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NUMBERS AND FUNCTIONS IN HILBERT'S FINITISM

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1. Introduction

Next to the logicist project of Frege and Russell of reducing mathematics (or at least arithmetic) wholesale to logic, Hilbert's program is the main contribution to the foundation of mathematics of the last century. Taking as his motivation the paradoxes of set theory and the revisionist attacks on classical mathematics by such predicativists as Poincaré and such intuitionists as Brouwer and the later Weyl, Hilbert formulated a new program for the grounding of arithmetic. The two basic tenets of the program were finitism and an instrumentalist formalism. Finitism proceeds from the conviction that certain basic statements of the mathematics of natural numbers, i.e., the finitistic statements, are firmly grounded in our intuition of finite sequences. These statements include equalities and inequalities between numbers and number terms built using certain finitistic functions (e.g., addition) and are closed under truth-functional connectives and bounded quantification. Statements involving unbounded quantification, on the other hand, are in general not finitistically meaningful. In particular, a universal statement is only finitistically meaningful in so far as it serves to convey a hypothetical judgement about arbitrary given numbers, and an existential statement only in so far as it is understood to consist in the exhibition of a witness or of a constructive method to find one. Under this interpretation the quantified statements are not capable of negation, and the principle of the excluded middle fails for them. But, "no one, though he speak with the tongues of angels, will keep people from negating arbitrary assertions, forming partial judgements, or using the principle of excluded middle. What, then, shall we do?"[1] Hilbert's solution is to round out the body of finitistic statements by introducing ideal elements. These ideal elements are given by the τ, and later, the ε functions, which in Hilbert's system of logic do duty for the quantifiers. In order to do this, Hilbert adopts a formalistic stance: Mathematics is to be formalized in an axiomatic system. Some formulas of this system, when interpreted, will correspond to finitistic statements—these are the *real* statements, all others are *ideal*. The axiomatic system in turn consists of formulas (finite strings of symbols) and derivations (finite strings of formulas). Thus, directly or indirectly via arithmetization of syntax, the formal system is amenable to mathematical investigation on a finitistic basis. The finitist investigation of formal systems is the object of metamathematics, and its goal is to establish the consistency of the formal systems. The particular way such consistency proofs are to be given establishes much more, namely: the conservativeness of the ideal over the real statements. This justifies the use of ideal methods in mathematics on a secure, that is finitist, basis.[2]

Today it is widely held that Gödel's incompleteness theorem destroys Hilbert's program.[3] For if there were a finitist consistency proof, to this proof would correspond a derivation in the formal system (using only the real part of that system) of the formalized statement of consistency. The possibility of such a derivation, is, however, excluded by the second incompleteness theorem. Despite this, the program was both fruitful in that it was the initial motivation for a wide area of mathematical research in proof theory, and interesting in its own right as a philosophical position.[4]

In the following, I will concentrate on the first tenet of the program, finitism. The two aspects thereof are the finitistic conception of number on the one hand, and finitistic reasoning on the other. Both have been object of analysis and criticism in the literature (most notably, by Kitcher 1976 and Tait 1981). I will try to give an overview of the issues involved and develop some new ideas on both aspects. A detailed historical treatment, however, is urgently needed. Hilbert did change his views between the first formulation of the program in 1922 and the early 1930s due to a number of influences. These influences include the technical work done by Hilbert's students, in particular Ackermann's attempted consistency proof, but also debates with a number of philosophers (in particular, Müller and Nelson), and finally the incompleteness theorems. Our understanding of finitism is incomplete until these influences are studied and the changes in the philosophical conceptions are analyzed. Mancosu (1998b) has emphasized one such important change, namely the shift from taking mathematical intuition to be essentially empirical to taking it to be a Kantian pure intuition. In this regard, unpublished material from Hilbert's and Bernays's papers in Göttingen and Zürich would provide sources, and Moore (1997) and Sieg (in press) have taken first steps in assessing this material.

2. Numbers and numerals

Hilbert and Bernays[5] conceived of the finitistic view of numbers in reaction and contrast to the logicist conception of number, which was supposed to yield a reduction of the number concept to broadly logical concepts. In Frege's view, numbers are extensions of certain concepts (For him, extensions of concepts, i.e., classes, are a logical notion). Russell took over this conception and tried to avoid the use of classes in *Principia* by his no-class theory.[6] Hilbert found this reduction of numbers to logical notions circular. In 1905 he writes:

> Arithmetic is often considered to be a part of logic, and the traditional fundamental logical notions are usually presupposed when it is a question of establishing a foundation for arithmetic. If we observe attentively, however, we realize that in the traditional exposition of the laws of logic certain fundamental arithmetic notions are already used, for example, the notion of set and, to some extent, also that of number. Thus we find ourselves turning in a circle, and that is why a partly simultaneous development of the laws of logic and of arithmetic is required if paradoxes are to be avoided. (Hilbert 1905, 131)

Although around 1917 Hilbert was leaning towards a logicist viewpoint, he later abandoned it. Bernays (1930, 243) made the disagreements between logicism and Hilbert's view very clear, and asserted that Frege's definition of cardinal numbers ("Numbers," *Anzahlen*) conceals the epistemologically essential characteristics of mathematics. Hilbert's finitistic viewpoint is not concerned with an analysis of the Number concept along the lines of the Fregean project, but with a methodological program that presents an analysis of a certain minimal mode of numerical reasoning which is epistemologically grounded and serves to secure higher mathematics. Like Wittgenstein's proverbial ladder, the finitistic viewpoint can be abandoned once it provides this secure foundation.[7] Securing higher mathematics through consistency proofs need not and cannot presuppose the logicist general analysis of Number. The Fregean analysis requires concepts and their extensions, entities which the finitist cannot consider; and Russell's solution was rejected because of the use of the axiom of reducibility. Hilbert attempts to account for numerical reasoning in terms of finite sequences, at first introduced as sequences of strokes. After 1923 Hilbert and Bernays are careful to distinguish between these "finitistic numbers" and the general concept of number (in either ordinal or cardinal sense), and usually refer to them as "numerals" (*Ziffern*).

Hilbert is interested in an account of elementary number theoretic reasoning which satisfies certain constraints of immediacy, intuitiveness, and certainty. These requirements of the methodological program translate into requirements on the subject matter of contentual finitistic arithmetic [*inhaltliche finite Arithmetik*]. First of all, Hilbert wants the numerals to be "concretely" given and surveyable by us. We have some immediate access to them which allows us to gain knowledge of finitary number-theoretic facts. The question now is: What are the numerals exactly, and how do we have knowledge of them? It seems clear that Hilbert wants some sort of intuition (in the Kantian sense) to be the source of knowledge, but is it pure or empirical? What is Hilbert's "primitive arithmetical intuition" and of what do we have an intuition when we engage in contentual arithmetical reasoning? Contemporary philosophy of mathematics would put the question in terms such as the following: Are the numerals physical objects? Mental constructions? Token or type? Abstract or concrete?

Some of the most fruitful sources on the topic of Hilbert's conception of finitism are his 1922 and 1926 papers, his collaborator Bernays's exchange with Müller (Müller 1923, Bernays 1923), as well as the relevant sections in Hilbert and Bernays (1934, 1939). In 1905, Hilbert gives a first account of finitistic number theory in terms of strokes and equality signs. We note here that no identification of certain (sequences of) signs with numbers is made, rather, the sequences of 1's and ='s are divided into two classes, the class of entities (these are the sequences of the form "1…1 = 1…1" with equal numbers of 1's on the left and right) and the class of nonentities; the former are the *true propositions*. Hence we have here a finitistic account, not of numbers, but of numerical truth.[8]

In the 1922 paper, Hilbert presents an explicit account of numbers as signs built up from 1's and +'s. He writes:

> The sign 1 is a number.
>
> A sign which begins with 1 and ends with 1, and such that in between + always follows 1 and 1 always follows +, is likewise a number […]

> These number-signs, which are numbers and which completely make up the numbers, are themselves the objects of our consideration, but have otherwise no *meaning* of any sort. (Hilbert 1922, 1122)

Just before this paragraph, Hilbert lays out the main conditions that govern these stroke symbols. They are "extra-logical concrete objects, which exist intuitively as immediate experience before all thought." Furthermore, they are capable of "being completely surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately, intuitively, as something that cannot be reduced to something else."[9]

Hilbert's account was criticized by the philosopher Aloys Müller, and some of the points he made were well taken. Following Müller's criticism, Hilbert (and Bernays) change the account slightly.

1. The term 'sign' connotes having a meaning, but the number-signs are supposed to have no meaning attached to them. To avoid ambiguity, Hilbert and Bernays subsequently use the term 'numeral' [*Ziffer*] instead of 'number-sign.' For, I suppose, similar reasons, they also cease to use the word 'number' in this context, and after (Hilbert 1922), no identification of numbers with numerals is made. This is in keeping with my remark above that Hilbert is not after an analysis of the number concept in general. Hilbert does, however, give an account of how numerals *function* as numbers in the sense of cardinal numbers, *Anzahlen*.[10]

2. The particular shape of the signs is immaterial. Bernays clarifies that what is important is that some objects of the same type are put together in a (finite) sequence.

> […T]he special shapes "1" and "1 + 1" are inessential. If we disregarded the connection to habit, it would even be advisable, in order to emphasize the principle, to take as numerical signs figures of the type
>
> . . . . . .
>
> (which are thus constituted merely of points). And, of course, stars, vertical strokes, circles and other shapes could just as well be chosen instead of points. One could also take a time sequence, say, of similar noises, instead of a spatial sequence.
>
> But it is essential that *specimens of equal shape be joined in the same sort of arrangement* [*Zusammensetzung*]. (Bernays 1923, 224)

How can we make sense of all this? Sometimes Hilbert's view is presented as if he had claimed that the numbers are signs on paper. It is important to stress that this is a misrepresentation, that the numerals are not physical objects in the sense that truths of elementary number theory are dependent only on external physical facts or even physical possibilities (e.g., on what sorts of stroke symbols it is possible to write down). Hilbert makes much of the fact that for all we know, neither the infinitely small nor the infinitely large are actualized in physical space and time. Hilbert must certainly hold that the number of strokes in a numeral is at least potentially infinite. It is also essential to the conception that the numerals are sequences of one kind of sign, and that they are somehow dependent on being grasped as such a sequence, that they do not exist independently of our intuition of them. Only our seeing or using '1111' as a sequence of 4 strokes as opposed to a sequence of 2 symbols of the form '11' makes '1111' into the numeral that it is. Would two stones lying side by side count as a numeral of the same kind as 11? If yes, then pretty much everything could be a numeral. Tait has pointed out that one can pass from the any token to another by a sequence of finitely many tokens, each of which we cannot distinguish from the next. Since the ability to distinguish the signs is central, he takes this to imply that the numerals cannot be physical tokens. The obvious alternative would be that numerals are mental constructions. However, Bernays denies also this, writing that "the objects of intuitive number theory, the number signs, are, according to Hilbert, also not 'created by thought.' But this does not mean that they exist independently of their *intuitive construction*, to use the Kantian term that is quite appropriate here." (Bernays 1923, 226). Kitcher considers this option as well. If the numerals were mental constructions,

> it seems that we shall have to accept many 3's (the array of three strokes I am currently contemplating, the array you are currently contemplating, the array I contemplated yesterday, etc.). So our discourse about numbers *simpliciter* should be replaced with talk about *X*'s number *n* at time *t*. Arithmetical knowledge is immediately vulnerable to all kinds of skepticism. Perhaps 'My 2 at *t* plus my 2 at *t* equals my 4 at *t*' holds for some past *t* (not all, for I have not always been alive, nor always awake). But what of the

> future? What of your 2's to which I am forbidden access? And what is the status of arithmetic before anyone ever constructed a stroke-symbol?[11]

Kitcher's alternative is to hold that, whatever the numerals are, the strokes on paper or the stroke sequences I am contemplating *represent* the numerals. The numerals are given in our representation, but they are not merely subjective "mental cartoons" (Kitcher's term).

> If we want […] the ordinal numbers as unique objects, free of all inessential ingredients, we must take the mere schema of each repetition figure [*Wiederholungsfigur*] as an object, which requires a very high abstraction. We are free, however, to represent these purely formal objects by concrete objects ("number signs"); these contain inessential and arbitrarily added properties, which, however, are readily comprehended as such. (Bernays 1930, 340)

One version of this view would be to hold that the numerals are *types* of stroke-symbols as represented in intuition. This is the interpretation that Tait (1981, 438–39) gives. At first glance, this seems to be a viable reading of Hilbert. It takes care of the difficulties that the reading of numerals-as-tokens (both physical and mental) faces, and it gives an account of how numerals can be dependent on their intuitive construction while at the same time not being created by thought. The reasoning that leads Tait to put forward his reading lies in several constraints that Hilbert and Bernays put on the numerals. Their *shape*[12] (but not they themselves) is supposed to be independent of place and time, independent of the circumstances of production, independent of inessential differences in execution, and capable of secure recognition in all circumstances (Hilbert 1922, 163). Tait infers from this that identity between numerals is type identity, and hence, that numerals should be construed as types of stroke symbols.

Types are usually considered to be abstract objects, however, not located in space or time. Taking the numerals as intuitive representations of sign types might commit us to taking these abstract objects as existing independently of their intuitive representation. That numerals are "space- and timeless" is a consequence that already Müller thought could be drawn from Hilbert's statements,[13] and that was in turn disavowed by Bernays. The reason is that a view on which numerals are space- and timeless objects existing independently of us would be committed to them existing simultaneously as a completed totality, and this is exactly what Hilbert is objecting to.

> It is by no means compatible […] with Hilbert's basic thoughts to introduce the numbers as ideal objects "with quite different determinations from those of sensible objects," "which exist entirely independent of us." By this we would go beyond the domain of the immediately certain. In particular, this would be evident in the fact that we would consequently have to assume the numbers *as all existing simultaneously*. But this would mean to assume at the outset that which Hilbert considers to be problematic. (Bernays 1923, 162)

This is not to say that it is *incoherent* to consider the numbers as being abstract objects, only that the finitistic viewpoint prohibits such a view. Bernays goes on to say:

> Hilbert's theory does not exclude the possibility of a philosophical attitude which conceives of the numbers [but not the finitist's numerals] as existing, non-sensible objects (and thus the same kind of ideal existence would then have to be attributed to transfinite numbers as well, and in particular to the numbers of the so-called second number class). Nevertheless the aim of Hilbert's theory is to make such an attitude dispensable for the foundation of the exact sciences. (Bernays 1923, 163)

Another open question in this regard is exactly what Hilbert means by "concrete." He very likely does not use it in the same sense as it is used today, that is, as characteristic of spatio-temporal physical objects in contrast to "abstract" objects. In that modern sense, sign types are not concrete. However, sign types certainly are different from full-fledged abstracta like pure sets in that all their tokens are concrete. Parsons takes account of this difference by using the term "quasi-concrete" for such abstracta. Tait, on the other hand, thinks that even the tokens are not concrete physical objects, but abstract themselves.

The considerations outlined so far should have convinced the reader by now that the view is not as easily made sense of as one might be inclined to think on a cursory reading of "On the infinite." On the one hand, for instance, the numerals are supposed to be objective, not merely created by thought; on the other hand they are not to be independent of their intuitive representation. They need to be concrete and surveyable, but they also cannot be physical objects. The situation is not alleviated by the fact that it is not even clear what Hilbert means by "intuition." Kitcher argues that it is pure sensuous intuition in the sense of Kant, and that this kind of intuition cannot meet all of Hilbert's requirements. Mancosu (1998b) has shown that Hilbert and Bernays at first held the intuition involved to be empirical, but later in the 1920s turned to pure intuition as the source of certainty in elementary contentual arithmetic.

Many of the problems discussed so far arise because Hilbert considers contentual finite mathematics to be about certain entities, the numerals, and we are puzzled by their epistemological and ontological status. Not only that, but Niebergall and Schirn (1998) argue that assumptions of infinity are implicitly made by Hilbert's finitism. If they are right, then the question of whether numerals are tokens or types is the least of Hilbert's problems. Their argument depends in part on assuming that Hilbert's contentual mathematics does have a standard referential semantics; that, say, "2 + 2 = 4" is true in virtue of the properties of the numerals that "2" and "4" refer to. What if we tried to make sense of finite mathematics without assuming standard semantics, without assuming that there are entities (the numerals "11" and "1111") that "2" and "4" refer to and which make "2 + 2 = 4" true?

A promising avenue which has been suggested by Kitcher (1976) is that in order to accommodate all of the finitist's requirements on numbers, this so-called standard account of mathematical truth must be abandoned. On the standard account, statements involving number terms are supposed to be analyzed in the same way as statements involving physical object terms, i.e., the terms refer and the truth conditions of sentences in which they occur are given by standard referential semantics. Benacerraf (1983, 403) argues that the virtue of the standard account is that it provides a "homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of the language." Such a theory is one of two requirements a theory of mathematical truth must fulfill, the other is that it "mesh with a reasonable epistemology." Given the foundational character of the finitist viewpoint and its explicit (and only) goal, namely to give an account of truth for (a fragment of) arithmetic which is *secure*, it is reasonable to allow the second requirement to be the more important one. As I will try to show, Hilbert and Bernays can be read as abandoning the standard view in their later writings.[14]

In many places Hilbert does seem to expound something like the standard account. For instance, in (Hilbert 1926, 377) he introduces first the numerals, which are supposed to "have no meaning at all by themselves." Contentual arithmetic, however, requires, besides the numerals, other signs "that mean something and serve to convey information, for example the sign 2 as an abbreviation for the numeral 11." There are also passages that suggests a non-standard reading, both have been mentioned before. In the early paper from 1905, the numerals are not introduced as an independent notion, but only in the context of identity statements. All we are given there are conditions of when an expression of the form "1…1 = 1…1" should count as a true proposition. In the response to Müller's criticism of Hilbert, Bernays introduces a distinction between the numerals [*Zahlzeichen*] and the notion of cardinal number [*Anzahl*]. He writes:

> It should also be noted that the contentual character of the *Number* [*Anzahl*] concept is indeed compatible with the purely figural character of the number signs. The figures are used as tools for *counting*, and by counting one arrives at the determination of cardinal number […]
>
> One here has to recognize that the cardinal numbers are only defined in the context of the entire *Number statement.* For example, it will not be explained what "the Number five" is, but only what it means for the Number five to apply to a given totality of things. (Bernays 1923, 225)

We find two very interesting ideas here. Bernays suggests a non-standard reading of Number statements quite explicitly. His formulation is very close to Russell's "meaning in use" of integral sign or class abstraction in *Principia Mathematica*. The other idea is that the numerals are tools for counting. Hand (1989, 1990) has presented a non-standard account of finitistic number theory which takes up this idea. Since in this account truth conditions for finitistic statements take center stage, let me now turn to a discussion of statements.

3. Statements

We can find in Hilbert's writings three distinctions of mathematical statements: (1) contentual vs. formal, (2) finitistic vs. infinitistic, and (3) real vs. ideal.[15] Contentual mathematics is comprised of those statements that we make based on our intuition of concrete objects. This includes statements we make about numerals, but also those about formulas and formal derivations. These are concrete objects given in intuition, we can come to know about them directly. Formal discourse starts where we leave the grounds of intuition and instead proceed from an axiomatic standpoint, when we deal with a formalized language. According to Hilbert, all mathematical discourse, "mathematics proper," [*die eigentliche Mathematik*] is to be formalized and thus becomes a "stock of provable formulas" (Hilbert 1922, 146). Whereas contentual mathematics deals with those statements and ways of reasoning that are justified on finite grounds, formalized mathematics appeals also to completed infinities, uses unbounded quantification, the unrestricted principle of the excluded middle, etc. The exact nature of the "finitistic standpoint" is subject to debate and historical analysis.[16] We find, however, a number of examples in Hilbert's and Bernays's writing of what falls within this standpoint. First there are the basic equalities and inequalities in the contentual sense between numerals; these are finitistically justified statements. We are also allowed to introduce certain computable functions, however, not as functions in the usual sense, i.e., sets of ordered pairs of numbers, but only as descriptions of methods of construction (see discussion in the next section). As a mode of reasoning, induction on propositions with "elementary intuitive content" is finitistically acceptable.[17] We are told which propositions can be finitistically interpreted. The most extensive discussion of this is in §2 of *Grundlagen* I:

> For the characterization of the finitistic standpoint we may emphasize some general considerations concerning the use of logical forms of judgment in finitist thought [*logische Urteilsformen im finiten Denken*], which we shall exemplify in the case of propositions about *numerals*.[18]
>
> A *universal* judgment about numerals can be interpreted finitistically only in a hypothetical sense, i.e., as a proposition about any given numeral. Such a judgment pronounces a law which must verify itself in each given particular case.
>
> An *existential sentence* about numerals, i.e., a sentence of the form "there is a numeral **n** with the property **A**(**n**),"[19] is to be understood finitistically as a "partial judgment," i.e., as an incomplete communication of a more specific proposition consisting in either a direct exhibition of a numeral with the property **A**(**n**), or the exhibition of a process to obtain such a numeral,—where part of the exhibition of such a process is a determinate bound for the sequence of action to be performed.
>
> Those judgments combining a universal proposition with an existential assertion have to be finitistically interpreted correspondingly. For instance, a sentence of the form "for each numeral **k** with the property **A**(**k**) there is a numeral **l**, for which **B**(**k**, **l**) holds," has to be finitistically understood as an incomplete communication of a process which makes it possible to find, for each given numeral **k** with the property **A**(**k**), a numeral **l** which stands in the relation **B**(**k**, **l**) to **k**. (Hilbert and Bernays 1934, 32–33)

This discussion is an expansion and clarification of the earlier exposition in "On the infinite"(Hilbert 1926, 377). It is clear that the negation of a finitistic statement need not be a finitistic statement, more precisely: A finitistic interpretation for a contentual statement about numerals does not, by itself, yield a finitistic interpretation of the negation of the statement. This is what it means that some finitist statements are "incapable of being negated:" From the inability, or even impossibility, to see that **A**(**k**) for each given numeral **k**, it does not follow that we have a witness **l**, or even a bound on such, for ¬**A**(**l**). At this point Hilbert and Bernays propose the formalization of mathematics and the introduction of ideal elements, i.e., transfinite quantifiers, to retain the simplicity of classical logic. (In formalized mathematics, we have real statements (roughly, these are the quantifier-free formulas), which admit of a contentual interpretation, and the ideal statements, which round out the theory, appeal to the infinite, and thus do not have a finite, contentual interpretation.)

Now how is the semantics of the basic finitistic statements explained? Hand (1990) discusses an *iterativistic* tendency in Hilbert's views on this issue. The basic idea is that we have the capacity to count intransitively, i.e., to count without counting any thing (in particular, not the natural numbers). The numerals are, so to speak, a crutch for us to remember when we are supposed to stop counting. An equality between numerals is then, e.g., to be understood as the assertion that in counting the strokes in both strings simultaneously, we will stop at the same point (in contrast to the view that identity is "figural correspondence"); an inequality as the statement that we can count further on one numeral than on another (in contrast to the idea that one numeral extends beyond the other).

I think this view has a certain appeal. After all, as a matter of developmental and historical fact, the concept of number had its origins in the human capacity to *count*. We count fingers, we count cattle, eventually, we just count. The phenomenon of counting without counting any thing, it seems to me, is much better explained as engaging in an iterative mental procedure than as naming, in succession, the members of the natural number sequence (whatever those are). Such a conception, argues Hand, is at work in some places in Hilbert's finitism. He supports his view mainly by textual evidence from Hilbert's (1905) and (1926). I believe that there are even stronger suggestions in this direction in *Grundlagen*.

For instance, in "On the infinite," the numerals are introduced in a very pictorial way, as sequences of strokes. A few years later, when even Hilbert is becoming convinced that empirical intuition will not do the trick of providing arithmetic with a secure and philosophically acceptable footing, Bernays writes:

> The ordinal number is in and of itself also not determined as object; it is only a place marker. We can objectively standardize it by choosing as a *place marker the simplest structure from those that originate in the form of the succession.* […W]e have an initial thing and a process; the objects are then the initial thing itself and further the objects one obtains, beginning with the initial thing, through a single or repeated application of that process. (Bernays 1930, 244)

In *Grundlagen* the idea is made more concrete:

> In number theory, we have an initial object and a process of succession [*Prozeß des Fortschreitens*]. For both we must settle on a particular intuitive representation. The particular kind of representation is inessential, but the choice, once made, must be retained throughout the whole of the theory. We choose as initial object the numeral 1 and as process of succession the attachment of 1.
>
> The objects which we obtain from the numeral 1 by applying the process of succession, such as
>
> 1, 11, 1111
>
> are figures of the following kind: The start with 1, they end in 1; each 1 which does not already form the end of the figure is followed by an attached 1. They are obtained through application of the process of succession, that is by an *assembling* [*Aufbau*] which concretely comes to an end, and this assembling can be undone by a stepwise *disassembling* [*Abbau*]. (Hilbert and Bernays 1934, 20–21)

The idea we are given here is that the object of number theory is an iterative process, and this iterative process is representable in intuition. The iterative process is basic; the numerals only help us, so to speak, to keep track of how far the iteration has proceeded. This is necessary since a crucial aspect of the iterative process is that it can be reversed. This is the basis for induction and recursion in finitistic mathematics.

The import the iterativist account has for semantics is that it no longer requires denotations for the terms of arithmetic. The numerals do not stand for anything, they just give us information on how to verify or falsify arithmetical statements. Equalities between numerals are true, not if the numerals are the same or they have the same form, but if they both give the same bound on the process of succession. If we can start counting and both numerals tell us to stop at the same time, they are equal.

4. Finitistic functions

The issue is a little more involved when considering more complex arithmetical terms, e.g., sums and products. If numbers are controls on iteration, then the functions of arithmetic should be defined by iteration. A finitistic function, or more precisely, its definition, is a way of communicating a certain procedure which allows us to obtain a numeral from certain other numerals. The operations that this procedure may appeal to are assembling and disassembling numerals and iterating (previously defined) operations. This is all that is needed to give truth conditions for equalities between arithmetical terms: Use the definition of the function to obtain an explicit bound on iteration and we have reduced the question to an equality between numerals.

How much of this is present in Hilbert's writings? In *Grundlagen*, the definition of addition still depends on the visual image of numerals as sequences of strokes:

> If a numeral **b** corresponds to a part of **a**, then the rest is again a numeral **c**; thus we obtain the numeral **a** by appending **c** to **b**, in the manner in which the 1 which starts **c** is appended to the 1 in which **b** ends according to the process of succession. We call this kind of composition of numerals *addition* and use for it the sign +. (Hilbert and Bernays 1934, 22)

However, the definition of multiplication agrees much more with the idea of definition of a function by iteration:

Multiplication can be defined as follows: **a** · **b** denotes the numeral obtained from **b** by replacing, during its construction, always the 1 by the numeral **a**, so that one first forms **a** and then appends **a** instead of each appending of 1 in the construction of **b**. (Hilbert and Bernays 1934, 24)

It is clear, however, that the general notion of finitistic function is based on iteration/recursion, and Hilbert could as well have given an explanation of addition and multiplication in these terms, without appeal to the picture of sequences of strokes and their geometrical manipulation. This is evident from the discussion of recursion:

[O]ne point still requires a basic discussion, the method of *recursive definition*. Let us see what this method consists in: A new function symbol, say φ, is introduced, and the [corresponding] function is defined by two equations. In the simplest case, these equations are of the form:

$$\varphi(1) = \mathbf{a}$$
$$\varphi(\mathbf{n} + 1) = \psi(\varphi(\mathbf{n}), \mathbf{n}).$$

Here, **a** is a numeral and ψ a function which is formed from previously known functions by composition, so that ψ(**b**, **c**) can be computed for given numerals **b**, **c** and gives another numeral as value. […]

It is not immediately clear, which sense may be assigned to this method of definition. For its elucidation we must first make the notion of function precise. A *function*, for us, is an intuitive instruction on the basis of which to each given numeral another numeral is assigned. A pair of equations of the above kind—called a "*recursion*"—is to be understood as an *abbreviated communication* of the following instruction:

Let **m** be any numeral. If **m** = 1, so let the numeral **a** be assigned to **m**. Otherwise, **m** has the form **b** + 1. One then writes down schematically:

$$\psi(\varphi(\mathbf{b}), \mathbf{b}).$$

Now if **b** = 1, so one replaces therein φ(**b**) by **a**; otherwise **b** again has the form **c** + 1, and one then replaces φ(**b**) by

$$\psi(\varphi(\mathbf{c}), \mathbf{c}).$$

Again, either **c** = 1 or **c** is of the form **d** + 1. In the former case one replaces φ(**c**) by **a**, in the latter case by

$$\psi(\varphi(\mathbf{d}), \mathbf{d}).$$

Repeating this process in any case terminates. For the numerals

$$\mathbf{b}, \mathbf{c}, \mathbf{d}, \ldots,$$

which we obtain one after the other, develop through the *disassembling of the numeral* **m**, and this must terminate just like the assembling of **m** does. When we arrive at 1 in this process of disassembling, then φ(1) is replaced by **a**; the sign φ does then no longer occur in the resulting figure. Rather, the only function symbol occurring, possibly in multiple superposition, is ψ and the innermost arguments are numerals. Thus we have arrived at a computable expression; for ψ was supposed to be a function already known. This computation must be executed from the inside out, and the numeral thus obtained shall be assigned to the numeral **m**. (Hilbert and Bernays 1934, 25–26)

We see how this conception of computation fits in with the iterativist conception of finitistic truth: An equation between arithmetical terms is evaluated, by iteration, until we arrive at an equation between numerals. The numerals themselves are mnemonic devices, in principle dispensable, for effecting this procedure. This corresponds precisely to Hand's "canonical verifications."

Recently, Parsons (1998) has taken up the account of finitistic functions from *Grundlagen*, in particular of primitive recursion, as successive replacement of terms. He argues that the account is not strong enough to support what he calls Hilbert's Thesis, namely, that finitistic proofs of propositions yield intuitive knowledge of them. In particular, he questions the subsidiary thesis that primitive recursion always yields intuitively well-defined functions (and thus that at least all the primitive recursive functions are intuitively seen to be well-defined). He claims that addition and multiplication are intuitively well-defined, but only because they are special cases. In general, primitive recursive functions can only be seen to be total using Σ_1-induction. If Parsons is right, then the non-standard approach based on iteration cannot achieve the intuitive certainty that finitism requires. Tait, for one, certainly would disagree with Parsons. For Tait, primitive recursion is one of the kinds of definition of which it is impossible to ask for a proof of their well-definedness. Even asking for a demonstration that primitive recursion yields well-defined functions would thus be misguided. Parsons even suggests that Hilbert and Bernays not only have not established Hilbert's Thesis, but that they cannot. Any argument that would establish that primitive recursion preserves intuitive well-definedness should be convertible into a proof that, say, exponentiation is well-defined using only unproblematic primitive recursive functions. He concludes that this is impossible, since Δ_0-induction is not strong enough to prove that exponentiation is total. Results of this kind, however, are very sensitive to syntactic considerations and vary with the formal system used. For instance, if Parsons thinks it reasonable to require a proof of the totality of exponentiation, he should also admit that it is not incoherent to ask for a proof that multiplication is total using only addition, or even for a proof that addition is well-defined using only the successor function. Such proofs are not possible for similar reasons as the impossibility of a proof that exponentiation is total using principles weaker than Σ_1-induction (in fact, the relevant statements cannot even be formalized), but surely this does not show that addition and multiplication cannot intuitively be seen to be well defined.

5. Finitism and PRA

Let me conclude with some remarks on Tait's claim that the finitistically acceptable functions are exactly the primitive recursive ones. There is no question that the primitive recursive functions are finitistic, since they are all given by recursive definitions of the above kind. But are they *all* the finitistic functions? Any description of an iterative procedure that allows us, given a term involving the function symbols introduced, to arrive at a numeral as "value" of the term, using only iteration and substitution (in particular, no unbounded search), should count as finitistic, if the primitive recursive ones do. The Ackermann function is an example of a function computable in this way, which is well known not to be primitive recursive.

Tait did consider the issue of the Ackermann function being finitistic, as part of an objection to Kreisel's (1960) characterization of finitistic functions as the provably total functions of first-order Peano arithmetic. The issue is that Kreisel points out[20] that in "On the infinite" (1926) Hilbert explicitly discusses the Ackermann function. Tait's argument against the conclusion, based on this fact, that we should regard it as finitistic is that Hilbert introduces the Ackermann function in the context of a theory of functions of higher types, and these higher types are certainly not finitistic. It is true that Hilbert introduces the Ackermann function by recursion on higher types, but it can also be introduced by nested recursion on two arguments. If it is correct that all that is needed for a function to be finitistic is that it be given by a process of recursion which allows the computation of the "value" by successive rewriting of numerals, then this definition would certainly make the Ackermann function finitistic. In fact, there is rather explicit evidence that the Ackermann function was considered to be finitistic, in *Grundlagen*. In §7 of the first volume, Hilbert and Bernays discuss (what is now called) primitive recursive arithmetic. This is a formal theory, all of whose statements, however, are *real*, i.e., finitistically meaningful:

> This recursive number theory is close to the intuitive number theory as considered in §2, insofar as its formulas all *admit of a finite contentual interpretation.* This contentual interpretability is a result of the verifiability of all derivable formulas of recursive number theory, [a fact] which we have already stated. Indeed, in this area verifiability has the character of a direct contentual interpretation, and it was because of this that the proof of consistency was so easy to give here.
>
> The difference of recursive number theory vis à vis intuitive number theory consists in its formal restrictions [*formale Gebundenheit*]; its only method of concept-formation, aside from explicit definition, is the schema of recursion, and also the methods of deduction are strictly limited.
>
> We may, however, admit certain *extensions of the schema of recursion* as well as of the induction schema, without taking away what is characteristic of the method of recursive number theory. We shall now discuss these briefly. (Hilbert and Bernays 1934, 325 [330], emphasis in the original)

Hilbert and Bernays then go on to introduce course-of-values recursion, simultaneous recursion (which can both be reduced to primitive recursion), nested recursion [*verschränkte Rekursion*], the Ackermann function, and nested induction, and prove that the Ackermann function is not primitive recursive. We find the most conclusive statements in the second volume:

> Certain methods of *finitistic mathematics which go beyond recursive number theory* (in the original sense) have been discussed in §7 [of volume I of *Grundlagen*], namely the introduction of functions by nested recursion [e.g., Ackermann's function] and the more general induction schemata. (Hilbert and Bernays 1939, 340 [354], my emphasis)

> The original narrow concept[21] of a finitistic proposition amounts in the field of number theory to admitting as finitistic number-theoretic propositions only such propositions which can be expressed in the formalism of [primitive] recursive number theory, possibly *including symbols for certain computable number-theoretic functions (of one or more arguments)*, but without use of formula variables, or which admit a stricter interpretation by a formula of such a form. (Hilbert and Bernays 1939, 348 [362], my emphasis)

Elsewhere, we read that "*contentual* finite arithmetic [is formalized by] recursive number theory" (Hilbert and Bernays 1939, 214 [224]).[22] This remark is made in the context of arithmetization of syntax, and I take its force to be that the methods used in arithmetization are primitive recursive and hence finitistic, rather than making a programmatic identification of "finitistic" with "primitive recursive." In a footnote, the reader is referred to the passage from p. 325 [330] of the first volume quoted above, where the relationship between contentual arithmetic and recursive arithmetic is discussed. That passage suggests that Hilbert and Bernays considered finite arithmetic as *partially* but not necessarily completely formalized by primitive recursive arithmetic. It also suggests that the *verifiability* of formulas is the criterion of finitistic meaningfulness. Verifiability here is defined as follows: Every true equality or inequality between numerals is verifiable. Every boolean combination of verifiable formulas is verifiable. A formula containing free individual variables (but no formula variables or bound variables) is verifiable if every instance resulting by substituting numerals for the free variables is verifiable (pp. 229 [228], 238 [237]). A closed formula containing primitive recursive function symbols is verifiable if the formula resulting from calculating the primitive recursive terms occurring in it is verifiable (p. 297 [297]). If the function can be calculated, it is finitistic? It is obvious that some restriction must be placed on the notion of calculability involved here. Without restrictions, every total general recursive function would be finitistic, and any formula containing symbols for total recursive functions would be verifiable in that sense. The restriction would most likely have to do with being able to see that the calculation process comes to an end, and this is precisely the issue in the question of whether the Ackermann function should be considered finitistic. Bernays was aware that there is a substantial difference between primitive and nested recursion in this respect, and the issue comes up when he proves that primitive recursion can be replaced by the μ-operator (Hilbert and Bernays 1934, 421–22 [430–31]).[23] Much later, he took nested recursions (in the sense of *verschränkte Rekursionen* considered in *Grundlagen* I) to be finitist on the grounds that they could be computed by a sequence of replacements of terms, the number of which is bounded. In a letter to Gödel from 1970, he writes:

> These nested recursions […] appear to me to be finite in the same sense as the primitive recursions, i.e., if one regards them as a statement of a computation procedure where one can recognize that the function defined by the respective process satisfies the recursion equations (for every system of numeral values [*Ziffernwerte*] of the arguments). Indeed, the computation of the value of a function according to a nested recursion, when the numeral values of the arguments are given, comes down to the application of several primitive recursions, the number of which is determined by a numeral argument [*Ziffernargument*].[24]

It is consistent with Hilbert's early writings that finitism, as originally conceived in the early 1920s, does not surpass primitive recursive methods. In all likelihood, Hilbert and Bernays did not think they had to address the issue explicitly. The Ackermann function had not been discovered when the finitistic standpoint was first formulated, and in any case it was probably thought initially that primitive recursive methods suffice for metamathematics. I hope to have shown, however, that there is considerable evidence in *Grundlagen* that Hilbert and, in particular, Bernays considered finitistic reasoning to go beyond the methods formalized or formalizable in primitive recursive arithmetic.

Notes

1. Hilbert (1926, 379). All page references are to the English translations, where available. All other translations are mine. For references to *Grundlagen der Mathematik* (Hilbert and Bernays 1934, 1939), page numbers in the second edition are given in brackets.
2. For a detailed discussion of the instrumentalism expressed here, see Kitcher (1976, 102–105).
3. Detlefsen (1986) disagrees. For Gödel's own opinions on the topic, see Sieg (1988, 342) and Giaquinto (1983, 125–26). Gödel's early assessment turns on a (questionable but then prevalent) identification of finitism with intuitionism and the question whether intuitionist reasoning is formalizable, of which Brouwer was skeptical.
4. For an overview and assessment of the developments spawned by the program, see Sieg (1988) and Simpson (1988). For a defense of the plausibility of the program in a historical and philosophical context, see Giaquinto (1983), especially §5.
5. While undoubtedly the ideas underlying the program were Hilbert's, in their details the views are in large part due to Hilbert's collaborator Bernays. One of Bernays' main contributions to the program was philosophical clarification of Hilbert's ideas; Bernays, in contrast to Hilbert, had philosophical training (see Mancosu 1998a, Section 4) for an outline of Bernays's contributions).
6. For a discussion of this effort, see Chihara (1973, Chapter 1).
7. This methodological point is made clear in a letter from Bernays to Rosza Péter, probably from 1940 (Bernays Papers, ETH Zürich Library/WHS, Hs. 975:3473).
8. See Hilbert (1905, 131–32). In a course at Göttingen, Hilbert went even further in the development of this idea, see Peckhaus (1990, Chapter 3).
9. Hilbert (1922, 1121), repeated almost verbatim in Hilbert (1926, 376)
10. This account is based on our ability to put finite collections of objects into one-to-one correspondences with the strokes making up a numeral. This ability accounts for the usefulness of contentual number theory. The account is indicated in passing by Bernays (1923, 225) and is developed in detail by Hilbert and Bernays (1934, 28–29).
11. Kitcher: (1976, 107–8). Frege (1884, §27) advanced essentially the same criticism against Schloemilch.
12. "Figures [i.e., numerals] *are* not shapes, they *have* a shape" Bernays (1923, 159).
13. "These objects must be […] space- and timeless […]'' (Müller 1923, 158)
14. Benacerraf (1983) also finds a non-standard account in Hilbert's view of mathematics. That account, however, does not concern the contentual mathematics we are interested in, but formalized mathematics. According to Benacerraf, Hilbert's account of formalized mathematics is non-standard since unbounded quantifiers—since they are finitistically meaningless—are not evaluated according to standard semantics, but based on the derivability of sentences containing them from axiom systems that have been shown to be consistent.
15. For a discussion of these distinctions, see Sinaceur (1993).
16. Hilbert and Bernays acknowledge that they have not drawn the distinction precisely: "[W]e have introduced the expression 'finitistic' [*finit*] not as a sharply delineated term, but only as the name of methodical guideline, which enables us to recognize certain kinds of concept-formations and ways of reasoning as definitely finitistic and others as definitely not finitistic. This guideline, however, does not provide us with a precise demarcation between those [concept-formations and ways of reasoning] which accord with the requirements of the finitistic method and those that do not.'' (Hilbert and Bernays 1939, 347–48 [361])
17. *Aussagen mit elementar anschaulichem Inhalt*. I propose to read this as: propositions which permit a finitistic interpretation (see below). See Bernays (1922, 169–70) for the distinction between the form of induction discussed here, "the narrower form of induction," and the full schema of induction on arbitrary formulas. This distinction is essential for the rebuttal, by Hilbert, of Poincaré's and Becker's charge of circularity in Hilbert's theory. For this, see Mancosu (1998b).
18. I take the word "exemplify" to imply that the same forms of judgment also apply to other finitistically acceptable concept-formations, e.g., functions defined by recursion.
19. Hilbert uses old German type for meta-language variables for numerals (lower case) and propositions/formulas (upper case). Boldface is used here.
20. "For instance, Tait refers to [Hilbert 1926] as a source concerning Hilbert's notion of a finitist proof, goes on to say 'it is difficult perhaps to determine what Hilbert really had in mind' and argues that Ackermann's enumeration of the primitive recursive functions is not finitist. But whatever else may be in doubt, Hilbert's own notion as *used in* (1926) certainly includes Ackermann's function since it is explicitly mentioned!" (Kreisel 1970, 514, n. 43). In a recent talk ("Some remarks about finitism," 13 December 1998, *Reflections* Symposium, Stanford), Tait argued

in detail that the use of the Ackermann function in Hilbert (1926) (in Hilbert's attempted proof of the continuum hypothesis) was not meant to be finitistic.

21. "Original concept of finitism" in contrast to some slight extensions that are discussed subsequently, in particular, admission of implications with a universal antecedent and inductions with premises of such a form. The passage occurs in the context of considering the question of whether there are finitistic principles which go beyond number theory *Z.*
22. This passage was pointed out by Tait in the talk cited in note 20.
23. For a discussion of nested recursion and the issues coming up in computing functions defined by nested recursion, see Tait (1961).
24. Bernays to Gödel, 7 January 1970. Bernays Papers, ETH Zürich Library/WHS, Hs. 975:1745.

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