We henceforth call both FR^1 and $FR^⊤$ frames R^l frames.

A *forcing* on an FR^1 frame is a relation ⊩ between nodes, propositional variables and formulas satisfying: for any propositional variable p,

(AHC) if b ≤ a and a ⊩ p, then b ⊩ p;
(max) the set {a ∈ RF : a ⊩ p} has a maximum; and

for the proposition constants **f**, **t**,

(0) a $\Vdash$ **f** if and only if a $\leq$ 0;
(1) a $\Vdash$ **t** if and only if a $\leq$ 1; and

for arbitrary formulas,

(~) a $\Vdash$ ~P if and only if ~a $\nVdash$ P;
($\wedge$) a $\Vdash$ P $\wedge$ Q if and only if a $\Vdash$ P and a $\Vdash$ Q;
($\vee$) a $\Vdash$ P $\vee$ Q if and only if a $\Vdash$ P or a $\Vdash$ Q;
($\rightarrow$) a $\Vdash$ P $\rightarrow$ Q if and only if for each b, c $\in$ RF, if Rbac and b $\Vdash$ P, then c $\Vdash$ Q.

For a forcing on an $FR^{\top}$ frame, the following two more conditions are needed:

(min) $\bot$ $\Vdash$ p for any propositional variable p;
($\bot$) a $\Vdash$ **F** if and only if a = $\bot$ for the propositional constant **F**.

Note that the condition below is redundant since the connective & is definable.

(&) a $\Vdash$ P & Q if and only if there are b, c $\in$ RF such that Rcba, b $\Vdash$ P, and c $\Vdash$ Q.

Definition 3.2 (R^l model) An *R^l model* is a pair (**RF**, $\Vdash$), where **RF** is an R^l frame and $\Vdash$ is a forcing on **RF**.

Definition 3.3 Given an R^l model (**RF**, ⊩), a node a of **RF** and a formula P, a is said to *force P* if a ⊩ P. P is said to be *true* in (**RF**, ⊩) if 1 ⊩ P; *valid* in the frame **RF** (denoted by **RF** ⊨ P) if P is true in (**RF**, ⊩) for any forcing ⊩ on **RF**.

Definition 3.4 An R^l frame **RF** is an $\mathbf{R}^l$ frame if all axioms of R^l are valid in **RF**. An R^l model (**RF**, ⊩) is an $\mathbf{R}^l$ *model* if **RF** is an $\mathbf{R}^l$ frame.

3.2 Soundness and completeness

We first introduce some lemmas

Lemma 3.5 (Yang (2020; 2012)) (Hereditary Lemma, HL) Let **RF** be an R^l frame.

(i) For any formula P and for each node a, b ∈ **RF**, if b ≤ a and a ⊩ P, then b ⊩ P.

(ii) Given a forcing ⊩ on an R^l frame and a formula P, the set {a ∈ RF : a ⊩ P} has a maximum.

Lemma 3.6 1 ⊩ P → Q if and only if for each a ∈ RF, if a ⊩ P, then a ⊩ Q.

Proof: (⇒) Since the operation * has the identity 1, using the condition (→) and (df_F), one has a ⊩ Q. (⇐) Using the condition (→), we prove this direction. Let Ra1b and a⊩ P. We need to verify that b ⊩ Q. Using (df_F) and Ra1b, one has that b $\leq$ 1 * a = a and so b $\Vdash$ Q by Lemma 3.5. (i). □

Proposition 3.7 (Soundness) If $\vdash_{R^l}$ P, then P is valid in any R^l frame.

Proof: We prove the validity of (DNE) and A13 as examples.

(DNE) To verify that 1 $\Vdash$ ~~P → P, by Lemma 3.6, we assume that a $\Vdash$ ~~P and prove that a $\Vdash$ P. The condition (~) ensures that a $\Vdash$ ~~P if and only if ~a $\nVdash$ ~P if and only if ~~a $\Vdash$ P. Then by p_{inv}, one has a $\Vdash$ P.

(A13) To verify that 1 $\Vdash$ **F** → P, as above, we assume that a $\Vdash$ **F** and prove that a $\Vdash$ P. The condition ($\perp$) ensures that a = $\perp$. Then, since R$\perp$1$\perp$ and $\perp$ $\Vdash$ P, one has a $\Vdash$ P. □

The following proposition ensures that the postulates for R^l frames are reducible to algebraic (in)equations for the structural theorems of R^l.

Proposition 3.8 Consider all the postulates for R^l frames introduced in Definition 3.1.

(i) The postulates p_s, p_t, $p_{\leq}$, and p^l together with (identity) a * 1 = a = 1 * a for each a $\in$ RF assure that (RF, $\leq$) forms a linear order.

(ii) The postulates p_e, p_a, p_c, and p_{inv} can be reduced to the (in)equations (commutitivity) a * b = b * a for each a, b $\in$ RF, (associativity) a * (b * c) = (a * b) * c for each a, b, c $\in$ RF, (contraction) a $\leq$ a * a for each a $\in$ RF, and (involution) a =

~~a for each a ∈ RF, respectively, which correspond to the structural theorems of R^l, (&-commutativity), (&-associativity), (&-contraction), and (DN), respectively, introduced in Proposition 2.2.

Proof: The definition (df_F) assures (i) and (ii).

For (i), we note that p_s, p_t, $p_{\leq}$, and (identity) ensure that (RF, ≤) is a partial order. Since (df_F) and p^l ensure that a ≤ b or b ≤ a for each a, b ∈ RF and so ≤ is connected, (RF, ≤) is a linear order.

For (ii), consider p_e. By (df_F), one has that c ≤ (a * b) implies c ≤ (b * a) for each a, b, c ∈ RF. This fact implies that a * b ≤ b * a and so a * b = b * a. Similarly, one can prove that p_a is reducible to (associativity). (df_F) assures that p_c is reducible to (contraction) a ≤ (a * a) for each a ∈ RF. p_{inv} is the same as (involution). □

An *R^l-chain* means a linearly ordered R^l-algebra. Now, we explain a relationship between R^l-chains and R^l framse.

Proposition 3.9 (i) The {1, 0, (⊤, ⊥,) *, ≤} reduct of an R^l chain $\mathcal{A}$ is an R^l frame.

(ii) For an R^l frame **RF** = (RF, 1, 0, (⊤, ⊥,) *, ≤). the structure $\mathcal{A}$ = (RF, 1, 0, (⊤, ⊥,) *, min, max, →) forms an R^l-algebra.

(iii) For the {1, 0, (⊤, ⊥,) *, ≤} reduct **RF** of an R^l chain $\mathcal{A}$ and an $\mathcal{A}$-evaluation v, let a ⊩ p if and only if a ≤ v(p) for any propositional variable p and for any a ∈ $\mathcal{A}$. (**RF**, ⊩) forms an R^l model, and one has: a ⊩ P if and only if a ≤ v(P) for any formula P and for any a ∈ $\mathcal{A}$,

(iv) For an R^l model (**RF**, ⊩) and the R^l-algebra $\mathcal{A}$ defined as in (ii), define v(p) = max{a ∈ RF : a ⊩ p} for any propositional variable p. One has that v(P) = max{a ∈ RF : a ⊩ P} for any formula P.

Proof: Here we consider (iii) since one can easily prove (i) and (ii) and using (iii) and Lemma 3.5 (ii) one can obtain (iv).

For (iii), one has to deal with the induction steps of P = ~Q, P = Q ∧ R, P = Q ∨ R, and P = Q → R.

P = ~Q: The condition (~) assures that a ⊩ ~Q if and only if ~a ⊮ Q. Then, by the induction hypothesis (IH), a ⊩ ~Q if and only if ~a ≰ v(Q), i.e., ~a > v(Q), and so only if ~~a ≤ ~v(Q); thus a ≤ ~v(Q). For the reverse direction, let a ⊮ ~Q. We prove that ~a ≤ v(Q). By The condition (~), one has ~a ⊩ Q, and so ~a ≤ v(Q) by IH.

P = Q ∧ R: The condition (∧) assures that a ⊩ Q ∧ R if and only if a ⊩ Q and a ⊩ R, and so by IH, if and only if a ≤ v(Q) and a ≤ v(R); hence, if and only if a ≤ v(Q) ∧ v(R).

P = Q ∨ R: The proof is analogous to the case P = Q ∧ R.

P = Q → R: The condition (→) assures that a ⊩ P → Q if and only if for any b, c ∈ RF, Rbac and b ⊩ Q imply c ⊩ R, hence by (df_F) and IH, if and only if c ≤ b * a and b ≤ v(Q) imply c ≤ v(R), and so if and only if a ≤ v(Q → R) =

v(Q) → v(R) since v(Q) * a ≤ v(R). □

Theorem 3.10 (Completeness) Let T be a theory over $R^l \in$ {$\mathbf{FR}^t$, $\mathbf{FR}^T$}, P be a formula and $\mathcal{R}^l$ a class of R^l frames.

$$T \vdash_{R^l} P \text{ if and only if } T \vDash_{R^l} P.$$

Proof: We obtain the claim using Proposition 3.9 and Theorem 2.6. □

4. Concluding Remarks

We investigated algebraic Routley-Meyer semantics for two fuzzy R systems. Namely, we indirectly provided completeness results for them using algebraic completeness. But we did not provide any direct completeness for them. To provide such completeness remains an open problem.

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퍼지 R 체계들과 대수적 루트리-마이어 의미론

양 은 석

이 논문에서 우리는 연관 논리 R의 두 퍼지 버전 $\mathbf{FR}^t$, $\mathbf{FR}^T$를 위한 대수적 루트리-마이어 의미론을 다룬다. 이를 위하여 먼저 R의 두 버전 $\mathbf{R}^t$, $\mathbf{R}^T$와 그것들의 퍼지 확장 $\mathbf{FR}^t$, $\mathbf{FR}^T$가 그것들의 대수적 의미론과 함께 논의된다. 다음으로 이 두 퍼지 확장을 위한 대수적 루트리-마이어 의미론이 소개된다. 마지막으로 이러한 체계들이 주어진 의미론에서 건전하고 완전하다는 것을 보인다.

주요어: 루트리 마이어 의미론, 연관 논리, 퍼지 논리, **FR**, $\mathbf{FR}^t$, $\mathbf{FR}^T$