# Renormalization group theory in physics and general science

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Renormalization group (RG) theory, while proposed to study particle physics, has found its usage in a large variety of topics over the years, including other physics branches like solid state physics, fluid mechanics, cosmology, machine learning and even non-physics fields like biology, epidemiology, economics, psychology, sociology and so on. The omnipresence of renormalization group theory thus raises the philosophical question of what are the common features of the systems that enable the employment of RG theory and what can be revealed by using the RG method, why renormalization group theory is the naturally approach to deal with these problems and what is the essential point that leads to organizing the system according to renormalization group flow, i.e., order and disorder. These question are addressed in this paper.

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## I. INTRODUCTION

Renormalization group theory stems from the study of scale transformation, dating back to Pythagoras, Euclid, and Galileo [1]. It was initially explored and employed to study particle physics, playing a significant role in quantum field theory. However, the range of application of renormalization group theory sees a rapid expansion. It was not only introduced to study condensed matter physics, fluid mechanics, solid state physics and cosmology[2], but also adapted to the fields such as biology[3, 4], epidemiology[5], economics[6], psychology[7], sociology[8, 9], deep learning[10] etc. The ability of renormalization group theory in so many different fields lead us to ponder deeply on the philosophical foundation of it.

The wide application of RG theory indicates that it does not simply describe structure formation in physics, but serves as a mathematical approach to handling systems that have abstract hierarchical structures, no matter the hierarchy is in physical systems, economic systems or biological systems etc. It can also be understood as a way of handling and understanding data, as in the end, RG comes down to analyzing data and extracting information from data. The hierarchical structure is manifested by the features of the data. View the picture in a different light, in many systems, from microscale to macro-scale, hierarchy structures naturally form according to what can be described by RG, in essence, this indicates a kind of underlying principle.

The most essential questions that occur to us when beholding the wide variety of scenarios that RG methods can be employed are: why is RG so powerful? What are the similarities of the systems where RG can be employed? What can RG tell us about the systems being studied? Why do the macroscopic systems naturally organize according to renormalization group flow?

We will address these questions in this paper.

## II. THE COMMON STRUCTURE OF RENORMALIZATION GROUP THEORY

First of all, let's look into the ontology of RG. RG itself is a methodological topic instead of a phenomenological one. From the phenomena that can be studied by RG, i.e., particle physics, condensed matter physics, fluid mechanics and even biology or economics, of the systems studied, we can observe the objective aspects of RG.

Usually, to study a physical system with RG, we start with finding out its Hamiltonian. However, that RG requires finding Hamiltonian as a first step [11] is just a special case for Hamiltonian systems. In diverse sciences that lack Hamiltonian, RG has long been employed implicitly[6]. Here we first discuss the application in physics in its most popular form and then discuss that in general science.

In studying physical phenomena with RG, the first step is to find out its Hamiltonian, i.e., the Hamiltonian of the microscale system $S_0$. After that, we figure out the way that the Microscale systems form larger systems $S_1$, as is the way that $S_1$s form larger $S_1$s, that is the so-called RG transformation. Given the Hamiltonian and the RG transformation, the RG flows and all the way up to $S_N$ level by level.

Take the simplest model for example, that is, the 1D Ising model. The starting Hamiltonian is

$$\beta H = -K \sum_{i=1}^{N} S_i S_{i+1} - h \sum_{i=1}^{N} s_i - c \tag{1}$$

where $K = \frac{J}{k_B T}$ and $h = \frac{\mu_B B}{k_B T}$, with $s_i = \pm 1$. The partition function is

$$Z = \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \sum_{s_3=\pm 1} \cdots \sum_{s_N=\pm 1} \exp(-\beta H) \tag{2}$$

Summing over the spins to form a block consisting of several spins, and carrying this process recursively until the macro-scale of the system, the properties of the macro-scale can be obtained. In this renormalization procedure, several steps are essential: 1. the Hamiltonian of the microscale element; 2. the recursive procedure of renormalization transformation; the final macroscopic picture.

This procedure applies similarly in most physical applications. There are exceptions as in fluid systems, in many cases the Hamiltonian is not well-defined, or the renormalization procedure is not carry out for the Hamiltonian[12, 13]. This reminds us to ponder over the essence of renormalization independent of physical scenarios. In the past few years, studies on the correspondence between deep learning and renormalization group [10, 14–17] reveal that it is the hierarchy structure of the organization of data that counts in the renormalization group procedure. Deep learning, which manifests no physical system, shows a pure RG manipulation of data. Take the example of Restricted Boltzmann Machine (RBM), it has an exact correspondence with RG. The ”Hamiltonian” of RMB is defined by summing over its nodes, here we use quotation marks as it is not a physical Hamiltonian. The RBM also defines the probability for obtaining the configuration of visible nodes and hidden nodes, where summing over all possible hidden and visible vectors yields the partition function. In RBM, subsequent layers of neurons are analogous to successive steps along the RG. Note that the neurons are stochastic neurons, and RBM is a special case of Markov random fields. In fact, the deep learning model shows most explicitly the essence of RG, as it is independent of specific scenarios for application.

Besides physics and RBM, we can look into the treatment of problems with RG in biology, economics and sociology etc. In the model of price variation, price variation with respect to time assumes the form of recursive peaks, RG explains the large deviation of the distribution from Gaussianity, and lead to fractional Brownian motion[6]. In political science, real space RG was used to study voting problem [9], the model consists of a self-directed pyramidal structure from bottom up to the top. This is a manifest hierarchy model. In the process, the self-elimination of population amounts to coarse-graining, and finally the result of voting is arrived at, with one winner from the initial candidates. As to the big topic of cultural evolution[8], RG presents in the decision making process where people iteratively update and refine the mental representation of their natural and social environment, this is made of the recursion of three steps: physical attributes input, concepts abstracted, and payoffs obtained. Unlike in physics, this renormalization is not carried along space, but along time. Cultural coherence, i.e., the overlap of mental representations can be characterized with an order parameter, dynamic phase transition here appears as the cultural coherence advances in the society. Even ”cultural explosion” can be interpreted as phase transition, with coherent culture as an order phase and no culture as a disordered phase. In biology, RG has been proposed to explain structure formation[3], where in addition to collective behavior in physics, cooperative behavior was proposed to characterize the more complex inter-scale interactions there, i.e., hydrogen bonds, twist, bend and fold. In epidemiology, some effective epidemiological models in space and time are related to each other through scaling transformations, which can be interpreted as RG transformations[5]; in the large scale limit, the microscopic details of the infection process become irrelevant and show a universal behavior. And in study consciousness[7], RG is employed in brain phase space dynamics and criticality, where the ontology of the subjective arises in the RG transformation of the brain’s phase space dynamics.

In fact, like any other branches of science, the development and application of physical theories amounts to the analysis and understanding of data. The essence just lies in that physical data are organized subject to physical laws, which manifest as rules of physics. Given the broad usage of RG, it is easy to perceive that RG is not a rule of physics that results from physical constraints, but a kind of coarse-graining process that applies to any self-similar hierarchy structures. The exact map between RBM and RG therefore offers a clear view point for how to look at RG as a data processing method. RBM manifests the usual properties of RG, with RG flow, fixed point, scaling dimension, correlation function, phase transition and randomness. Basically, the reason why these data can be manipulated with RG method and yield meaningful results is due to the common properties of the data. They all amounts to extracting macro-scale information (smaller data set) from microscale information (large data set) through coarse-graining.

Having discussed what are really essential for RG rather than its representation in a specific field, here we give a detailed review of the structure of RG method.

First of all, in RG process, no matter what the scenario is, there must be a quantity that is being renormalized which plays the role of Hamiltonian in physical problems. Here we call it RG kernel. The RG kernel is going to be recursively renormalized from the large data starting point to the small data end point, corresponding to the microscopic scale and the macroscopic scale in physics. Usually, the scales are intrinsic and have manifest meaning. For example, in the block spin problem, the microscale is characterized by a single spin, whereas the macroscale is the scale at which the properties of the system become fixed with respect to renormalization process; and in the problem of turbulence, the microscale is the scale of a single eddy, and the macroscale is the energy injection scale of the whole system; in deep learning neural network, the microscale is the input layer of neurons, the macroscale is then the output layer of neurons. The process of renormalization requires a RG transformation that determines how the RG kernel transforms from one level to the next level. This RG transformation is what encodes the properties of the system, which captures the hierarchy structure of the problem. By RG transformation, lower level effective elements, such as effective spin blocks, neurons, combinations to form a higher level effective element. RG kernel, RG transformation, RG interval (in between the starting scale and end scale of RG transformation) are therefore the pillar elements of RG no matter in which area it works. The process of renormalization from the large data set side to the small data set side induces a RG flow, where the RG flow equation is an essential ingredient of RG. When we organize the data on a manifold which is parameterized by a Riemannian metric, we can define a map from the space of data to the manifold, the change of state variable induces the change of coupling constants, which is characterized by the $\beta$ function. $\beta$ function induces the renormalization flow on the space of the coupling constants, i.e., running couplings in physics. RG flow equation encodes fixed points of the theory, which can give the small data side, that is the macroscopic scale in physics. In fact, the fixed points in RG flow of any problem describe the scale invariant theories therein, whereas for other generic theories of the same problem, following RG flow, they will approach the scale-invariant theories defined at the fixed points. This is the so-called universality, i.e., the properties of the system is independent of the details of the microscale. In fact, in the momentum shell RG approach, the separation of fast modes and slow modes indicate the decoupling of the dynamics of the two scales.

## III. THE COLLECTION OF RG THEORIES

Having discussed the general structure of RG, we turn to the different approaches to RG. Due to the diverse concern and specialty of different scenarios, a large variety of RG theories have been proposed to the resolution of different problems. For example, in physics, there are perturbative RG[18] and non-perturbative RG[19], dealing with perturbative and nonperturbative problems respectively. There are real space RG and momentum shell RG, with respect to where RG is performed. Concerning the studied system is evolving or in a steady state, RG could be dynamical or not[20]. And with respect to the property of the procedure, iterative RG[12] and recursive RG[21] both play important roles.

Hence let us discuss the diversity of RG from the above mentioned perspectives, i.e., perturbative or nonperturbative, the parameter space of RG procedures, dynamical or not, and the type of RG procedures.

Though both perturbative RG and nonperturbative RG are being extensively explored, the renormalization group, in principle and in practice, is designed for nonperturbative analysis. This is obvious from its original usage in studying phase transition which is a highly nonperturbative process, particularly the highly nonperturbative QCD, turbulent flows and quantum gravity. The nonperturbative RG equation is a functional partial differential equation[22], evolving the functionals $\Gamma_k[M]$ and $\Gamma_k^{(2)}[M]$ (Gibbs free energy of the system), whose solution is not known in general. Green function approach and the derivative expansion is usually used to find approximations of the solution. Whereas perturbative RG[18] performs series expansion in a small parameter, nonperturbative RG solves the RG equation in a restricted functional space. Functional RG(FRG) is therefore being immensely explored [23], where for highly nonlinear problems like fully developed turbulence FRG is able to give a description at all scales [24]. Variational RG, with its standard form formulated in Hamiltonian formalism, has always been employed to study physical problems. Due to its elegant structure, it has also been the first example where the exact map between restricted Boltzmann machine and RG was discovered[14]. The applications of RG to economics, biology, social sciences and etc, are intrinsically nonperturbative, and therefore the RG employed there are usually nonperturbative. In contrast to the prevalence of nonperturbative RG, the usage of perturbative RG i.e., $\epsilon$-expansion, is rather limited. Successful applications were found in high energy physics and condensed matter physics [18, 25, 26], where there are no phase transitions or critical behaviors.

The parameter space in which RG is studied, though diverge with respective to diverse scenarios, can be roughly classified into two categories: 1. RG in the space of the phenomenological parameter space where the problem was originally raised. 2. RG in the parameter space where the patterns are more prominent. In physics, the most characteristic example is real space RG[27–29] and momentum shell RG[30, 31]. When first introduced by Wilson, RG was originally formulated in momentum space. It can be understand that where RG is implemented depends on where the structure of RG is prominent. For example, in the spin block model, it is the spin space instead of the real space that the hierarchy structure exists. And in spectral analysis, like waves, turbulence and energy spectrum, momentum space RG is naturally chosen to be the solution to the problems. Real space RG, on the other hand, is prevalent in cases where RG hierarchy structure is manifest in the real space. Examples can be seen as percolation problems where the structure is in the real space[32], tight-binding model [29], the chain-like models and lattice models naturally manifest hierarchy structure in real space, such as Heisenberg model [33], XYZ model and Ising model [34]. As in real space boundary counts, real space RG had a hard time dealing with boundary conditions [29]. On the other hand, momentum space RG plays a much more important role as spectral analysis is crucial in the analysis of problems in almost all branches. The application of momentum space RG can be found in density matrix RG (DMRG) [31], the calculation of entanglement in QFT [35], where the momentum vector is chosen as the basis element for renormalization. It can be seen from these examples that momentum space RG, or more generally, RG for spectral analysis, deals more with the dynamic part of the problem. Real space RG, on the other hand, are more straightforward in describing the macroscopic quantities of the system.

Most of the literature are focused on ordinary RG, where the RG equations describe a certain state. However, when it comes to the evolution of the system, dynamical RG (DRG) was introduced[4, 20, 36–38]. Instead of calculating the constant observables, i.e., expectation values and other correlators that are crucial to the problem, in DRG the quantities to be calculated are dynamical. This is particularly important when it comes to nonequilibrium relaxation, and in other science branches where the dynamic process is important, i.e., the evolving of swarms which is never in a steady state. Essentially, DRG comes down to the cases when the RG structure itself is not constant, but it still exists and the predictions of it makes sense. Unlike that in phase transition, the evolution here is not drastic but continuous. This is why the RG structure is preserved though not constant. This DRG, however, does not apply straightforwardly to cases where RG is applied to the time structure, such as the the cultural evolution and price variation. In those cases, nevertheless, the variation of RG over other parameters, like country or region, can similarly be tackled by DRG is the region being analyzed is varied continuously, which unfortunately is not the common case in real life.

Although iteration and recursion both call for repeating some certain procedures, the disparity between them leads to considerable difference in coding complexity. In RG procedure, there are subtleties between the two as well. Essentially, in view of the system being studied, iterative RG and recursive RG captures different structures. Recursive RG is much more prevalent as its logic structure highly resembles the hierarchy structure of RG. Recursive RG is therefore very commonly seen in many different scenarios, such turbulence [39], spin models [40], hydrodynamics [41], Ginsburg-Landau model[42], where the RG procedures are formulated in recursion relations. Iterative RG, on the contrary, found its usage much more limited. In physics problems, iterative RG usually transform the Hamiltonian iteratively until the Hamiltonian stay fixed by the iterative transformation, i.e., fixed point. It was introduced in quantum fluids [43] as well as fluid turbulence [12, 44]. In deep learning, the RG flow arises from iterative probability transformation in a neural network. Recursive RG and iterative RG, as they capture general structures of RG rather than specific phenomena, are sure to be found in RG applications in all areas.

## IV. THE PREDICTIONS MADE BY RENORMALIZATION GROUP THEORY

Having discussed the basic structure and the collection of theories of RG in the previous sections II III, we turn to the predictions made by RG theory. In general, RG provides three different kinds of predictions: 1. the predictions of the macroscopic properties, such as the bare mass of electron in QED, the total magnetization of the spin model and the final decision of the voting problem. 2. the phase transition of the macroscopic system, the examples are the transition between the magnetized phase and the non-magnetized phase, order-to-chaos transition in neural network, and the cultural explosion. 3. the correlation functions to calculate the expectation values of observables, which give the observables we are concerned with.

RG predictions in different scenarios share common structure which results from the common structure of RG hierarchy model. However, due to the differences in the contexts, in different problems we are concerned with different observables, i.e., in physics, we are usually concerned with various quantities that can be computed, where correlation function is one of the basic quantities that serve as the building block for the computation of other quantities; in the selection model in sociology, however, only the final result counts; phase transition is important when the dynamics of the system is important. Furthermore, the differences lie not only in the representations, but in the fact that the constraints on different models diverge, in many physics models, there are well-defined Hamiltonians of the systems, however, in some RG models of physics and in RG models in other science branches, Hamiltonian is unavailable. Absence of Hamiltonian indicates that the system may lack some conservation properties, equation of motion and symplectic theory structure. In the RG procedure itself, no constraint is encoded to ensure that the system is physical, biological or economical etc. Thus the constraints on the RG theories are not intrinsic but have to be imposed manually, which are not predictable from the RG theory. Instead of giving specific predictions for micro-scales, RG is efficient for identifying universal properties. For theories that flow to the same RG fixed point, relevant observables are shared in common, whereas differences in phenomena among fine-scale components are determined by irrelevant observables that can be integrated out. From the intensively explored RG-neural network correspondence[45, 46], it is clear that universality transfers among mappings between RG in different systems. One field theory model, i.e., the field theory at the fixed point, can correspond to a large class of deep neural network model, properties of a fixed point in a given theory therefore offer insights to corresponding theories in other theories. On the other hand, starting from the critical points of a RG theory, many predictions can be made. First of all, the power law scaling properties of observables can be calculated, which give clear predictions of relevant observables. The critical point is also where first order phase transition terminates. As the theories at the critical points manifest conformal symmetry, they can be solved exactly.

## V. WHY RENORMALIZATION GROUP THEORY IS OMNIPRESENT

Having seen the prevalence of RG, we are naturally confronted with a question: why is RG so omnipresent, with its presence in a large variety of sciences? Here we answer this question from several perspectives.

First of all, as we have mentioned already, RG is a kind of data processing tool. It does not contain in itself any information about the fields to be employed in. And on the other hand, for any field of science, when it comes down to analyze the problems theoretically, we will finally reach the point of dealing with the abstract data, and finding the rules for the organization of data. In this sense, as long as the problem manifests hierarchy structure in its data, it is possible that RG may come to help.

Secondly,the emergence of macroscopic properties from microscopic ones are universal. In any system, a large range of different scales exist naturally. The relationship between the macroscopic properties and the microscopic properties can be either: they are strongly coupled and have relevant interactions; or they are decoupled and the the microscopic degrees of freedom are irrelevant to the macroscopic properties which can be integrated out. And even in the first case, when we look at a larger scale range, we will finally find a scale where the macroscopic properties decouple from the microscopic ones. This is the essential reason for the prevalence of RG.

## VI. SUMMARY

In this paper, we discussed the common structure of RG that enables it to play significant roles in many science branches, the collection of representations of RG theories which are aimed to tackle different problems according to context and motivation, the predictions that can be made from RG descriptions which is its main function and the reason for the prevalence of RG over different disciplines. In general, RG is prevalent due to the omnipresence of emergence of macroscopic systems from microscopic systems, the predictions made by RG results from universality, and universality derive from the decoupling of the macro-scale dynamics and the microscale dynamics. The wide application of renormalization group theory indicates that the mathematical structures, equations are more universal than phenomenological theories. Symmetry governs over a wider range than detailed dynamics.

Despite the extensive studies on RG in various sciences, the inverse RG, i.e., how to reconstruct the microscopic components from the macroscopic theory has not been fully studied yet [47, 48]. This indicates new perspectives from the philosophical side, which shall be discussed in the future.


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