*Internal Logic* brings together several threads of Yvon Gauthier's work on the foundations of mathematics and revisits his attempt to, as he puts it, radicalize Hilbert's Program. A radicalization of Hilbert's Program, I take it, is supposed to take Hilberts' finitary viewpoint more seriously than other attempts to salvage Hilbert's Program have. Such a return to the "roots of Hilbert's metamathematical idea" will, so claims Gauthier, enable him to save Hilbert's Program (p. 47).

Gauthier's "radicalization" has both a positive and a negative part. The negative part consists in a critique, interspersed throughout the book, of rival attempts to salvage Hilbert's Program, in particular, attempts to justify Gentzen's consistency proof for Peano Arithmetic on finitary grounds, but also Gödel's *Dialectica* interpretation. The positive part consists in the outline of a consistency proof of arithmetic in Chapter 5.

Both negative and positive part of Professor Gauthier's foundational discussion rely on his view that the method of infinite descent is priviledged over complete induction. This is an interesting view closely related to the distinction between potential and completed infinity. In reasoning by infinite descent, a proposition is proved to hold for an arbitrary number $n$ by showing that the supposition that it does not hold for $n$ implies the existence of $m < n$ for which it also does not hold. But since there can be no infinite descending sequence of natural numbers, this is impossible.

Professor Gauthier's discussion unfortunately does not succeed in elucidating the relationship he takes there to be between infinite descent and induction. Professor Gauthier tells us that one distinguishing characteristic of infinite descent is that "it does not require a universal (classical) quantification, but [only?] an unlimited or 'effinite' quantification over indefinite, potentially infinite sequences or Brouwer's indefinitely proceeding sequences..." (p. 57). Professor Gauthier's "effinite" quantifier plays a crucial role here.

In classical terms, one would straightforwardly formalize the principle of infinite descent as $\forall x(\neg A(x) \to \exists y < x \neg A(x)) \to \forall x A(x)$. This, again classically (by contraposing the conditional in the antecedent), is equivalent to $\forall x(\forall y < x A(x) \to A(x)) \to \forall x A(x)$. Professor Gauthier's formalization using the effinite quantifier $\mp x$ reads (in this context):

$$\mp x\{[\neg A(x) \land \exists y < x \neg A(y)] \to \exists y \mp z < y \neg A(z)\} \to \mp x A(x)$$

I must admit that I am not able to make sense of this formula, and why it constitutes a formalization of the principle of infinite descent. This may be due to my incomplete grasp of the meaning of the effinite quantifier. $\mp x A(x)$ supposedly means “there are effinitely many $x$ so that $A(x)$,” where “effinitely many” means “for the infinitely proceeding sequence of natural numbers.” Professor Gauthier’s finitary scruples suggest that the effinite quantifier is some kind of finitary universal quantifier, a way of saying $thatA(x)$ holds for all numbers without supposing that there is a totality of natural numbers. If that is granted, one would assume that replacing $\text{Ǝ}xA(x)$ by $\forall xA(x)$ would result in something that is classically equivalent to induction. But

$$\forall x\{[\neg A(x) \wedge \exists y < x \neg A(y)] \to \exists y \forall z < y \neg A(z)\} \to \forall x A(x),$$

as is easily seen, is not true for all $A$. On the other hand, in the discussion of the derivation rules for the effinite quantifier (p. 87), Professor Gauthier says that it behaves like universal quantification in positive occurrences and like existential quantification in negative ones. So perhaps, the classical equivalent should instead be:

$$\exists x\{[\neg A(x) \wedge \exists y < x \neg A(y)] \to \exists y \exists z < y \neg A(z)\} \to \forall x A(x).$$

But this fares even worse: this sentence is false whenever $\forall xA(x)$ is false.

On p. 81 Professor Gauthier states that “from a (classical) logical point of view, infinite descent is identified with the least number principle,” and on p. 57, after introducing the formalized version of infinite descent, he states that the least number principle is only classically equivalent to complete induction. Are we to suppose then that, from the standpoint of Kronecker-Fermat arithmetic, infinite descent somehow is closer, or perhaps even *amounts* to the least number principle $\exists xA(x) \to \exists x(A(x) \wedge \forall y < x \neg A(x))$? What counts in favor of this suspicion is the fact that the least number principle does not contain unbounded universal quantifiers of the kind Professor Gauthier finds objectionable in the schema of complete induction. On the other hand, Professor Gauthier seems to aim for some position close to intuitionism. Yet, in intuitionistic arithmetic, the induction principle is an axiom whereas the least number principle cannot be proved.[1]

Perhaps the difficulties in the discussion of the formalization of infinite descent could be overcome by providing a semantics and proof theory for the logic in which Professor Gauthier’s Kronecker-fermat arithmetic is to be couched. This Professor Gauthier aims to do in Chapter 4. One would expect here a precise formulation of a logical calculus, of a semantics, and a proof of soundness and perhaps completeness of the former with respect to the latter. Unfortunately, no

[1] See e.g., A. S. Troelstra and D. van Dalen, *Constructivism in Mathematics*. vol. I, Amsterdam: North-Holland, 1988, p. 129.

such proofs are offered. Indeed, I have serious difficulty understanding Professor Gauthier's proposed semantics (p. 88). The notion of a structure $M$ looks familiar enough: a domain $D_M$ (in practice, the set of natural numbers), and interpretations of predicates and function symbols. Then Professor Gauthier offers a definition of a function $\varphi_M$ which maps closed formulas to either 0 or 1. Evidently, $\varphi_M(A)$ is supposed to give the truth value of $A$. But the definition of $\varphi$ is somewhat confusing. For instance, the clause for conjunction is

$$\varphi_M(A \wedge B)[n \times m] = 1 \text{ iff } A_n \in D_M \text{ and } B_n \in D_M.$$

If $\varphi_M$ is a function from formulas to $\{0,1\}$, then what is the role of the extra argument $n \times m$? And what are $A_n$ and $B_m$? Perhaps they are the "valuators" Professor Gauthier hints at several lines above, "a number which locates the formula in the arithmetical universe." If $A_n$ and $B_m$ are numbers, and $D_M$ is the set of natural numbers, it seems to me, the left-hand side of the definition is always true—but surely not every conjunction can evaluate to 1?

These and other shortcomings of the presentation of the technical details make it difficult to ascertain what exactly Professor Gauthier is attempting to achieve in his consistency proof. A mapping of formulas and proofs to polynomials is important, and infinite descent on some ordering of polynomials is then applied to obtain the consistency result. I suspect that this approach to the problem is at its root not so very different from the approaches of the consistency proofs by Ackermann and Gentzen. They, too, proceed by establishing a mapping from proofs to polynomials, but in their case polynomials in which $\omega$ works as a variable, and in which exponents may themselves be polynomials. In other words, there is a natural way in which the Cantor normal form of an ordinal less than $\varepsilon_0$—ordinal notations as used by Gentzen and Ackermann—can be seen as a polynomial. Gentzen and Ackermann's proofs do not require any kind of transfinite induction in the set-theoretical sense. The induction—indeed, in these cases, it is infinite descent that is used, and not induction—proceeds over such $\omega$-polynomials. Cantor's normal form theorem is not at all a "basic ingredient" (p. 48) of Gentzen and Ackermann's work. Perhaps Professor Gauthier is right in doubting the subsistence of the limit ordinals $< \varepsilon_0$—not they, but instead ordinal *expressions* play a role in the consistency proofs.

What is at stake in the discussion of the status of consistency proofs like Gentzen and Ackermann's is not whether they employ infinitary objects, i.e., ordinals—they obviously do not—but whether the finitist is entitled to the insight that there are no infinitely descending sequences of ordinal notations of the kind used. Professor Gauthier seems to think that this insight cannot be finitarily justified, and the argument (on p. 64) seems to rest on the undeniable circumstance that "limit ordinals" have no immediate predecessor in the relevant order. It seems to me, however, that the same can be said of the polynomials Professor Gauthier uses in his consistency proof. What is the immediate predecessor of $x^2 + x$ in the ordering of polynomials along which his method of infinite descent proceeds? He seems to be in a situation vis-a-vis his polynomials that entirely parallels that of Gentzen and Takeuti vis-a-vis ordinal notations. Without providing a more detailed and precise account of the basic framework in which his consistency proof is carried out, and the proof methods used, it will be hard to shake the suspicion that the complications of his proof obscure, but do not avoid, the difficulties faced by Gentzen's proof.