Are Geometrical Trinity of Gravity Underdetermined?

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Abstract

General relativity (GR) describes gravity through the curvature of spacetime. However, there are two equivalents of GR that describe flat spacetimes with gravitational effects attributed to torison or non-metricity. These theories, together with GR, are known as the geometrical trinity of gravity and are said to present a case of underdetermination by Wolf et al. (2024). In this article, I argue against this stance by examining the empirical equivalence and possible interpretations of the trinity. I propose a new framework where the trinity emerge as different gauge-fixed versions of a unifying theory. Thus I contend that the apparent disagreements on spacetime ontology arise from different gauge choices without physical significance, thereby breaking down the underdetermination.

1 Introduction

As a geometric theory of gravitation, GR describes spacetime as curved, with the inertial motions of matter affected by the curvature of spacetime. The idea is famously summarized by John Wheeler: "Spacetime tells matter how to move; matter tells spacetime how to curve" (Misner et al., 1973). This conception of gravity is so prevalent that it has become a common practice to interpret gravity as the manifestation of the curvature of spacetime. However, the teleparallel equivalent of general relativity (TEGR) and the symmetric teleparallel equivalent of general relativity (STEGR), while sharing equivalent dynamics with GR, describe flat spacetimes that appear fundamentally different from GR. This discrepancy has sparked recent debates (Knox, 2011; Mulder and Read, 2024; Wolf et al., 2024) on whether these theories present a case of underdetermination, thereby challenging the realism about GR.

TEGR and STEGR are special cases within a broader framework of teleparallel gravity. Teleparallel gravity was proposed by Einstein following the discovery of GR as an attempt to unify gravitation and electromagnetism (Aldrovandi and Pereira, 2013). This approach was based on a flat spacetime with a parallel transport structure defined by a tetrad, a set of vector fields that constitute a basis at any point, hence the name 'teleparallel', meaning that the tetrad is defined to be parallel (at least) locally (Sauer, 2006). However, the attempt did not succeed. In the 1960s, the concept of teleparallelism was revived by a large group of physicists, leading to the discovery of TEGR (Cho, 1976) and STEGR (Nester and Yo, 1999). In TEGR, spacetime is flat but torsionful, with the effects of gravity attributed to torsion instead of curvature. In STEGR, however, spacetime is flat but the connection is not metric-compatible, resulting in a different way to ascribe gravity. These two theories together with GR are known as the geometrical trinity of gravity in the literature (Jimenez et al., 2019), presenting three different geometries that are designed to be dynamically equivalent in gravitational effects. Given that the trinity (1) render the same empirical predictions, so there is no possible empirical data to differentiate them, and (2) postulate different geometrical structures in their descriptions of gravity, it appears that the trinity satisfy the conditions of underdetermination.

In this article, I argue that the discrepancy in spacetime ontology is superficial and that there is no genuine underdetermination in the trinity. In Section 2, I briefly introduce each theory within the trinity and some fundamental elements therein. In Section 3, I analyze the conditions under which the empirical equivalence stands. In Section 4, I examine the plausibility of putative interpretations about spacetime ontology and discuss previous debates. In addition, I propose a unifying framework where the trinity emerge as different gauge-fixed versions of a more fundamental theory, allowing them to be interpreted as different facets of the same underlying ontology.

2 Geometrical trinity of gravity

Since my discussion concerns the theoretical structure of the geometrical trinity of gravity, it's necessary to give a brief introduction to the theories within and set the stage for a more sophisticated analysis (see Bahamonde et al. (2023), Jimenez et al. (2019) and references therein for further elaboration).

2.1 General relativity

Within the domain of general relativity, spacetime is elegantly portrayed by a 4-dimensional manifold $\mathcal{M}$, equipped with a Lorentzian metric $g_{\mu\nu}$ and an affine connection $\Gamma^{\rho}{}_{\mu\nu}$. The connection adheres to two criteria: (i) symmetry and (ii) metric compatibility. Consequently, the metric uniquely determines the connection, known as the Levi-Civita connection $\mathring{\Gamma}^{\rho}{}_{\mu\nu}$:

$$\mathring{\Gamma}^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_{\mu} g_{\sigma\nu} - \partial_{\sigma} g_{\nu\mu} + \partial_{\nu} g_{\mu\sigma}) \tag{1}$$

where the circle above the symbol signifies that the symbol represents the Levi-Civita connection or an associated object.

The curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$ is defined as

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\tau}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\tau}\Gamma^{\tau}{}_{\mu\sigma} \tag{2}$$

which measures the change of a vector after being transported around an infinitesimal loop. Contracting with itself, the curvature tensor yields a scalar $\mathcal{R} = R_{\mu\nu}g^{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}g^{\mu\nu}$ known as Ricci curvature scalar, which decodes the dynamics of spacetime through the Hilbert action:

$$S_{\mathrm{GR}} = \frac{1}{2\kappa^2}\int \mathring{\mathcal{R}}(g)\sqrt{-g}\,\mathrm{d}^4x \tag{3}$$

where $\kappa^2 = 8\pi G$ is the Einstein gravitational constant. Therefore, in GR, the effect of gravity is attributed to the curvature of spacetime. Since the Levi-Civita connection is entirely determined by the metric and the Ricci scalar is constructed upon this connection, the action has only one dynamical variable: the metric tensor.

The variation with respect to the metric yields the Einstein tensor:

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{GR}}}{\delta g^{\mu\nu}} = \mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{\mathcal{R}}g_{\mu\nu} \tag{4}$$

In conjunction with the energy-momentum tensor $t_{\mu\nu}$ resulting from the variation of the action of matter fields, one derives the Einstein field equations:

$$\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{\mathcal{R}}g_{\mu\nu} = \kappa^2 t_{\mu\nu} \tag{5}$$

This elegant equation encapsulates the profound principle that the energy-momentum of matter induces the curvature of spacetime.

2.2 Teleparallel equivalent of general relativity

While agreeing with GR on the Lorentzian metric, teleparallel spacetime is equipped with a flat connection, meaning that the associated curvature tensor vanishes, $R^{\rho}{}_{\sigma\mu\nu} = 0$. This condition forces the connection to take the form (Jimenez et al., 2019):

$$\Gamma^{\rho}{}_{\mu\nu} = e_a{}^{\rho}\partial_{\mu}\theta^a{}_{\nu} \tag{6}$$

Where $e_a{}^{\mu}$ are the coefficients of a (non-orthonormal) tetrad e_a with respect to the coordinate basis ∂_{μ}, while its dual vector is identified as $\theta^a = \theta^a{}_{\mu}dx^{\mu}$, satisfying $\theta^a(e_b) = \theta^a{}_{\sigma}e_b{}^{\sigma} = \delta^a_b$. Note that the tetrad is determined by the connection up to a global $\mathrm{GL}(4,\mathbb{R})$ transformation $\theta^a \to \Lambda^a{}_b\theta^b$ since e_a transform inversely with respect to θ^a, thereby canceling out the transformation matrix $\Lambda^a{}_b$.

While the introduction of a tetrad is not necessary for formulating the theory, it turns out to be convenient. By definition, the covariant derivative, known as *Weitzenbock gauge derivative*, of e_a vanishes:

$$\nabla e_a = 0 \tag{7}$$

Hence the tetrad is parallel with respect to the Weitzenbock connection. Consequently, parallel transport is equivalent to having the components with respect to the tetrad remain unchanged.

Moreover, TEGR requires abandoning the symmetry criterion of the Levi-Civita connection, while retaining the metric compatibility of connection, forcing the tetrad to satisfy (Jiménez Cano, 2021):

$$g_{\mu\nu} = \theta^a{}_\mu \theta^b{}_\nu c_{ab} \tag{8}$$

where c_{ab} are constants. Given the interest in Lorentzian metric, it's natural to choose $c_{ab} = \eta_{ab}$, so the tetrad is orthonormal $g(e_a, e_b) = \eta_{ab}$, thereby fixing the tetrad up to a local Lorentz transformation. In this convention, the metric can be expressed solely by the tetrad.

With the geometrical description of spacetime in place, we now turn to the dynamics of TEGR. The antisymmetric part of connection $T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} - \Gamma^\rho{}_{\nu\mu}$, known as torsion tensor, measures the failure of two vectors to form a parallelogram after being transported along each other. The deviation of the connection from the Levi-Civita one is characterized by *contorsion* tensor $K^\rho{}_{\mu\nu}$,

$$\Gamma^\rho{}_{\mu\nu} = \mathring{\Gamma}^\rho{}_{\mu\nu} + K^\rho{}_{\mu\nu} \tag{9}$$

which is entirely determined by torsion. With this remarkable decomposition, we could split the Ricci scalar into two parts (Bahamonde et al., 2023):

$$\mathring{\mathcal{R}} = \mathcal{R} + \mathcal{T} + 2\mathring{\nabla}_\mu T_\nu{}^{\nu\mu} \tag{10}$$

where torsion scalar $\mathcal{T}$ are constructed from the torsion tensor:

$$\mathcal{T} = -\frac{1}{4} T_{\mu\nu\rho} T^{\mu\nu\rho} - \frac{1}{2} T_{\mu\nu\rho} T^{\mu\rho\nu} + T^\alpha_{\ \mu\alpha} T^{\beta\mu}_{\ \ \ \beta} \tag{11}$$

It's important to note that the metric is involved to raise and lower the indices. Given the vanishing curvature, the action of TEGR is given by:

$$S_{\text{TEGR}} = \frac{1}{2\kappa^2} \int \mathcal{T}(e)\sqrt{-g}\ \mathrm{d}^4x \tag{12}$$

which differs from the Hilbert action only by a total derivative, thus identically recovering the dynamics of GR.

2.3 Symmetric teleparallel equivalent of general relativity

TEGR is not the end of the story, since there is another equivalent theory within flat spacetime called symmetric teleparallel equivalent of general relativity (STEGR). The teleparallel condition presumes a vanishing curvature which, again, leads to a connection of the form (6) with the introduction of a tetrad. As the name suggests, STEGR retains the symmetry condition of connection but lifts the constraint of metric compatibility, resulting in a holonomic tetrad (non-orthonormal in general) parameterized by a set of functions ξ^a (Jiménez Cano, 2021):

$$\theta^a{}_\mu = \partial_\mu \xi^a \tag{13}$$

where ξ^a could be interpreted as a new coordinate system $x'^\mu = \xi^\mu$ known as the *coincident gauge*, within which the connection coefficients vanish. Moreover, the absence of metric compatibility gives rise to a non-vanishing non-metricity tensor:

$$Q_{\alpha\mu\nu} = \nabla_\alpha g_{\mu\nu} \tag{14}$$

which measures the change in the length of a vector during parallel transport.

Similarly, the deviation of the connection is known as *disformation* tensor $L^\rho{}_{\mu\nu}$:

$$\Gamma^\rho{}_{\mu\nu} = \mathring{\Gamma}^\rho{}_{\mu\nu} + L^\rho{}_{\mu\nu} \tag{15}$$

which is constructed upon the non-metricity tensor. Again, the Ricci scalar can be decomposed into two parts (Bahamonde et al., 2023):

$$\mathring{\mathcal{R}} = \mathcal{R} + \mathcal{Q} + \mathring{\nabla}(Q_\nu{}^{\nu\mu} - Q^{\mu\nu}{}_\nu) \tag{16}$$

where the non-metricity scalar $\mathcal{Q}$ are constructed from the non-metricity tensor:

$$\mathcal{Q} = -\frac{1}{4} Q_{\mu\nu\rho} Q^{\mu\nu\rho} + \frac{1}{2} Q_{\mu\nu\rho} Q^{\nu\mu\rho} + \frac{1}{4} Q_{\mu\alpha}{}^\alpha Q^{\mu\beta}{}_\beta \tag{17}$$

Note, again, that the metric is involved to raise and lower the indices. Consequently, the action of STEGR is formulated as:

$$S_{\text{STEGR}} = \frac{1}{2\kappa^2} \int \mathcal{Q}(g, \xi) \sqrt{-g}\ \mathrm{d}^4 x \tag{18}$$

It differs from the Hilbert action by a total derivative, thereby reproducing the dynamics of GR.

2.4 Remarks

The three theories introduced above form a trinity of equivalent theories of gravity, known as the geometrical trinity of gravity. They appear to attribute the effects of gravity to different geometric quantities—curvature in GR, torsion in TEGR, and non-metricity in STEGR—leading to seemingly distinct spacetime ontologies. All these quantities are constructed upon the affine connection, so their disagreements are rooted in the adoption of different connections, but they all share the same metric. It's quite natural since to be equivalent, the trinity have to agree on the measured lengths, which are direct manifestations of the metric.

Therefore, it seems that the trinity satisfy the two conditions of underdetermination. Firstly, their actions differ from each other only by a total derivative, typically resulting in the same dynamical equations. In this regard, it seems reasonable to say they are empirically equivalent. Secondly, they are genuinely distinct theories, postulating different spacetime geometries. However, as I am about to demonstrate, the empirical equivalence isn't as straightforward as it appears, instead, it only stands under certain conditions. Besides, the apparent disagreements on spacetime ontology are not valid and can be resolved under a unifying framework.

3 Empirical equivalence of the trinity

Before delving into the analysis of the ontologies of the trinity, their empirical equivalence must be examined with greater rigor. Two theories in question are considered empirically equivalent if they have the same range of applicability regarding the empirical scenarios, and provide indistinguishable predictions for said scenarios (Wolf et al., 2024). However, the notion remains too general in practice. A widely-accepted sufficient condition for empirical equivalence is the ability to yield identical dynamical equations. For instance, Knox (2011) asserted that the equivalent Lagrangians ensure both GR and TEGR yield Einstein field equations, thereby granting the empirical equivalence between them. This makes sense, as in most cases, the empirical predictions are derived from the solutions to a set of specific dynamical equations. However, the subtleties regarding the disposal of the total derivative mentioned above deserve special attention, and the action of the matter part remains an ongoing topic of discussion.

3.1 Boundary term

Following the formulation in the last section, the trinity are considered equivalent because their actions differ only by a total derivative, or a boundary term after integration by parts. Consequently, when the boundary term vanishes, the variation with respect to the metric field yields identical equations—Einstein field equations (5). According to the said criterion for empirical equivalence, the identical dynamics described by the Einstein field equations underscores the empirical equivalence. Hence, the equivalence critically relies on the ability to throw away the boundary term.

The omission of the boundary term is reasonable when considering a manifold without boundary, for instance, to model cosmological solutions, which represent the entire universe with infinite and boundaryless space. In such a case, the boundary term automatically vanishes.

In the context of a manifold with boundary, the disposal could also be achieved given specific boundary conditions. For example, it's of considerable physical interest to investigate isolated systems where the region far away from the system appears asymptotically flat, thus enabling the imposition of the Dirichlet boundary condition $\delta g|_{\partial\mathcal{M}} = 0$ at spatial infinity. It has been argued that it's only through the notion of isolated system can one integrate GR with other branches of physics and deal individually with subsystems in the universe (Geroch, 1977). However, while TEGR and STEGR seem to work perfectly well under such boundary condition (Oshita and Wu, 2017; Jimenez et al., 2019), the variation of the Hilbert action is problematic due to the involvement of the derivative of the metric on the boundary, hence it should be supplemented with the Gibbons-Hawking-York term to function properly (Gibbons and Hawking, 1977).

As emphasized by Wolf and Read (2023), ensuring empirical equivalence requires not only identical dynamics but also the appropriate boundary condition. Moreover, it remains questionable whether non-equivalence will arise if quantum effects are taken into account, e.g. the Casimir effect (Bahamonde et al., 2023; Hassan et al., 2022). For the time being, I will focus on classical physics, where, assuming the boundary term is properly addressed, the trinity do yield identical empirical predictions.

3.2 Matter coupling

The previous sections primarily focus on the action of the gravity part, with limited discussion on the matter part. It's necessary to devote attention to this part as well, since the dynamical equations encompass not only the Einstein field equations (5), with the energy-momentum of matter on the right-hand side, but also the Euler-Lagrange equations for matter fields, both of which critically depend on how matter fields couple to gravity. In GR, the matter fields couple to gravity through the minimal coupling prescription:

$$\eta_{\mu\nu} \to g_{\mu\nu}, \qquad \partial_\mu \to \mathring{\nabla}_\mu \tag{19}$$

In TEGR, a natural analogy might suggest a coupling prescription that replaces the ordinary derivative with the torsionful Weitzenbock derivative ∇. However, serious problems arise in this context. For instance, the field strength of a photon field is then given by:

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} \;\to\; F_{\mu\nu} = \nabla_{[\mu} A_{\nu]} = \partial_{[\mu} A_{\nu]} + T^{\rho}{}_{\mu\nu} A_\rho \tag{20}$$

The potential directly couples to the torsion, thereby breaking the $U(1)$ symmetry, which is in stark contrast to the case in GR.

Furthermore, since the Levi-Civita connection depends solely on the metric, the only dynamical variable in GR is the metric tensor. Consequently, the variation of matter's action with respect to the independent connection, known as *hypermomentum*, vanishes automatically:

$$\Delta_\rho{}^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{Matt}}}{\delta \Gamma^\rho{}_{\mu\nu}} = 0 \tag{21}$$

Allowing the matter fields to couple to the connection results in a non-vanishing hypermomentum, which serves as a gravitational source just like the energy-momentum tensor does, contributing to the evolution of spacetime. It has been demonstrated that the test bodies with non-vanishing hypermomentum couple to non-Riemannian effects, such as torsion and non-metricity, leading to observable phenomena that deviate from the predictions of GR (Puetzfeld and Obukhov, 2008, 2014; Iosifidis and Hehl, 2023). Conversely, to ensure empirical equivalence, coupling to connection must be prohibited, thereby forcing the hypermomentum to vanish.

When discussing the operationalisability of the torsion tensor, i.e., the ability of the torsion to "be coordinated in a systematic way with an empirical/observable process" (Mulder and Read (2024), p.126), Mulder and Read, citing Hehl (1971), suggest that the torsion is correlated in the equations of motion governing a test particle with spin. However, to achieve this, the spin must directly couple to the contorsion, resulting in a non-vanishing hypermomentum. The coupling inevitably breaks the empirical equivalence, thus they must give up the argument for the underdetermination of GR and TEGR to support their perspective on operationalisability, which, I believe, is not their intention.

For the empirical equivalence to hold, the minimal coupling prescription must be retained in both TEGR and STEGR so as to produce identical energy-momentum tensor and identical evolution of the matter fields. In order to avoid invoking the Levi-Civita connection, the prescription can be reformulated in a neutral manner:

$$\partial_\mu V^\rho \;\to\; \nabla_\mu V^\rho - K^\rho{}_{\mu\nu} \backslash L^\rho{}_{\mu\nu} V^\nu \tag{22}$$

A direct implication of this is that a freely falling particle, instead of following the geodesic[1] determined by ∇, follows equations analogous to Lorentz force law:

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} + \Gamma^\rho{}_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = K^\rho{}_{\mu\nu} \backslash L^\rho{}_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \tag{23}$$

In conclusion, the empirical equivalence of the trinity holds only if the boundary term is appropriately addressed and the minimal coupling prescription is retained. However, this doesn't diminish the significance of the underdetermination, since these conditions can be easily satisfied and are indeed prevalent in most practical scenarios (see, for example, Wolf et al. (2024) for tests of GR along with two other theories). Consequently, realism faces a significant challenge, as multiple theories, while equally supported by empirical data, offer distinct ontological commitments. There seems to be no definite answer to what the geometry of spacetime and the nature of gravity truly are. Nonetheless, the argument hinges on an important premise that the trinity exhibit disagreements over spacetime ontology, which leads us to a careful examination of their ontological commitments in the next section.

4 Spacetime ontology of the trinity

In this section, I aim to determine whether the trinity propose distinct ontology commitments, thereby leading to ontological underdetermination. A general solution is to examine the theoretical equivalence among the trinity, determining whether they are different formulations of a single theory and share the same underlying ontology. A classic example is the two formulations of quantum mechanics—Heisenberg's matrix mechanics and Schrodinger's wave mechanics. Although they use seemingly incompatible mathematical apparatuses, von Neumann proved their mathematical equivalence and unified them under the framework of abstract Hilbert space (von Neumann and BEYER, 2018).

Several criteria exist for theoretical equivalence, including formal approaches such as definitional equivalence and categorical equivalence (Weatherall, 2019a,b). For instance, Barrett (2019) applied categorical equivalence to Lagrangian mechanics and Hamiltonian mechanics, demonstrating their theoretical equivalence by establishing an isomorphism between the categories of models for both theories. While these formal criteria are powerful and precise, they often fall short in terms of interpretation. It's one thing to demonstrate the equivalence between two formal structures, but another, though intertwined, to infer an identical ontology from them. Therefore, in our case, it's more relevant to apply interpretative equivalence. Here is Coffey on interpretive equivalence:

> Two theoretical formulations are theoretically equivalent exactly if they say the

[1]There are two ways to define geodesics. One definition describes them as trajectories minimizing the length, which relies on the metric. The other defines them as the trajectories whose tangent vectors are parallel transported along themselves, related to the affine connection. Hence the latter is sometimes referred to as 'autoparallel curves', but I will keep using 'geodesic' exclusively for this concept in this article.

same thing about what the physical world is like, where that content goes well beyond their observable or empirical claims. Theoretical equivalence is a function of interpretation. It's a relation between completely interpreted formulations. (Coffey (2014), p.834-835)

This directly addresses the interpretive framework of theories, aiming to determine whether they tell the same story about the physical world, thereby aligning with my objective of resolving the ontological underdetermination suggested by the trinity. Therefore, this section will be devoted to discussing several possible interpretations of the trinity, examining their plausibility, and determining whether the trinity share the same interpretation. But before that, let's review the previous work on this topic.

Knox (2011) posits that, just like in GR, the Weitzenbock connection used in TEGR represents the inertial structure of the theory, which determines a set of inertial reference frames. It's generally postulated that in these frames, the connection coefficients vanish. However, no such frame exists in TEGR because of the antisymmetric nature of the connection, leading to a conclusion that the inertial structure is determined by the Levi-Civita connection instead:

> [...]the minimal coupling prescription in TPG [teleparallel gravity] uses the teleparallel covariant derivative (which is simply a recasting of the Levi-Civita covariant derivative), and not the Weitzenbock covariant derivative. As a result, all other fundamental forces take their simplest form relative to the inertial frames picked out by the Levi-Civita connection. Moreover, freely falling bodies still follow geodesics of the metric, which, along with the Levi-Civita connection, is still present in the theory, albeit in disguised form. (Knox (2011), p.273)

Therefore she concludes that despite being expressed in different mathematical formulations, GR and TPG propose the same spacetime ontology. She seems to adopt an interpretive strategy where the structure of spacetime is fully captured by its function, and for Knox, the function is its inertial structure. This metaphysical stance is known as *spacetime functionalism* as Knox (2017) elaborated:

> [...]spacetime is whatever serves to define a structure of inertial frames, where inertial frames are those in whose coordinates the laws governing interactions take a simple form (that is universal insofar as curvature may be ignored), and with respect to which free bodies move with constant velocity. (Knox (2017), p.122)

The validity of this interpretation hinges on this specific philosophical viewpoint that remains controversial (see, e.g., Baker (2020)'s and Menon and Read (2019)'s critiques), which inevitably weakens the overall argument. Moreover, the argument could not be applied to STEGR since there are well-defined inertial frames where the connection coefficients vanish, allowing it to have a different inertial structure. As I will demonstrate, there is no need to appeal to a contentious ontological approach to spacetime in order to break down the underdetermination.

4.1 Extra degrees of freedom

As illustrated by Knox (2011)'s argument mentioned above, the choice of interpretive framework is quite liberal. The most natural approach is perhaps the interpretation that takes the face value of the theory's structure. This interpretation basically asserts that the mathematical structure directly represents the structure of the world, allowing one to straightforwardly read off the ontological commitments from the mathematical framework (Barrett, 2019). In the case of TEGR, a face value interpretation yields:

> **Ontological commitment 1**: Spacetime is a 4-dimensional manifold $\mathcal{M}$, equipped with a Lorentzian metric $g_{\mu\nu}$ and an affine connection identified as the Weitzenbock connection $\Gamma^{\rho}{}_{\mu\nu}$. Consequently, spacetime is flat and torsionful regarding the properties of the affine connection. The effects of gravity are encoded in torsion scalar through the action (12), with matter minimally coupling to gravity.

which is directly derived from the formulation in Section 2.2. One can derive the face value interpretations of GR and STEGR in similar ways. It highlights the intuition that TEGR is distinct from GR since they stipulate different affine connections for spacetime and attribute gravitational effects to different geometric objects.

However, this interpretation encounters significant issues. TEGR is well known for its Lorentz gauge invariance, meaning that the torsion scalar $\mathcal{T}$ is invariant under the local Lorentz transformation of tetrad (Cho, 1976). The symmetry of the action of TEGR passes on to its solutions, so the tetrad field can only be determined up to a local Lorentz transformation. However, as noted in Section 2.2, the flat connection is only invariant under a global linear transformation. Given a solution $\Gamma^{\rho}{}_{\mu\nu}$, a local Lorentz transformation $\theta^a \to \Lambda^a{}_b\theta^b$ leads to another solution of connection:

$$\Gamma^{\rho}{}_{\mu\nu} = e_a{}^{\rho}\partial_\mu\theta^a{}_\nu \;\to\; \tilde{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + e_c{}^{\rho}\theta^b{}_\nu(\Lambda^{-1})^c{}_a\partial_\mu\Lambda^a{}_b \tag{24}$$

Please don't confuse it with the transformation law for connection coefficients. The connection coefficients are still expressed in the same coordinate basis, though it is no longer the original connection. Therefore, the transformation of the tetrad will result in a change in the Weitzenbock connection, thereby altering the ontology of spacetime. Such a move would be unacceptable to both physicists and philosophers.

Similar issues can also be identified in STEGR. The flat and symmetric connection in STEGR is constructed from a set of functions ξ^a, representing the coordinates in which the connection coefficients vanish.

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{\partial x^\rho}{\partial\xi^a}\frac{\partial^2\xi^a}{\partial x^\mu\partial x^\nu} \tag{25}$$

Again, the action of STEGR is invariant under the transformation of ξ^a (Blixt et al., 2024), resulting in a symmetry group of ξ^a identified as a (local) diffeomorphism group. It's important to note that this diffeomorphism invariance pertains to ξ^a instead of the manifold $\mathcal{M}$ itself. However, the connection itself, while interpreted to correctly reflect the affine structure of spacetime, changes under such gauge transformation $\xi^a \to \zeta^a$:

$$\Gamma^{\rho}{}_{\mu\nu} = \frac{\partial x^\rho}{\partial\xi^a}\frac{\partial^2\xi^a}{\partial x^\mu\partial x^\nu} \;\to\; \tilde{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + \frac{\partial x^\rho}{\partial\zeta^a}\frac{\partial^2\zeta^a}{\partial x^\mu\partial\xi^b}\frac{\partial\xi^b}{\partial x^\nu} \tag{26}$$

In summary, the issues in both TEGR and STEGR originate from their extra degrees of freedom (dofs). As stated in Section 2.1, the only dynamical variable in GR is the metric field. Due to the symmetry of the metric tensor, it has $\frac{4\times(4+1)}{2} = 10$ dofs in total. However, for TEGR, while the only dynamical variable is the tetrad, it possesses $4 \times 4 = 16$ dofs. To eliminate the surplus structure, the extra 6 dofs must be removed by imposing the invariance under local Lorentz transformation, which has exactly 6 dofs. For STEGR, the extra 4 dofs introduced by ξ^a are eliminated by imposing diffeomorphism invariance specific to ξ^a (Jimenez et al., 2019). Therefore, all quantities that are not invariant under the gauge transformations, including the connection, the torsion tensor, and the non-metricity tensor, should be considered gauge-dependent. Consequently, they do not warrant ontological commitments. Warning: I am not claiming that the the extra dofs undermine the ontology of TEGR compared to GR as Knox (2011) suggests. Instead, I fully appreciate the significance of gauge freedom, so I merely argue that the presence of extra dofs in TEGR and STEGR challenges the validity of the face value interpretations.

Referring back to Mulder and Read (2024), although they do not explicitly state their interpretation of TEGR, it seems Mulder and Read consider the torsion tensor to be an objective quantity when discussing its operationalisability. They claim that the effects of torsion can be observed, in summary, by the failure of two vectors to form a parallelogram after being transported along each other. The method is correct regarding the definition of the torsion tensor, but the torsion is changeable during tetrad transformation. This leads to a paradoxical situation, where one observer sees two vectors successfully forming a parallelogram, while another observer, using a different tetrad, does not, consequently leading to different results. Moreover, the observation appears purely tautological since it depends on the knowledge of the connection, which can only be specified after the tetrad is fixed. Once the tetrad is fixed, the torsion tensor is entirely determined, eliminating the need for observation.

Furthermore, Mulder and Read claim that making use of identities (2), (9), and the Bianchi identity (Bahamonde et al., 2023):

$$R^{\mu}{}_{\nu\rho\sigma} = \nabla_{[\nu} T^{\mu}{}_{\rho\sigma]} + T^{\mu}{}_{\omega[\nu} T^{\omega}{}_{\rho\sigma]} \tag{27}$$

one could establish the relation between the curvature tensor associated with the Levi-Civita connection and the torsion tensor[2], by which “the gravitational gradiometer [able to read out curvature components] will also be capable of reading out torsion components and their derivatives” (Mulder and Read (2024), p.126). Nonetheless, this cannot succeed. The curvature tensor associated with the Levi-Civita connection is obviously invariant under tetrad transformation, whereas the desired outcomes, including torsion components and their derivatives, are not. Consequently, there is no way to derive the components of torsion by merely measuring curvature. Ultimately, their argument for the operationalisability of the torsion collapses.

So far I have demonstrated that it’s not plausible to interpret TEGR and STEGR at face value. However, undermining the ontological status of certain aspects of these theories doesn’t make them more similar to GR. Although the connection is gauge-dependent, the

[2]It seems that they make a mistake on this. There is a more direct way to establish such relation, merely making use of the definitions of curvature and contorsion.

properties shared by all possible connections used to model a physical state in a theory are, nevertheless, gauge-invariant. In this regard, it's still plausible to commit to specific features of them, such as the flatness and torsionfulness in TEGR, though not all properties. Moreover, since a symmetric, metric-compatible, and flat connection can exist only in Minkowski space[3], the connection adopted by any two of the trinity would not coincide in general cases. Consequently, the geometric structures of the trinity are distinct in the sense that, given a physical state, the sets of all possible connections used by the models of the trinity are so different that they don't even intersect, thereby leading to the postulations of distinct invariant features of spacetime.

How can the proponents of the underdetermination learn from this section? Well, they can acknowledge that to avoid the issues raised by gauge transformation, the ontological commitments should be limited to those invariant structures, such as the curvature tensor, the torsion scalar, and the non-metricity scalar. A more reasonable interpretation will treat the connection as a gauge choice without physical significance, while the changeable torsion tensor still yields a non-vanishing torsion scalar and spacetime remains flat regarding the Weitzenbock connection, which is nevertheless substantially distinct from the ontology of GR. Therefore, the ontological commitment of TEGR can be revised:

> **Ontological commitment 2**: Spacetime is a 4-dimensional manifold $\mathcal{M}$, equipped with a Lorentzian metric $g_{\mu\nu}$. The affine connection of $\mathcal{M}$, however, is a gauge choice under the local Lorentz transformation of tetrad. Spacetime is flat and torsionful in the sense that the curvature tensor vanishes while the torsion scalar is non-vanishing. The effects of gravity are encoded in the torsion scalar through the action (12), with matter minimally coupling to gravity.

To resolve the underdetermination presented by the new interpretation, an interpretational bridge is needed to connect the seemingly distinct geometries of the trinity, which isn't easier than it was at the beginning of this section. In the next section, I will propose a new theoretical framework allowing us to interpret the trinity as different perspectives of the same ontology, thereby breaking the underdetermination.

4.2 Towards unification

As stated before, the trinity all agree on the metric but disagree on the affine connection, leading to different geometries of spacetime. It appears that to unify the trinity, it's necessary to explain why they can impose distinct affine structures of spacetime while yielding identical empirical predictions. The previous section has demonstrated that even within the same theory, the choice of connections is not unique but rather a matter of gauge freedom. Hence, I wonder whether it's possible to expand the gauge freedom to further encompass all the possible connections adopted by the trinity such that the disagreements on the connection can be interpreted as gauge choices without physical significance.

Inspired by the metric-affine gravitation theory advocated by Hehl et al. (1995), I find a way to achieve the desired effects. I start with metric-affine geometry, which is composed of three fundamental objects: a manifold $\mathcal{M}$, a metric structure $g_{\mu\nu}$ and an affine connection

[3] To be precise, a space locally isomorphic to Minkowski space.

$\Gamma^{\rho}{}_{\mu\nu}$. Unlike the geometries adopted by the trinity, there is no *a priori* requirement on the connection, especially no relation between the metric and the connection, hence the name 'metric-affine'. Nonetheless, the deviation of the connection from the Levi-Civita one is quantified by the *distorsion* tensor $N^{\rho}{}_{\mu\nu}$, which is, surprisingly, identified as the sum of contorsion and disformation as defined in (9) and (15):

$$\Gamma^{\rho}{}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\mu\nu} + N^{\rho}{}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu} + L^{\rho}{}_{\mu\nu} \tag{28}$$

Using the decomposition and the definition of curvature tensor (2), the Ricci scalar can be split (Bahamonde et al., 2023):

$$\mathring{\mathcal{R}} = \mathcal{R} + \mathcal{T} + \mathcal{Q} + \mathcal{C} + \mathring{\nabla}_{\mu}(2T_{\nu}{}^{\nu\mu} + Q_{\nu}{}^{\nu\mu} - Q^{\mu\nu}{}_{\nu}) \tag{29}$$

where $\mathcal{T}$ and $\mathcal{Q}$ are the torsion scalar and the non-metricity scalar respectively, and $\mathcal{C}$ is a scalar describing the coupling of torsion and non-metricity:

$$\mathcal{C} = -Q_{\mu\nu\rho}T^{\rho\mu\nu} + Q_{\mu\alpha}{}^{\alpha}T^{\beta\mu}{}_{\beta} - Q^{\alpha}{}_{\alpha\mu}T^{\beta\mu}{}_{\beta} \tag{30}$$

Let $\mathcal{G} = \mathcal{R}+\mathcal{T}+\mathcal{Q}+\mathcal{C}$, I construct the action of *metric-affine equivalent of general relativity* (MAEGR):

$$S_{\mathrm{MAEGR}} = \frac{1}{2\kappa^2}\int \mathcal{G}(g,\Gamma)\sqrt{-g}\ \mathrm{d}^4x \tag{31}$$

which differs from the Hilbert action by a total derivative. Assuming the boundary condition discussed in Section 3 is satisfied, and the minimal coupling is retained:

$$\partial_{\mu}V^{\rho} \ \rightarrow \ \nabla_{\mu}V^{\rho} - N^{\rho}{}_{\mu\nu}V^{\nu} \tag{32}$$

MAEGR is empirically equivalent to GR.

The variation of S_{MAEGR} with respect to the metric is, without doubt, the same as in GR, yielding the Einstein tensor as defined in (4). In a similar fashion, although it seems that the independent connection is necessarily involved in the computation of $\mathcal{G}$, the variation of S_{MAEGR} with respect to the connection vanishes due to its equivalence to the Hilbert action:

$$\frac{\delta S_{\mathrm{MAEGR}}}{\delta\Gamma^{\rho}{}_{\mu\nu}} = \frac{\delta S_{\mathrm{GR}}}{\delta\Gamma^{\rho}{}_{\mu\nu}} = 0 \tag{33}$$

Therefore, the dynamical equations do not specify connection, leaving the choice of affine structure entirely arbitrary and allowing us to attain the maximum gauge freedom of connection. The trinity emerge as different gauge-fixed versions of MAEGR. As an example, I will demonstrate how to derive TEGR from MAEGR.

Firstly, enforce the vanishing of curvature by introducing a Lagrangian multiplier $\lambda_{\rho}{}^{\sigma\mu\nu}$ in the action:

$$S_{\parallel} = \int (\frac{1}{2\kappa^2}\mathcal{G}(g,\Gamma)\sqrt{-g} + \lambda_{\rho}{}^{\sigma\mu\nu}R^{\rho}{}_{\sigma\mu\nu})\mathrm{d}^4x \tag{34}$$

Instead of directly deriving the variation, we start by solving the constraint. The vanishing curvature forces the connection to be parametrized by a tetrad as in (6). By enforcing the constraint, we fix the connection to be flat, resulting in a flat geometry. Consequently, the action is expressed by:

$$S_{\mathrm{GTEGR}} = \frac{1}{2\kappa^2}\int(\not{\mathcal{R}}+\mathcal{T}+\mathcal{Q}+\mathcal{C})\sqrt{-g}\,\mathrm{d}^4x = \frac{1}{2\kappa^2}\int(\mathcal{T}(g,e)+\mathcal{Q}(g,e)+\mathcal{C}(g,e))\sqrt{-g}\,\mathrm{d}^4x \tag{35}$$

which is the action of a theory known as the *general teleparallel equivalent of general relativity* (Beltrán Jiménez et al., 2020).

To recover TEGR, we further enforce the vanishing of non-metricity:

$$S_{\mathrm{TEGR}'} = \int(\frac{1}{2\kappa^2}\mathcal{G}(g,\Gamma)\sqrt{-g} + \lambda_{\rho}{}^{\sigma\mu\nu}R^{\rho}{}_{\sigma\mu\nu} + \mu^{\alpha\mu\nu}Q_{\alpha\mu\nu})\mathrm{d}^4x \tag{36}$$

Solve the non-metricity condition, which further restricts the tetrad to take the form in (8), resulting in the action of TEGR:

$$S_{\mathrm{TEGR}} = \frac{1}{2\kappa^2}\int(\not{\mathcal{R}}+\mathcal{T}+\not{\mathcal{Q}}+\not{\mathcal{C}})\sqrt{-g}\,\mathrm{d}^4x = \frac{1}{2\kappa^2}\int\mathcal{T}(e)\sqrt{-g}\,\mathrm{d}^4x \tag{37}$$

Similarly, the actions of GR and STEGR can be derived by enforcing the appropriate constraints. This demonstrates that MAEGR is an example of what Le Bihan and Read (2018) call *overarching theories*. The overarching theory approach embeds the empirical equivalent theories into a new framework, allowing us to interpret these theories as different perspectives of an underlying ontology. By admitting larger degrees of freedom, MAEGR successfully incorporates the trinity into a single framework. Therefore, the trinity emerge as the specializations of MAEGR to more restricted geometries.

However, as Le Bihan and Read (2018) point out, merely embedding the solution space of the empirically equivalent theories into that of a "deeper" theory can not resolve the underdetermination, an important property of MAEGR should be acknowledged: the solutions of the trinity do not exhaust the solutions of MAEGR, instead, they are merely a small portion of the latter. Beyond the trinity, there are infinitely many equivalent versions of GR, all of which can be derived by imposing constraints on the connection. They adopt different geometries resulting from the various choices of affine connections and attribute gravitational effects to different geometric objects. The trinity are not special in this regard, as their geometries might merely be the most natural choices given our current understanding of the affine structure.

Consequently, the revised ontological commitment of TEGR simply collapses: if there are infinitely many equivalent but distinct theories like TEGR, why believe that TEGR correctly describes the flat and torsionful nature of the physical world? Here, the logic of underdetermination helps us fight against itself. The underdetermination of the trinity is correct in that the we can never know whether spacetime is curved or flat, torsionful or symmetric, since the presence of other equivalent theories undermines the legitimacy of the question "is spacetime curved?" as stated in the title of Mulder and Read (2024). According to this view, the answer depends on a choice of gauge, reflecting no objective feature of the world. However, this does not challenge scientific realism, because MAEGR unifies the trinity and nevertheless provides us with substantial knowledge of spacetime:

Ontological commitments 3: Spacetime is a 4-dimensional manifold $\mathcal{M}$ equipped with a Lorentzian metric $g_{\mu\nu}$. The affine connection of $\mathcal{M}$ is a pure gauge choice and this gauge freedom passes on to curvature, torsion, and non-metricity. The effects of gravity are encoded in the combined contributions of curvature, torsion, and non-metricity through the action (31), with matter minimally coupling to gravity.

This replaces the revised ontological commitment of TEGR and becomes the shared interpretation of the trinity, where the underdetermination collapses.

The unification also explains why it's possible to use seemingly different geometries to describe spacetime while maintaining empirical equivalence. Firstly, although the connection is involved in the computation of the action, its contributions accidentally cancel out, leading to a vanishing variation with respect to the connection. Of course, it wouldn't be surprising if one acknowledges the construction process of the action, which is deliberately manipulated to differ from the Hilbert action by a total derivative. Hence, the action doesn't specify the connection other than the explicit constraints imposed by the Lagrangian multipliers. Consequently, one can adopt different connections—whether it is flat or curved, symmetric or torsionful—since all these connections are valid solutions of the theory.

Secondly, in the matter part, the minimal coupling prescription is imposed such that a freely falling particle with 4-velocity u^ρ follows the rule:

$$\nabla u^\rho = N^\rho{}_{\mu\nu} u^\mu u^\nu \tag{38}$$

which resembles the Lorentz force law, identically matching the effects of the Levi-Civita connection. If we consider inertial motion to be the motion following one of the geodesics defined by the connection, the choice of connection appears to represent a choice of how gravity and inertia are split. All the connections on the manifold comprehensively represent all consistent ways to separate gravity and inertia in spacetime, both of which are affected by the configuration of matter as described by the field equations. In this regard, GR is unique compared to other possible equivalents, as the distorsion tensor identically vanishes only in GR, representing the unification of gravity and inertia. However, I must admit that the conclusion I draw here regarding the gravity/inertia split is too premature and far from definitive, serving at best as a hint for future pursuits.

5 Conclusion

Problems of underdetermination might not undermine scientific realism, but they do shake the validity of the common belief that realism about a theory is *prima facie*. As demonstrated in the article, the interpretation of a theory necessitates careful inspection, which is not easier than the discovery and the development of the theory itself. The inspection conducted in this article is based on a straightforward criterion: the invariance of physical structure. A similar idea can be found in North (2009)'s analysis of the structure of physics. According to North, the structure of the world comprises those objects least needed to state the theory in an invariant way under allowable transformation. From the perspective of MAEGR, the choice of connection appears gauge-variant and superfluous in the description of gravity.

Consequently, connection, along with the geometric objects pertaining to it such as curvature and torsion, should not be considered fundamental and objective features of the world. Therefore, the underdetermination of the trinity is broken, highlighting that the apparent disagreements are non-physical and that the underlying physical structures remain invariant regardless of the chosen formalism.

However, more research could be done on this topic. For instance, it remains perplexing why curvature, torsion and non-metricity, despite differing in their mathematical origins, could describe identical gravitation effects. Future pursuit based on MAEGR could be devoted to unifying various geometric objects in a more intuitive way, potentially prompting fresh insights into the nature of gravity and spacetime. Moreover, regarding the mysterious freedom of the gravity/inertia split, the interpretation of MAEGR is still incomplete, pending further exploration. It is also of great interest to perform a similar construction in non-relativistic gravity, potentially shedding new light on the debates surrounding Newton-Cartan theory. A quick conjecture is that Maxwell gravitation, as discussed in March et al. (2024), can be identified as the non-relativistic limit of MAEGR. I plan to elaborate on this in another paper.

References

Aldrovandi, R. and Pereira, J. G. (2013). *Fundamentals of Teleparallel Gravity*, pages 39–50. Springer Netherlands, Dordrecht.

Bahamonde, S., Dialektopoulos, K. F., Escamilla-Rivera, C., Farrugia, G., Gakis, V., Hendry, M., Hohmann, M., Levi Said, J., Mifsud, J., and Di Valentino, E. (2023). Teleparallel gravity: from theory to cosmology. *Reports on Progress in Physics*, 86(2):026901.

Baker, D. J. (2020). Knox's inertial spacetime functionalism. *Synthese*, 199(S2):277–298.

Barrett, T. W. (2019). Equivalent and inequivalent formulations of classical mechanics. *British Journal for the Philosophy of Science*, 70(4):1167–1199.

Beltrán Jiménez, J., Heisenberg, L., Iosifidis, D., Jiménez-Cano, A., and Koivisto, T. S. (2020). General teleparallel quadratic gravity. *Physics Letters B*, 805:135422.

Blixt, D., Golovnev, A., Guzman, M.-J., and Maksyutov, R. (2024). Geometry and covariance of symmetric teleparallel theories of gravity. *Physical Review D*, 109(4).

Cho, Y. M. (1976). Einstein lagrangian as the translational yang-mills lagrangian. *Phys. Rev. D*, 14:2521–2525.

Coffey, K. (2014). Theoretical equivalence as interpretative equivalence. *British Journal for the Philosophy of Science*, 65(4):821–844.

Geroch, R. (1977). *Asymptotic Structure of Space-Time*, pages 1–105. Springer US, Boston, MA.

Gibbons, G. W. and Hawking, S. W. (1977). Action integrals and partition functions in quantum gravity. *Phys. Rev. D*, 15:2752–2756.

Hassan, Z., Ghosh, S., Sahoo, P. K., and Bamba, K. (2022). Casimir wormholes in modified symmetric teleparallel gravity. *The European Physical Journal C*, 82(12).

Hehl, F. (1971). How does one measure torsion of space-time? *Physics Letters A*, 36(3):225–226.

Hehl, F. W., McCrea, J., Mielke, E. W., and Ne'eman, Y. (1995). Metric-affine gauge theory of gravity: field equations, noether identities, world spinors, and breaking of dilation invariance. *Physics Reports*, 258(1):1–171.

Iosifidis, D. and Hehl, F. W. (2023). Motion of test particles in spacetimes with torsion and nonmetricity.

Jimenez, J. B., Heisenberg, L., and Koivisto, T. S. (2019). The geometrical trinity of gravity.

Jiménez Cano, A. (2021). *Metric-affine Gauge theories of gravity. Foundations and new insights*. PhD thesis, Granada U., Theor. Phys. Astrophys.

Knox, E. (2011). Newton-cartan theory and teleparallel gravity: The force of a formulation. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 42(4):264–275.

Knox, E. (2017). Physical relativity from a functionalist perspective. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 67:118–124.

Le Bihan, B. and Read, J. (2018). Duality and ontology. *Philosophy Compass*, 13(12):e12555. e12555 10.1111/phc3.12555.

March, E., Wolf, W. J., and Read, J. (2024). On the geometric trinity of gravity, non-relativistic limits, and maxwell gravitation. *Philosophy of Physics*.

Menon, T. and Read, J. (2019). The limitations of intertial frame spacetime functionalism. *Synthese*, 1(Suppl 2):229–251.

Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*. W. H. Freeman, San Francisco.

Mulder, R. and Read, J. (2024). Is spacetime curved? assessing the underdetermination of general relativity and teleparallel gravity. *Synthese*, 204(4):1–29.

Nester, J. M. and Yo, H.-J. (1999). Symmetric teleparallel general relativity.

North, J. (2009). The "structure" of physics. *Journal of Philosophy*, 106(2):57–88.

Oshita, N. and Wu, Y.-P. (2017). Role of spacetime boundaries in a vierbein formulation of gravity. *Physical Review D*, 96(4).

Puetzfeld, D. and Obukhov, Y. N. (2008). Probing non-riemannian spacetime geometry. *Physics Letters A*, 372(45):6711–6716.

Puetzfeld, D. and Obukhov, Y. N. (2014). Equations of motion in metric-affine gravity: A covariant unified framework. *Physical Review D*, 90(8).

Sauer, T. (2006). Field equations in teleparallel space–time: Einstein's fernparallelismus approach toward unified field theory. *Historia Mathematica*, 33(4):399–439.

von Neumann, J. and BEYER, R. T. (2018). *Mathematical Foundations of Quantum Mechanics: New Edition*. Princeton University Press, ned - new edition edition.

Weatherall, J. O. (2019a). Part 1: Theoretical equivalence in physics. *Philosophy Compass*, 14(5):e12592. e12592 10.1111/phc3.12592.

Weatherall, J. O. (2019b). Part 2: Theoretical equivalence in physics. *Philosophy Compass*, 14(5):e12591. e12591 10.1111/phc3.12591.

Wolf, W. J. and Read, J. (2023). Respecting boundaries: Theoretical equivalence and structure beyond dynamics. *European Journal for Philosophy of Science*, 13(4):1–28.

Wolf, W. J., Sanchioni, M., and Read, J. (2024). Underdetermination in classic and modern tests of general relativity. *European Journal for Philosophy of Science*, 14(4):1–41.