Kurt Gödel, 'Über formal unentscheidbare Sätze der *Principia mathematica* und verwandter Systeme I' (1931)

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1 Gödel's Life and Work

Gödel's incompleteness results are two of the most fundamental and important contributuions to logic and the foundations of mathematics. Gödel showed that no axiomatizable formal system strong enough to capture elementary number theory can prove every true sentence in its language. This theorem is an important limiting result regarding the power of formal axiomatics, but has also been of immense importance in other areas, e.g., the theory of computability.

Kurt Gödel was born on April 28, 1906 in Brünn, the capital of Moravia, then part of the Austro-Hungarian Empire and now Brno, Czech Republic. His father was a well-to-do part-owner of a textile company. Gödel attended the German *Gymnasium* in Brünn and in 1923 followed his elder brother to study at the University of Vienna. Gödel first studied physics, but Philipp Furtwängler's lectures on number theory so impressed him that he switched to mathematics in 1926. His teachers quickly realized Gödel's talent, and upon the recommendation of Hans Hahn, who later became his supervisor, Gödel was invited to join the group of philosophers around Schlick known as the *Vienna Circle*. Gödel regularly attended until 1928, and later remained in close contact with some members of the circle, especially Rudolf Carnap. His interest in logic and the foundations of mathematics was sparked around that time, mainly through Carnap's lectures on logic, two talks which L. E. J. Brouwer gave in Vienna in 1928, and Hilbert and Ackermann's *Grundzüge der Theoretischen Logik* (1928).

One of the open problems posed in Hilbert and Ackermann (1928) was that of the completeness of the axioms of the *engere Funktionenkalkül*, the first-order predicate calculus. Gödel solved this problem in his dissertation, which was submitted to the University of Vienna in 1929 and appeared as (1930). Gödel then set to work on the main open problem in Hilbert's foundational program, that of finding a finitary consistency proof for formalized mathematics. This led him to the discovery of his first incompleteness theorem. In September 1930, following a report on his dissertation work, he gave the first announcement of his new result in a discussion of the foundations of mathematics at the *Tagung für Erkenntnislehre der exakten Wissenschaften* in Königsberg. John von Neumann, who was in the audience, immediately recognized the significance Gödel's result had for Hilbert's program. Shortly thereafter, von Neumann wrote to Gödel with a sketch of the second incompleteness theorem about the unprovability of the consistency of a system within that system. By that time, Gödel had also obtained this result and published an abstract of it. The second result showed that Hilbert's program could not be carried out, and gave a negative solution to the second problem in Hilbert's famous 1900 list of mathematical questions: Gödel proved that there can be no finitary consistency proof for arithmetic. The full paper (1931) was submitted for publication on November 17, 1930 and appeared in January of 1931. It was also accepted as Gödel's *Habilitationsschrift* in 1932, and he was made *Privatdozent* (unpaid lecturer) at the University of Vienna in 1933.

Throughout the 1930s, Gödel worked on topics in logic and the foundations of mathematics, andlectured often on his incompleteness results. In particular, he gave a course on these results during his first visit to Princeton during the 1933/34 academic year which exerted a significant influence on the logicians there, especially Alonzo Church and his student Stephen C. Kleene. During the 1930s, Gödel settled a subcase of the decision problem for first-order logic (proving the decidability of the so-called Gödel-Kalmar-Schütte class), showed that intuitionistic logic cannot be characterized by finitely many truth values (and in the process inventing the family of Gödel logics), gave an interpretation of classical arithmetic in intuitionistic arithmetic (thus showing the consistency of the former relative to the latter), and established some proof-theoretic speed-up results.

After the annexation of Austria by Nazi Germany in 1938, during a second visit to Princeton, the title of *Privatdozent* was abolished. Gödel's application for *Dozent neuer Ordnung* was delayed, and he was deemed fit for military duty. He and his wife Adele, whom he had married in 1938, obtained U.S. visas and emigrated in 1940. From that date on, Gödel held an appointment at the Institute for Advanced Study at Princeton University. 1940 also saw the publication of his third major contribution to mathematical logic, the proof of the consistency of the axiom of choice and of the continuum hypothesis with the other axioms of set theory. This work was also inspired by a problem set by Hilbert: The first in his famous 1900 list of problems had asked for a proof of Cantor's continuum hypothesis. Gödel's result, together with Paul Cohen's 1963 proof of the consistency of the negation of the axiom of choice and of the continuum hypothesis, gave a negative solution to Hilbert's first problem: the axioms of set theory do not decide the continuum hypothesis one way or the other.

From 1943 onward, Gödel became increasingly interested in philosophy and relativity theory. In 1944, he contributed a study of Russell's mathematical logic (see §X–*Principia*) to the Russell volume in the *Library of Living Philosophers*. In the 1950s, he published several contributions to general relativity theory around 1950. In 1958, Gödel's consistency proof of arithmetic by an interpretation using functionals of heredetarily finite type, the so-called *Dialectica* interpretation, appeared in print. Much of his post-1940 work, however, remained unpublished, including his modal-logical proof of the existence of God.

In the last ten years of his life, Gödel was in poor health, both physical and mental. He suffered from depression and paranoia, to the point at which fear of being poisoned kept him from eating. He died of "malnutrition and inanition" in Princeton on January 14, 1978. (For more on Gödel's life and work, see Feferman 1986 and Dawson 1997.)

2 Hilbert's Program, Completeness, and Incompleteness

Gödel's groundbreaking results were obtained against the backdrop of the foundational debate of the 1920s. In 1921, reacting in part to calls for a "revolution" in mathematics by the intuitionist L. E. J. Brouwer and his own student Hermann Weyl, Hilbert had proposed a program for a new foundation of mathematics. The program called for (i) a formalization of all of mathematics in an axiomatic systems followed by (ii) a demonstration that this formalization is consistent, i.e., that no contradiction can be derived from the axioms of mathematics. Partial progress had been made by Wilhelm Ackermann and John von Neumann, and Hilbert in 1928 claimed that consistency proofs had been established for first-order number theory. Gödel's results would later show that this assessment was too optimistic; but he had himself set out to with the aim of contributing to this program.

According to Wang (1987), Gödel attempted to give a consistency proof for analysis relative to arithmetic. For this, he needed a definition of the concept of truth in arithmetic to verify (in arithmetic itself) the truth of the axioms of analysis. But Gödel soon realized that the concept of truth for sentences of arithmetic cannot be defined in arithmetic. The was led to this result by considerations similar to the liar paradox, thus anticipating later work by Tarski. But *provability* of a sentence from the axioms of arithmetic *is* representable in arithmetic, and combining these two facts enabled Gödel to prove that every consistent axiomatic system in which provability was representable must contain true, but unprovable sentences. Gödel had apparently obtained this result in the Summer of 1930. At the time, he represented symbols by numbers, and formulas and proofs by sequences of numbers. Sequences of numbers can be straightforwardly formalized in systems of type theory or set theory. At the occasion of the announcement of his incompleteness result in the discussion at Königsberg, von Neumann asked if it was possible to construct undecidable sentences in number theory. This suggested a possible simplification to Gödel, and indeed he subsequently succeeded in arithmetizing sequences by an ingenious use of the Chinese Remainder Theorem.

It had been assumed by Hilbert that first-order number theory is complete in the sense that any sentence in the language of number theory would be either provable from the axioms or refutable (i.e., its negation would be provable); indeed, he asked for a proof of this in his lecture on problems in logic Hilbert (1929). Gödel's first incompleteness theorem showed that this assumption was false: it states that there are sentences of number theory which are neither provable nor refutable. The first theorem is general in the sense that it applies to any axiomatic theory which is ω-consistent, has an effective proof procedure, and is strong enough to represent basic arithmetic. The system for which Gödel proved his results is a version of the system of *Principia Mathematica*. In this system, the lowest type of variables ranges over numbers, the usual defining axioms for successor, plus comprehension are available as axioms. However, practically all candidates for axiomatizations of mathematics, such as first-order Peano Arithmetic, the full system of *Principia mathematica*, and Zermelo-Fraenkel set theory satisfy these conditions, and hence are incomplete.

3 An Outline of Gödel's Results

Gödel's (1931) paper is organized in four sections. Section 1 contains an introduction and an overview of the results to be proved. Section 2 contains all the important definitions and the statement and proof of the first incompleteness theorem. In Section 3, Gödel discusses strengthenings of this result. Section 4 is devoted to a discussion of the second incompleteness theorem.

In Section 2, Gödel first sets up some necessary definitions, gives the axioms of the variant P of the system of *Principia Mathematia* which he uses, introduces the machinery necessary for the arithmetization of metamathematics (Gödel numbering), and proves four theorems (I–IV) about recursive function and relations. The language of the system P consists of the usual logical symbols, 0 and the successor function f, as well as a repository of simply typed variables. Variables of the lowest type range over natural numbers, variables of the next type range over classes of numbers, variables of the third type range over classes of classes of numbers, and so on. The axioms of the system are the usual logical axioms, the comprehension axiom $(\exists u)(\forall v)(u(v) \equiv A(v))$ (where u is a variable of type $n+1$, v a variable of type n, and A a formula not containing u free), and the extensionality schema $(\forall v)(x(v) \equiv y(v)) \rightarrow x = y)$.

One of the novel methods Gödel uses is the arithmetization of syntax, now called ‘Gödel numbering’. In order to be able to formalize reasoning about formulas and proofs in system P—which is, after all, a system for number theory—Gödel defines a mapping of the symbols in the language of P to numbers. In Gödel’s original paper, the mapping is given by $0 \mapsto 1$, $f \mapsto 3$, $\sim\,\mapsto 5$, $\vee \mapsto 7$, $\forall \mapsto 9$, $(\mapsto 11$, $) \mapsto 13$ and the k-th variable of type n is mapped to p^n, where p is the k-th prime > 13 (e.g., the first variable of lowest type is coded by 17). A sequence of symbols (e.g., a formula) with codes $n_1, \ldots, n_k$ is then mapped to the number $2^{n_1} \cdot 3^{n_2} \cdot \ldots \cdot p_k^{n_k}$.

What Gödel calls recursive functions and relations would now be called *primitive* recursive functions (and relations); Gödel used the terminology in use at the time. A function ϕ is primitive recursive if there is a sequence of functions each of which is either the successor function $x+1$, a constant function, or results from two functions ψ, μ occurring previously in the sequence by the schema of primitive recursion,

$$\begin{aligned} \phi(0, x_2, \ldots, x_n) &= \psi(x_2, \ldots, x_n) \\ \phi(k+1, x_2, \ldots, x_n) &= \mu(k, \phi(k, x_2, \ldots, x_n), x_2, \ldots, x_n). \end{aligned}$$

A relation between natural numbers is primitive recursive if it can be defined by $\phi(x_1, \ldots, x_n) = 0$, where ϕ is a primitive recursive function.

Gödel’s Theorem I states that primitive recursive functions are closed under substitution and primitive recursion. Theorem II states that recursive relations are closed under complement and union. Theorem III states that if two functions ϕ, ψ are primitive recursive, then so is the relation defined by $\phi(\overline{x}) = \psi(\overline{x})$. Theorem IV, finally, establishes that primitive recursive relations are closed under bounded existential generalization, i.e., if $\phi(x)$ and $R(x, \overline{y})$ are primitive recursive, then so is the relation defined by $(\exists z)(z \leq \phi(x) \,\&\, R(z, \overline{y}))$.

Gödel next defines 46 functions and relations needed for the arithmetization of syntax and provability, of which the first 45 are primitive recursive. These definitions culminate in the definition of (45) xBy (‘x is a proof of y’) and (46) $\mathrm{Bew}(x)$ (‘x is a provable formula’). $\mathrm{Bew}(x)$ is not primitive recursive, since it is obtained from B by *un*bounded existential generalization (i.e., as $(\exists y)yBx$). These definitions use the arithmetization of syntax introduced earlier in the sense that, e.g., the relation $\mathrm{Bew}(x)$ holds of a *number* x if it is the code of a provable formula.

Gödel then sketches the proof of Theorem V, which states that whenever R is a primitive recursive n-ary relation, then there is a formula A with n free variables, so that if $R(k_1, \ldots, k_n)$, then $A(\overline{k}_1, \ldots, \overline{k}_n)$ is provable, and when not $R(k_1, \ldots, k_n)$, then $\sim A(\overline{k}_1, \ldots, \overline{k}_n)$ is provable. (Here, $\overline{k}$ is 0 preceded by k f’s). A formula A which is obtained from the primitive recursive definition of R in the way outlined in the proof is called a primitive recursive formula. Since the proof is only sketched, this is not an explicit definition of what a primitive recursive formula is. In particular, in the system P, the most natural way to formalize primitive recursion is by higher-order quantification over sequences of numbers. Gödel explicitly uses such a second-order quantifier in the proof of Theorem VII discussed below. The method of constructing a formula of P which satisfies the conditions of Theorem V for a given primitive recursive relation R—a formula which 'numeralwise represents' R—yields such a formula for each of the 46 functions and relations defined earlier.

Following the proof of Theorem V, Gödel introduces the notion of ω-consistency. Roughly, an axiomatic system is ω-consistent if it does not both prove $A(\overline{n})$ for all n and $\sim(\forall x)A(x)$. Theorem VI then is the first-incompleteness theorem. Suppose κ is a primitive recursive predicate that defines a set of (codes of) formulas, which we might add as axioms to system P. Then we can, in a similar way as before, define the relation $xBw_\kappa y$ (x is a proof of y in P_κ) and the predicate $\mathrm{Bew}_\kappa x$ (x is provable in P_κ). Theorem VI states that if P_κ is ω-consistent, then there is a primitive recursive formulas $A(x)$ so that neither $(\forall x)A(x)$ nor $\sim(\forall x)A(x)$ are provable in P_κ. By an ingenious trick combining diagonalization and the arithmetization of syntax (especially, Theorem V), Gödel proves that there is a formula $A(x)$ so that $(\forall x)A(x)$ is provably equivalent in P_κ to $\sim\mathrm{Bew}_\kappa(\overline{p})$, where p is the Gödel number of $(\forall x)A(x)$ itself. Hence, $(\forall x)A(x)$ in a sense says of itself that it is unprovable.

The paper continues in Section 3 with a number of strengthenings of Theorem VI. The formula A whose existence was proved in Theorem VI may contain quantifiers over higher-type variables. A relation which can be defined using only quantification over individual variables, and also $+$ and $\cdot$ as additional functions (addition and multiplication) is called *arithmetical*. Theorem VII states that every primitive recursive relation is arithmetical. Furthermore, the equivalence of recursive relation with arithmetical relations is formalizable in the system, i.e., if A is a primitive recursive formula, then system P proves that A is equivalent to an arithmetical formula (one containing $+$, $\cdot$, but no quantification over variables of higher type). It then follows from Theorem VI that every ω-consistent axiomatizable extension of P contains undecidable arithmetical sentences (Theorem X).

The final section is devoted to the second incompleteness theorem (Theorem XI), which says that the formalization of consistency of of an extension P_κ of P is not provable in P_κ. In this context, the formula formalizing consistency of P_κ is taken to be $\mathrm{Wid}_\kappa \equiv (\exists x)(\mathrm{Form}(x)\,\&\sim\mathrm{Bew}_\kappa(x))$ ('there is an unprovable formula'). The proof of Theorem XI is only sketched. The argument for the first half of Theorem VI, namely, that $(\forall x)A(x)$ is unprovable in P_κ, uses only the consistency of P_κ, but not its ω-consistency. By formalizing this proof in P_κ itself, we see that P_κ proves the implication $\mathrm{Wid}_\kappa \rightarrow \sim\mathrm{Bew}_\kappa(\overline{p})$, where p is the Gödel number of the unprovable $(\forall x)A(x)$. But, as noted above, $\sim\mathrm{Bew}_\kappa(p)$ is equivalent, in P_κ, to $(\forall x)A(x)$. So if Wid_κ were provable, then $(\forall x)A(x)$ would be provable as well. (For in-depth treatments of the technical results, see Smoryński 1977 or Hájek and Pudlák 1993).

4 Importance and Impact of the Incompleteness Theorems

The main results of Gödel's paper, the first (Theorem VI) and second (Theorem XI) incompleteness theorems stand as two of the most important in the history of mathematical logic. Their importance lies in their generality: Although proved specifically for extensions of system P, the method Gödel used is applicable in a wide variety of circumstances. Any ω-consistent system for which Theorem V holds will also be incomplete in the sense of Theorem VI. Theorem XI applies not as generally, and Gödel only announced a second paper in which this was going to be carried out for systems which are not extensions of P. However, the validity of the result for other systems was soon widely recognized, and the announced paper was never written. Hilbert and Bernays (1939) provided the first detailed proof of the second incompleteness theorem, and gave some sufficient conditions on the provability predicate Bew in order for the theorem to hold (see §X–*Grundlagen der Mathematik*).

One important aspect of the undecidable sentence $(\forall x)A(x)$ is that, although it is neither provable nor refutable *in P*, it is nevertheless readily seen to be *true*. For what it states is that it itself is not provable in P, and by the first incompleteness theorem, this is precisely the case. Since it is also not refutable, i.e., its negation is also unprovable in P, the existence of undecidable sentences like $(\forall x)A(x)$ shows the possibility of axiomatic systems which are ω-inconsistent. The system resulting from P by adding $(\forall x)A(x)$ as an additional axiom is one example. It proves $A(\overline{n})$ for all n, and also $(\exists x) \sim A(x)$. Although by Theorem VI, there will also be true, but unprovable statements in *this* system, the existence of undecidable sentences is left open. Rosser (1936) weakened the assumptions of Theorem VI and showed that also ω-inconsistent, but consistent systems of the type discussed by Gödel will contain independent sentences.

The immediate effect of Gödel's theorem, and in particular, of Gödel's second theorem, was that the assumptions of Hilbert's program were challenged. Hilbert assumed quite explicitly that arithmetic was complete in the sense that it would settle all questions that could be formulated in its language—it was an open problem he was confident could be given a positive solution. The second theorem, however, was more acutely problematic for Hilbert's program. As early as January 1931, in correspondence between Gödel, Bernays, and von Neumann, it became clear that the consistency proof developed by Ackermann must contain errors (see Zach 2003). Both Bernays and von Neumann accepted that the reasoning in Gödel's proof can be readily formalized in systems such as P; on the other hand, a consistency proof should, by Gödel's own methods, also be formalizable and yield a proof in P of the sentence expressing P's consistency. The errors in the consistency proof were soon found. It fell to Gentzen (1936) to give a correct proof of consistency using methods that, of necessity, could not be formulated in the system proved consistent. Although Gödel's results dealt a decisive blow to Hilbert's program as originally conceived, they led to Gentzen's work, which opened up a wide range of possible investigations in proof theory. (For more on the reception of Gödel's theorems, see Dawson 1989 and Mancosu 1999).

As mentioned above, up to 1930 it was widely assumed that arithmetic, analysis, and indeed set theory could be completely axiomatized, and that once the right axiomatizations were found, every theorem of the theory under consideration could be either proved or disproved in the object-language theory itself. Gödel's theorem showed that this was not so, and that once a sharp distinction between the object- and metatheory was drawn, one could always formulate statements which could be decided in the metatheory, but not in the object theory itself. The first incompleteness theorem shows that object-level provability is always outstripped by meta-level truth. Gödel's proof, by example as it were, also showed how carefully object- and meta-language have to be distinguished in metamathematical considerations. A few years later, Tarski's work on truth and semantic paradoxes pointed to the same issue, showing that truth cannot be defined in the object-level theory (provided the theory is strong enough).

Gödel's results had a profound influence on the further development of the foundations of mathematics. One was that it pointed the way to a reconceptualization of the view of axiomatic foundations. Whereas prevalent assumption prior to Gödel—and not only in the Hilbert school—was that incompleteness was at best an aberrant phenomenon, the incompleteness theorem showed that it was, in fact, the norm. It now seemed that many of the open questions of foundations, such as the continuum problem, might be further examples of incompleteness. Indeed, Gödel (1940) himself succeeded not long after in showing that the axiom of choice and the continuum hypothesis are not refutable in Zermelo-Fraenkel set theory; Cohen (1966) later showed that they were also not provable. The incompleteness theorem also played an important role in the negative solution to the decision problem for first-order logic by Church (1936). The incompleteness phenomenon not only applies to provability, but, via the representability of recursive functions in formal systems such as *P*, also to the notion of computability and its limits.

Perhaps more than any other recent result of mathematics, Gödel's theorems have ignited the imagination of non-mathematicians. They inspired Douglas Hofstadter's bestseller *Gödel, Escher, Bach*, which compares phenomena of self-reference in mathematics, visual art, and music. They also figure prominently in the work of popular writers such as Rudy Rucker. Although they have sometimes been misused, as when, self-described Postmodern writers claim that the incompleteness theorems show that there are truths that can never be known, the theorems have also had an important influence on serious philosophy. Lucas, in his paper “Minds, machines, and Gödel” (1961) and more recently Penrose in *Shadows of the Mind* (1994) have given arguments against mechanism (the view that the mind is, or can be faithfully modeled by a digital computer) based on Gödel's results. It has also been of great importance in the philosophy of mathematics. Gödel himself, for instance, saw them as an argument for Platonism (see, e.g., Feferman 1984).

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