<Draft, December 25th, 2017.>

The Necessity of Mathematics[1]

Juhani Yli-Vakkuri and John Hawthorne

> 'As soon as these questions were squarely faced, a wide range of new phenomena were discovered, including quite simple ones that had passed unnoticed.'
> —Noam Chomsky, *Knowledge of Language*, p. 7[2]

It is a commonplace that statements of pure mathematics are necessarily true if true at all. But why should we think this? A cursory investigation of the practice of mathematics itself presents something of a puzzle here. Mathematicians do not appear to make use of the language of metaphysical necessity and possibility in their own investigations. Of course they do use the modal idioms 'might' and 'must' and their cognates. However, their use of these idioms does not provide much evidence that metaphysical modality is in play in any serious way. On the one hand, many of their uses seem to be metaphorical. As Wilfrid Hodges points out, when a mathematician says, for example, that one system 'can be embedded' in another, this is little more than a colorful way of saying that there is an embedding of one into the other. What the modal 'can' adds is

> a certain human colouring, by suggesting that part of the mathematics is carried out by a human being. This adds nothing to the mathematical content, but somehow it helps the readability (Hodges 2013: 6).

On the other hand, many uses of modals in mathematics express epistemic modality. For example, when mathematicians say at some point in their investigations, 'Various answers might be correct', they are not giving voice to a perceived metaphysical contingency in mathematical reality, but signaling that which answer is correct is an open question at the relevant stage in the process of mathematical discovery. And similarly, when they say, 'Only one answer can be correct', they are talking about what has been established at the relevant stage, not about what is metaphysically necessary: if it turns out that two answers are epistemically live at the time of speaking or writing, then the 'must' claim will be reckoned false. Also similarly, when a mathematician says that 'Given that *A*, it must be that *B*', it is arguable that the 'must' again expresses a kind of epistemic modality.[3] (One of us has explored elsewhere the behavior of epistemic modals embedded in logically complex

[1] We would like to thank Richard Heck, Richard Price, Margot Strohminger, Timothy Williamson, the students in John Hawthorne and Timothy Williamson's graduate seminar at the University of Oxford in Trinity Term 2017, and the participants of an Academia.edu session on a draft of this paper in June-July 2017 for helpful comments and discussions.

[2] This is also the epigraph to Hodges (2013). We did not, however, choose it as an allusion to that paper, but simply because we could think of no better epigraph for our own paper.

[3] We will not defend this take over Hodges' own gloss on these 'must's as 'formatting to guide the reader through the structure of the reasoning' (Hodges 2007: 12).

sentences.[4]) And it is far from clear that there is any modality left once we set aside the metaphorical and epistemic occurrences of modals in mathematical texts.[5] *Prima facie*, then, it seems that mathematical practice is silent on the question of the status of mathematical truths *vis-à-vis* metaphysical modality.

It is tempting to take these observations to support the view that the doctrine that mathematics is necessary is something that has been added by philosophers to the body of knowledge provided by mathematics itself, which can proceed just fine without taking on any metaphysically modal commitments. This view has prominent defenders. For example, in 'Modality in Mathematics' Wilfrid Hodges tells us:

> Mathematicians are pleased to know that
>
> (1) Every finite field is commutative.
>
> or that
>
> (2) $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.$$
>
> The fact that these statements are necessarily true might attract the attention of a philosopher of mathematics, and some mathematicians dream about such things in idle moments. But adding 'Necessarily' to either (1) or (2) would introduce nothing of any mathematical significance (Hodges 2013: 1-2).

Hodges begins another paper, 'Necessity in Mathematics', by prominently displaying the following 'fact':

> FACT A: Mathematics contains no modal notions (Hodges 2007: 1).

He continues:

[4] See Dorr and Hawthorne (2013).

[5] As Timothy Williamson pointed out in conversation, Church's thesis, which asserts the equivalence of the intuitive notion of computability with the formal ones, is an interesting test case here. The key issue is whether the modality expressed by the suffix '-able' in 'computable', in its intuitive sense, is some kind of an objective modality (in the sense of Williamson 2017). (There are various inequivalent glosses on the intuitive notion. Turing [1939: 8], for example, says he 'shall use the expression 'computable function' to mean a function calculable by a machine'. Another kind of gloss, common in textbooks, speaks of computability by a human or some other kind of agent. For example, Boolos, Burgess, and Jeffrey [2007: 23] say that a function is computable iff 'there are definite, explicit rules by following which one could in principle compute its value for any given argument'.) Certainly the modality in play is not epistemic, but is it metaphysical or otherwise objective? One reason for being cautious here is that mathematicians tend to make free use of Church's thesis without ever worrying about whether it is metaphysically possible to build a certain kind of machine or for a human or other agent to perform certain kinds of tasks. (Hence the frequently occurring weasel words 'in principle'. More things may be 'in principle possible' than are possible *simpliciter*. 'In principle' does not appear to be a factive operator.) This suggests to us that the mathematicians' intuitive notion of 'computable' might be similar to their intuitive notion of 'provable', in that what makes something computable in the intuitive relevant sense is simply that *there is* a certain kind of procedure for computing it, and this is unrelated to the difficult question of whether it is metaphysically possible for any mathematical creature to implement that procedure. We will leave further exploration of this issue to others.

> Of course mathematics is full of necessary truths, for example this theorem of analysis:
>
> $$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$$
>
> But only philosophers are interested in the fact that this theorem is necessarily true. Mathematicians are content if they can show that it is true (*ibid.*).

Having gotten this far, one might think that, insofar as we are justified in thinking that mathematics is necessary at all, such justification will proceed not from the practices of mathematicians but from the practices of the armchair philosophers. Stories about how such practices confer justification or knowledge are many and varied. Most crudely, one might posit cognitive phenomenology that forcefully presents the necessity of a certain propositions, including those of pure mathematics.[6] On the heels of this picture, one might tack on the epistemological principle, popular in some circles, that one is '*prima facie* justified' in believing any proposition for which one has an 'intuition' or an 'intellectual seeming'. And one might then hope to run the gauntlet of candidate 'defeaters' in order to emerge with justification *simpliciter*. Alternatively, one might take a conventionalist approach. Perhaps, one might think, there is no 'joint in reality' that is picked out by the idioms of metaphysical modality, and the truths of mathematics get to be necessary simply on account of our having decided in a quasi-stipulative way that they belong to a special 'list'.[7] Or alternatively, and perhaps most intriguingly, one might claim that mathematics is necessary on the basis of its purported reducibility to logic in combination with the necessity of logic itself. This is the so-called (neo-)logicist program adapted to the role of proving the necessity of mathematics.[8],[9] (The other main approaches to the foundations of mathematics—intuitionism and formalism—are less obviously well equipped to provide any compelling story about the necessity of mathematics.) The picture-thinking is clear enough: since it is not that mysterious that logic is necessary, by reducing mathematics to logic we also render the necessity of the mathematics unmysterious.

For what it's worth, we find the neologicist approach to our question more promising than the other two mentioned in the previous paragraph. (And we are not alone. When we asked a variety of philosophers why they thought we should think that mathematics is necessary, some variant of 'Logicism' was by far the most common answer.) Here is how that approach would work. Neologicists maintain that some decent-sized axiomatizable mathematical theory—typically, the chunk of arithmetic characterized by the Peano axioms—is reducible to logic on account of its axioms being derivable from an abstraction principle in some axiomatic system of second-order logic. By supplementing their favorite system of second-order logic with the 'necessitation' rule

$$\frac{A}{\Box A},$$

[6] See Bealer (2002).

[7] See Sider (2011: ch. 12).

[8] See Hale and Wright (2001).

[9] A particularly radical and technically untaxing form of the view that all truths of mathematics follow by logic from analytic truths is the view all truths of mathematics are analytic. Timothy Williamson gestured at this view in conversation, and pointed out that at least it bypasses worries concerning truths of mathematics that are not provable from standard axioms. (Of course, as an analyticity-skeptic, Williamson himself does not endorse any version of the view that mathematics is analytic.) A proponent of any such view cannot think of analytic truths as truths that one immediately assents to upon understanding them, but arguably even ordinary neologicists have to distance themselves from any such conception of analyticity.

which ensures that $\Box A$ is a theorem whenever A is, and the standard **K** axiom

$$\Box(A \to B) \to (\Box A \to \Box B),$$

they will obtain a system S that can be shown to be sound (but not complete) on a natural generalization of standard possible worlds semantics for modal logic to the second-order case.[10] Then, for any theorem T of the mathematical theory whose axioms are provable from the abstraction principle α in the original system of non-modal second order logic, $\Box\alpha \to \Box T$ will be a theorem of S. Thus, given that the abstraction principle is necessary, and given that the system is (informally) sound, it follows that each theorem of the reduced mathematical theory is also necessary.

Yet an appeal to neologicism as a general answer to our question still does not seem very promising. For one thing, that strategy has certain inherent limitations. It can, at best, establish only the necessity of those mathematical truths that are provable in whatever axiomatic system it uses. By Gödel's first incompleteness theorem, we know that these cannot even include all truths of first-order arithmetic. (That result was, after all, the downfall of the original logicist program of Frege and Russell. Hence the prefix 'neo'. Neologicists are content to reduce some but not all of mathematics to logic.) We also have two more general philosophical qualms about the approach. First, neologicists cannot even establish the necessity of the axiomatizable fragments of mathematics they target unless they can establish the necessity of the abstraction principles they assume. But, insofar as there is any neologicist story about why the abstraction principles are necessary, it tends to proceed via the claim that they are 'analytic' or that they are 'conceptual truths'—ideology that we find problematic for broadly Williamsonian reasons.[11] Second, many neologicists seem to be motivated by a commitment to the view that logic is in some sense metaphysically neutral or innocent (since they often write as if a reduction to logic would purge mathematics of metaphysical tendentiousness).[12] But logic isn't metaphysically neutral.[13] What makes logic logic is not its neutrality but its generality. If one wants to play it safe from an ontological point of view, sticking to logic as the foundation of both (some) mathematics and the source of its necessity is not a good game plan. (Of course we don't expect these brief remarks to convince die-hard neologicists, but we put them forward in the hope that they will clue the reader into our own orientation in the philosophy of logic and mathematics. The original contribution of this paper is the alternative picture it develops and not its critique of extant accounts of the necessity of mathematics.)

In our view, the supposition that mathematics is silent on questions of metaphysical modality is completely wrong-headed. Hodges' claim that mathematics does not directly deploy idioms of metaphysical necessity and possibility is certainly plausible. However, we shall argue, mathematics makes use of the counterfactual conditional, which in both ordinary and mathematical English is paradigmatically expressed by the subjunctive conditional construction

If … (then) - - - would _ _ _.[14]

[10] For example, Williamson's (2013*a*: §5.5) semantics for his system ML_P.

[11] See Williamson (2007: Chs. 3-4).

[12] Hale and Wright are our paradigms. See the Introduction to Hale and Wright (2001), and see Raatikainen (forthcoming) for further discussion.

[13] See Williamson (2013*b*).

[14] Arguably the distinctive hallmark in English is the occurrence in the antecedent of the conditional of 'fake past tense', a layer of tense that has nothing to do with temporal past. See, e.g., Iatridou (2000).

The use of counterfactual conditionals is by no means a marginal feature of mathematical discourse. (We will later explain why it is not dispensable.) In fact, the pattern of their deployment encodes a commitment to the necessity of all mathematics. These facts put the thesis of the necessity of mathematics on a firmer footing, since they show that challenging that thesis requires challenging the practice of mathematics itself.

Admittedly, some foundational questions remain open. Grounding-lovers will still wish to inquire after what grounds the necessity of mathematics. We will not undertake to defend the claim that the necessity of mathematics is grounded in counterfactual facts. And epistemologists may wish to inquire after how mathematicians are justified in thinking and saying the things that commit them to the necessity of mathematics. We are not going to address these further questions in this paper. Our ambitions more modest.

In §1 we introduce our assumptions about the language of mathematics. In §2 we show that mathematical practice is committed to the necessity of all provable mathematical truths in virtue of its commitment to the acceptability of a certain inference pattern involving counterfactuals. In §3 we turn to mathematical truths that are not provable. Here we establish an even stronger result: characteristic patterns of inference involving counterfactuals in mathematics manifest a commitment to the thesis that all mathematical truths, whether provable or not, are necessary. In §4 we argue that the modal commitments of mathematics extend even further than we found in §3: mathematics, it turns out, is committed to all theorems and rules of inference of the modal system **S5** that are expressible in mathematical language—**S5** being the system that is widely thought to capture the logic of metaphysical modality.

1. The language of mathematics

By the 'language of mathematics' we mean the language of pure (i.e., not applied) mathematics that one finds in textbooks and professional journals. In what follows we are going to make some fairly modest assumptions about the logical constants that are present in that language. First, we will assume that the language has at least the standard truth-functional connectives, including $\bot$. ($\bot$ is the 0-place connective—i.e., sentence constant—that is a truth-functional contradiction.) Second, we assume that the language has the counterfactual conditional connective $\Box\!\!\rightarrow$.

Admittedly, our assumption about the presence of the standard truth-functional connectives in the language of mathematics is a little idealized. For example, the use of a primitive contradiction symbol is far from a pervasive feature of mathematical texts. Yet the idealization is harmless enough. For example, one could define $\bot$ as an abbreviation for some paradigmatic truth-functional contradiction.

Similarly, while the language of mathematics doesn't contain a single expression with the syntactic type and the meaning of $\Box\!\!\rightarrow$, it does contain the resources for expressing everything that can be expressed by $\Box\!\!\rightarrow$. Mathematics is rife with counterfactual conditionals, although these are often not in the standard form 'If … (then) - - - would _ _ _'. For example, in the canonical contemporary text on mathematical logic, we encounter the following sentence early on:

> Suppose there were a machine computing *t*. It would have some number *k* of states (Boolos, Burgess, and Jeffrey 2007: 41).[15]

[15] Note that the syntactic analysis Kratzer (1986) would give this discourse would make 'would' a restricted modal operator, where the restriction is supplied by the preceding sentence together with the 'modal base'

A casual survey of indirect proofs in virtually any classic mathematical text yields many more examples. To pick an example virtually at random, here is one from a classic text on computability theory:

> THEOREM 6.1. *The set of all Gödel numbers of Turing machines Z, for which $\Psi_Z(x)$ is total, is not recursively enumberable.*
>
> PROOF. Let us designate the set of all such Gödel numbers by R, and let us suppose that R is recursively enumerable. Then, since $R \neq \varnothing$, there would be a recursive function $f(n)$ whose range is R.
>
> The function $U(\min_y T(f(n), x, y)$ would be total, and hence recursive. Hence $U(\min_y T(f(n), x, y) + 1$ would be recursive. Hence, by the very definition of $f(n)$, there would be a number n_0 such that
>
> $$U(\min_y T(f(n), x, y) + 1 = U(\min_y T(f(n_0), x, y).$$
>
> Setting $x = n_0$ yields a contradiction (Davis 1958: Ch. 5, p 78).

Since the reader can easily carry out the exercise of collecting a long list of similar examples,[16] we will not devote any more space to them.

We assume that the common construction 'Suppose … . (Then) - - - would _ _ _' is simply a reader-friendly way of expressing a counterfactual conditional. Mathematical writing often splits conditionals, both subjunctive and indicative, into two or more sentences, which makes it easier to parse conditionals with logically complex consequents and antecedents. (Note that 'Suppose *A*. Then *B*' is an entirely standard way of stating a theorem. The theorem so stated is just this: if *A* then *B*.) One also often encounters in mathematical texts sentences whose main verb is 'would' with no relevant 'suppose' preceding it. In such cases, at least typically, there is either a preceding sentence that is meant to be understood as the antecedent of a counterfactual or some preceding sentences whose conjunction is meant to be so understood.

We take it that our assumptions about the character of the language of mathematics are not especially tendentious. What has been overlooked in discussions of the modal status of mathematics is that these assumptions entail that that language has the resources for asserting the necessity of every mathematical proposition expressible in it. After all, the following definition of the metaphysical necessity operator $\Box$ falls out of both of the two standard semantics for counterfactuals (Stalnaker 1968 and Lewis 1973).[17]

Definition 1. $\Box A =_{df} (\neg A \Box\!\!\rightarrow \bot)$

While the semantics of Lewis and Stalnaker provide a powerful motivation for Definition 1, a perhaps even more powerful motivation for it is supplied by a proof-theoretic observation due to Timothy Williamson. Williamson (2007: 155-58) observes that the

---

supplied by the context of speech. We find her treatment of the underlying logical forms of natural language counterfactuals plausible, but we are not going to rely on it here.

[16] Here is one way to do so: find a canonical mathematics text in Google Books, and search for occurrences of 'would' within it.

[17] Note that those who treat counterfactuals as strict conditionals (like Kratzer 1986) are also committed to the equivalence. Of course, strict conditional-lovers will inevitably think that counterfactuals and associated modals are context-sensitive in that different contexts supply different domains of worlds or 'modal bases' for the counterfactuals and modals to generalize over. We address the interaction of our discussion of this kind of context-dependence below.

material equivalence of the two sides of Definition 1 is derivable from the following two principles in an extremely weak modal logic.

NECESSITY: $\Box(A \rightarrow B) \rightarrow (A \mathrel{\Box\!\!\rightarrow} B)$

POSSIBILITY: $(A \mathrel{\Box\!\!\rightarrow} B) \rightarrow (\Diamond A \rightarrow \Diamond B)$

These principles are pretty hard to disagree with. NECESSITY says that strict implication implies counterfactual implication—or, in other words, that, whenever it is necessary that if *A* then *B*, it is also the case that, if it had been the case that *A*, then it would have been the case that *B*. Or, equivalently, using the suppositional idiom:

> Suppose that it is necessary that if *A* then *B*. Then, if it had been the case that *A*, it would have been the case that *B*.

POSSIBILITY says that anything counterfactually implied by a possible proposition is also possible—or, equivalently, in suppositional language:

> Suppose that if it were the case that *A* then it would be the case that *B*. Then it is possible that *B* only if it is possible that *A*.

The derivation of $\Box A \leftrightarrow (\neg A \mathrel{\Box\!\!\rightarrow} \bot)$ from NECESSITY and POSSIBILITY requires nothing more than the weakest normal modal logic **K**. Given the validity of both NECESSITY and POSSIBILITY and the soundness of **K**, it follows that $\Box A$ and $(\neg A \mathrel{\Box\!\!\rightarrow} \bot)$ are logically equivalent, wherefore we may treat the former as an abbreviation for the latter, just as we do in Definition 1.

The language of mathematics can thus assert the necessity of any mathematical proposition it can express. But it remains to be shown that mathematical practice is committed to the necessity of all mathematical truths. That is the task of the next two sections.

2. Provable mathematical truths

In this section we will argue that mathematics is committed to the necessity of every provable mathematical truth. The notion of provability that we are working with is what philosophers of mathematics often call 'informal provability', as opposed to the system-relative notion of 'formal provability', i.e. provability in a given formal system. The basic idea is that a statement is informally provable just in case there is a proof of it in the sense of 'proof' operative in actual mathematical practice, as opposed to philosophers' formalizations of mathematical theories. We will use the symbol '$\vdash$' to express the relation of informal provability. Thus '$A_1, \ldots, A_n \vdash B$' says that *B* is informally provable from $A_1, \ldots, A_n$, and '$\vdash A$' says that *A* is informally provable *simpliciter*.

Two features of informal provability are important for our discussion. First, informal provability differs from formal provability in that it always preserves truth. As a special case, a statement that is informally provable *simpliciter* is true. In contrast, there are formal systems—ones with false axioms or unsound rules of inference—in which provability does not preserve truth or in which falsehoods are provable. (The system of Frege's *Grundgesetze der Arithmetik*, in which everything turned out to be provable, is a famous example.) Second, the informal provability of *B* from $A_1, \ldots, A_n$ does not imply that *B* is a logical consequence of $A_1, \ldots, A_n$. As a special case, the informal provability of *A* does not imply that *A* is a logical truth. (Of course, a (neo-)logicist may wish to claim that the informal provability of B from $A_1, \ldots, A_n$ implies that B is a logical consequence of $A_1, \ldots, A_n$ together with some analytic or conceptual truths.) An informal proof of a mathematical truth B typically consists in a logically valid argument for B from some assumptions $A_1, \ldots, A_n$, each of which is informally provable. When such a proof is given, we say that $\vdash B$, and not merely that $A_1, \ldots, A_n \vdash B$. Indeed, the mere possibility of giving such a proof, whether or not anyone ever actually gives it, implies that $\vdash B$, since '$\vdash B$' expresses informal prov*ability* of B, not merely that B has been (or is or will be) informally proved.[18] In what follows, we will simply use the word 'provability' and its cognates for informal provability and related notions, since no other varieties of provability will be at issue.[19]

We will now argue that mathematics is committed to the necessary truth of any provable mathematical truth.

To establish this, we need only make two assumptions about provability. The first is the following standard principle.

REDUCTIO. $\quad \vdash A$ iff $\neg A \vdash \bot$.

(In fact, we will only need the left-to-right direction of REDUCTIO in our arguments.)

The second assumption is that the principle of 'deduction within conditionals'—a completely standard principle of counterfactual logic[20]—also holds for $\vdash$:

DEDUCTION. $\quad$ If $A_1, \ldots, A_n \vdash B$ then $A_1, \ldots, A_{n-1} \vdash A_n \mathrel{\Box\!\!\rightarrow} B$.

Of course, the fact that DEDUCTION becomes wholly unremarkable when '$\vdash$' is interpreted as expressing provability in a standard system of counterfactual logic does not quite suffice for establishing DEDUCTION, which is a claim about provability in an absolute and informal sense. To argue for DEDUCTION we must argue that mathematical practice

[18] However—to return to the theme of note 5—it is not obvious that the converse holds. Perhaps 'provability' here should be not be understood as implying the metaphysical possibility of some agent or machine constructing a proof. It *may* be sufficient for $A_1, \ldots, A_n \vdash B$ that *there is* an informal proof of B from $A_1, \ldots, A_n$—i.e., a certain kind of structure of sentences of the language of mathematics. Such a structure may exist even if it is, for whatever reason, metaphysically impossible for anyone to inscribe, speak, or think a token of it. In this respect informal provability may turn out to resemble formal provability.

[19] We note in passing that mathematics also contains the practice we call *very informal proof*: that of supporting statements of pure mathematics by appeal to contingent facts about what mathematicians have and have not achieved. Very informal proofs often contain occurrences of counterfactuals: for example, the claim that a theory T is consistent might be supported by statements like: 'If T were inconsistent, then someone would have derived a contradiction from T by now'. In the context in which such an argument is given it is typically known that no one has derived a contradiction from T by the time of speech of writing. In such a context the indicative 'If T is inconsistent, then someone derived a contradiction from T' would be a just as strange as a speech by a doctor, made in the manifest presence of spots: 'If he doesn't have disease X, then he has no spots'. Thus it is no linguistic accident that we resort to the counterfactual construction in very informal proofs. We shall not, however, be giving very informal proofs a starring (or indeed any) role in our discussion, as it will be widely (and plausibly) held that such 'proofs' are easily dispensable from pure mathematics proper.

[20] This is Lewis's (1973: 132) label for a more general principle (the label has stuck); see also Stalnaker (1968: 105-16). Certain indexicals make trouble for DEDUCTION as a principle of counterfactual logic. For example, DEDUCTION would be a disastrous principle for a language that contains the standard indexical actuality operator @, because (by $A \vdash @A$) it would imply $\vdash A \mathrel{\Box\!\!\rightarrow} @A$, and $A \mathrel{\Box\!\!\rightarrow} @A$ is false whenever A is false but possible (see Davies and Humberstone 1980 and Kaplan 1989: 539, n. 65). But this is no objection to DEDUCTION on its intended interpretation, which concerns informal provability in the language of *pure* mathematics, which, we assume, contains no indexicals. (Of course, mathematical texts do contain indexicals—'*We* shall now prove…', and so on—but they do not occur in statements of *pure* mathematics.)

displays a commitment to it. But this is not at all difficult. In mathematics it is commonplace, having supposed A_n, in addition to any assumptions $A_1, \ldots, A_{n-1}$ one has previously made, to conclude that B would be true (if A_n were true), on the basis of a recognition that B is provable from $A_1, \ldots, A_n$. In such cases, antecedent of the counterfactual is often left implicit. Consider this example from the passage we already quoted in Davis's *Computability and Unsolvability*:

> Let us designate the set of all such Gödel numbers by R, and let us suppose that R is recursively enumerable. Then, since $R \neq \varnothing$, there would exist a recursive function $f(n)$ whose range is R. (Davis 1958: Ch. 5, p. 78).

Here Davis supposes that a certain set R is recursively enumerable, recognizes that the existence of a recursive function whose range is R is provable from that supposition and his earlier assumptions, and concludes that there would be such a function. The implicit antecedent is that R is recursively enumerable.

To complete the argument, note that REDUCTIO and DEDUCTION imply:

If $\vdash A$ then $\vdash (\neg A \;\Box\!\!\rightarrow \bot)$,

which, by Definition 1, is none other than:

($\Box$) If $\vdash A$ then $\vdash \Box A$.

($\Box$) implies that all provable mathematical truths are necessary, but it says something even stronger than that: that all provable mathematical truths are *provably* necessary. (Because provability implies truth, provable necessity implies necessity.)

It is significant that our result is not merely that $\vdash A$ implies that A is necessary, but that $\vdash A$ implies $\vdash \Box A$. What this means is that mathematics is committed to the necessity of its provable truths in the precise sense that the necessity of any given provable mathematical truth is itself provable *in* mathematics. There is no metaphysical division of labor here. Philosophers may still wish to debate the modal status of provable mathematical truths, but our result shows that in doing so they are calling into question a commitment of mathematics itself.

3. Unprovable mathematical truths

So far we have only argued that mathematics is committed to the necessity of all provable mathematical truths. But of course this does not straightforwardly establish the claim that mathematics is committed to the necessity of all mathematical truths, since it says nothing about the unprovable ones. And no doubt some deviants will flirt with the view that, while all provable mathematical truths are necessary, some or all of the unprovable ones are contingent.

The above paragraph presupposes that there are some unprovable mathematical truths. It is worth recalling again that the topic of our discussion is provability in the *informal* sense. The presupposition that there are informally unprovable mathematical truths is by no means beyond question. Consider Gödel's famous incompleteness result. It establishes that, for any consistent formal first-order system whose theorems include the Peano axioms, there are true sentences of first-order arithmetic that are not among its theorems. Nothing about what is *informally* provable follows from this result about *formal* provability. As Timothy Williamson has observed, insofar as a given truth of mathematics A is knowable, there is no obstacle to A being treated as an axiom by mathematicians in the activity of informal proof, and, if A were so treated, A would, *qua* axiom, be a limiting case of informal provability.[21] Given that what is informally provable is a non-contingent matter, it follows that A is actually informally provable. Since one cannot straightforwardly derive any constraint on mathematical knowledge from Gödel's result, one also cannot straightforwardly derive any constraint on informal provability from it. For this kind of reason, philosophers of mathematics tend to think that it is a difficult open question whether every mathematical truth is informally provable.

However, we will not assume that all mathematical truths are informally provable. As we are arguing that a commitment to the necessity of all mathematical truths is implicit in mathematical practice, it would be inappropriate to rely on that assumption. There is no reason to think that mathematical practice is committed to the informal provability of all mathematical truths.

Instead, we would like to call attention to certain facets of the interaction of counterfactuals with informal proof in mathematics that manifest a commitment not merely to the necessity of informally provable mathematical truths but to the necessity of all mathematical truths. To begin, here is a fact about informal provability:

(1) $A \vdash B \mathrel{\Box\!\!\rightarrow} (A \wedge B)$

As in the case of DEDUCTION, it is difficult to find instances of reasoning having *exactly* the form (1) in mathematical texts, because such instances are too obvious to state explicitly. For example, the following is such an instance, but it is so glaringly obvious that it could hardly play a useful role in a proof:

> x is in S. Therefore, if y were in S, both x and y would be in S.

But it is not at all difficult to find proofs that manifest a commitment to (1) in mathematical texts. And this is just what one should expect: mathematics does not exhibit its commitment to obvious structural facts about provability by stating those facts, but rather by reasoning in accordance with them. (This is not, of course, a special feature of mathematical practice. Extremely obvious principles of inference are rarely stated explicitly in informal reasoning in any field.) Here are two examples from a single proof in Martin Davis's classic *Computability and Unsolvability*:

> THEOREM 3.2. *Let* $f(\xi^{(n)})$, $g(\xi^{(n+2)})$ *be total functions. Then, there exists a total function* $h(\xi^{(n+1)})$ *that satisfies* (1).
>
> PROOF. We consider sets of $(n + 2)$-tuples $(y, \xi^{(n)}, u)$ of numbers. Such a set S of $(n + 2)$-tuples will be called *satisfactory* if
>
> (a) For each choice of $\xi^{(n)}$, $(0, \xi^{(n)}, f(\xi^{(n)})) \in S$; and
> (b) For each choice of $\xi^{(n)}$, if $(z, \xi^{(n)}, f(\xi^{(n)})) \in S$, then
>
> $$(z + 1, \xi^{(n)}, g(z, y, \xi^{(n)})) \in S$$
>
> Let Ω be the class of all satisfactory sets S. Then Ω is nonempty, since the set of *all* $(n + 1)$-tuples of numbers is satisfactory. Let S_0 be the intersection of all sets that are members of Ω. Then it is easy to see that S_0 is not satisfactory and that S_0 is contained in every satisfactory set.

[21] See Williamson (2016).

Next, suppose that $(0, \xi^{(n)}, u) \in S_0$. Then we have $u = f(\xi^{(n)})$. For otherwise the set obtained on deleting $(0, \xi^{(n)}, u)$ from S_0 would be satisfactory and would not contain S_0 (*ibid*: 47).

Later in the same proof we have the following.

Since S_0 is satisfactory,

$$(z + 1, \xi^{(n)}, g(z, y, \xi^{(n)})) \in S_0.$$

Suppose that

$$(z + 1, \xi^{(n)}, v) \in S \qquad v \neq g(z, u, \xi^{(n)})$$

Then the set obtained by deleting $(z + 1, \xi^{(n)}, v)$ from S_0 would clearly be satisfactory and would not contain S_0 (*ibid*: 48).

Here, having laid down criteria for satisfactoriness, Davis takes for granted that, under the counterfactual suppositions of the proof, those criteria of satisfactoriness would still be in effect. So, for example, he assumes that, supposing that $(0, \xi^{(n)}, u) \in S_0$, a set would be satisfactory only if (a) and (b) held. And, even more obviously, he assumes that, if $(0, \xi^{(n)}, u) \in S_0$ had held, then $(0, \xi^{(n)}, u) \in S_0$ would have held. Implicit here is the thought that, given (a) and (b), if it were the case that $(0, \xi^{(n)}, u) \in S_0$, then it would be the case that both $(0, \xi^{(n)}, u) \in S_0$ and (a) and (b). And that is an instance of our principle. Of course this does not show that the general principle (1) is at work, but given the myriad instances of similar reasoning, it is extremely natural to think that a commitment to (1) is implicit in mathematical practice. (The situation is similar with the general principle of conjunction elimination.)

Note that (1) is not a principle of *logic*. For example, it does not logically follow from the fact that Williamson is teaching a seminar in the Ryle Room now that if Williamson were taking a bath now, then he would be both taking a bath and teaching a seminar in the Ryle Room now. Yet reasoning in accordance with (1) will never lead one from truth to falsehood if both *A* and *B* are non-contingent (in fact, it suffices that *A* is non-contingent).

In spite of its being no principle of logic, mathematicians are perfectly happy to reason in accordance with (1). That is to say, they are committed to (1) holding for any statements *A* and *B* of pure mathematics. (They are also committed to (1) holding when doing applied mathematics from pure mathematical premises, i.e., when *A* is pure and *B* isn't. This will be important later.) To emphasize, *A* and *B* here need not be provable mathematical truths; after all, one can reason about what follows from unprovable statements. This fact alone, together with our previous observation that reasoning in accordance with (1) never leads from truth to falsehood when the reasoning concerns non-contingent subject matter, might be thought to manifest a kind of mathematical commitment to the non-contingency of mathematics. But that will not be our argument.

Rather, our argument will be this: the necessity of all mathematical truths, provable or not, follows from (1), and indeed is provable from (1) *in* mathematics, so the necessity of all mathematical truths, provable or not, is provable in mathematics.

To establish this, we will have to make one further assumption about provability, which we take to be so obvious as to require no supporting argument:

CLASSICALITY. (i) $A_1, \ldots, A_n \vdash B$ whenever B follows from $A_1, \ldots, A_n$ by classical truth-functional logic.

(ii) $\vdash$ obeys the principle of proof by cases.

(iii) If $A_1, \ldots, A_n \vdash B$, then $A_1^*, \ldots, A_n^* \vdash B^*$, where $A_1^*, \ldots, A_n^* \vdash B^*$ results from replacing any number of occurrences of sentences in $A_1, \ldots, A_n \vdash B$ by occurrences of classically truth-functionally equivalent sentences.

We are aware, of course, that some mathematicians, operating within an intuitonistic framework, are explicitly committed to denying each of the three clauses of CLASSICALITY. But it is classical mathematics that concerns us here, and we will not be discussing which of our results carry over to an intuitionistic setting.

Now, for the proof, note first that the following instance of (1)

$$A \vdash \neg A \mathrel{\Box\!\!\rightarrow} (A \wedge \neg A)$$

is equivalent by CLASSICALITY(iii) to $A \vdash \neg A \mathrel{\Box\!\!\rightarrow} \bot$, which, by Definition 1, is none other than:

(*) $A \vdash \Box A$

By CLASSICALITY(i) and (ii) it follows that:

($\Box$*) $\vdash \Box A \vee \Box \neg A$

That is to say, every mathematical statement is provably non-contingent—i.e., is provably either necessarily true or necessarily false. In committing itself to (1), mathematics commits itself to the necessity of all mathematical truths in the precise sense that the claim that each mathematical truth is necessary—or, equivalently, that each mathematical statement is non-contingent—is itself provable in mathematics.

While (1) is suffices for our argument, we suspect that there is nothing special about it. While our textual investigations have focused on (1) and delivered a positive verdict, we suspect that mathematicians would also be happy with a variety of other inference patterns that manifest a commitment to ($\Box$*). Here are five examples of such inference patterns, and there are many more:

(2) If $A \vdash C$ and $B \vdash D$, then $A \vdash B \mathrel{\Box\!\!\rightarrow} (C \wedge D)$

(3) If $A \vdash B$ and $C \vdash \neg B$, then $A \vdash C \mathrel{\Box\!\!\rightarrow} \bot$

(4) $A \vee B \vdash \neg A \mathrel{\Box\!\!\rightarrow} B$

(5) If $A \vdash B$ then $A \vdash C \mathrel{\Box\!\!\rightarrow} (B \wedge C)$

(6) $A \mathrel{\Box\!\!\rightarrow} B \vdash \neg B \mathrel{\Box\!\!\rightarrow} \neg A$[22]

[22] Here are, respectively, the instances of (2)-(6) from which ($\Box$*) can be proved:

(2′) If $A \vdash A$ and $\neg A \vdash \neg A$, then $A \vdash \neg A \mathrel{\Box\!\!\rightarrow} (A \wedge \neg A)$

4. Metaphysical modal logic within mathematics

How deep do the modal commitments of mathematics run? So far we have argued against views that posit an epistemic division of labor in which mathematics supplies the truths and philosophy supplies their necessity. We have argued that the non-contingency of all mathematical truths is provable within mathematics itself. This still leaves room for a kind of epistemic division of labor: one might think that, while mathematics supplies both the truths and their necessity, it stops short of telling us anything further about necessity. In the new division of epistemic labor, it is up to philosophers to supply any further truths about the modal status of mathematical statements, such as that any mathematical statement that is necessarily true is necessarily necessarily true. According to this picture, to put it in a slogan, mathematicians supply both the truths and their necessity, and philosophers supply the logic of their necessity.

In fact, it already follows from what has been said that this picture cannot be correct. Mathematical practice turns out to be highly opinionated about the application of principles of modal logic to mathematics. By replacing 'A' in ($\Box$) with '$\Box A$' we get

4: $\vdash \Box A \to \Box\Box A$.

Of course, we can also replace 'A' in ($\Box$) with '$\Diamond A$', so we also have:

5: $\vdash \Diamond A \to \Box\Diamond A$.

It is clear that $\neg A \mathrel{\Box\!\!\to} \bot \vdash A$, which implies that $\vdash (\neg A \mathrel{\Box\!\!\to} \bot) \to A$, so, by Definition 1:

T: $\vdash \Box A \to A$

It should come as no surprise that the **K** axiom $\Box(A \to B) \to (\Box A \to \Box B)$ is also provable in mathematics.[23] Finally, the key observation of §2 was, in effect, that provability in mathematics obeys the principle of *Necessitation*: if A is provable, then so is $\Box A$. Given CLASSICALITY, it follows that the necessity of any truth-functional tautology is provable in mathematics, as well as that, whenever $\Box A$ is provable in mathematics and $A \to B$ is a truth-functional tautology, $\Box B$ is provable in mathematics.

The previous paragraph's observations amount to no more and no less than this:

(3′) If $A \vdash A$ and $\neg A \vdash \neg A$, then $A \vdash \neg A \mathrel{\Box\!\!\to} \bot$

(4′) $A \vee \bot \vdash \neg A \mathrel{\Box\!\!\to} \bot$

(5′) $A \vdash A$ then $A \vdash \neg A \mathrel{\Box\!\!\to} (\neg A \wedge A)$

(6′) $\neg\bot \mathrel{\Box\!\!\to} A \vdash \neg A \mathrel{\Box\!\!\to} \neg\neg\bot$

(($\Box$*) follows from (6′) by $A \vdash \top \mathrel{\Box\!\!\to} A$, which is entailed by CLASSICALITY and DEDUCTION.)

[23] But the proof is a little more involved than those of the other axioms. Since $\neg A \vee B$ is tautologically equivalent to $A \to B$, to show that the **K** axiom is provable in mathematics, it will suffice to show that:

(K*) $\neg(\neg A \vee B) \mathrel{\Box\!\!\to} \bot, \neg A \mathrel{\Box\!\!\to} \bot \vdash \neg B \mathrel{\Box\!\!\to} \bot$

(K*) seems completely unexceptionable. Consider this speech: 'Suppose that, (i) if A were true and B were false (which is tautologically equivalent to $\neg(\neg A \vee B)$), then a contradiction would be true. And suppose that (ii) a contradiction would be true if A were false. Then (it is immediate from (ii)) that A is true. It is immediate from this and (i) that both A is true and, if A were true and B were false, then a contradiction would be true. So, a contradiction would be true if B were false.' If the foregoing speech counts as an informal proof, and we maintain that it does, and our mathematician informants agree, then (K*) is true.

*Every theorem of the modal system **S5** that is expressible in the language of mathematics is provable in mathematics*.[24]

In effect, mathematics contains within itself the system **S5**. Since the logic of metaphysical modality is at least as strong as **S5**,[25] and no one (as far as we know) thinks it is stronger than **S5**, this means that mathematics is as opinionated as it could be without going against philosophical consensus on the application of principles of modal logic to mathematics. Indeed, mathematics has locked on to the characteristic feature of metaphysical modality, which is the collapse of all iterated modalities: in **S5**, both $\Box A$ and $\Diamond A$ are equivalent to $\Box/\Diamond A$, where $\Box/\Diamond$ is any finite string of boxes and diamonds. From ($\Box$5) and the above observations it follows that:

($\Box$5*) $\vdash \Box/\Diamond A \vee \Box/\Diamond\neg A$.

That is to say, each mathematical statement is such that it is provable in mathematics that either it or its negation is necessarily necessarily true, necessarily possibly true, necessarily necessarily necessarily true, and so on for all finite sequences of 'necessarily's and 'possibly's. There is no epistemic division of labor even of the modest kind envisaged at the beginning of this section.

5. Objections and replies

We shall now consider three objections to the foregoing.

The first objection concerns Definition 1, which entails that all counterfactuals with metaphysically impossible antecedents—so-called *counterpossibles*—are true.[26] Yet many philosophers maintain that there are false counterpossibles,[27] and they will accordingly reject Definition 1. Naturally they will also reject Williamson's argument for the logical equivalence of the two sides of Definition 1. For our own part, we think that there are good reasons to think that all counterpossibles are true.[28] However, we need not assume either that principle or Definition 1, which entails it, in order to establish ($\Box$*) or (*a fortiori*) ($\Box$). Our arguments for both relied on only one direction of the putative definitional equivalence of $\Box A$ and $(\neg A \mathrel{\Box\!\!\rightarrow} \bot)$: the entailment of $\Box A$ by $(\neg A \mathrel{\Box\!\!\rightarrow} \bot)$. And that direction is typically not questioned by philosophers who think there are false counterpossibles. Such philosophers typically reject NECESSITY while accepting POSSIBILITY, from which $(\neg A \mathrel{\Box\!\!\rightarrow} \bot) \rightarrow \Box A$ is derivable in **K**.[29] To make vivid why rejecting the entailment of $\Box A$ by $(\neg A \mathrel{\Box\!\!\rightarrow} \bot)$ would be

---

[24] **4** is a redundant step on the way to this conclusion. It is a theorem, not an axiom, of standard axiomatizations of **S5**.

[25] That is, the logic of metaphysical modality without indexicals. When indexical operators (such as 'actually') are present, Necessitation must be restricted. But there are no indexical operators in the language of mathematics. See note 18?? for further discussion.

[26] Here is a proof. Suppose for a *reductio* that there is a false counterpossible. Then, for some A and B, $\Box\neg A$ is true and $A \mathrel{\Box\!\!\rightarrow} B$ is false. Then, by Definition 1, $A \mathrel{\Box\!\!\rightarrow} \bot$ is true. But if $A \mathrel{\Box\!\!\rightarrow} \bot$ is true then so is $A \mathrel{\Box\!\!\rightarrow} C$, for any C. (Everything is a truth-functional consequence of $\bot$, and any truth-functional consequence of a proposition counterfactually implied A is also counterfactually implied by A.) So, in particular, $A \mathrel{\Box\!\!\rightarrow} B$ is true, contrary to hypothesis.

[27] E.g., Lowe (2012), Brogaard and Salerno (2013), and Berto et al. (forthcoming).

[28] See Williamson (forthcoming *a*, forthcoming *b*, and 2010).

[29] See Strohminger and Yli-Vakkuri (2017: §3) for review. Berto, French, Priest, and Ripley's (forthcoming) development of a logic that allows counterpossibles to be false but nevertheless validates POSSIBILITY is a recent representative example. Similarly, Lowe (2012) maintains that there are false counterpossibles and

a bad idea, consider the issue in terms of possible worlds semantics. In any standard possible worlds semantics, the existence of a counterexample to the entailment of $\Box A$ by $\neg A \ \Box\!\!\rightarrow \bot$ would require the existence of a world at which $\bot$ is true, which in turn would require the truth of $\Diamond\bot$. But clearly no truth-functional contradiction is possibly true.[30],[31]

The second objection proceeds from a thesis we'll call DISPENSABILITY. DISPENSABILITY says that any counterfactuals in pure mathematics are dispensable in that they contribute nothing to the *content* of mathematics, in the sense that what mathematics contributes to our total body of knowledge it would still contribute (albeit perhaps in a less reader-friendly way) if its counterfactuals were replaced by indicative or (if these are different) material conditionals.[32] DISPENSABILITY is consistent with, and can be motivated by, a variety of views about the role of counterfactual conditionals in mathematics. For example, an advocate of DISPENSABILITY might think that there are, contrary to appearances, no counterfactuals in mathematical texts, and that the subjunctive conditional construction has a different semantics in mathematical and other contexts. Or he might think that mathematicians are simply being careless when they use subjunctive conditionals, and that what they really mean to express by these conditionals, when push comes to shove, are the corresponding indicative or material conditionals. Or, rather more plausibly, she might concede that, as in the case of epistemic modals, there are good reasons for the occurrence of counterfactuals in mathematical texts, and that they are even indispensable for some (e.g., pragmatic) purposes, but, like epistemic modals, counterfactuals are nevertheless dispensable to what mathematics contributes to our knowledge. According to this perspective, we are making the same kind of mistake that would be made by someone who takes pure mathematics to be committed to various epistemic claims on account of the ubiquity of epistemic modals in mathematical texts. Clearly that would be a mistake: whatever role epistemic modals play in those texts, mathematics is not in the business of teaching us anything about knowledge: the actual *axioms*, *proofs*, and *results* of mathematics, when strictly and literally stated, do not concern knowledge,[33] no matter what epistemic language one finds in standard mathematical texts. Similarly, the objection goes, those same axioms,

---

accepts POSSIBILITY, with, however, less promising results (see Strohminger and Yli-Vakkuri 2017: §4 for criticisms).

[30] A subtle complication arises when we retreat to versions of our arguments that do not rely on Definition 1, for then we can no longer assume that the language of mathematics can express the necessity of any mathematical statement: without Definition 1, we have no argument that $\Box A$ is in the language of mathematics whenever A is. To derive ($\Box$) and ($\Box$*) we must either consider a mathematical language enriched with $\Box$ or interpret '$A_1, \ldots, A_n \vdash B$' to mean something like 'For some $C_1, \ldots, C_n$, each of $C_1, \ldots, C_n$ is informally provable from $A_1, \ldots, A_n$ and B is formally provable in **K** from $C_1, \ldots, C_n$ and POSSIBILITY'. The second option strikes us as particularly attractive. It still delivers a result that displays the commitment of mathematics to the necessity of provable mathematical truths. Since **K** is sound and POSSIBILITY is valid, the result becomes that mathematics is committed to the necessity of its provable truths in the sense that (i) whenever A is provable in the mathematics itself, so are some statements from which $\Box A$ logically follows, and that (ii), whatever mathematical statement A may be, $\Box A \vee \Box\neg A$ is logically follows from statements provable in mathematics.

[31] Granted, certain proponents of the view that there are false counterpossibles may take the further radical step of not merely rejecting the view that DEDUCTION holds for *logical* consequence but also that it holds for informal provability in mathematics. We are not going to engage with such radicalism, except to remind the reader that DEDUCTION is a part of the practice of mathematics, and so the radical step is tantamount to revisionism about that practice.

[32] Quine was a prominent defender of this view. In fact, he held the even stronger view that counterfactuals were dispensable to all of science. See, e.g., Quine (1994: 149-50).

[33] Some areas of mathematics—game theory, for example, and certainly epistemic logic, if it counts as part of mathematics—might be thought to concern knowledge, but, even if they do, it would be a mistake to treat the presence of *epistemic modals* as evidence of the commitment of mathematics to claims about knowledge.

proofs, and results, strictly and literally stated, do not concern counterfactual matters,[34] no matter what counterfactual-sounding language one finds in mathematical texts.

A great deal could be said about possible motivations for DISPENSABILITY (we find the hypotheses of carelessness and special semantics to be especially implausible[35]), but there is no need to engage with its motivations when we can attack DISPENSABILITY directly, and we will.

In fact, counterfactuals are absolutely indispensable to what mathematics contributes to our total body of knowledge—which is to say, DISPENSABILITY is false. Note first that myriad applications of mathematics to the hustle and bustle of both everyday life and engineering require our knowing that mathematical truths would remain true even if things had gone differently in various ways. For example, in justifying a particular engineering solution, one often appeals to mathematical truths in reasoning about how things would have gone if one had opted for an alternative solution. In doing so one assumes—and if one is successful, one knows—that those mathematical truths would have been true even if one had opted for the alternative solution. Note second that, as the queen of the sciences, mathematics is primed for application in any area of objective inquiry, whether it be the science of electromagnetism, the theory of rook and pawn endings, or natural language semantics. Each of these disciplines deploys its own counterfactuals, and in applying mathematics to them one must know that the truths of mathematics would remain true also under their counterfactual suppositions. Notably such counterfactual suppositions often include ones that are *nomologically* impossible and yet are not treated as ones from which anything whatsoever follows. For example, when doing physics, we are perfectly happy to hold the truths of mathematics fixed when suppositionally reasoning about the behavior of particles under various permutations of the standard model, and our comfort level does not at all diminish when reasoning about models that we take to be nomologically impossible. The success of our practice of applying mathematics to anything whatsoever requires that we know that mathematical truths would remain true under any counterfactual suppositions whatsoever[36]—which is to say, it requires that we know that mathematical truths would remain true no matter what, or, equivalently, that they are necessary truths.

The third objection turns on the point that, for any restricted necessity $\Box'$, one can define an associated 'restricted counterfactual' $\Box\!\!\rightarrow'$ that obeys the principle

$$\Box' A \leftrightarrow (\neg A \;\Box\!\!\rightarrow' \bot).$$

Even granting that mathematics deploys counterfactual discourse, it may be suggested that the counterfactuals in play are restricted in some way, and consequently mathematics is not committed to the necessity *simpliciter* of mathematical truths, but only to their necessity in

[34] As in the case of knowledge, one might think that this claim is undermined by the existence of counterfactual logic as a *bona fide* area of mathematical inquiry, but, in any case, it would be hard to argue from mathematical commitment to some kind of counterfactual logic to the claims we wish to argue for. After all, our claims concern informal provability, and our premises include claims about counterfactuals that are not principles of any reasonable counterfactual logic.

[35] The short story is that the hypothesis of careless ascribes implausible deficits in semantic processing to sophisticated mathematicians, and the hypothesis of special semantics is supported by no linguistic data. And it is no more plausible to suggest that, like Hodges' example 'can be embedded', counterfactuals in mathematics are being used non-literally.

[36] This sentence contains our reply to Gideon Rosen's (2002) parable of the two tribes who disagree about whether it is metaphysically contingent that there are numbers. Rosen finds it hard to tell which tribe is right. Our view is that the 'modally deviant' tribe (to use his term) would have a hard time applying mathematics as widely as we do. And if they did apply mathematics as widely, they would be in the awkward position of not knowing that various of their applications were correct.

some restricted sense. By analogy, suppose you are playing craps and you throw a red die and a green die, and the first comes up six and the second three. It is then natural for you to say: 'If the green die had come up six, I would have had boxcars'. It's pretty clear here that you are somehow restricting the domain of the possibilities your counterfactual generalizes over to ones that match the actual world with respect ones in which the red die comes up six. (Without this restriction, what you say may well be false: perhaps, for all you know, if the green die had come up six, it would have done so by bumping into the red die and causing it to come up one.) One might similarly suspect that the counterfactuals of pure mathematics involve a *de jure* restriction to (e.g.) possibilities that in which the actual truths of pure mathematics are true. And if that is right, then counterfactuals of pure mathematics would be unimpugned by possibilities in which the truths of mathematics were different from what they actually are, and facts like (1) would not, after all, manifest a mathematical commitment to the necessity of mathematical truths, but only to their truth. (One could imagine various other versions of the view that mathematical counterfactuals are restricted in some way; any of them would undermine our arguments. We only offer this one as an illustration.)

This objection fails for pretty much the same reason as the previous one did. If the counterfactuals of pure mathematics were restricted in the way just described, or indeed in any other way, then it would be a mystery how we get to know that our applications of mathematics in everyday life, engineering, and other fields of objective inquiry are correct. It would be a mystery what entitles us to deploy mathematics under the counterfactual suppositions of all of these other areas of thought.[37] Some explanation would be needed of why mathematical counterfactuals, in spite of being restricted, are nevertheless at least as unrestricted as any of the counterfactuals of any of the areas of thought to which we routinely and successfully apply mathematics. By far the most natural explanation is the one that we, in effect, have assumed is correct: that the counterfactuals of pure mathematics are completely unrestricted.

This epistemological issue is not very different from the analogous epistemological issue concerning the perfectly general logical laws articulable in an indexical-free second-order language, such as

$$\forall p(p \vee \neg p).$$

What we said about provable mathematical truths in §2 carries over smoothly to the case of provable logical laws, and what we said about unprovable mathematical truths in §3 carries over smoothly to the case of unprovable logical laws. And here too one might worry that the counterfactuals we deploy in our informal proofs of laws like $\forall p(p \vee \neg p)$ are somehow restricted and therefore do not manifest a logical commitment to the metaphysical necessity of (indexical-free) second-order logic. And here too it would be correct to reply that if those counterfactuals were restricted, then it would be a complete mystery where we get the license to apply the logical laws in question under any counterfactual suppositions concerning any subject matter whatsoever.

[37] Using lingo popular among Kratzerians, if the modal base of pure mathematical counterfactuals were not known to be the domain of all possibilities, then one could not with security deploy pure mathematics under counterfactual suppositions made in contexts that involve a wider modal base.

References

Bealer, George (2002). 'Modal Epistemology and the Rationalist Renaissance'. In Gendler and Hawthorne (2002).

Berto, French, Priest, and Ripley (forthcoming). 'Williamson on Counterpossibles', *Journal of Philosophical Logic.*

Boolos, George S., John P. Burgess and Richard C. Jeffrey (2007). *Computability and Logic*, 5th ed. Cambridge: Cambridge University Press.

Brogaard, Berit and Joe Salerno (2013). 'Remarks on Counterpossibles', *Synthese*, 190: 639-660.

Davis, Martin (1958). *Computability and Unsolvability*. New York: McGraw-Hill.

Davies, Martin and Lloyd Humberstone (1980). 'Two Notions of Necessity', *Philosophical Studies*, 38:1-31.

Dorr, Cian and John Hawthorne (2013). 'Embedding Epistemic Modals,' *Mind*, 122: 867-913.

Gendler, Tamar Szabó and John Hawthorne, eds., (2002). *Conceivability and Possibility*. Oxford: OUP.

Hale, Bob and Crispin Wright (2001). *The reason's proper study: Essays toward a neo-Fregean philosophy of mathematics*. Oxford: OUP.

Hodges, Wilfrid (2013). 'Modality in mathematics', *Logique et Analyse*, 56: 5-23. Online at http://wilfridhodges.co.uk/semantics12.pdf, accessed July 6, 2017. Page references to the latter.

Hodges, Wilfrid (2007). 'Necessity in Mathematics', manuscript, online at http://wilfridhodges.co.uk/semantics06.pdf, accessed July 6, 2017.

Iatridou, Sabine (2000). 'The Grammatical Ingredients of Counterfactuality', *Linguistic Inquiry*, 31(2): 231-270.

Kaplan, David (1989). 'Demonstratives.' In J. Almog at al., eds., *Themes from Kaplan*. Oxford: Oxford OUP.

Kratzer, Angelika (1986). 'Conditionals', *Chicago Linguistics Society* 22: 1-15.

Lewis, David (1973). *Counterfactuals.* Oxford: Blackwell.

Lowe, E. J. (2012). 'What is the source of our knowledge of modal truths?' *Mind*, 121: 919–50.

Quine, W. V. (1994). 'Promoting Extensonality', *Synthese*, 98: 143-15.

Raatikainen, Panu (forthcoming). 'Neo-logicism and its logic'. *History and Philosophy of Logic.*

Rosen, Gideon (2002). 'A Study in Modal Deviance'. In Gendler and Hawthorne (2002).

Sider, Theodore (2011). *Writing the book of the world*. Oxford: Oxford University Press.

Stalnaker, R. 1968. A theory of conditionals. In *Studies in Logical Theory*, ed. N. Rescher, 98–112. Oxford: Blackwell.

Strohminger, Margot and Juhani Yli-Vakkuri (2017). 'The Epistemology of Modality', *Analysis*, 77: 825-838

Turing, Alan M. (1939), *Systems of Logic Based on Ordinals* Ph.D. thesis, Princeton University.

Williamson, Timothy (2007). *The Philosophy of Philsophy*. Oxford: OUP.

Williamson, Timothy (2010). 'Modal Logic within Counterfactual Logic'. In B. Hale and A. Homan (eds.), *Modality: Metaphysics, Logic, and Epistemology*. Oxford: Oxford University Press.

Williamson, Timothy (2013*a*). *Modal Logic as Metaphysics*. Oxford: OUP.

Williamson, Timothy (2013*b*). 'Logic, Metalogic, and Neutrality', *Erkenntnis*, 79, supplement 2: 211-231.

Williamson, Timothy (2016). 'Absolute Provability and Safe Knowledge of Axioms'. In Leon Horsten and Philip Welch, eds, *Gödel's Disjunction*. Oxford: OUP.

Williamson, Timothy (2017). 'Modal Science'. In Yli-Vakkuri and McCullagh (2017).

Williamson, Timothy (forthcoming *a*). 'Counterpossibles', *Topoi.*

Williamson, Timothy (forthcoming *b*). 'Counterpossibles in metaphysics'. In B. Armour-Garb and F. Kroon, eds., *Philosophical Fictionalism*. Forthcoming.

Williamson, Timothy (2007). *The Philosophy of Philosophy*. Oxford: Oxford University Press.

Yli-Vakkuri, Juhani and Mark McCullagh (2017). *Williamson on Modality*. London: Routledge.