# Is quantum mechanics real or complex?

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It has been long debated whether quantum mechanics is real or complex. Local experiments have been carried out confirming the complex nature of quantum mechanics in the standard formalism. Nevertheless, recent theoretical work demonstrated that in a closed universe, quantum mechanics is real. We discuss the philosophical implications of whether quantum mechanics is real or complex.

## I. INTRODUCTION

Ever since the establishment of quantum mechanics, the introduction of complex variables raised the question whether quantum mechanics is real or complex[1, 2]. It is always believed that in its usual formulation, i.e., state vectors in Hilbert space, real quantum mechanics is irreconcilable with the tensor-product structure of the Hilbert space of composite quantum systems. The physical observables are of real values regardless of the formulation of the theory. Whereas the phases in quantum mechanics are described by complex variables and have experimental manifestation as quantum phases. Whether quantum mechanics can be formulated with pure real numbers in the state vector formulation is therefore a question that have to be answered for understanding the nature of quantum mechanics. Despite the long-lasting debate in the last century, this question remains controversial. Over years, this question has been investigated theoretically as well as experimentally, where experimental physicists seem to have finally ruled out the possibility of real-valued formalism of quantum mechanics in its usual formulation[3, 4].

Soon after the success of experimental physicists in our local universe, theorist, however, proved that in a closed universe, quantum mechanics can and should be real[5]. In a closed universe, quantum gravity demands that there be no global symmetry. Gauging spacetime reversion symmetry leads to new conclusions to our old beliefs. One of the surprising new findings is that the Hilbert space of quantum gravity in a closed universe must be a real vector space, and thus quantum gravity in a closed universe must be real.

In this paper, we review the history of the debate on whether quantum mechanics is real or complex, and discuss the implications of real quantum gravity in a closed universe.

## II. THE HISTORY OF THE DEBATE

Although physical observables assume the form of real numbers, i.e., quantities that can be obtained experimentally, quantum mechanics has been the first theory that necessitates the complex formulation[1, 6]. Since the origin of quantum mechanics, the question whether complex numbers are essential has been brought out and Lorentz, Planck and Schrodinger debated on this question in their correspondence [7]. Pioneer physicists did not try to formulate a real quantum mechanics because they deemed it impractical. Later on, the question was sharpened and a clear-cut answer demands establishing a formal construction of real quantum mechanics or ruling it out definitely. The complex nature of quantum mechanics is rooted in the foundation of the Hilbert space formulation where vectors in the complex-valued Hilbert space represent states. On the other hand, for a single, isolated quantum system real description of Hilbert space is available straightforwardly, i.e., double the dimension of the Hilbert space and the complex space is then isomorphic to a 2D real plane. This is, in fact, the underlying physical foundation for purification[8].

It was found that quantum correlations, among which the most intensively studied being entanglement, can violate the principle of local realism, i.e., physical properties exist independently of measurement and information cannot travel faster than the speed of light. And the violation of local realism necessitates the complex formalism of quantum mechanics. This violation can be captured by Bell inequality, and experiments have thus been designed to detect the violation of Bell tests to find proofs for whether quantum mechanics is real or complex. For the same reason, although still unexplored by experimental physicists, other quantities that characterize quantum correlation, such as complexity and OTOC, should exert bound on identifying the complex nature of quantum mechanics as well.

On the other hand, to explore the possibility of real quantum mechanics, theoretical works have been done to test the ability of real numbers to describe quantum mechanics. By simulation [9], the evolution and measurement has been studied for quantum system whose states and operators lie in a real Hilbert space. The essential point of this work is to find whether one can double some or all the Hilbert spaces and let each party manipulate the phases independently of each other. It successfully reproduced the statistics of any standard Bell experiment, including multipartite quantum states and continuous-time evolution. Although the result seems to support the view that complex numbers are not necessary, lack of general proof makes its conclusion limited.

In contrast to the inability to general proof for real

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quantum mechanics, success has been made on the opposite side. Experiments in ordinary quantum systems has ruled out the possibility of real quantum mechanics, claiming the success of the debate [10–12]. We will discuss the details in the next section.

Rather astonishingly, while experimental physicists have confirmed the success of complex quantum mechanics in local universe, a scenario that never occurred in previous debate, quantum mechanics in a closed universe lead this question into a new avenue[5]. Whereas complex quantum mechanics is testified in ordinary quantum systems, putting the quantum system in a quantum gravity background lead to a totally different conclusion that the Hilbert space of quantum states in a closed universe is real. This question, therefore, has not yet been fully explored yet, a large research gap has just emerged. Theoretical and experimental researches are remain to be done for the understanding of the question in greater depth and wider scope.

## III. THE EXPERIMENTS FOR COMPLEX QUANTUM MECHANICS

In order to find out whether complex numbers are really needed in quantum formalism, [12] proved that real and complex quantum theory make different predictions in network scenarios comprising independent states and measurements. The question is transformed into a game between two players, the "real" quantum physicist Regina and the "complex" quantum physicist Conan. In the case of experiments involving a single quantum system, a real quantum explanation is always available. That is, for any complex quantum system used in the experiment, there is a density matrix representation $\rho$. The complex matrix can be represented by real formulation as

$$\begin{aligned}\tilde{\rho} &= \mathrm{Re}(\rho) \otimes \tfrac{I}{2} + \mathrm{Im}(\rho) \otimes \tfrac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \tfrac{1}{2}(\rho \otimes |{+i}\rangle \langle{+i}| + \rho^* \otimes |{-i}\rangle \langle{-i}|)\end{aligned} \tag{1}$$

where $|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ and $*$ denotes complex conjugation. The operator $\tilde{\rho}$ is real and positive semi-definite, thus a real quantum state. Real quantum theory cannot be falsified in this scenario. Besides, [13] demonstrated that real numbers are sufficient for maximal violation of all bipartite Bell inequalities. Whereas [9] proved that real numbers are sufficient for maximal violation of Bell inequalities involving any number of parties.

On the other hand, in experiments involving several distant labs, phenomena like entanglement and Bell nonlocality can manifest. In the case of two separate labs, a source emits two particles in a state $\rho_{AB}$, each being measured by different observers, i.e., Alice and Bob. By Bell inequality [14], there exist quantum experiments where the observed correlations, encapsulated by the measured probabilities $P(a, b\,|x, y)$, are such that they cannot be reproduced by any local deterministic model. An experimental realization of such correlations disprove the universal validity of local classical physics.

The mere observation of a Bell violation is insufficient to disprove real quantum theory. By Clause-Horne-Shimony-Holt (CHSH) Bell inequality $\mathrm{CHSH}(x_1, x_2; y_1, y_2) := \langle A_{x_1} B_{y_1}\rangle + \langle A_{x_1} B_{y_2}\rangle + \langle A_{x_2} B_{y_1}\rangle - \langle A_{x_2} B_{y_2}\rangle \leq 2$. The inequality is derived for a Bell experiment where Alice and Bob perform two measurements with outcomes $\pm 1$, and $A_x$, $B_y$ denote the results by Alice and Bob when performing measurement $x$, $y$. The maximal quantum violation of this inequality is $\beta_{\mathrm{CHSH}} = 2\sqrt{2}$ and Alice and Bob can attain it using real measurements on a real two-qubit state.

For more complex CHSH inequalities

$$\begin{aligned}\mathrm{CHSH}_3 := \mathrm{CHSH}(1, 2; 1, 2) + \mathrm{CHSH}(1, 3; 3, 4) \\ + \mathrm{CHSH}(2, 3; 5, 6) \leq 6\end{aligned} \tag{2}$$

In this scenario, Alice and Bob perform three and six measurements respectively. The maximal violation of inequality is $2\beta_{\mathrm{CHSH}} = 6\sqrt{2}$.

None of these Bell inequalities works. Real quantum Bell experiments [9, 13, 15] can reproduce the statistics of any quantum Bell experiment, even if conducted by more than two separate parties.

Other no-go theorems in quantum theory, Pusey-Barrett-Rudolph construction[16] involves states prepared in independent labs subject to joint measurements. Such scenarios can also be explained with real quantum mechanics.

Consider experimental scenarios where independent sources prepare entangled states to several parties, who in turn conduct independent measurements. Network corresponding to a standard entanglement-swapping scenario is proposed, which consists of two independent sources and three observers: Alice, Bob and Charlie. The two sources prepare two maximally entangled states of two qubits, the first one $\bar{\sigma}_{AB_1}$ distributed to Alice and Bob, and the second $\bar{\sigma}_{B_2C}$, to Bob and Charlie. Bob performs a standard Bell-state measurement on the two particles that he received from the two sources. This measurement swaps the entanglement from Alice and Bob and Bob and Charlie to Alice and Charlie: for each of Bob' four possible outcomes, Alice and Charlie share a two-qubit entangled state. Alice and Charlie implement the measurements leading to the maximal violation of the $\mathrm{CHSH}_3$ inequality. For these measurements, the state shared by Alice and Charlie, conditioned on Bob's result, maximally violates the inequality or a variant thereof produced by simple relabelings of the measurement outcomes.

No construction by the real theory can reproduce the measurement probabilities $\bar{P}(a, b, c\,|x, z)$. The marginal state shared by Alice and Charlie at the beginning of the experiment cannot be decomposed as a convex combination of real product states. The impossibility of real simulation also holds for non-maximal violations of the inequality between Alice and Charlie.

For experimental test, Bell-type functional $\mathcal{J}$ was proposed, which is defined by the sum of the violations of (the variants of) the $\mathrm{CHSH}_3$ inequality for each of Bob's measurement outputs, weighted by the probability of the output. In the ideal entanglement-swapping realization with two-qubit maximally entangled states, the maximal quantum value of $\mathrm{CHSH}_3$, equal to $6\sqrt{2}$, is obtained for each of the four outputs by Bob and therefore $\mathcal{J}$ reaches its maximal value as well. To violate the real bound on $\mathcal{J}$, the two distributed states should have a visibility beyond $\sqrt{7.66/6\sqrt{2}}$ respectively, which relies on the implementation of a two-qubit entanglement measurement.

Complex quantum theory outperforms real quantum theory when the non-local game is played in the entanglement swapping scenario. This game can be interpreted as an extension of the adaptive CHSH game. The measurement statistics generated in certain finite-dimensional quantum experiments involving causally independent measurements and state preparations do not admit a real quantum representation.

In this way, [12] actually proved conceptually that if quantum mechanics is formulated with the real formalism, the direct product of two Hilbert spaces fails. Based on the entanglement swapping scheme, using quantum resource theories where imaginary is crucial for state discrimination, [10] proved the necessity of complex numbers in quantum mechanics experimentally. [11] further improved the result by using independent quantum preparations and independent measurement, which closed locality loopholes under strict locality conditions.

## IV. REAL HILBERT SPACE IN CLOSED UNIVERSE

After all those experiments, it seems that conclusion could be drawn safely that quantum theory is intrinsically complex. However, the story doesn't end here. In investigating the consequences of gauging spacetime inversion symmetry in quantum gravity, Harlow [5] opened the door to a new world. Although in local universe complex quantum mechanics has been experimentally demonstrated, analysis in closed universe presents different results. In a closed universe, there are some essential properties of quantum gravity that leads to novelties therein. First of all, no global symmetry conjecture in quantum gravity indicates that all physical states should be gauge invariant.

CRT is a gauge symmetry that reverses time in quantum gravity. In quantum mechanics any symmetry that reverses time can be represented by an anti-unitary operator $\Theta$, and the states invariant under an anti-unitary operator form a real vector space. i.e., if $\Theta\left|\psi\right\rangle = \left|\psi\right\rangle$ then $\Theta i\left|\psi\right\rangle = -i\left|\psi\right\rangle$. The Hilbert space of quantum gravity in a closed universe is therefore a real vector space.

In our usual experience, that is at finite $L$, the standard rules of quantum mechanics with a complex Hilbert space are only approximations to a new set of rules on a real Hilbert space. In the limit that the universe is big, i.e., the size of the universe $L$ goes to infinity, the usual predictions of quantum mechanics can be recovered. The infinite size renders the interaction between the observer and the measured system negligible.

The essential difference between local mechanics and quantum mechanics in a closed universe is that in a closed universe, the observer is included in the universe. Whereas in ordinary quantum mechanics, an external observer is assumed to make the measurements. In the former case, the observer and the system that he is in can not be factorized. Whereas in the latter, the observer is presumed to be extrinsic to the system and is represented as tensor products. The observer, or the clock must have no interactions with the system being measured, otherwise the measurement $\Theta_S$ would not be a symmetry.

## V. THE IMPLICATION OF REAL QUANTUM MECHANICS IN A CLOSED UNIVERSE AND HOW TO CONCEIVE EXPERIMENTAL TEST

From the argument for real quantum mechanics in a closed universe [5], it can be seen that gauging $\mathcal{CRT}$ symmetry leads to the conclusion of real Hilbert space in closed universe, and the distant observer which only exists in the limit the size of the universe $L$ goes to infinity which is not interacting with the system ensures the measurement.

This provide clues for further investigation on the question of real quantum mechanics in a closed universe. From the symmetry perspective, as symmetry properties may determine whether the Hilbert space in a local system is real or not, we may construct systems that are real under symmetry constraint and gradually violate the symmetries. In the mean time, carry out the quantum games and see if the violations of the inequalities grow as the symmetry violation grows. For example, as time reversal symmetry leads to real Hilbert space, it is consistent with the fact that for an isolated single particle Hilbert space, it can be fully described by real numbers. On the other hand, non-interacting quantum systems definitely have time-reversal symmetry, and no interaction enable real description as well. From the observer perspective, changing the distance between the observer and the system may influence the degree of violation of the entanglement inequalities. Controlling the interaction between the observer and the system may lead to similar conclusion.

In a sense, the fact that Hilbert space is real globally and complex locally is somewhat similar to dissipation, i.e., the energy may be dissipated in a local system and is not invertible, but for a closed system, energy is never increased or decreased. This indicates some concept related to dissipation but more general than it, which is to be explored in the future.

[1] P. A. M. Dirac, *The principles of quantum mechanics*, 27 (Oxford university press, 1981).
[2] E. Schrödinger, Physical review **28**, 1049 (1926).
[3] Z.-D. Li, Y.-L. Mao, M. Weilenmann, A. Tavakoli, H. Chen, L. Feng, S.-J. Yang, M.-O. Renou, D. Trillo, T. P. Le, *et al.*, Physical Review Letters **128**, 040402 (2022).
[4] M.-C. Chen, C. Wang, F.-M. Liu, J.-W. Wang, C. Ying, Z.-X. Shang, Y. Wu, M. Gong, H. Deng, F.-T. Liang, *et al.*, Physical Review Letters **128**, 040403 (2022).
[5] D. Harlow and T. Numasawa, arXiv preprint arXiv:2311.09978 (2023).
[6] J. Von Neumann, *Mathematical foundations of quantum mechanics: New edition*, Vol. 53 (Princeton university press, 2018).
[7] C. N. Yang, *Selected Papers of Chen Ning Yang II* (World Scientific Publishing, 2013) pp. 94–105.
[8] A. Kent, Physical review letters **81**, 2839 (1998).
[9] M. McKague, M. Mosca, and N. Gisin, Physical review letters **102**, 020505 (2009).
[10] K.-D. Wu, T. V. Kondra, S. Rana, C. M. Scandolo, G.-Y. Xiang, C.-F. Li, G.-C. Guo, and A. Streltsov, Physical Review Letters **126**, 090401 (2021).
[11] D. Wu, Y.-F. Jiang, X.-M. Gu, L. Huang, B. Bai, Q.-C. Sun, X. Zhang, S.-Q. Gong, Y. Mao, H.-S. Zhong, *et al.*, Physical Review Letters **129**, 140401 (2022).
[12] M.-O. Renou, D. Trillo, M. Weilenmann, T. P. Le, A. Tavakoli, N. Gisin, A. Acín, and M. Navascués, Nature **600**, 625 (2021).
[13] K. F. Pál and T. Vértesi, Physical Review A—Atomic, Molecular, and Optical Physics **77**, 042105 (2008).
[14] J. S. Bell, Physics Physique Fizika **1**, 195 (1964).
[15] T. Moroder, J.-D. Bancal, Y.-C. Liang, M. Hofmann, and O. Gühne, Physical review letters **111**, 030501 (2013).
[16] M. F. Pusey, J. Barrett, and T. Rudolph, Nature Physics **8**, 475 (2012).