Continuity for the Maximal Bochner-Riesz operators on the weighted Weak Hardy spaces

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Abstact: In this papers ,we generalize some results of other authors to weighted spaces and gain the boundedness of maximal Bochner-Riesz operator on weighted Herz-Hardy spaces,weighted Hardy spaces and weighted weak Hardy spaces ,where $\omega \in A_1$.

Key words: Weighted Herz-Hardy spaces, Maximal Bochner-Riesz operator,Weighted weak Hardy space

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1 Introduction

Jiang Yinsheng and other authors [1,2] discuss the boundednes of maximal Bochner-Riesz and it$'s$ commutator .Recentily ,Tongseng Quek [3]proved some C-Z type operators are bounded on weighted Hardy spaces and weighted weak Hary spaces . Inspiring by the above and the paper[4] ,we will prove the boundedness of the maximal Bochner-Riesz operator on these spaces.

Let maximal Bochner-Riesz operator B_δ^* be

$$T_\delta^* f(x) = \sup_{t>0} |B_\delta^t(f)(x)|,$$

where$B_\delta^t(x) = t^{-n}B_\delta(x/t)$ is the kernel of T_δ^t, and satisfies $|\frac{\partial^\beta}{\partial x^\beta}B_\delta(x)| \leq C(1+|x|)^{(\delta+(n+1)/2)}$ for any $x \in \mathbb{R}^n$ and multi-index $\beta \in \mathbb{Z}_+^n$.

Suppose $0 < p < \infty$, we denote the weak $L_\omega^p(\mathbb{R}^n)$ by $WL_\omega^p(\mathbb{R}^n)$ and set

$$\|f\|_{WL_\omega^p(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda[\omega(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})]^{1/p},$$

where, and in what follows, $\omega(E) = \int_E \omega(x)dx$.

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2 $[\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$,$H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)]$-Type Continuity.

Let us first recall the definition of Herz spaces .For $k\in\mathbb{Z}$, let $B_k=\{x\in\mathbb{R}^n:|x|\leq 2^k\}$,$C_k=B_k\setminus B_{k-1}and\chi_k=\chi_{C_k}$.

Definition 1[4] Let $0\leq\alpha<\infty$, $1<q<\infty$,$0<p<\infty$, and ω_1,ω_2 be non-negative weight function. The homogeneous weighted Herz spaces $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ are defined by

$$\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)=\{f\in L_{loc}^q(\mathbb{R}^n\setminus\{0\},\omega_2):\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}<\infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}=\{\sum_{k=-\infty}^{\infty}\omega_1(B_k)^{\frac{\alpha p}{n}}\|f\chi_k\|_{L_{\omega_2}^q}^p\}^{1/p}.$$

Definition 2[4] Let $\alpha\in\mathbb{R}^n$, $1<q<\infty$,$0<p<\infty$ and ω_1,ω_2 be non-negative weight function. The homogeneous weighted Herz-Hardy spaces are defined by $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)=\{f\in S':G(f)\in\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)\}$ and

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}=\|G(f)\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)},$$

where $G(f)$ is usually called the grand maximal function of f.

Definition 3[4] Let $\omega_1,\omega_2\in A_1$, $n(1-\frac{1}{q})\leq\alpha<\infty$ and $s\geq[\alpha+n(\frac{1}{q}-1)]$ be non-negative integral. The function $a(x)$ is called a center atom of $(\alpha,q,\omega_1,\omega_2)-type$,if $a(x)$ satisfies
(1) $supp\ a_k\subset B_k=B(0,2^kr)$,
(2) $\|a_k\|_{L_{\omega_2}^q}\leq\omega_1(B_k)^{\frac{-\alpha}{n}}$, (3)$\int a(x)x^\beta dx=0,|\beta|\leq s$.

Theorem 1 Let $\delta>\frac{n-1}{2}$, $0<p<\infty$, $1<q<\infty$, and $n(1-\frac{1}{q})\leq\alpha<n(1-\frac{1}{q})+\varepsilon$, $\varepsilon=\min\{1,(\delta-\frac{n-1}{2})\}$, $\omega_1,\omega_2\in A_1$,Then T_δ^* is a bounded operator from $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ to $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$.

Proof of Theorem 1 Since $\omega_1,\omega_2\in A_1(\mathbb{R}^n)$, a temperate distribution of f can be written as $f=\sum_{-\infty}^{\infty}\lambda_ja_j$ and $(\sum_{-\infty}^{\infty}|\lambda_j|^p)^{\frac{1}{p}}\leq C\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}$, where a_j is the central atom of $(\alpha,q,\omega_1,\omega_2)-type$. Let $n(1-\frac{1}{q})\leq\alpha<n(1-\frac{1}{q})+\varepsilon$ and $f\in HK_q^{\alpha,p}(\omega_1,\omega_2)$. Then we have

$$\|T_\delta^t(f)\|_{HK_q^{\alpha,p}(\omega_1,\omega_2)}=\{\sum_{k=-\infty}^{\infty}\omega_1(B_k)^{\frac{\alpha p}{n}}\|T_\delta^t(f)\chi_k\|_{L_{\omega_2}^q}^p\}^{\frac{1}{p}}$$

$$\leq C\{\sum_{k=-\infty}^{\infty}\omega_1(B_k)^{\frac{\alpha p}{n}}(\sum_{j=-\infty}^{k-3}|\lambda_j|\|T_\delta^t(a_j)\chi_k\|_{L_{\omega_2}^q})^p\}^{\frac{1}{p}}$$

$$+C\{\sum_{k=-\infty}^{\infty}\omega_1(B_k)^{\frac{\alpha p}{n}}(\sum_{j=k-2}^{\infty}|\lambda_j|\|T_\delta^t(a_j)\chi_k\|_{L_{\omega_2}^q})^p\}^{\frac{1}{p}}=D_1+D_2.$$

For D_2, when $0 < p \le 1$. We notice $\frac{\omega(E)}{\omega(B)} \le C(\frac{|E|}{|B|})^{\xi}$ as $E \subset B$ and $0 < \xi < 1$. By the boundedness of operator T^*_δ on $L^q_{\omega_2}(\mathbb{R}^n)$[5] , we have

$$
\begin{aligned}
D_2 &\le \{\sum_{k=-\infty}^{\infty} \omega_1(B_k)^{\frac{\alpha p}{n}}(\sum_{j=k-2}^{\infty}|\lambda_j|^p\|T^t_\delta(a_j)\chi_k\|^p_{L^q_{\omega_2}})\}^{\frac{1}{p}} \\
&\le C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p[\sum_{k=\infty}^{j-1}(\frac{\omega_1(B_k)}{\omega_1(B_j)})^{\frac{\alpha p}{n}} + 1 + (\frac{\omega_1(B_{j+1})}{\omega_1(B_j)})^{\frac{\alpha p}{n}} + (\frac{\omega_1(B_{j+2})}{\omega_1(B_j)})^{\frac{\alpha p}{n}}]\}^{\frac{1}{p}} \\
&\le C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p[\sum_{k=\infty}^{j-1}(\frac{|B_k|}{|B_j|})^{\frac{\alpha p\xi}{n}} + 1 + 2^{\alpha p} + 2^{2\alpha p}]\}^{\frac{1}{p}} \le C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\}^{\frac{1}{p}}..
\end{aligned}
$$

When $p > 1$, by $H\ddot{o}lder's$ inequality , we have

$$
\begin{aligned}
&D_2 \le \{\sum_{k=-\infty}^{\infty}\omega_1(B_k)^{\frac{\alpha p}{n}}[\sum_{j=k-2}^{\infty}|\lambda_j|\omega_1(B_j)^{\frac{-\alpha}{n}}]^p\}^{1/p} \\
&\le \{\sum_{k=-\infty}^{\infty}(\sum_{j=k-2}^{\infty}|\lambda_j|^p\omega_1(B_j)^{\frac{-\alpha p}{2n}}\omega_1(B_k)^{\frac{\alpha p}{2n}})(\sum_{j=k-2}^{\infty}\omega_1(B_j)^{\frac{-\alpha p'}{2n}}\omega_1(B_k)^{\frac{\alpha p'}{2n}})^{\frac{p}{p'}}\}^{\frac{1}{p}} \\
&\le \{\sum_{k=-\infty}^{\infty}[\sum_{j=k-2}^{\infty}|\lambda_j|^p(\frac{\omega_1(B_k)}{\omega_1(B_j)})^{\frac{\alpha p}{2n}}[\sum_{j=k+1}^{\infty}2^{\frac{(k-j)\alpha p'\xi}{2}} + 1 + 2^{\frac{\alpha p'}{2}} + 2^{\alpha p'}]^{\frac{p}{p'}}\}^{\frac{1}{p}} \\
&\le \{\sum_{j=-\infty}^{\infty}|\lambda_j|^p(\sum_{k=-\infty}^{j+2}(\frac{\omega_1(B_k)}{\omega_1(B_j)})^{\frac{\alpha p}{2n}})\}^{\frac{1}{p}} \le \{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+2}2^{\frac{(k-j)\alpha p\xi}{2}}\}^{\frac{1}{p}} \le C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\}^{\frac{1}{p}}
\end{aligned}
$$

Now,let us turn to the estimate for D_1. For any j respect to a fixed k satisfied $j \le k-3$,let $B_j = B(0, 2^j r) = B(0, r_1)$, $B_k = B(0, 2^{k-j}r_1)$. As $x \in B_k, y \in B_j$, we know $|x-y| \sim |x-0|$ .We will consider two cases for D_1.

case 1 $t < r_1$. By $H\ddot{o}lder's$ inequality and the condition of core B^t_δ, we have

$$
\begin{aligned}
&\{\int_{B_k}|\int_{B_j}B^t_\delta(x-y)a_j(y)dy|^q\omega_2(x)dx\}^{\frac{1}{q}} \\
&\le C\{\int_{B_k}|t^{-n}\int_{B_j}(1+\frac{|x-y|}{t})^{-(\delta+\frac{n+1}{2})}a_j(y)dy|^q\omega_2(x)dx\}^{\frac{1}{q}} \\
&\le Ct^{\delta-\frac{n-1}{2}}(2^{(k-j)}r_1)^{-(\delta+\frac{n+1}{2})}|B_j|(\frac{\omega_2(B_k)}{\omega_2(B_j)})^{\frac{1}{q}}(\int_{B_j}|a_j(y)|^q\omega_2(y)dy)^{\frac{1}{q}} \\
&\le C2^{(j-k)(\delta+\frac{n+1}{2}-\frac{n}{q})}\|a_j\|_{L^q_{\omega_2}} \le C2^{(j-k)(\delta+\frac{n+1}{2}-\frac{n}{q})}\omega_1(B_j)^{-\frac{\alpha}{n}}.
\end{aligned}
$$

When $0<p\leq 1$,we gain

$$D_1\leq C\{\sum_{k=-\infty}^{\infty}(\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{(j-k)p(\delta+\frac{n+1}{2}-\frac{n}{q})}(\frac{\omega_1(B_k)}{\omega_1(B_j)})^{\frac{\alpha p}{n}})\}^{\frac{1}{p}}$$
$$\leq C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p(\sum_{k=j+3}^{\infty}2^{(j-k)p(\delta+\frac{n+1}{2}-\frac{n}{q}-\alpha)}\}^{\frac{1}{p}}\leq C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\}^{\frac{1}{p}}.$$

When $p>1$ .By $H\ddot{o}lder^{'}s$ inequality , we have

$$D_1\leq C\{\sum_{j=-\infty}^{\infty}(\sum_{k=j+3}^{\infty}|\lambda_j|^p2^{(j-k)\frac{p}{2}(\delta+\frac{n+1}{2}-\frac{n}{q}-\alpha)})(\sum_{k=j+3}^{\infty}2^{(j-k)\frac{p^{'}}{2}(\delta+\frac{n+1}{2}-\frac{n}{q}-\alpha)})^{\frac{p}{p^{'}}}\}^{\frac{1}{p}}$$
$$\leq C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p(\sum_{k=j+3}^{\infty}2^{(j-k)\frac{p}{2}(\delta+\frac{n+1}{2}-\frac{n}{q}-\alpha)})\}^{\frac{1}{p}}\leq C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\}^{\frac{1}{p}}.$$

case 2 $t\geq r_1$. By the mean value theorem and the vanishing moment condition of atom a_j , we have

$$(\int_{B_k}|\int_{B_j}B_\delta^t(x-y)a_j(y)dy|^q\omega_2(x)dx)^{\frac{1}{q}}$$
$$=(\int_{B_k}|\int_{B_j}[B_\delta^t(x-y)-B_\delta^t(x-0)]a_j(y)dy|^q\omega_2(x)dx)^{\frac{1}{q}}$$
$$\leq C\{\int_{B_k}|t^{-(n+1)}\int_{B_j}(1+\frac{|x-0|}{t})^{-(\delta+\frac{n+1}{2})}ya_j(y)dy|^q\omega_2(x)dx\}^{\frac{1}{q}}$$

(1) When $\delta<\frac{n+1}{2}$.

$$(\int_{B_k}|\int_{B_j}B_\delta^t(x-y)a_j(y)dy|^q\omega_2(x)dx)^{\frac{1}{q}}$$
$$\leq C(\frac{r_1}{t})^{\frac{n+1}{2}-\delta}2^{(j-k)(\delta+\frac{n+1}{2})}[\frac{\omega_2(B_k)}{\omega_2(B_j)}]^{\frac{1}{q}}\|a_j\|_{L^q_{\omega_2}}$$
$$\leq C2^{(j-k)(\delta+\frac{n+1}{2}-\frac{n}{q})}\omega_1(B_j)^{-\frac{\alpha}{n}}.$$

When $0<p\leq 1$,we obtain

$$D_1\leq C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p(\sum_{k=j+3}^{\infty}2^{(j-k)p(\delta+\frac{n+1}{2}-\frac{n}{q}-\alpha)}\}^{\frac{1}{p}}\leq C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\}^{\frac{1}{p}}.$$

When $p>1$. By $H\ddot{o}lder^{'}s$ inequality , we also have

$$D_1\leq C\{\sum_{k=-\infty}^{\infty}(\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{(j-k)\frac{p}{2}(\delta+\frac{n+1}{2}-\frac{n}{q})}(\frac{\omega_1(B_k)}{\omega_1(B_j)})^{\frac{\alpha p}{2n}})\}^{\frac{1}{p}}\leq C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\}^{\frac{1}{p}}.$$

(2) When $\delta \geq \frac{n+1}{2}$. By the mean value theorem and the vanishing moment condition of atom a_j , we have

$$(\int_{B_k} |\int_{B_j} B^t_\delta(x-y)a_j(y)dy|^q \omega_2(x)dx)^{\frac{1}{q}} \leq C2^{(j-k)(n+1-\frac{n}{q})}\omega_1(B_j)^{-\frac{\alpha}{n}}$$

As the above, we consider D_1 under the condition of $0 < p \leq 1$ at first.We obtian

$$D_1 \leq C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p(\sum_{k=j+3}^{\infty}2^{(j-k)p(n+1-\frac{n}{q}-\alpha)}\}^{\frac{1}{p}} \leq C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\}^{\frac{1}{p}}.$$

Also, we secondly consider D_1 under the condition of $p > 1$.By $H\ddot{o}lder's$ inequality , we have

$$D_1 \leq C\{\sum_{k=-\infty}^{\infty}(\sum_{j=-\infty}^{k-3}|\lambda_j|^p 2^{(j-k)\frac{p}{2}(n+1-\frac{n}{q}-\alpha)}\}^{\frac{1}{p}} \leq C\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\}^{\frac{1}{p}}.$$

Thus,we have

$$\|T^t_\delta(f)\|_{\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)} \leq C\|T^t_\delta(f)\|_{H\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)}.$$

We take the supreme of the left side respect to the inequality above for any $t > 0$, and obtain desirable result.

3 $[L^p_\omega(\mathbb{R}^n)$,$H^p_\omega(\mathbb{R}^n)]$-Type Continuity

Definition 4 Let $\omega_\infty(\mathbb{R}^n)$ and $p \in (0,1]$. The weighted Hardy spaces $H^p_\omega(\mathbb{R}^n)$ is defined by

$$H^p_\omega(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : \phi_t * (f)(x) = \sup_{t>0}|\phi_t * f(x)| \in L^p_\omega(\mathbb{R}^n)\},$$

where $\phi \in S(\mathbb{R}^n)$ is a fixed function with $\int_{\mathbb{R}^n}\phi(x)dx \neq 0$ and $\phi_t = t^{-n}\phi(y/t)$ for any $t > 0$. Moreover ,we define$\|f\|_{H^p_\omega(\mathbb{R}^n)} = \|\phi * (f)\|_{L^p_\omega(\mathbb{R}^n)}$.

Definition 5 Let $\omega \in A_\infty$, $p \in (0,1]$. A p-atom with respect to ω is a function a supported in a ball B such that

$$\|a\|_{L^\infty(\mathbb{R}^n)} \leq \omega(B)^{-\frac{1}{p}}$$

$\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for every multi-index α with $|\alpha| \leq [n(q_\omega/p-1)]$, where ,and in what follows, [s] denotes the greatest integer less than or equal to s.

Theorem 2 Let $\omega \in A_1$, $\delta > \frac{n-1}{2}$, $\min\{\frac{n}{\delta+(n+1)/2}, \frac{n}{n+1}\} < p \leq 1$, Then T^*_δ is a bounded map from $H^p_\omega(\mathbb{R}^n)$ into $L^p_\omega(\mathbb{R}^n)$.

Proof of Theorem 2 We only need to show that for any p-atom a with respect to ω, $\|T^*_\delta(a)\|_{L^p_\omega(\mathbb{R}^n)} \leq C$ with C independent of a .Suppose supp $a \subset B(x_0,r)$.Let $\omega \in A_{q_0}(\mathbb{R}^n)$ with . We choose $p_0 > 1$, and write

$$\|T^t_\delta(a)\|^p_{L^p_\omega(\mathbb{R}^n)} \leq \int_{B(x_0,4r)}|T^t_\delta(a)(x)|^p\omega(x)dx + \int_{\mathbb{R}^n\setminus B(x_0,4r)}|T^t_\delta(a)(x)|^p\omega(x)dx = L_1 + L_2.$$

By the boundedness of operator T^*_δ on L^q_ω[5], we then have

$$L_1 \le C(\int_{B(x_0,4r)} |T^t_\delta(a)(x)|^{p_0}\omega(x)dx)^{\frac{p}{p_0}}(\int_{B(x_0,4r)} \omega(x)dx)^{1-\frac{p}{p_0}}$$
$$\le C(\int_{B(x_0,r)} |a(x)|^{p_0}\omega(x)dx)^{\frac{p}{p_0}}\omega(B(x_0,r))^{1-\frac{p}{p_0}} \le C,$$

where C is independent of a.

Noticing $y \in B(x_0, r)$ and $x \in \mathbb{R}^n \setminus B(x_0, 4r)$, we gain $|x - x_0| \sim |x - y|$.

case 1 $t < r$ By the vanishing moments of a-atom,we gain

$$T^t_\delta(a)(x) \le Ct^{-n}\int_{B(x_0,r)} (1 + \frac{|x-y|}{t})^{-(\delta+\frac{n+1}{2})}|a(y)|dy$$
$$\le Ct^{\delta-\frac{n-1}{2}}|x - x_0|^{-(\delta+\frac{n+1}{2})}\|a\|_\infty|B| \le Ct^{\delta-\frac{n-1}{2}}|x - x_0|^{-(\delta+\frac{n+1}{2})}\omega(B)^{-\frac{1}{p}}|B|$$
$$L_2 \le Ct^{p(\delta-\frac{n-1}{2})}\omega(B)^{-1}|B|^p\int_{\mathbb{R}^n\setminus B(x_0,4r)} |x - x_0|^{-p(\delta+\frac{n+1}{2})}\omega(x)dx$$
$$\le Ct^{p(\delta-\frac{n-1}{2})}\omega(B)^{-1}|B|^p\sum_{k=2}^{\infty}\int_{B_{k+1}\setminus B_k} |x - x_0|^{-p(\delta+\frac{n+1}{2})}\omega(x)dx$$
$$\le C(\frac{t}{r})^{p(\delta-\frac{n-1}{2})}\sum_{k=2}^{\infty}(\frac{\omega(B_{k+1})}{\omega(B)})2^{-kp(\delta+\frac{n+1}{2})} \le C\sum_{k=2}^{\infty}2^{-k[p(\delta+\frac{n+1}{2})-n]} \le C$$

case 2 $t \ge r$.By the vanishing moments of a-atom and the mean value theorem ,we have

$$T^t_\delta(a)(x) \le Ct^{-(n+1)}\int_{B(x_0,r)} (1 + \frac{|x-x_0|}{t})^{-(\delta+\frac{n+1}{2})}|y - x_0||a(y)|dy$$

When $\delta < \frac{n+1}{2}$, we obtain

$$L_2 \le C(\frac{r}{t})^{p(\frac{n+1}{2}-\delta)}\sum_{k=2}^{\infty}2^{-kp(\delta+\frac{n+1}{2})}\frac{\omega(B_{k+1})}{\omega(B)} \le C\sum_{k=2}^{\infty}2^{-k[p(\delta+\frac{n+1}{2})-n]} \le C.$$

When $\delta \ge \frac{n+1}{2}$, we obtain

$$T^t_\delta(a)(x) \le |x - x_0|^{-(n+1)}r^{n+1}\|a\|_\infty$$
$$L_2 \le r^{p(n+1)}\omega(B)^{-1}\sum_{k=2}^{\infty}\int_{B_{k+1}\setminus B_k} |x - x_0|^{-p(n+1)}\omega(x)dx \le C\sum_{k=2}^{\infty}2^{-k[p(n+1)-n]} \le C.$$

Thus,we have

$$\|T^t_\delta(a)\|^p_{L^p_\omega} \le C$$

For any $t > 0$,we take the supreme of the left side and can gain the desire result.

4 $[WL^p_\omega(\mathbb{R}^n),\ WH^p_\omega(\mathbb{R}^n)]$-Type Continuity .

Lemma 1 Let $p \in (0,1]$ and $\omega \in A_\infty(\mathbb{R}^n)$. For $f \in WH^p_\infty(\mathbb{R}^n)$, there exists a sequence $\{f_k\}_{k=-\infty}^{\infty}$ of bounded measurable functions such that

$$f = \sum_{k=-\infty}^{\infty} f_k \quad inS^{'}(\mathbb{R}^n). \tag{1}$$

Each f_k can be further decomposed into $f_k = \sum_i b_{ki}$, where the sequence $\{b_{ki}\}_i$ satisfies

$$suppb_{ki} \subset Q_{ki} and Q_{ki} is a cube, \tag{2}$$

$$\sum_i \omega(Q_{ki}) \leq c_1 2^{-kp}; \sum_i \chi_{Q_{ki}}(x) \leq c_1,$$

χ_E being the characteristic function of the set E, c_1 a constant and $c_1 \leq C\|f\|^p_{WH^p_\omega(\mathbb{R}^n)}$;

$$\|b_{ki}\|_{L^\infty(\mathbb{R}^n)} \leq C2^k \quad and \quad \int_{\mathbb{R}^n} b_{ki}(x)x^\alpha dx = 0, for |\alpha| \leq [n(q_\omega/p-1)].$$

Conversely, if $f \in S^{'}(\mathbb{R}^n)$ has a decomposition satisfying (1) and (2) ,then $f \in WH^p_\omega(\mathbb{R}^n)$ and $\|f\|^p_{WH^p_\omega(\mathbb{R}^n)} \leq Cc_1$, where C is a constant.

Theorem 3 Let $\delta > \frac{n-1}{2}$, $\max\{\frac{n}{\delta+(n+1)/2}, \frac{n}{n+1}\} < p \leq 1$, $\omega \in A_q$,$q \geq 1$, Then T^*_δ is a bounded operator from $WH^p_\omega(\mathbb{R}^n)$ to $WL^p_\omega(\mathbb{R}^n)$.

Proof of Theorem 3 For any $\lambda > 0$, let $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \lambda < 2^{k_0-1}$. Let $\omega \in A_q(\mathbb{R}^n)$ and $f \in WH^p_\omega(\mathbb{R}^n)$. Then by Lemma 1,we write

$$f = \sum_{k=-\infty}^{\infty} f_k = \sum_{k=-\infty}^{\infty}\sum_i b_{ki} = \sum_{k=-\infty}^{k_0}\sum_i b_{ki} + \sum_{k=k_0+1}^{\infty}\sum_i b_{ki} = F_1 + F_2,$$

where $b^{'}_{k,i}s$ are as in Lemma 1. Suppose $A_k = suppf_k$,then $A_k = \cup_i Q_{ki}$ and $\omega(A_k) \leq \sum_i \omega(Q_{ki}) \leq C2^{-kp}\|f\|^p_{WH^p_\omega}$.Since B^δ_* is bounded on L^q_ω spaces, we have

$$\begin{aligned}
&\omega(\{x \in \mathbb{R}^n : |T^t_\delta(F_1)(x)| > \lambda\}) \leq C\|T^t_\delta(F_1)\|^2_{L^2_\omega}/\lambda^2 \\
&\leq C\|F_1\|^2_{L^2_\omega}/\lambda^2 \leq C[\sum_{k=-\infty}^{k_0}\|f_k\|_{L^2_\omega}]^2/\lambda^2 \leq C[\sum_{k=-\infty}^{k_0}(\sum_i\int_{B_{ki}}|b_{ki}|^2\omega(x)dx)^{\frac{1}{2}}]^2/\lambda^2 \\
&\leq C\{\sum_{k=-\infty}^{k_0} 2^k[\sum_i \omega(B_{ki})]^{\frac{1}{2}}\}^2/\lambda^2 \leq C[\sum_{k=-\infty}^{k_0} 2^{k(1-p/2)}]^2\|f\|^p_{WH^p_\omega}/\lambda^2 \leq C\|f\|^p_{WH^p_\omega}/\lambda^p.
\end{aligned}$$

Now set $Q^*_{ki} = Q(x_{ki}, (5\sqrt{n})r_{ki})$ and $A_{k_0} = \bigcup_{k=k_0+1}^{\infty}\bigcup_i Q^*_{ki}$, where Q^*_{ki} is the cube with the same x_{ki}-center as Q_{ki} and side length $5\sqrt{n}$ times the side length of Q_{ki}. We get

$$\omega(Q_{k_0}) \leq \sum_{k=k_0+1}^{\infty}\sum_i \omega(Q^*_{ki}) \leq \sum_{k=k_0+1}^{\infty}\sum_i \omega(Q_{ki}) \leq \sum_{k=k_0+1}^{\infty} 2^{-kp}\|f\|^p_{WH^p_\omega} \leq \|f\|^p_{WH^p_\omega}/\lambda^p.$$

To finish the proof, we still need to estimate

$$\omega(\{x \in A_{k_0}^c : |T_\delta^t(F_2)(x)| > \lambda)\}) \le C\|T_\delta^t(F_1)\|_{L_\omega^p}^p/\lambda^p.$$

For $x \in (A_{k_0}^*)^c$,$y \in Q_{ki}$, we know $|x-y| \sim |x-x_{ki}|$.

case 1 $t < r_{ki}$,We then obtain

$$\int_{A_{k_0}^c} |T_\delta^t(b_{ki})(x)|^p\omega(x)dx = \int_{A_{k_0}^c} \{\int_{Q_{ki}} t^{-n}(1+\frac{|x-y|}{t})^{-(\delta+\frac{n+1}{2})}|b_{ki}(y)|dy)\}^p\omega(x)dx$$
$$\le C\sum_{j=2}^\infty \int_{Q_{ki}^{j+1}\setminus Q_{ki}^j} t^{p(\delta-\frac{n-1}{2})}|x-x_{ki}|^{-p(\delta+\frac{n+1}{2}}2^{kp}|Q_{ki}|^p\omega(x)dx$$
$$\le C\sum_{j=2}^\infty 2^{-jp(\delta+\frac{n+1}{2})}(\frac{t}{r_{ki}})^{p(\delta-\frac{n-1}{2})}2^{kp}\omega(Q_{ki}^{j+1}) \le C\sum_{j=2}^\infty 2^{-j[p(\delta+\frac{n+1}{2})-n]}2^{kp}\omega(Q_{ki}) \le C2^{kp}\omega(Q_{ki}),$$

where Q_{ki}^j is the cube with the same x_{ki}-center as Q_{ki} and side length 2^j times the side length-r_{ki} of Q_{ki}.

case 2 $t \ge r_{ki}$.

we consider $\delta < \frac{n+1}{2}$ at first. By the vanishing moments of b_{ki} and the mean value theorem, we have

$$\int_{A_{k_0}^c} |T_\delta^t(b_{ki})(x)|^p\omega(x)dx = \int_{A_{k_0}^c} |\int_{Q_{ki}} [B_\delta^t(x-y) - B_\delta^t(x-x_{ki})]b_{ki}(y)dy|^p\omega(x)dx$$
$$\le C\sum_{j=2}^\infty \int_{Q_{ki}^{j+1}\setminus Q_{ki}^j} t^{p(\delta-\frac{n+1}{2})}(2^j r_{ki})^{-p(\delta+\frac{n+1}{2})}r_{ki}^p 2^{kp}|Q_{ki}|^p\omega(x)dx$$
$$\le C\sum_{j=2}^\infty (\frac{r_{ki}}{t})^{p(\frac{n+1}{2}-\delta)}2^{-j[p(\delta+\frac{n+1}{2})-n]}2^{kp}\omega(Q_{ki}) \le C2^{kp}\omega(Q_{ki})$$

Secondly,we consider $\delta \ge \frac{n+1}{2}$,

$$\int_{A_{k_0}^c} |T_\delta^t(b_{ki})(x)|^p\omega(x)dx = \int_{Q_{k_0}^c} |\int_{Q_{ki}} [B_\delta^t(x-y) - B_\delta^t(x-x_{ki})]b_{ki}(y)dy|^p\omega(x)dx$$
$$\le C\sum_{j=2}^\infty \int_{Q_{ki}^{j+1}\setminus Q_{ki}^j} (\int_{B_{ki}} |x-x_{ki}|^{-(n+1)}r_{ki}|b_{ki}(y)|dy)^p\omega(x)dx$$
$$\le C\sum_{j=2}^\infty 2^{-j[p(n+1)-n]}2^{kp}\omega(Q_{ki}) \le C2^{kp}\omega(Q_{ki})$$

These yields the desired results .

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