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Imagination, Mereotopology, and Topic Expansion*

Aybüke Özgün[1] and A. J. Cotnoir[2]

[1]ILLC, University of Amsterdam, The Netherlands
[2]Arché, University of St. Andrews, United Kingdom

Abstract

In the *topic-sensitive* theory of the logic of imagination due to Berto (2018a), the topic of the imaginative output must be contained within the imaginative input. That is, imaginative episodes can never expand what they are *about*. We argue, with Badura (2021), that this constraint is implausible from a psychological point of view, and it wrongly predicts the falsehood of true reports of imagination. Thus the constraint should be relaxed; but how? A number of direct approaches to relaxing the controversial content-inclusion constraint are explored in this paper. The core idea is to consider adding an expansion operator to the mereology of topics. The logic that results depends on the formal constraints placed on topic expansion, the choice of which are subject to philosophical dispute. The first semantics we explore is a topological approach using a closure operator, and we show that the resulting logic is the same as Berto's own system. The second approach uses an inclusive and monotone increasing operator, and we give a sound and complete axiomatisation for its logic. The third approach uses an inclusive and additive operator, and we show that the associated logic is strictly weaker than the previous two systems, and additivity is not definable in the language. The latter result suggests that involved techniques or a more expressive language is required for a complete axiomatization of the system, which is left as an open question. All three systems are simple tweaks on Berto's system in that the language remains propositional, and the underlying theory of topics is unchanged.

Keywords: *topic-sensitivity, intensional modals, imagination, mereotopology*

Intentional modals have recently received *topic-sensitive* treatment (Berto, 2018b, 2022). One application involves the logic of *imagination*. According to a prominent treatment (Berto, 2018a), the topic of the imaginative output must be contained within the topic of the imaginative input. That is, imaginative episodes can never expand what they are *about*. This constraint is prima facie too strong, and has been recently criticised by Badura (2021) as wrongly predicting the falsehood of true reports of imagination. It is also implausible from the perspective of recent philosophical theories of imagination. Thus the constraint should be relaxed; but how?

One option for relaxing the constraint involves reconsidering the general theory of what propositions are *about*. The recent proposal due to Badura (2021) follows this route, generalising from a propositional language to a first-order one, and combining this with Hawke's (2018) *issue-based* theory of aboutness. The underlying theory of aboutness generates a plethora of possible content-overlap relations that could be appropriate for the application to imagination, in that imaginative jumps to new topics are permitted insofar as the new topic shares content (in a relevant sense) to prior topics.

*The author order is not alphabetical to reflect the proportion of contributions of the respective authors.

Another option is simply to generalise the controversial content-inclusion constraint directly, without modifying the underlying theory of what propositions are about. The frameworks proposed here follow this second approach, making only minimal revisions to Berto's logic in a way that is more adequate to the application to imagination. The resulting theories are simpler in that the language remains propositional, and the theory of topics simply adds an expansion function (topological or otherwise) to Berto's mereological framework. The guiding principle is simply that output of imaginative episode should have a topic that is *connected* to the topic of its input.

In §1, we introduce the logic of imagination according to the Theory of Topic-Sensitive Intentional Modals (TSIM) and criticise the content inclusion constraint. In §2, we give a new mereotopological approach to topics, explore the logic that results from requiring the topic of imaginative outputs to be contained in the *closure* of the topic of imaginative inputs. We then investigate the expressive differences of the object language over the models with and without the closure operator. Surprisingly, we show that relaxing the content inclusion constraint in this way does not affect the overall logic of imagination, and so this was merely an incidental feature of Berto's semantics. However, unsurprisingly, the same language has expressive differences over the two semantics. In §3, we motivate and explore the logics that result from weaker operators on topics, including a *preclosure* expansion operator, as well as *inclusive* and *monotone increasing* expansion operators. We provide modal definability results concerning the properties of expansion operators, and give sound and complete axiomatizations for the logics of some the operators that are weaker than preclosure. Difficulties around obtaining a sound and complete axiomatization for the preclosure operator over the same language is explained and left for future work.

1 Imagination & TSIM

The topic-sensitive treatment of intentional modals proceeds by taking seriously their intentionality — by focusing on what they are about. These modals have *topics*, and the space of topics is usefully structured in terms of part-whole.

> The insight behind TSIM theory is that we should take at face value the view of belief, knowledge, (cognitive) information, but also of other notions less explored in formal logic, like imagination and mental simulation, as (propositional) representational mental states bearing intentionality, that is, being about states of affairs, issues, situations, or circumstances which make for their contents. I will generically call these things *topics*, and provide a simple formal mereology for them. The semantics for our TSIMs will be given in a kind of conditional logic framework, with an added mereology of topics. (Berto, 2018b, p. 33)

Following Berto (2018b), we write $X^{\varphi}\psi$, to be generically read as 'Given φ, the agent Xs that ψ', where X is some mental state or act. Such TSIMs are *variably strict modals* and encompass a *topicality* or *aboutness* filter capturing their standing for intentional mental states. There are various possible interpretations of such Xs:

- '$X^{\varphi}\psi$', relabeled as '$K^{\varphi}\psi$', as expressing a notion of *knowability relative to information.*[1]
- '$X^{\varphi}\psi$', relabeled as '$B^{\varphi}\psi$', as expressing a hyperintensional *conditional belief*, or (static) *belief revision* operator ('Conditional on φ, one believes ψ', or 'After revising by φ, one believes ψ').[2]

[1]See Berto and Hawke (2018). Cf Hawke et al. (2020) for a related framework.

[2]See Berto (2019) and Özgün and Berto (2021).

- '$X^{\varphi}\psi$', relabeled as '$I^{\varphi}\psi$', as expressing an imagination or mental simulation operator ('In an act of imagination starting with input φ, one imagines that ψ').[3]

In this paper, we focus on the last of these interpretations.

By using TSIM theory to model imagination, the target is a kind of rational mental simulation operation. Following Berto (2018a), the idea is that imaginative episodes have an *input* and *output* structure. The inputs of the episode behave something like premises, the outputs like conclusions which are generated by closing the inputs under some process of mental *reasoning*. The internal logic of this rational activity is not purely anarchic — it is guided by rules. But nor are we merely attempting to model some idealised agent whose imaginations automatically output all and only the logical consequences of their inputs. By treating imagination as topic-relative, we can filter out quite a lot of (irrelevant) logical consequences from the outputs. This approach gives imaginative episodes much of the hyperintensional behaviour theorists have sought.

The treatment of imagination as reality-oriented mental simulation fits with a wide array of empirical and theoretical work on the subject. Following Canavotto et al. (2020), we can list a number of empirically-motivated constraints on imagination:

- Imagination is agentive and episodic.
- Acts of imagination have deliberate starting points, given by an input.
- Inputs are integrated with contextual background information.
- Imagination is constrained by topic and relevance.
- Imaginative acts are goal-driven and question-based.

So far so good.

A major conflict with the TSIM approach concerns how to implement the topic-relativity of imaginative inputs and outputs. How should the topic of an input be related to the topic of an output of an imaginative episode? The background TSIM theory has a ready-made answer: for $X^{\varphi}\psi$ to come out true at w we ask for two things to happen:

(TC) ψ must be true at all worlds w' one looks at, via the accessibility determined by φ.

(AP) ψ must be fully on topic with respect to φ.

The first constraint is the truth-conditional component (TC) making $X^{\varphi}\psi$ a variably strict quantifier over worlds. The second constraint is the aboutness preservation component (AP): *the topic of the output must be contained in the topic of the input.*

As one might suspect, the AP has been controversial when applied to the logic of imagination. It seems intuitive that imaginative episodes can clearly involve expansions of the topic at hand. Of course, not just any topic is relevant, but surely an imaginative jump to relevant topics not *contained* in the input topic seems perfectly legitimate. As (Badura, 2021, p. 524) puts it:

> Consider the following example, where Gwenny is a dog, and Helena is concerned with taking her to the lake: "In an act of imagining that Gwenny is at her favourite lake, Helena imagines that Gwenny swims in her (Gwenny's) favourite lake". This expresses a perfectly legitimate imaginative episode. . . . Problematically, Berto's account does not predict this episode as true since linguistic intuition suggests that the content of "Gwenny swims in her favourite lake" is not a part of the content of "Gwenny is at her favourite lake".

[3] See Berto (2018a).

We can extend the list of examples by, e.g., adapting some of the intuitively acceptable indicative conditionals presented in (Berto and Özgün, 2021, p. 3708) to the case of imagination: 'In an act of imagining that we keep burning fossil fuels at this pace, Helena imagines that the polar ice melts.' As argued in Berto and Özgün (2021), even though the topic of 'we keep burning fossil fuels at this pace' does not include the topic of 'the polar ice melts' (the former does not talk about polar ice at all), the imaginative episode initiated by the given input takes on board contextually determined relevant background information such as the emission of carbon, raise in global temperatures etc. and expands the topic of the imaginative input accordingly.

So it seems that AP is too strong. But how should we restrict it? We should want to preserve intuitive imaginative jumps to nearby topics, as in the case of Gwenny. But we should also want to restrict topicality in such a way that rules out arbitrary or anarchic topic shifts. In imagining Gwenny swimming in the lake, I don't imagine scenarios where Gwenny is swimming in the lake or Gwenny is swimming in the Orion nebula.

It seems then, in any realistic attempt to model imagination, the aboutness preservation constraint must fail in constrained ways. But to see clearly how to adjust AP, we need to look more closely at the mereology of topics presupposed in the TSIM framework.

1.1 Formal Framework

We work with a sentential language $\mathcal{L}$ with a countable set $\mathcal{L}_{AT}$ of atomic formulas, p, q, r $(p_1, p_2, \dots)$, negation $\neg$, conjunction $\wedge$, a normal (global) modal operator $\Box$, and a two-place (intentional modal) operator X, and round parentheses as auxiliary symbols. We use φ, ψ, η $(\varphi_1, \varphi_2, ...)$, as metavariables for formulas of $\mathcal{L}$. The well-formed formulas of the language $\mathcal{L}$ are given by the following grammar in BNF form:

$$p \mid \neg\varphi \mid (\varphi \wedge \psi) \mid \Box\varphi \mid X^{\varphi}\psi$$

where $p \in \mathcal{L}_{AT}$. We use $\vee$ for disjunction, $\supset$ for the material conditional, and $\equiv$ for material equivalence, defined in the usual manner as $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \supset \psi := \neg\varphi \vee \psi$, and $\varphi \equiv \psi := (\varphi \supset \psi) \wedge (\psi \supset \varphi)$. We follow the usual rules for the elimination of the parentheses.

Definition 1 (Topic-sensitive Frame/Model for $\mathcal{L}$)**.** *A* topic-sensitive frame *(ts-frame) for $\mathcal{L}$ is a tuple $\mathcal{F} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus\rangle$ where*

1. *W is a non-empty set of* possible worlds*,*
2. *$R_\varphi \subseteq W \times W$,*
3. *$\mathcal{T}$ is a non-empty set of* topics*,*
4. *$\oplus : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$,* topic fusion*, is a binary operation that satisfies*
   - *(a) $x \oplus x = x$ (idempotence),*
   - *(b) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ (associativity), and*
   - *(c) $x \oplus y = y \oplus x$ (commutativity).*

   $\mathcal{T}$ is closed under $\oplus$: $\forall xy \in \mathcal{T} \exists z \in \mathcal{T}(z = x \oplus y)$.
   Topic parthood $\leq$ is defined in the usual way: $\forall xy \in \mathcal{T}(x \leq y$ iff $x \oplus y = y)$.
   Topics overlap when they share a part: $\forall xy \in \mathcal{T}(x \circ y$ iff $\exists z(z \leq x \wedge z \leq y))$.

A topic-sensitive model *(ts-model) for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, t, V\rangle$ where $\langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus\rangle$ is a ts-frame, $V : \mathcal{L}_{AT} \to \mathcal{P}(W)$ is a valuation map, and*

5. $t : \mathcal{L}_{AT} \to \mathcal{T}$ *is a topic assignment function.* t *extends to* $\mathcal{L}$ *by*

$$t(\varphi) = \oplus\mathfrak{At}(\varphi) = t(p_1) \oplus \cdots \oplus t(p_n),$$

where $\mathfrak{At}(\varphi) = \{p_1, \ldots, p_n\}$ *is the set of atomic formulas occurring in* φ.

The topic of a complex sentence φ, defined from its primitive components in $\mathfrak{At}(\varphi)$, makes all the logical connectives and modal operators in $\mathcal{L}$ topic-transparent:

- $t(\neg\varphi) = t(\Box\varphi) = t(\varphi)$;
- $t(\varphi \wedge \psi) = t(X^{\varphi}\psi) = t(\varphi) \oplus t(\psi)$.

Observe that $(\mathcal{T}, \oplus)$ is a *join-semilattice* of topics, and $(\mathcal{T}, \leq)$ a *poset*. (Berto, 2018a, p. 1877) commits firmly to a *mereological* reading of the structure of topics, interpreting $\oplus$ as a topic *fusion* operator, and showing how topic parthood $\leq$ behaves as a partial order on $\mathcal{T}$. A minor matter worth highlighting: having a join-semilattice together with a partial order does not suffice for a 'mereology'; notably missing is any kind of decomposition principle.[4] Still, treating bare join-semilattices as 'quasi-mereological' for the purposes of formal semantics is commonplace going back to Sharvy (1983); Link (1983), and complicating this structure is beyond the scope of this paper.[5]

Definition 2 (Semantics for $\mathcal{L}$ ($\Vdash$))**.** *Given a ts-model* $\mathcal{M} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, t, V\rangle$ *and a state* $w \in W$*, the* $\Vdash$-semantics *for the language* $\mathcal{L}$ *is defined recursively as:*

$$\begin{array}{lll} \mathcal{M}, w \Vdash p & \textit{iff} & w \in V(p) \\ \mathcal{M}, w \Vdash \neg\varphi & \textit{iff} & \textit{not } \mathcal{M}, w \Vdash \varphi \\ \mathcal{M}, w \Vdash \varphi \wedge \psi & \textit{iff} & \mathcal{M}, w \Vdash \varphi \textit{ and } \mathcal{M}, w \Vdash \psi \\ \mathcal{M}, w \Vdash \Box\varphi & \textit{iff} & \textit{for all } w' \in W(\mathcal{M}, w' \Vdash \varphi) \\ \mathcal{M}, w \Vdash X^{\varphi}\psi & \textit{iff} & \textit{for all } w' \in W(\textit{if } wR_\varphi w' \textit{ then } \mathcal{M}, w' \Vdash \psi) \textit{ and } t(\psi) \leq t(\varphi). \end{array}$$

The *intension of* φ with respect to $\mathcal{M}$ is $|\varphi|_{\mathcal{M}} := \{w \in W : \mathcal{M}, w \Vdash \varphi\}$. We omit the subscript $\mathcal{M}$ and write $|\varphi|$ when the model is contextually clear. To ease notation in proofs, we define $R_\varphi(w) = \{w' \in W : wR_\varphi w'\}$. Note the clause for $X^{\varphi}\psi$ to be true in $\mathcal{M}$ at w. This clause requires both that (i) ψ must be true at all φ-accessible worlds w', and (ii) that the topic of ψ be part of the topic of φ. The latter corresponds to the aboutness preservation (AP) and is subject to concerns raised above.

Logical consequence is defined in the usual way, as truth preservation at all worlds of all models. With $\Sigma \subseteq \mathcal{L}$,

Logical consequence $\Sigma \Vdash \varphi$ iff for all models $\mathcal{M} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, t, V\rangle$ and all $w \in W$: if $\mathcal{M}, w \Vdash \psi$ for all $\psi \in \Sigma$, then $\mathcal{M}, w \Vdash \varphi$.[6]

[4]Decomposition principles — the most well known of which are *Weak* or *Strong Supplementation* — come in many varieties. However they are important for well-behaved mereological structures; without them overly-simplistic models are possible. For example, take any finite linear order. Or consider a large set of atoms, topped by a single universal element such that for any x and y, $x \oplus y = \top$. See (Cotnoir and Varzi, 2021, Ch. 4) for more details.

[5]A simple approach might involve supplementing the lattice with a *residuation* operator $x \ominus y$, read as 'x without y' satisfying the usual residuation condition: $x \ominus y \leq z$ iff $x \leq y \oplus z$. This mereological 'subtraction' would add some decompositional structure to the semilattice, bringing it much more into line with standard mereology. See Cotnoir and Varzi (2019).

[6]We follow the usual conventions. For single-premise entailment, we write $\psi \Vdash \varphi$ for $\{\psi\} \Vdash \varphi$. As a special case, *logical validity*, $\Vdash \varphi$, truth at all worlds of all models, is $\emptyset \Vdash \varphi$, logical consequence of the empty set of premises.

2 A Mereotopological Approach to Topics

What is needed, then, is a way of freeing up the modal operators $X^{\varphi}\psi$ a little by placing less strict constraints on the topics involved. The core insight of the approach we will follow is that imaginers should be free to move to other topics that are *connected* to the topics of the inputs, but not necessarily contained within them. This suggests adding some structure to the set of topics by *connectedness*; that is, it suggests adding *topology* to the mereology of topics.

2.1 Mereotopology

Our approach supplements the quasi-mereological structure of $(\mathcal{T}, \oplus)$ with a topological closure operation $f : \mathcal{T} \rightarrow \mathcal{T}$ satisfying the following so-called 'Kuratowski' axioms.

Inclusion $x \leq f(x)$

Additivity $f(x \oplus y) = f(x) \oplus f(y)$

Idempotence $f(f(x)) = f(x)$

These three constraints entail that the operation f is monotone increasing with respect to topic parthood.

Monotone Increasingness if $x \leq y$ then $f(x) \leq f(y)$

This constraint will become important in section 3 .

Intuitively, f takes a topic of an imaginative input and maps it to (possibly) another topic that integrates the topics of the contextual, relevant background information that is taken on board in the imaginative episode initiated by the input, allowing for imaginative jumps to relevant topics. For example, given the input that 'Gwenny is at her favourite lake', f maps the topic of the input to a topic that includes topics of relevant background information such as 'Gwenny likes swimming', which, in turn, includes the topic of 'Gwenny swims in her favourite lake'. Given the input that 'we keep burning fossil fuels at this pace', f maps the topic of this input to another one that includes also the topics of, e.g., 'burning fossil fuels increases the emission of CO_2', 'more CO_2 in the air raises the global temperatures', 'ice melts in high temperatures' etc., which will, in turn, include the topic of 'the polar ice melts'.[7]

Given this intuitive reading of f, the Kuratowski axioms are well-motivated from the philosophical standpoint of this paper. Inclusion ensures that the closure of a topic is only ever an expansion; it will never take us completely away from the topic of the input, but only broaden its scope. We allow for imaginative jumps from the topic of an imaginative input to a larger topic by integrating the topics of the contextual, relevant background information.

Additivity ensures that the result of closing the topic of a whole sentence φ is not different from the result of closing the topics of the atoms within φ and then fusing those topics. Expansions of the whole never outstrip the expansions of its parts.

Finally, Idempotence ensures that the expansion by imagination can't be repeated unless given different inputs. So for example, we might take an input, expand the topic by imagining, and then once in a new topic find new inputs and go again. But absent new inputs, the possible outputs of repeated imaginings cannot change.

[7]A similar idea of topic expansion via the topological closure operator is investigated in Berto and Özgün (2021) for a proposal of probabilistic acceptability conditions for simple indicative conditionals. This framework is incompatible with ours in that it is based on a probabilistic framework that focuses only on simple conditionals (conditionals with no conditionals embedded) but the way it models topic expansion and inclusion is similar.

A new proposal emerges for how to understand what happens to topics in imagination. The idea is that the topics of a given input can be expanded to other, distinct, but *connected* topics in the output of imaginings. Not just any connection will do, only the *points of connection* — those topics most closely related to the inputs — will be permitted. That is, the topic of outputs of imagination must be within the *closure* of the topic of the input. To formalize this idea, we endow the ts-models with an additional function f and then place constraints on it.

Definition 3 (Topic Sensitive Frames/Models with Functions)**.** *A* ts-frame with functions *is a tuple* $\mathcal{E} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f\rangle$ *where* $\langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus\rangle$ *is a ts-frame and* $f : \mathcal{T} \to \mathcal{T}$ *is a function defined on* $\mathcal{T}$*. A* ts-model with functions *is a tuple* $\langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f, t, V\rangle$ *where* $\langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f\rangle$ *is a ts-frame with functions,* t *is a topic assignment function, and* $V : \mathcal{L}_{AT} \to \mathcal{P}(W)$ *is a valuation map.*

$X^\varphi\psi$ is then interpreted in ts-models with functions as

$$w \models X^\varphi\psi \text{ iff (for all } w' \text{ if } wR_\varphi w' \text{ then } w' \models \psi) \text{ and } t(\psi) \leq f(t(\varphi)). \tag{F-Sem}$$

These generic models place no constraints on f, and will become important later in section 3 where we provide a sound and complete axiomatization for them. For the time being, we add the topological constraints on f as follows.

Definition 4 (Mereotopological Topic Sensitive Model/Frame for $\mathcal{L}$)**.** *A ts-frame with functions* $\langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f\rangle$ *is called a* topo-ts-frame *when* $f : \mathcal{T} \to \mathcal{T}$ *is a closure operator satisfying :*

- $x \leq f(x)$ *(inclusion)*
- $f(x \oplus y) = f(x) \oplus f(y)$ *(additivity)*
- $f(f(x)) = f(x)$ *(idempotence)*

By convention, we will re-write 'f*' as '*c*' whenever it satisfies all the Kuratowski axioms to bring to mind the notion of a closure operation. A mereotopological topic sensitive model (topo-ts-model) for* $\mathcal{L}$ *is a tuple* $\mathcal{X} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, c, t, V\rangle$ *where* $\langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, c\rangle$ *is a topo-ts-frame and* t *and* V *are as in Definition 3.*

Similarly $X^\varphi\psi$ is then interpreted in topo-ts-models as

$$w \models X^\varphi\psi \text{ iff (for all } w' \text{ if } wR_\varphi w' \text{ then } w' \models \psi) \text{ and } t(\psi) \leq c(t(\varphi)). \tag{C-Sem}$$

Imaginative episodes can therefore lead to proper expansions of the subject matter — even to output topics that fail to mereologically overlap input topics. This can happen when connected output topics are contained within the closure of the input topic.

2.2 Resulting logic for the topological closure operator

What logic results from this new mereotopological approach? Let's now compare the logic of ts-models wrt the semantics given in Definition 2 to the logic of topo-ts-models wrt the so-called *closure semantics* as in (C-Sem). For the sake of presentation, some formal definitions and notational conventions are given below.

The semantic clauses of the other components of $\mathcal{L}$ in topo-ts-models are as given in Definition 2 and we denote the *intension of* φ with respect to topo-ts-model $\mathcal{X}$ by $[\![\varphi]\!]_\mathcal{X} := \{w \in W : \mathcal{X}, w \models \varphi\}$.

The notion of logical consequence wrt the C-Sem in topo-ts-models is defined standardly (as in Section 1.1) and we write $\Gamma \models \varphi$ when φ is a logical consequence of Γ wrt the semantics given in Definition 2 with C-Sem in topo-ts-models. Recall that $\Gamma \Vdash \varphi$ says φ is a logical consequence of Γ wrt the class of ts-models (*without* a closure operator). The notion of *logical validity* in topo-ts-models, $\models \varphi$, is standardly defined as entailment by the empty set of premises ($\emptyset \models \varphi$). To be precise, we take the logic of ts-models wrt the semantics given in Definition 2 to be the set of all logical consequence relations, namely the set $\{(\Gamma, \varphi) \in \mathcal{P}(\mathcal{L}) \times \mathcal{L} \mid \Gamma \Vdash \varphi\}$. Similarly, the logic of all topo-ts-models wrt the C-Sem is defined as $\{(\Gamma, \varphi) \in \mathcal{P}(\mathcal{L}) \times \mathcal{L} \mid \Gamma \models \varphi\}$. In the remainder of this section, we will show that these two sets are equivalent.

It is easy to see that every ts-model is a topo-model: take the closure operator c on $\mathcal{T}$ as the identity function, i.e., define $c : \mathcal{T} \to \mathcal{T}$ such that $c(x) = x$ for all $x \in \mathcal{T}$. Such a c trivially satisfies *inclusion*, *idempotence*, and *additivity*. Moreover, given a topo-ts-model $\mathcal{X} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, c, t, V\rangle$, we can construct a ts-model $\mathcal{M}_\mathcal{X} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, t_\mathcal{X}, V\rangle$ where $t_\mathcal{X}(\varphi) = c(t(\varphi))$ for all $\varphi \in \mathcal{L}$.

Lemma 1. *For any* $(\mathcal{T}, \oplus, c)$ *and* $x, y \in \mathcal{T}$, $x \leq c(y)$ *iff* $c(x) \leq c(y)$.

Proof. Right-to-left follows since c satisfies inclusion and $\leq$ is transitive. Left-to-right follows since c is idempotent and additive. □

Lemma 2. *Given a topo-ts-model* $\mathcal{X} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, c, t, V\rangle$, *tuple* $\mathcal{M}_\mathcal{X} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, t_\mathcal{X}, V\rangle$ *is a ts-model. Moreover, for all* $w \in W$ *and* $\varphi \in \mathcal{L}$, $\mathcal{X}, w \models \varphi$ *iff* $\mathcal{M}_\mathcal{X}, w \Vdash \varphi$ *(i.e.,* $[\![\varphi]\!]_\mathcal{X} = |\varphi|_{\mathcal{M}_\mathcal{X}}$*).*

Proof. We first need to show that $\mathcal{M}_\mathcal{X} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, t_\mathcal{X}, V\rangle$ is a ts-model:

- $t_\mathcal{X}$ is well-defined: let $\varphi, \psi \in \mathcal{L}$ such that $t_\mathcal{X}(\varphi) \neq t_\mathcal{X}(\psi)$. This means, by the definition of $t_\mathcal{X}$, that $c(t(\varphi)) \neq c(t(\psi))$. Since c is well-defined, we obtain $t(\varphi) \neq t(\psi)$. Similarly, since t is well-defined, we conclude that $\varphi \neq \psi$.
- for any $\varphi \in \mathcal{L}$, $t_\mathcal{X}(\varphi) = \oplus\mathfrak{At}(\varphi) = t_\mathcal{X}(p_1) \oplus \cdots \oplus t_\mathcal{X}(p_n)$ where $\mathfrak{At}(\varphi) = \{p_1, \ldots, p_n\}$:

$$\begin{aligned} t_\mathcal{X}(\varphi) &= c(t(\varphi)) && \text{(by the defn. of } t_\mathcal{X}) \\ &= c(t(p_1) \oplus \cdots \oplus t(p_n)) && \text{(by the defn. of } t) \\ &= c(t(p_1)) \oplus \cdots \oplus c(t(p_n)) && \text{(by additivity of } c) \\ &= t_\mathcal{X}(p_1) \oplus \cdots \oplus t_\mathcal{X}(p_n) && \text{(by the defn. of } t_\mathcal{X}) \end{aligned}$$

The proof of $[\![\varphi]\!]_\mathcal{X} = |\varphi|_{\mathcal{M}_\mathcal{X}}$ follows by induction on the structure of φ, where cases for the atomic formulas, the Boolean connectives, and $\varphi := \Box\psi$ are trivial. So assume inductively that the result holds for ψ and η, and show that it holds also for $\varphi := X^\psi\eta$. Let $w \in W$:

$$\begin{aligned} \mathcal{X}, w \models X^\psi\eta \text{ iff } & R_\psi(w) \subseteq [\![\eta]\!]_\mathcal{X} \text{ and } t(\eta) \leq c(t(\psi)) && \text{(C-Sem)} \\ \text{iff } & R_\psi(w) \subseteq |\eta|_{\mathcal{M}_\mathcal{X}} \text{ and } c(t(\eta)) \leq c(t(\psi)) && \text{(induction hyp. and Lemma 1)} \\ \text{iff } & R_\psi(w) \subseteq |\eta|_{\mathcal{M}_\mathcal{X}} \text{ and } t_\mathcal{X}(\eta) \leq t_\mathcal{X}(\psi) && \text{(by the defn. of } t_\mathcal{X}) \\ \text{iff } & \mathcal{M}_\mathcal{X}, w \Vdash X^\psi\eta && \text{(Defn. 2)} \end{aligned}$$

□

We then obtain the following result:

Theorem 3. *For all* $\Gamma \subseteq \mathcal{L}$ *and* $\varphi \in \mathcal{L}$, $\Gamma \models \varphi$ *iff* $\Gamma \Vdash \varphi$.

Proof. Let $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$:

($\Leftarrow$) Suppose that $\Gamma \Vdash \varphi$ and, toward contradiction, that $\Gamma \not\models \varphi$. The latter means that there is a topo-ts-model $\mathcal{X} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, c, t, V\rangle$ and $w \in W$ such that $\mathcal{X}, w \models \psi$ for all $\psi \in \Gamma$ but $\mathcal{X}, w \not\models \varphi$. Then, by Lemma 2, we have that $\mathcal{M}_\mathcal{X}, w \Vdash \psi$ for all $\psi \in \Gamma$ and $\mathcal{M}_\mathcal{X}, w \nVdash \varphi$, where $\mathcal{M}_\mathcal{X} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, t_\mathcal{X}, V\rangle$, contradicting $\Gamma \Vdash \varphi$.

($\Rightarrow$) Suppose that $\Gamma \models \varphi$ and, toward contradiction, that $\Gamma \nVdash \varphi$. The latter means that there is a model $\mathcal{M} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, t, V\rangle$ and $w \in W$ such that $\mathcal{M}, w \Vdash \psi$ for all $\psi \in \Gamma$ but $\mathcal{M}, w \nVdash \varphi$. Note that $\mathcal{X}_\mathcal{M} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, id, t, V\rangle$, where $id : \mathcal{T} \to \mathcal{T}$ is the identity function, is a topo-ts-model such that $[\![\varphi]\!]_{\mathcal{X}_\mathcal{M}} = |\varphi|_\mathcal{M}$. We therefore obtain that $\mathcal{X}_\mathcal{M} \models \psi$ for all $\psi \in \Gamma$ and $\mathcal{X}_\mathcal{M}, w \not\models \varphi$, contradicting $\Gamma \models \varphi$. $\square$

Theorem 3 shows that the addition of the closure operator does not change the resulting logic. From a purely formal perspective, the theorem shows that the aboutness preservation requirement (AP) in Berto's logic *is not essential to the system.* The logic defined on mereotopological models, where AP clearly fails, results in the same logical system.

The result is also philosophically significant. An extremely natural and simple way of relaxing AP does not ultimately change the logic of imagination. So criticisms of AP (including Badura's (2021)) do not ultimately refute Berto's *logic* of the imagination operator.[8] However, these criticisms do display that certain sentences (e.g.'In an act of imagining that Gwenny is at her favourite lake, Helena imagines that Gwenny swims in the lake') should be satisfiable, but simply aren't on Berto's semantics.[9] There is no model of them with topic assignments that allow for the required expansion. In our view, this simply refutes the semantic framework as an empirically adequate proposal for modelling the relevant truth conditions. Our semantic proposal arguably does a better job on this front, if one accepts the underlying philosophical contention that 'swimming in the lake' is a topic connected closely enough to the topic 'Gwenny's favourite lake' as to be contained in its closure.[10]

2.3 Expressivity

As noted, TSIMs are useful for more than just imagination; in other applications where the aboutness preservation constraint (AP) is more plausible, there is at least one potential benefit to Berto's semantics over the closure semantics we have given above: Berto's semantics permits topic parthood to be expressible in the language, while our mereotopological semantics does not. We show this expressive advantage below. On the flipside (and unsurprisingly) the mereotopological semantics can express topic-parthood and topic equivalence *under the topic expansion operator*, as shown in what follows.

The following abbreviation will be useful in the following: we will use '$\overline{\varphi}$' to denote the tautology $\bigwedge_{p\in\mathfrak{At}(\varphi)}(p \vee \neg p)$[11], following a similar idea in Giordani (2019). The reader should not

[8]Indeed Berto himself seems to rest his case for his logic of imagination on this point. Citing the (unpublished) results in this paper, (Berto, 2022, p. 131f) claims, 'So, whereas the semantics using the closure operator may be philosophically more satisfactory, giving us a sense of how subject matters can be suitably expanded without going off-topic[...] it does not change one bit the logic [...] I take this to be a good result for my simple setting, in spite of doubts on the draconian nature of plain topic-inclusion.'

[9]Thanks to an anonymous referee for emphasising this point.

[10]We admit that intuitive judgments about 'closing' a topic might not always be robust, though perhaps intuitions about topic expansions along the lines proposed in section 3 avoid this issue.

[11]In order to have a unique definition of each $\overline{\varphi}$, we set the convention that elements of $\mathfrak{At}(\varphi)$ occur in $\bigwedge_{p\in\mathfrak{At}(\varphi)}(p \vee \neg p)$ from left-to-right in the order they are enumerated in $\mathcal{L}_{AT} = \{p_1, p_2, \dots\}$. For example, for $\varphi := \Box(p_3 \to p_2) \vee X^{p_1}p_5$, $\overline{\varphi}$ is $(p_1 \vee \neg p_1) \wedge (p_2 \vee \neg p_2) \wedge (p_3 \vee \neg p_3) \wedge (p_5 \vee \neg p_5)$, and not $(p_5 \vee \neg p_5) \wedge (p_1 \vee \neg p_1) \wedge (p_3 \vee \neg p_3) \wedge (p_2 \vee \neg p_2)$ or $(p_3 \vee \neg p_3) \wedge (p_5 \vee \neg p_5) \wedge (p_2 \vee \neg p_2) \wedge (p_1 \vee \neg p_1)$ etc. This convention will eventually not matter since our logics cannot differentiate two conjunctions of different

identify $\overline{\varphi}$ with $\top$: they will turn out to be logically equivalent for all $\varphi \in \mathcal{L}$, but our bimodal operator X will discern them.

Notice that, since the topic component of the semantic clause for '$X^{\varphi}\psi$' uses the closure operator, $\mathcal{L}$ with respect to C-Sem is not expressive enough to speak of parthood relations. Instead it is expressive enough to speak of topic-parthood under closure. That is, $X^{\psi}\overline{\varphi}$ expresses, with respect to C-Sem, 'The topic of φ is included in the closure of the topic of ψ' and also, via Lemma 1, 'The closure of the topic of φ is included in the closure of the topic of ψ':

$$\begin{aligned}\mathcal{X}, w \models X^{\psi}\overline{\varphi} &\text{ iff } R_{\psi}(w) \subseteq [\![\overline{\varphi}]\!] \text{ and } t(\overline{\varphi}) \leq c(t(\psi))\\ &\text{ iff } R_{\psi}(w) \subseteq W \text{ and } t(\varphi) \leq c(t(\psi)) && (t(\overline{\varphi}) = t(\varphi), \text{ since } \mathfrak{At}(\overline{\varphi}) = \mathfrak{At}(\varphi))\\ &\text{ iff } t(\varphi) \leq c(t(\psi))\\ &\text{ iff } c(t(\varphi)) \leq c(t(\psi)) && \text{(Lemma 1)}\end{aligned}$$

On the other hand, in $\mathcal{L}$ we cannot say things like 'The topic of φ is included in the topic of ψ', or 'φ and ψ have exactly the same topic'. The opposite is the case in Berto (2018a): since the proposal in Berto (2018a) does not accommodate the closure operator, $X^{\psi}\overline{\varphi}$ *there* states precisely that the topic of φ is included in that of ψ (as also observed in Özgün and Berto (2021)):

$$\begin{aligned}\mathcal{M}, w \Vdash X^{\psi}\overline{\varphi} &\text{ iff } R_{\psi}(w) \subseteq |\overline{\varphi}| \text{ and } t(\overline{\varphi}) \leq t(\psi)\\ &\text{ iff } R_{\psi}(w) \subseteq W \text{ and } t(\varphi) \leq t(\psi) && (t(\overline{\varphi}) = t(\varphi), \text{ since } \mathfrak{At}(\overline{\varphi}) = \mathfrak{At}(\varphi))\\ &\text{ iff } t(\varphi) \leq t(\psi).\end{aligned}$$

To see that $\mathcal{L}$ is not expressive enough to state 'The topic of φ is included in the topic of ψ' with respect to C-Sem, consider the models $\mathcal{X}_1 = \langle \{w\}, \{R_{\varphi} \mid \varphi \in \mathcal{L}\}, \{x_1, y_1, z_1\}, \oplus_1, c_1, t_1, V\rangle$ and $\mathcal{X}_2 = \langle \{w\}, \{R_{\varphi} \mid \varphi \in \mathcal{L}\}, \{x_2, y_2, z_2\}, \oplus_2, c_2, t_2, V\rangle$, where $R_{\varphi} = \{(w, w)\}$ for all $\varphi \in \mathcal{L}$, $V(p) = V(q) = \{w\}$, and $(\{x_1, y_1, z_1\}, \oplus_1, t_1)$ and $(\{x_2, y_2, z_2\}, \oplus_2, t_2)$ are as given in Figure 1, and both c_1 and c_2 are constant functions such that $c_i(x_i) = c_i(y_i) = c_i(z_i) = z_i$ for $i \in \{1, 2\}$.[12] We have 'The topic of q is included in the topic of p' true in $\mathcal{X}_1$ at w (since $t_1(q) = x_1 \leq_1 y_1 = t_1(p)$) and false in $\mathcal{X}_2$ at w (since $y_2 = t_2(q) \not\leq_2 x_2 = t_2(p)$). However, as shown in Lemma 4, $\mathcal{X}_1, w$ and $\mathcal{X}_2, w$ are modally equivalent wrt C-Sem for the language $\mathcal{L}$.

Lemma 4. *For all $\varphi \in \mathcal{L}$, $\mathcal{X}_1, w \models \varphi$ iff $\mathcal{X}_2, w \models \varphi$.*

Proof. The proof follows by induction on the structure of φ, where cases for the atomic formulas, the Boolean connectives, and $\varphi := \Box\psi$ are trivial. So assume inductively that the result holds for ψ and η, and show that it holds also for $\varphi := X^{\psi}\eta$. For the direction left-to-right, suppose that $\mathcal{X}_1, w \models X^{\psi}\eta$. This means that $\{w\} \subseteq [\![\eta]\!]_{\mathcal{X}_1}$ and $t_1(\eta) \leq_1 c_1(t_1(\psi))$, i.e., $t_1(\eta) \leq_1 z_1$. Observe that no matter what the topics of η and ψ are, as $c_2(t_2(\psi)) = z_2$ is the top element in $\mathcal{T}_2$, we have $t_2(\eta) \leq_2 c_2(t_2(\psi))$. Moreover, the induction hypothesis implies that $[\![\eta]\!]_{\mathcal{X}_2} = \{w\}$, hence, $R_{\psi}(w) = \{w\} \subseteq [\![\eta]\!]_{\mathcal{X}_2}$. Therefore, $\mathcal{X}_2, w \models X^{\psi}\eta$. The other direction follows analogously. □

To see that $\mathcal{L}$ is not expressive enough to state 'φ and χ have exactly the same topic' with respect to C-Sem, compare the model $\mathcal{X}_1$ given in Figure 1a with $\mathcal{X}_3 = \langle \{w\}, \{R_{\varphi} \mid \varphi \in$

---

order: $\varphi \wedge \psi$ provably and semantically equivalent to $\psi \wedge \varphi$.

[12]One might worry that the join-semilattices given in these models do not suffice for a 'mereology' as they do not satisfy (weak or strong) supplementation. However, similar models can easily be given based on a mereology with strong supplementation by adding extra redundant elements. In $\mathcal{X}_i$, add topic $u_i \leq y_i$ disjoint from x_i and add topic $v_i \leq z_i$ disjoint from y_i. Both models will then satisfy (strong and weak) supplementation. We thank one of the anonymous referees for pressing this point.

(a) $\mathcal{X}_1$ (b) $\mathcal{X}_2$

Figure 1: Models $\mathcal{X}_1$ and $\mathcal{X}_2$. (In figures of models, circles represent possible worlds, diamonds represent possible topics. Lines between topics represent the parthood relation going upwards (e.g., $x \leq z$).Valuation and topic assignment are given by labelling each node with atomic formulas. We omit labelling when a node is assigned every element in $\mathcal{L}_{AT}$. The same conventions apply to all our diagrams.)

$\mathcal{L}\}, \{x_3, y_3\}, \oplus_3, c_3, t_3, V\rangle$, where $(\{x_3, y_3\}, \oplus_3, t_3)$ is as given in Figure 2 and $c_3(x_3) = c_3(y_3) = y_3$. It is then easy to see that 'p and q have exactly the same topic' is true in $\mathcal{X}_3$ at w (since $t_3(p) = x_3 = t_3(q)$), whereas it is false in $\mathcal{X}_1$ at w (since $t_1(q) = x_1 \neq y_1 = t_1(p)$). However, $\mathcal{X}_1, w$ and $\mathcal{X}_3, w$ are modally equivalent with respect to the language $\mathcal{L}$, that is, for all $\varphi \in \mathcal{L}$, $\mathcal{X}_1, w \models \varphi$ iff $\mathcal{X}_3, w \models \varphi$ (the proof follows similarly to the proof of Lemma 4).[13]

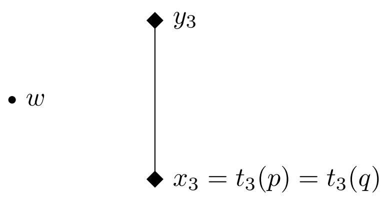

Figure 2: Model $\mathcal{M}_3$.

These results suggest one possible reason to prefer Berto's semantics for TSIMs over ours in applications where the aboutness preservation constraint (AP) is not in question and topic-parthood is desired to be expressible in the modal language $\mathcal{L}$.

3 Expanding further

The move to a mereotopology of topics, and using the closure operator was a natural suggestion. Surprisingly, it turns out not to affect the logic for somewhat technical reasons. However, one might wonder whether there are other simple generalisations that do have an affect on the logic of imagination.

Recall that the closure operator was postulated to satisfy the Kuratowski constraints, and hence also satisfied monotone increasingness. We motivated these constraints by thinking of the expansion operator topologically, but there is room for philosophical dispute over whether these constraints are empirically adequate to imaginative jumps. We began with a range of

[13]As before, one might object that the join-semilattice in $\mathcal{X}_3$ is not sufficiently mereological. One can again add a redundant topic $u_i \leq y_i$ incomparable to x_i to give a supplmented model.

empirically motivated constraints on imagination from Canavotto et al. (2020), putting pressure on AP. Some theories of imagination directly address imaginative jumps to new topics. Stich and Nichols (2000, p. 119) provide evidence that imaginative episodes of the sort under discussion display what they call *non-inferential embellishment* of an imagined scenario. The output of an imaginative episode is typically an expansion to contents 'that are not dictated by the pretense premise, or by the scripts and background knowledge that the pretender brings to the pretense episode' (Stich and Nichols, 2000, p. 127) and which outstrips what is inferentially available. Their theory of pretense posits an underlying cognitive mechanism – the *script elaborator* – that serves this function.[14] This theory of imagination can be used to question some of the Kuratowski constraints.

The most robustly motivated Kuratowski constraint with respect to imagination is clearly *inclusion*, allowing for imaginative jumps from the topic of an imaginative input to a larger topic — we motivated this initially via integrating the topics of the contextual, relevant background information. One might instead motivate topic expansion via non-inferential embellishment. Since an embellishment of a topic only broadens it, we can firmly accept inclusion.

On the other hand, additivity (as a constraint in addition to inclusion) can be contested. What is imagined in an act of imagining the input φ can go beyond the totality of what is imagined in an act of imagining the atoms within φ separately. Consider now that Helena lives in New York City and she has a friend named John who often moves from one city to another because of his job. In an act of imagining that she is on her way to meet John, she imagines the activities she will be doing with John (having lunch, going to the movies etc.). In an act of imagining that John is currently residing in Boston, she imagines how much John likes Boston, how cold Boston is in winter etc. However, in an act of imagining that she is on her way to meet John and John is currently residing in Boston, she imagines that she is driving to New England. The latter imagination is arguably triggered by the connection between driving to meet John and the location of John's current residence, which might be missing when topics of 'Helena is on her way to meet John' and 'John is currently residing in Boston' are first expanded and then fused.[15] In other words, some ways of elaborating the script of the whole φ do in fact outstrip the permitted elaborations of script for each atomic parts within it.

The least well-motivated of these with respect to non-inferential embellishment theories of imagination is clearly *idempotence*, which states that the closure of the closure of any topic is its closure: $c(c(x)) = c(x)$. We motivated this by suggesting that once a topic is 'expanded' by integrating the topics of the contextual relevant background information, all such information is exhausted and additional expansions would require further inputs. But that is clearly not supported by the idea that *non-inferential embellishment* is possible within an imagined scenario. On these theories, embellishment or elaboration might be repeatedly expanded upon given the same imaginative inputs.

Finally, there is room for potential dispute over *monotone increasingness*. Where topic x is part of topic y, there is some question as to whether the permitted embellishments of x are thereby contained within the permitted embellishments of y. Imagine that Laura is a 35 year-old woman running for a seat in the State Senate. There are many ways to non-inferentially embellish this scenario, including that she wins on a platform of supporting gun control. However, if we expand the initial subject matter by including that Laura is a Republican whose campaign was

[14]In recent work, Özgün and Schoonen (2024) present a logic of imagination in pretense that models many common features of a number of cognitive theories of pretense and imagination, including the one by Stich and Nichols (2000). The logic of Özgün and Schoonen (2024) has a topic sensitive *unary* imagination operator that is intended to capture "what the agent has imagined up until the current moment".

[15]This example is inspired by Lewis' famous Nassau Street example (Lewis, 1982) and another one in (Braddon-Mitchell and Jackson, 1996, p. 199), which are often used to argue for fragmentation of mental states such as knowledge and belief (see, e.g., Fagin and Halpern (1987); Lewis (1982); Hawke et al. (2020)).

financed by the National Rifle Association, this embellishment is no longer permitted. Some expansions of subject matter rule out embellishments permitted by parts.[16]

These considerations suggest several choice points that depend centrally on the details of the underlying theory of imagination. In what follows, we explore a range of weaker 'topic expansion' operators $f : \mathcal{T} \to \mathcal{T}$ that satisfy only *some* of the Kuratowski constraints. To this end, we investigate three more consequence relations for the topic expansion operators:

(a) $\models_{incl}$ for ts-models with an inclusive function f;

(b) $\models_{incl+mon}$ for ts-models with inclusive and monotone increasing function f;

(c) $\models_{pre}$ for ts-models with a preclosure (inclusive and additive) function f.

We provide strongly sound and complete axiomatizations for $\models_{incl}$ and $\models_{incl+mon}$. With respect to $\models_{pre}$ we show that additivity is not definable in the language $\mathcal{L}$ suggesting that obtaining a complete axiomatization requires either more involved techniques or a more expressive language.

To prove these results, we rely on the ts-frames/models with functions introduced in Definition 3. We interpret $X^{\psi}\varphi$ on these models as in (F-Sem).

We call a ts-model with functions an *inclusive ts-model with functions* if f satisfies inclusion, an *inclusive and monotone increasing ts-model with functions* if f is also monotone increasing, and a *preclosure ts-model with functions* if f satisfies inclusion and additivity.[17] Each notion of logical consequence and validity $\models_{incl}$, $\models_{incl+add}$, $\models_{pre}$ are defined standardly with respect to the relevant class of models, analogous to section 1.1.

3.1 Definability of the properties of f

In this section, we prove that inclusion and monotone increasingness are both definable in $\mathcal{L}$, while additivity is not. To do so, we need to define a few auxiliary notions and a notion of frame definability for ts-frames with functions, adapting the corresponding standard definitions of modal logic (see, e.g., (Blackburn et al., 2001, Chapter 3) and (Pacuit, 2017, Chapter 2.5)).

Given a ts-frame with functions $\mathcal{E} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f\rangle$, a topic assignment function $t : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, and a valuation function $V : \mathcal{L}_{AT} \to \mathcal{P}(W)$, the tuple $\mathcal{X}^{\mathcal{E}} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f, t, V\rangle$ is called a ts-model with functions *based on* $\mathcal{E}$. We say a formula $\varphi \in \mathcal{L}$ is *valid* in $\mathcal{E}$, denoted by $\mathcal{E} \models \varphi$ if $\mathcal{X}^{\mathcal{E}}, w \models \varphi$ for all models $\mathcal{X}^{\mathcal{E}}$ based on $\mathcal{E}$ and $w \in W$.

Definition 5 (Definability)**.** *Given $\varphi \in \mathcal{L}$ and a class $\mathfrak{C}$ of ts-frames with functions, φ* defines *$\mathfrak{C}$ if for all ts-frames $\mathcal{E}$, $\mathcal{E} \in \mathfrak{C}$ iff $\mathcal{E} \models \varphi$. A class of ts-frames is* definable in $\mathcal{L}$ *if there is a formula in $\mathcal{L}$ that defines it.*

Lemma 5. *Given a ts-frame with functions $\mathcal{E} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f\rangle$:*

1. *f is inclusive iff $\mathcal{E} \models X^{\varphi}\overline{\varphi}$*

2. *f is monotone increasing iff $\mathcal{E} \models X^{\psi}\overline{\varphi} \supset X^{\eta}\overline{\varphi}$ for $\mathfrak{At}(\psi) \subseteq \mathfrak{At}(\eta)$.*

Therefore, both inclusion and monotone increasingness are definable in $\mathcal{L}$.

Proof. Let $\mathcal{E} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f\rangle$ be a ts-frame with functions.

[16] Similar examples can be constructed from the famous examples of the probabilistic conjunction fallacy in Kahneman and Tversky (1982), such as Linda the feminist bank teller.

[17] Inclusive and additive operators have been called 'preclosure', 'praclosure', or sometimes 'Čech' operators (Arkhangelskii and Pontryagin, 1990, p. 25).

1. ($\Rightarrow$) Suppose that f satisfies inclusion, let $\mathcal{X}^{\mathcal{E}} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f, t, V\rangle$ be a model based on $\mathcal{E}$ and $w \in W$. Since f satisfies inclusion, we have $t(\varphi) \leq f(t(\varphi))$. As $t(\varphi) = t(\overline{\varphi})$, we obtain $t(\overline{\varphi}) \leq f(t(\varphi))$. Since $\overline{\varphi}$ is true in every world, by the semantics of X, we obtain that $\mathcal{X}^{\mathcal{E}}, w \models X^{\varphi}\overline{\varphi}$. Therefore, $\mathcal{E} \models X^{\varphi}\overline{\varphi}$.

   ($\Leftarrow$) Suppose that f does not satisfy inclusion, that is, there is $a \in \mathcal{T}$ such that $a \not\leq f(a)$. Take the model $\mathcal{X}^{\mathcal{E}} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f, t, V\rangle$ based on $\mathcal{E}$ such that $t(p) = a$. We then have that $t(\overline{p}) = t(p) = a \not\leq f(a) = f(t(p))$. Therefore, for every $w \in W$, we have $\mathcal{X}^{\mathcal{E}}, w \not\models X^{p}\overline{p}$, implying that $\mathcal{E} \not\models X^{\varphi}\overline{\varphi}$.

2. ($\Rightarrow$) Suppose that f satisfies the property Monotone Increasingness, let $\mathcal{X}^{\mathcal{E}} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f, t, V\rangle$ be a model based on $\mathcal{E}$ and $w \in W$. Moreover, let $\psi, \eta \in \mathcal{L}$ such that $\mathfrak{At}(\psi) \subseteq \mathfrak{At}(\eta)$ and suppose that $\mathcal{X}^{\mathcal{E}}, w \models X^{\psi}\overline{\varphi}$. This means that $t(\varphi) \leq f(t(\psi))$. Also observe that, since $\mathfrak{At}(\psi) \subseteq \mathfrak{At}(\eta)$, by the definition of t, we have that $t(\psi) \leq t(\eta)$. Then, by the property Monotone Increasingness, we obtain $f(t(\psi)) \leq f(t(\eta))$. Transitivity of $\leq$, $t(\varphi) \leq f(t(\psi))$, and $f(t(\psi)) \leq f(t(\eta))$ together imply that $t(\varphi) \leq f(t(\eta))$. This means, by the semantics of X, that $\mathcal{X}^{\mathcal{E}}, w \models X^{\eta}\overline{\varphi}$. We then conclude that $\mathcal{E} \models X^{\psi}\overline{\varphi} \supset X^{\eta}\overline{\varphi}$.

   ($\Leftarrow$) Suppose that f does not satisfy the property Monotone Increasingness, that is, there are $a, b \in \mathcal{T}$ such that $a \leq b$ but $f(a) \not\leq f(b)$. Consider the formulas $X^{p}\overline{q}$ and $X^{p\wedge r}\overline{q}$, and take the model $\mathcal{X}^{\mathcal{E}} = \langle W, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \mathcal{T}, \oplus, f, t, V\rangle$ based on $\mathcal{E}$ with $t(p) = a$, $t(q) = f(a)$ and $t(r) = b$. Therefore, for any $w \in W$, $\mathcal{X}^{\mathcal{E}}, w \models X^{p}\overline{q}$ (since $t(q) = f(a) \leq f(a) = f(t(p))$). However, $\mathcal{X}^{\mathcal{E}}, w \not\models X^{p\wedge r}\overline{q}$ (since $t(q) = f(a) \not\leq f(b) = f(a \oplus b) = f(t(p) \oplus t(r)) = f(t(p \wedge r))$). Therefore, $\mathcal{E} \not\models X^{\psi}\overline{\varphi} \supset X^{\eta}\overline{\varphi}$ for $\mathfrak{At}(\psi) \subseteq \mathfrak{At}(\eta)$.

□

Lemma 6. *Additivity is not definable in $\mathcal{L}$.*

Proof. Suppose, toward contradiction, that additivity is definable in $\mathcal{L}$, that is, there is a formula φ such that for all frames $\mathcal{E}$, $\mathcal{E} \models \varphi$ iff f in $\mathcal{E}$ satisfies additivity. Now consider the following two frames $\mathcal{E}_1 = \langle \{w\}, \{R_\psi | \psi \in \mathcal{L}\}, \mathcal{T}_1, \oplus_1, f_1\rangle$ and $\mathcal{E}_2 = \langle \{w\}, \{R_\psi | \psi \in \mathcal{L}\}, \mathcal{T}_2, \oplus_2, f_2\rangle$ where $(\mathcal{T}_1, \oplus_1, f_1)$ and $(\mathcal{T}_2, \oplus_2, f_2)$ are as given in Figure 3a and 3b, respectively, such that $f_1(x_1) = x_1, f_1(y_1) = y_1, f_1(z_1) = f_1(u_1) = u_1$ (represented by the red arrows in Figure 3a), f_2 is the identity function (represented by the blue arrows in Figure 3b).[18] $R_\psi = \{(w, w)\}$ for all $\psi \in \mathcal{L}$.

(a) $(\mathcal{T}_1, \oplus_1, f_1)$ (b) $(\mathcal{T}_2, \oplus_2, f_2)$

Figure 3: Topic components of $\mathcal{E}_1$ and $\mathcal{E}_2$. (In figures of models, arrows represent the expansion operator f (e.g., the arrow from z_1 to u_1 means $f(z_1) = u_1$.)

[18]The model $\mathcal{E}_1$ satisfies 'quasi-supplmentation' (Cotnoir and Varzi, 2021, p. 132) but not weak or strong supplementation. The model can be expanded by adding a topic $v_1 < u_1$ disjoint from z_1 and such that $f(v_1) = u_1$. The model is strongly supplemented and f still fails to be additive (and for the same reason).

It is easy to verify that f_2 is additive whereas f_1 is not ($f_1(x_1 \oplus y_1) = f(z_1) = u_1 \neq z_1 = f(x_1) \oplus f(y_1)$.) Therefore, it should be that $\mathcal{E}_2 \models \varphi$ but $\mathcal{E}_1 \not\models \varphi$, as φ is assumed to define additivity. The latter means that there is a model $\mathcal{X}^{\mathcal{E}_1} = \langle \{w\}, \{R_\psi | \psi \in \mathcal{L}\}, \mathcal{T}_1, \oplus_1, f_1, t_1, V_1 \rangle$ based on $\mathcal{E}_1$ such that $\mathcal{X}^{\mathcal{E}_1}, w \not\models \varphi$. However, as shown below, for all $\beta \in \mathcal{L}$, $\mathcal{X}^{\mathcal{E}_1}, w \models \beta$ iff $\mathcal{X}^{\mathcal{E}_2}, w \models \beta$, where $\mathcal{X}^{\mathcal{E}_2} = \langle \{w\}, \{R_\psi | \psi \in \mathcal{L}\}, \mathcal{T}_2, \oplus_2, f_2, t_2, V_2 \rangle$ such that $V_1 = V_2$ and for $\varphi \in \mathcal{L}$:

$$t_2(\varphi) = \begin{cases} x_2, \text{ if } t_1(\varphi) = x_1 \\ y_2, \text{ if } t_1(\varphi) = y_1 \\ z_2, \text{ if } t_1(\varphi) = z_1 \text{ or } t_1(\varphi) = u_1 \end{cases}$$

(It is easy to verify that t_2 is a well-defined topic assignment function.) Therefore, $\mathcal{X}^{\mathcal{E}_2}, w \not\models \varphi$, contradicting $\mathcal{E}_2 \models \varphi$ (as $\mathcal{X}^{\mathcal{E}_2}$ is a model based on $\mathcal{E}_2$). All that is left to show is the above mentioned equivalence claim of $\mathcal{X}^{\mathcal{E}_1}, w$ and $\mathcal{X}^{\mathcal{E}_2}, w$. We prove this by induction on the complexity of β. The cases for the propositional variables, Booleans, and $\varphi := \Box\psi$ are standard. We here prove the case $\beta := X^\psi \eta$. Moreover, since $R_\psi(w) \subseteq |\eta|_{\mathcal{X}^{\mathcal{E}_1}}$ iff $R_\psi(w) \subseteq |\eta|_{\mathcal{X}^{\mathcal{E}_2}}$ follows by the induction hypothesis and the fact that the possible-worlds components of both models are the same, we only need to show that $t_1(\eta) \leq f_1(t_1(\psi))$ iff $t_2(\eta) \leq f_2(t_2(\psi))$.

($\Rightarrow$) Suppose that $t_1(\eta) \leq f_1(t_1(\psi))$. We then have three cases:

(Case 1) $f_1(t_1(\psi)) = x_1$: This means that $t_1(\psi) = x_1$ and $t_1(\eta) = x_1$. Then, by the defn. t_2 and f_2, we have that $t_2(\eta) = x_2 \leq x_2 = f_2(x_2) = f_2(t_2(\psi))$.

(Case 2) $f_1(t_1(\psi)) = y_1$: Similar to the above case.

(Case 3) $f_1(t_1(\psi)) = u_1$: This means that $t_1(\psi) = z_1$ or $t_1(\psi) = u_1$. No matter which is the case, by the defn. of t_2 and f_2, $t_2(\psi) = z_2 = f_2(t_2(\psi))$. Therefore, by the structure of $(\mathcal{T}_2, \oplus_2, f_2)$, we obtain that $t_2(\eta) \leq f_2(t_2(\psi))$.

($\Leftarrow$) Suppose that $t_2(\eta) \leq f_2(t_2(\psi))$. We again have three cases:

(Case 1) $f_2(t_2(\psi)) = x_2$: This means that $t_2(\psi) = x_2$ and $t_2(\eta) = x_2$. Then, by the defn. of t_2 and f_2, we have that $t_1(\eta) = x_1 \leq x_1 = f_1(x_1) = f_1(t_1(\psi))$.

(Case 2) $f_2(t_2(\psi)) = y_2$: Similar to the above case.

(Case 3) $f_2(t_2(\psi)) = z_2$: This means that $t_2(\psi) = z_2$. Then, by the defn. of t_2 and f_2, we have that $t_1(\psi) = z_1$ or $t_1(\psi) = u_1$. No matter which one is the case, we have $t_1(\eta) \leq u_1 = f_1(t_1(\psi))$. □

3.2 Logics for weaker expansion operators

3.2.1 Inclusion and Monotone Increasing

In this section we provide strongly sound and complete axiomatizations for the logics of inclusive ts-models with functions, and inclusive and monotone increasing ts-models with functions. Soundness is a matter of routine validity check so we skip its proof. We prove the completeness results via a canonical model construction and present the proofs in full detail. Note that our canonical model is in fact more general and provides a completeness proof with respect to *all ts-models with functions.*

A sound and complete axiomatization Log_{incl} of the logic of inclusive ts-models with functions is presented in Table 1.

The notion of derivation, denoted by $\vdash_{incl}$, in Log_{incl} is defined as usual (Blackburn et al., 2001, p. 192). For any set of formulas $\Gamma \subseteq \mathcal{L}$ and any $\varphi \in \mathcal{L}$, we write $\Gamma \vdash_{incl} \varphi$ if there exists finitely many formulas $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash_{incl} (\varphi_1 \wedge \cdots \wedge \varphi_n) \supset \varphi$. We say that Γ is Log_{incl}-*consistent* if $\Gamma \not\vdash_{incl} \bot$, and Log_{incl}-*inconsistent* otherwise. A sentence φ is Log_{incl}-consistent with Γ if $\Gamma \cup \{\varphi\}$ is Log_{incl}-consistent (or, equivalently, if $\Gamma \not\vdash_{incl} \neg\varphi$). Finally, a set of formulas Γ is a *maximally* Log_{incl}-*consistent set* (or, in short, *mcs*) if it is Log_{incl}-consistent and

(CPL)	all classical propositional tautologies and Modus Ponens
($\mathsf{S5}_\Box$)	$\mathsf{S5}$ axioms and rules for $\Box$
	(I) Axioms for X:
($\mathsf{Ax1}$)	$X^{\psi}\overline{\varphi}$ if $\mathfrak{At}(\varphi) \subseteq \mathfrak{At}(\psi)$
($\mathsf{Ax2}$)	$(X^{\psi}\varphi \wedge X^{\psi}\eta) \equiv X^{\psi}(\varphi \wedge \eta)$
($\mathsf{Ax3}$)	$X^{\psi}\overline{\varphi} \supset X^{\eta}\overline{\varphi}$ if $\mathfrak{At}(\psi) = \mathfrak{At}(\eta)$
($\mathsf{Ax4}$)	$X^{\psi}\varphi \supset X^{\psi}\overline{\varphi}$
	(II) Axioms connecting $\Box$ and X:
($\mathsf{Ax5}$)	$X^{\psi}\overline{\varphi} \supset \Box X^{\psi}\overline{\varphi}$
($\mathsf{Ax6}$)	$(X^{\psi}\eta \wedge \Box(\eta \supset \varphi) \wedge X^{\psi}\overline{\varphi}) \supset X^{\psi}\varphi$

Table 1: Axiomatization Log_{incl} for the logic of inclusive ts-models with functions.

any set of formulas properly containing Γ is Log_{incl}-inconsistent (Blackburn et al., 2001). We drop mention of the logic Log_{incl} when it is clear from the context and also simply write $\vdash$ for $\vdash_{incl}$. Similar definitions also hold for the logic $\mathsf{Log}_{incl+mon}$ of inclusive and monotone increasing ts-models with functions obtained by replacing $\mathsf{Ax3}$ in Table 1 by the following stronger principle:

$$\mathsf{sAx3}: \; X^{\psi}\overline{\varphi} \supset X^{\eta}\overline{\varphi} \text{ if } \mathfrak{At}(\psi) \subseteq \mathfrak{At}(\eta).$$

Lemma 7. *The following is derivable in Log_{incl}: $X^{\psi}\overline{\varphi} \supset X^{\psi}\overline{\eta}$, if $\mathfrak{At}(\eta) \subseteq \mathfrak{At}(\varphi)$.*

Proof. Follows from $\mathsf{Ax2}$ and CPL. □

Lemma 8. *For every mcs Γ of Log_{incl} and $\varphi, \psi \in \mathcal{L}$, the following hold:*

1. $\Gamma \vdash \varphi$ *iff* $\varphi \in \Gamma$,
2. *if* $\varphi \in \Gamma$ *and* $\varphi \supset \psi \in \Gamma$ *then* $\psi \in \Gamma$,
3. *if* $\vdash \varphi$ *then* $\varphi \in \Gamma$,
4. $\varphi \in \Gamma$ *and* $\psi \in \Gamma$ *iff* $\varphi \wedge \psi \in \Gamma$,
5. $\varphi \in \Gamma$ *iff* $\neg\varphi \notin \Gamma$.

Proof. Standard. □

Lemma 9 (Lindenbaum's Lemma)**.** *Every Log_{incl}-consistent set can be extended to a maximally Log_{incl}-consistent one.*

Proof. Standard. □

Canonical Model Construction: Our canonical model is similar to the one presented in Giordani (2019) except for the construction of the topic relevant components. We follow Siemers (2021) for the construction of $\mathcal{T}^c$ and $\oplus^c$ and, to the best of our knowledge, component f^c is novel.

Let W^c be the set of all maximally Log_{incl}-consistent sets. For each $\Gamma \in W^c$, define

$$\begin{aligned} \Gamma[\Box] &:= \{\varphi \in \mathcal{L} : \Box\varphi \in \Gamma\}, \\ \Gamma[X^{\psi}] &:= \{\varphi \in \mathcal{L} : X^{\psi}\eta \wedge \Box(\eta \supset \varphi) \in \Gamma \text{ for some } \eta \in \mathcal{L}\}, \text{ and} \\ \Gamma/\psi &:= \{\varphi \in \mathcal{L} : X^{\psi}\varphi \in \Gamma\}. \end{aligned}$$

Moreover, we define $\sim_{\Box}$ and $\rightarrow^c_{\varphi}$ on W^c, respectively, as

$$\begin{aligned} \Gamma \sim_{\Box} \Delta &\text{ iff } \Gamma[\Box] \subseteq \Delta, \\ \Gamma \rightarrow^c_{\psi} \Delta &\text{ iff } \Gamma[X^{\psi}] \subseteq \Delta. \end{aligned}$$

Since $\Box$ is an $\mathsf{S5}$ modality, it is easy to see that $\sim_{\Box}$ is an equivalence relation (Blackburn et al., 2001). For any maximally Log_{incl}-consistent set Γ, we denote by $[\Gamma]_{\Box}$ the equivalence class of Γ induced by $\sim_{\Box}$, i.e., $[\Gamma]_{\Box} = \{\Delta \in W^c : \Gamma \sim_{\Box} \Delta\}$. Moreover, as shown by the lemma below, $\rightarrow^c_{\psi} \subseteq \sim_{\Box}$.

Lemma 10. *For all $\psi \in \mathcal{L}$ and $\Gamma, \Delta \in W^c$, if $\Gamma \rightarrow^c_{\psi} \Delta$, then $\Gamma \sim_{\Box} \Delta$, i.e., $\rightarrow^c_{\psi} \subseteq \sim_{\Box}$*

Proof. Let $\psi \in \mathcal{L}$ and $\Gamma, \Delta \in W^c$ such that $\Gamma \rightarrow^c_{\psi} \Delta$, i.e., that $\Gamma[X^{\psi}] \subseteq \Delta$. Let $\varphi \in \Gamma[\Box]$. This means that $\Box\varphi \in \Gamma$. Then, by $\mathsf{S5}_{\Box}$ (since $\vdash \Box\varphi \equiv \Box(\overline{\psi} \rightarrow \varphi)$) and Lemma 8, we have that $\Box(\overline{\psi} \supset \varphi) \in \Gamma$. Moreover, by $\mathsf{Ax1}$, we have that $X^{\psi}\overline{\psi} \in \Gamma$. Then, by the definition of $\Gamma[X^{\psi}]$, we conclude that $\varphi \in \Gamma[X^{\psi}]$. Then, by the first assumption that $\Gamma[X^{\psi}] \subseteq \Delta$, we obtain $\varphi \in \Delta$. Therefore, $\Gamma[\Box] \subseteq \Delta$, i.e., $\Gamma \sim_{\Box} \Delta$. □

Given a mcs Γ_0 of Log_{incl}, the canonical model for Γ_0 is a tuple $\mathcal{X}^c = \langle [\Gamma_0]_{\Box}, \{R^c_{\psi} \mid \psi \in \mathcal{L}\}, \mathcal{T}^c, \oplus^c, f^c, t^c, V^c \rangle$ where

- $[\Gamma_0]_{\Box}$ is as described above.
- $R^c_{\psi} = \rightarrow^c_{\psi} \cap ([\Gamma_0]_{\Box} \times [\Gamma_0]_{\Box})$
- $\mathcal{T}^c := \mathcal{P}(\mathcal{L}_{AT})$
- $\oplus^c : \mathcal{T}^c \times \mathcal{T}^c \rightarrow \mathcal{T}^c$ such that for all $A, B \in \mathcal{P}(\mathcal{L}_{AT})$, $A \oplus^c B = A \cup B$.
- $t^c : \mathcal{L} \rightarrow \mathcal{T}^c$ such that, for all $\varphi \in \mathcal{L}$, $t^c(\varphi) = \mathfrak{At}(\varphi)$.
- $f^c : \mathcal{T}^c \rightarrow \mathcal{T}^c$ such that

$$f^c(A) = \begin{cases} \bigcup\{t(\varphi) : X^{(\bigwedge A)}\overline{\varphi} \in \Gamma_0\}, & \text{if } A \text{ is finite} \\ \mathcal{L}_{AT}, & \text{otherwise} \end{cases}$$

- $V^c : \mathcal{L}_{AT} \rightarrow \mathcal{P}([\Gamma_0]_{\Box})$ such that $V^c(p) = \{\Gamma \in [\Gamma_0]_{\Box} : p \in \Gamma\}$

The canonical topic-parthood relation is the subset relation on $\mathcal{P}(\mathcal{L}_{AT})$: $A \oplus^c B = B$ iff $A \cup B = B$ iff $A \subseteq B$.

Lemma 11. *Given a mcs Γ_0, the canonical model $\mathcal{X}^c = \langle [\Gamma_0]_{\Box}, \{R^c_{\psi} \mid \psi \in \mathcal{L}\}, \mathcal{T}^c, \oplus^c, f^c, t^c, V^c \rangle$ for Γ_0 is an inclusive ts-model with functions.*

Proof. Observe that $\oplus^c$ and t^c are well-defined. Moreover, for all $\varphi \in \mathcal{L}$, $t^c(\varphi) = \mathfrak{At}(\varphi) = \{t^c(p) : p \in \mathfrak{At}(\varphi)\} = \oplus^c\mathfrak{At}(\varphi)$. Finally, f^c satisfies inclusion, that is, for all $A \in \mathcal{T}^c$, $A \subseteq f^c(A)$: if A is infinite, by definition, $A \subseteq f^c(A) = \mathcal{L}_{AT}$. Let A be finite and $p \in A$. We know, by $\mathsf{Ax1}$, that $\vdash X^{(\bigwedge A)}\overline{p}$, thus, $X^{(\bigwedge A)}\overline{p} \in \Gamma_0$. This means, by the definition of f^c and t^c, that $t^c(p) = \{p\} \subseteq f^c(A)$. We then conclude that $A \subseteq f^c(A)$. □

Inclusivity of f^c is guaranteed by $\mathsf{Ax1}$ (it is indeed guaranteed by the weaker theorem $\vdash X^{\varphi}\overline{\varphi}$). Therefore, the axiomatization given in Table 1 without $\mathsf{Ax1}$ is a strongly sound and complete axiomatization for all ts-models with functions (corresponding completeness proof follows the same steps as the proof presented in this section). Since we are interested in topic expansion operators, we focus on the logics of at least inclusive ts-models.

Lemma 12. *For any mcs Γ and $\psi, \varphi \in \mathcal{L}$, $X^{\psi}\overline{\varphi} \in \Gamma$ iff $X^{\psi}\overline{p} \in \Gamma$ for all $p \in \mathfrak{At}(\varphi)$.*

Proof. The direction from left-to-right follows from Lemma 7. For the opposite direction, let $\mathfrak{At}(\varphi) = \{p_1, \ldots, p_n\}$ and observe that $\overline{\varphi} := \overline{p_1} \wedge \cdots \wedge \overline{p_n}$. If $X^{\psi}\overline{p_i} \in \Gamma$ for all $p_i \in \{p_1, \ldots, p_n\}$, then $\bigwedge_{i \leq n} X^{\psi}\overline{p_i} \in \Gamma$ (by Lemma 8.4). Then, by $\mathsf{Ax2}$, we obtain that $X^{\psi}(\bigwedge_{i \leq n} \overline{p_i}) \in \Gamma$, i.e., $X^{\psi}\overline{\varphi} \in \Gamma$. ☐

Corollary 13. *Given the canonical model $\mathcal{M}^c = \langle [\Gamma_0]_\Box, \{R^c_\psi \mid \psi \in \mathcal{L}\}, \mathcal{T}^c, \oplus^c, f^c, t^c, V^c \rangle$ for Γ_0, $\varphi, \psi \in \mathcal{L}$, and $\Gamma \in [\Gamma_0]_\Box$, $X^{\psi}\overline{\varphi} \in \Gamma$ iff $t^c(\varphi) \subseteq f^c(t^c(\psi))$.*

Proof.

$$
\begin{aligned}
X^{\psi}\overline{\varphi} \in \Gamma \text{ iff } & X^{\psi}\overline{p} \in \Gamma \text{ for all } p \in \mathfrak{At}(\varphi) && \text{(Lemma 12)}\\
\text{iff } & X^{\psi}\overline{p} \in \Gamma_0 \text{ for all } p \in \mathfrak{At}(\varphi) && (\mathsf{Ax5},\ \mathsf{S5}_\Box, \text{ and } \Gamma \in [\Gamma_0]_\Box)\\
\text{iff } & X^{\psi}\overline{\varphi} \in \Gamma_0 && \text{(Lemma 12)}\\
\text{iff } & X^{\bigwedge \mathfrak{At}(\psi)}\overline{\varphi} \in \Gamma_0 && (\mathsf{Ax3})\\
\text{iff } & t^c(\varphi) \subseteq f^c(t^c(\psi)) && \text{(defn.s of } f^c \text{ and } t^c)
\end{aligned}
$$

☐

Lemma 14. *Given a mcs Γ, for all finite $\Phi \subseteq \Gamma[X^{\psi}]$, we have $\bigwedge \Phi \in \Gamma[X^{\psi}]$.*

Proof. Let $\Phi = \{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma[X^{\psi}]$. This means that, for each φ_j with $1 \leq j \leq n$, there is a $\eta_j \in \mathcal{L}$ such that $X^{\psi}\eta_j \wedge \Box(\eta_j \supset \varphi_j) \in \Gamma$. Thus, $\bigwedge_{1 \leq j \leq n} X^{\psi}\eta_j \wedge \bigwedge_{1 \leq j \leq n} \Box(\eta_j \supset \varphi_j) \in \Gamma$. Then, by $\mathsf{Ax2}$, we obtain that $X^{\psi}(\bigwedge_{j \leq n} \eta_j) \in \Gamma$. By $\mathsf{S5}_\Box$, we also have $\Box(\bigwedge_{j \leq n} \eta_j \supset \bigwedge_{j \leq n} \varphi_j) \in \Gamma$. Therefore, $\bigwedge \Phi \in \Gamma[X^{\psi}]$. ☐

Lemma 15. *For every mcs Γ and $\varphi \in \mathcal{L}$, if $\Gamma[X^{\psi}] \vdash \varphi$ and $X^{\psi}\overline{\varphi} \in \Gamma$, then $X^{\psi}\varphi \in \Gamma$.*

Proof. Suppose that $\Gamma[X^{\psi}] \vdash \varphi$ and $X^{\psi}\overline{\varphi} \in \Gamma$. The first assumption means that there is a finite $\Phi \subseteq \Gamma[X^{\psi}]$ such that $\vdash \bigwedge \Phi \supset \varphi$. By Lemma 14, we know that $\bigwedge \Phi \in \Gamma[X^{\psi}]$. This means that there is a η such that $X^{\psi}\eta \wedge \Box(\eta \supset \bigwedge \Phi) \in \Gamma$. Moroever, $\vdash \bigwedge \Phi \supset \varphi$ entails, by $\mathsf{S5}_\Box$, that $\vdash \Box(\bigwedge \Phi \supset \varphi)$, thus, $\Box(\bigwedge \Phi \supset \varphi) \in \Gamma$. Therefore, by CPL, $\mathsf{S5}_\Box$, and our first assumption, we obtain $X^{\psi}\eta \wedge \Box(\eta \supset \varphi) \wedge X^{\psi}\overline{\varphi} \in \Gamma$. Hence, by $\mathsf{Ax6}$, $X^{\psi}\varphi \in \Gamma$. ☐

Lemma 16 (Truth Lemma)**.**
Let Γ_0 be a mcs of Log_{incl} and $\mathcal{X}^c = \langle [\Gamma_0]_\Box, \{R^c_\psi \mid \psi \in \mathcal{L}\}, \mathcal{T}^c, \oplus^c, f^c, t^c, V^c \rangle$ be the canonical model for Γ_0. Then, for all $\varphi \in \mathcal{L}$ and $\Gamma \in [\Gamma_0]_\Box$, we have $\mathcal{X}^c, \Gamma \models \varphi$ iff $\varphi \in \Gamma$.

Proof. The proof follows by induction on the structure of φ. The cases for the propositional variables, Booleans, and $\varphi := \Box\psi$ are standard. We here prove the case $\varphi := X^{\psi}\eta$.

($\Leftarrow$) Suppose $X^{\psi}\eta \in \Gamma$. Since $X^{\psi}\eta \in \Gamma$, by $\mathsf{Ax4}$, $X^{\psi}\overline{\eta} \in \Gamma$. Thus, by Corollary 13, $t^c(\eta) \subseteq f^c(t^c(\psi))$. Now let $\Delta \in [\Gamma_0]_\Box$ such that $\Gamma R^c_\psi \Delta$. As $X^{\psi}\eta \in \Gamma$ and $\Box(\eta \supset \eta) \in \Gamma$ (the latter is by $\mathsf{S5}_\Box$), we have that $\eta \in \Gamma[X^{\psi}]$. Therefore, since $\Gamma R^c_\psi \Delta$, we have $\eta \in \Delta$. Then, by the induction hypothesis, we have $\mathcal{X}^c, \Delta \models \eta$. As Δ has been chosen arbitrarily, we obtain that $\mathcal{X}^c, \Gamma \models X^{\psi}\eta$.

($\Rightarrow$) Suppose $\mathcal{X}^c, \Gamma \models X^\psi \eta$, i.e., for all $\Delta \in [\Gamma_0]_\Box$ such that $\Gamma R^c_\psi \Delta$, $\mathcal{X}^c, \Delta \models \eta$ and $t^c(\eta) \subseteq f^c(t^c(\psi))$. By Corollary 13, the latter means that $X^\psi \overline{\eta} \in \Gamma$. Moreover, the former, by the induction hypothesis, implies that $\eta \in \Delta$ for all $\Delta \in [\Gamma_0]_\Box$ with $\Gamma R^c_\psi \Delta$. This implies that $\Gamma[X^\psi] \vdash \eta$. Otherwise, $\Gamma[X^\psi] \cup \{\neg\eta\}$ would be consistent, thus, by Lemma 9, there exists a mcs Δ' such that $\Gamma[X^\psi] \cup \{\neg\eta\} \subseteq \Delta'$. As $\Gamma[X^\psi] \subseteq \Delta'$, we have $\Gamma \rightarrow^c_\psi \Delta'$. Since $\rightarrow^c_\psi \subseteq \sim_\Box$ (Lemma 10) and $\Gamma \in [\Gamma_0]_\Box$, we obtain that $\Delta' \in [\Gamma_0]_\Box$, therefore, $\Gamma R^c_\psi \Delta'$. Hence, that $\neg\eta \in \Delta'$ contradicts with the assumption that $\eta \in \Delta$ for all Δ with $\Gamma R^c_\psi \Delta$. Since $X^\psi \overline{\eta} \in \Gamma$, by Lemma 15, we obtain that $X^\psi \eta \in \Gamma$. ☐

Corollary 17. Log_{incl} *is strongly complete with respect to the class of inclusive ts-models with functions.*

Proof. Let $\Phi_0 \subseteq \mathcal{L}$ be a Log_{incl}-consistent set of formulas. Then, by Lindenbaum's Lemma (Lemma 9), there exists a mcs Γ_0 such that $\Phi_0 \subseteq \Gamma_0$. We can then construct a canonical model $\mathcal{X}^c = \langle [\Gamma_0]_\Box, \{R^c_\psi \mid \psi \in \mathcal{L}\}, \mathcal{T}^c, \oplus^c, f^c, t^c, V^c \rangle$ for Γ_0 as described above. Then, by Lemma 16, we obtain that $\mathcal{X}^c, \Gamma_0 \models \varphi$ for all $\varphi \in \Phi_0$. ☐

Corollary 18. $\mathsf{Log}_{incl+mon} = \mathsf{Log}_{incl} + \mathsf{sAx3}$ *is strongly complete with respect to the class of inclusive and monotone increasing ts-models with functions.*

Proof. The proof follows similarly to the proof for Log_{incl}, by constructing the corresponding canonical model in the same way. The only additional step we need to show is that f^c in the canonical model for $\mathsf{Log}_{incl+mon}$ is monotone increasing and this follows from Lemma 5.2. ☐

3.2.2 Preclosure

As shown by Lemma 6, additivity is not definable in $\mathcal{L}$. This suggests that obtaining a complete axiomatization for a class of models satisfying additivity requires either more involved techniques or a more expressive language. We leave the investigations of a complete axiomatization for ts-models with functions satisfying additivity to future work. Nevertheless, we here show that the logic of preclosure ts-models with functions is strictly weaker than the logic of topo-ts-models and, in turn, Berto's logic of imagination in (Berto, 2018a).

In the following theorem, $\models_{pre}\varphi$ denotes that φ is valid in preclosure ts-models with functions, that is, it is true in all worlds of all preclosure ts-models.

Theorem 19. *For all $\varphi \in \mathcal{L}$, if $\models_{pre}\varphi$ then $\Vdash \varphi$. However, there is a $\psi \in \mathcal{L}$ such that $\Vdash \psi$ but $\not\models_{pre} \psi$.*

Proof. The first part immediately follows from Theorem 3 and the fact that $\models_{pre} \subseteq \models$. To show the latter, consider the following sentence:

$$\psi := (X^p \overline{q} \wedge X^{p \wedge q} \overline{r}) \supset X^p \overline{r}.$$

It is then easy to see that we have $\Vdash \psi$. However, ψ is falsified by the preclosure ts-model $\mathcal{X} = \langle \{w\}, \{R_\varphi \mid \varphi \in \mathcal{L}\}, \{x, y, z, z'\}, \oplus, f, t, V \rangle$ where $(T, \oplus, t)$ is as given in Figure 4 and $f(x) = f(z) = f(z') = z'$ and $f(y) = z$, $R_\varphi = \{(w, w)\}$ for all $\varphi \in \mathcal{L}$, $V(p) = V(q) = V(r) = \{w\}$: we have that $\mathcal{X}, w \models X^p \overline{q}$ (since $t(q) = x \leq z = f(t(p)) = f(y)$) and $\mathcal{X}, w \models X^{p \wedge q} \overline{r}$ (since $t(r) = z' = f(z) = f(t(p) \oplus t(q))$). However, $\mathcal{X}, w \not\models X^p \overline{r}$ (since $t(r) = z' \not\leq z = f(t(p))$. It is easy to verify that $\mathcal{X}$ is indeed a preclosure ts-model, in particular, f satisfies inclusion and additivity.[19] ☐

[19]To make this model supplemented, add another element u to the given join-semilattice such that $u < z'$ and does not overlap z and extend f with $f(u) = z'$. This structure satisfies strong supplementation, and f still satisfies inclusion and additivity, and violates idempotence. We thank one of the anonymous referees for pressing this point.

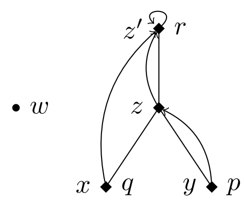

Figure 4: Counterexample for the invalidity of ψ

A few notes about the invalidated formula in the proof of Theorem 19 seem to be warranted. First of all, it is closely related to the so-called Cautious Transitivity principle $((X^{\varphi}\psi \wedge X^{\varphi\wedge\psi}\eta) \supset X^{\varphi}\eta)$; in fact, by replacing q in '$X^{p\wedge q}\overline{r}$' by $\overline{q}$, we obtain an instance of Cautious Transitivity that is invalidated by the same counterexample. Cautious Transitivity (also called 'Special Transitivity' in Berto (2018a)) has come to be controversial in the literature. On the one hand, it is sometimes seen as part of minimal conditional logics for non-monotonic inference. (Berto, 2018a, p. 1883) provides a constraint on R_{φ} that validates Cautious Transitivity but leaves open the possibility that there might be intuitive counterexamples against the principle. In Berto and Hawke (2018), Cautious Transitivity is invalidated due to the (lack of) constraints imposed on R_{φ}, whereas our models violate Cautious Transitivity due to the topicality component as shown in Theorem 19.

We think that there is an intuitive case against Cautious Transitivity at least where TSIMs have application to imagination. Here is a version of the Helena example from Section 3 which seems to violate Cautious Transitivity. In an act of imagining that Helena is on her way to meet John, she imagines that they go to the movies when they meet. In an act of imagining that Helena is on her way to meet John and they go to the movies together, she imagines buying some popcorn at the movie theatre. However, in an act of imagining that Helena is on her way to meet John, she might not imagine that she buys some popcorn at the movie theatre. The imaginative input 'they go to the movies together' in addition to 'Helena is on her way to meet John' might lead to further expansion of the topic of the latter even when the former is imagined in an act of imagining the latter.

Such failures of Cautious Transitivity can be connected to empirical theories of imagination. The core thought is that *non-inferential embellishment* has topical limits. Where $X^{\varphi}\psi$ one imagines that ψ via embellishing the input φ. The topic of input φ has a limited range of expansion, whereas when ψ is added to the inputs, embellishing $\varphi \wedge \psi$ might take us topically further afield. This further embellishment licenses the inference to η in $X^{\varphi\wedge\psi}\eta$, an inference that would not have been licensed simply by embellishing φ without additional inputs. Where non-inferential embellishment permits topic-expansion, it need not permit such expansions arbitrarily, in contrast to adding an explicit imaginative input.

4 Conclusion

In the application to imagination as reality-oriented mental simulation, the content inclusion constraint should be dropped. We have shown that a simple and natural way of doing so has no affect on the resulting logical system, though it does affect what content relations can be expressed in the language, and boosts the philosophical and empirical plausibility of the semantics. The approach we took involved a mereotopology of topics, and required that the topic of imaginative outputs be contained in the *closure* of the topic of the imaginative inputs. We also explored a number of further generalisations of the idea of topic expansion, motivated by a philosophical and empirical view of imaginative jumps as involving non-inferential embellishments. We examined three formal implementations to model weaker topic expansion operators: either (i) inclusive, (ii) inclusive and monotone increasing, or (iii) preclosure operators. Each of these approaches change the resulting logical system. We provided sound and complete axiomatisations Log_{incl} and $\mathsf{Log}_{incl+mon}$ for the first two semantics, respectively, and showed that additivity is not expressible in the language.

We think these systems are promising avenues for the logic of imagination, depending on which empirical theory of the imagination one starts with. Moreover, we hope these formal systems will have applications to other TSIMs, though of course which system works best for a given TSIM application remains a matter for future philosophical debate.

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