INDIVISIBLE PARTS AND EXTENDED OBJECTS:

Some Philosophical Episodes from Topology's Prehistory*

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Physical boundaries and the earliest topologists

Topology has a relatively short history; but its 19th century roots are embedded in philosophical problems about the nature of extended substances and their boundaries which go back to Zeno and Aristotle. Although it seems that there have always been philosophers interested in these matters, questions about the boundaries of three-dimensional objects were closest to center stage during the later medieval and modern periods. Are the boundaries of an object actually existing, less-than-three-dimensional parts of the object — that is, are solids bounded by two-dimensional surfaces, surfaces by one-dimensional "edges" or "physical lines", edges by dimensionless "simples"? If not, how does a perfectly spherical object manage to touch a perfectly flat object — what part of the sphere is in immediate contact with the plane, if the sphere has no unextended parts? But if such parts be admitted, are we not then saddled with "actual infinities" of simples, lines, and surfaces spread throughout each continuous object — the boundaries of all the object's internal parts? Does it help any to say that these internal boundaries exist only "potentially"?

These questions were still in the air as mathematicians and natural philosophers developed the notions which were to become the basis for topology.[1] In Bernard Bolzano's account of continuity, for example, we have "the first attempt at a mathematical formulation of the topological notion of connected."[2] And Bolzano's definition is meant to apply to physical substances as well as to space and time. Indeed, as we shall see, he uses his analysis of continuity to answer the traditional questions about physical boundaries listed above.[3]

Dimension theory further illustrates the close ties between topology and the philosophical problems of physical boundaries. Poincaré's suggestion of a purely topological definition of zero-, one-, two-, and three-dimensionality led to a more general account of dimension which was one of the earliest great successes of topology;[4] and Poincaré's definition is a clever development of a notion which was at home in the medieval dispute about physical boundaries: namely, the idea that points are "things completely indivisible", lines are "things divisible only in one dimension", and surfaces are "things divisible in two dimensions".[5]

The importance of Poincaré's purely topological account of dimension was made clear by the failure of earlier attempts to draw clear distinctions between the different dimensions. In particular, Cantor's proof that a line and a plane have the same number of points, and Peano's proof that a point could traverse a "space-filling curve" (so that a line and a square must contain the same number of points) raised difficulties for earlier assumptions about dimension.[6] These proofs resemble in important ways a series of influential arguments from the older discussions of physical boundaries.[7] Indeed, the scholastic tradition — with which Cantor, at least, was familiar — even includes space-filling curves![8]

Although topology may have been closely tied to physical notions of connectedness and physical continua at its origins, the old philosophical debates clearly had no place in the purely mathematical point-set topology that emerged at the beginning of this century. But just as topology was leaving behind the perennial questions about the boundaries of physical objects, so was everyone else. At one time the existence of indivisible boundaries was so hotly disputed that the question could split the entire faculty of an academy into rival factions, as at the school of Paris in the 14th century[9] and the Royal Academy of Berlin in the 18th.[10] Why did the air suddenly go out of these debates around the beginning of this century?

No doubt many factors conspired together to make the traditional problems about physical boundaries seem obsolete. I suspect, however, that interest faded primarily because the concept of an extended object with sharp boundaries presupposed by the old debate seemed outmoded in the light of newfangled conceptions of matter. In the final section I offer some thoughts about the nature and significance of these changes, and indicate why I think these questions are still worth the attention of metaphysicians. In fact, whether they deserve it or not, the problems of physical boundaries have received a little bit of attention during the latter half of this century; but, although often of a high caliber, recent work is typically quite disconnected from the older debate I will describe below.[11] This paper is meant as a contribution to the current discussion; I shall be digging up some nearly-forgotten arguments and doctrines that should be of interest to anyone attempting to answer metaphysical questions about the nature of extended objects and their boundaries. I hope that those who think it is no longer respectable to ask such questions may still find something of historical interest in the story I tell.

Brentano and Whitehead — last heroes of the old debate

There are, broadly speaking, three doctrines about physical boundaries to be found in the medieval and modern debate; I shall call them "indivisibilism", "moderate indivisibilism", and "anti-indivisibilism". In section II I describe these views, and mention some of their better-known proponents. Then indivisibilism, moderate indivisibilism, and anti-indivisibilism each receives a section of its own. Although this paper will not pretend to offer a decisive answer to the question which of these three (if any) is in fact correct, it is intended to serve as a sort of historical propaedeutic to the consideration of this question.

Along the way, I shall try to show that the most promising version of moderate indivisibilism was being developed by Franz Brentano at the same time Alfred North Whitehead was providing mortar to fill the holes remaining in earlier versions of anti-indivisibilism. Whitehead's contribution to anti-indivisibilism is his famous method of extensive abstraction, first developed in a series of papers written between 1914 and 1917.[12] Curiously enough, these were the very years during which Brentano — near the end of his life and by then completely blind — dictated his works on boundaries and continua.[13] Brentano admits physical points, lines, and surfaces as real parts of extended bodies; and, just as Whitehead's work advances the anti-indivisibilist cause, Brentano's is a step forward for moderate indivisibilism.

Brentano's work on boundaries trickled out ever so slowly, the lion's share remaining unpublished until 1976.[14] Furthermore, although Whitehead discussed the philosophical problems about boundaries in his first exposition of the method of extensive abstraction, he did not realize that the method contributed to their resolution;[15] and his better known later works omit discussion of these problems altogether. For these reasons, the continuity between the older debate and the contributions of Whitehead and Brentano is easy to miss.

I. The medieval and modern background

Naturally enough, it is Aristotle's discussion of continuity, contiguity, and boundaries which set the stage for later developments. But he left much to be settled: for example, whether indivisibles (simples, lines, surfaces) actually exist as parts of bodies; and what exactly it means to say that a point, line, or plane "exists potentially". By the 13th and 14th century, these sorts of questions about the reality of indivisibles had become prominent; and mathematical arguments with no precursors in Aristotle were added to the debate.[16]

Participants in the medieval debate fall into three categories. (1) There are extreme <u>indivisibilists</u>, who recognize <u>only</u> indivisible physical substances and wholes compounded out of them. Medieval indivisibilists of this stripe included Henry of Harclay, Walter Chatton, and Gerard of Odo.[17] (2) There are <u>moderate indivisibilists</u>, who admit the existence of indivisibles of all three sorts, but who also accept Aristotle's conclusion that extended objects cannot be composed of indivisibles alone. As a result, moderate indivisibilists like John Duns Scotus and Francisco Suarez[18] must recognize two kinds of parts in every extended body: (a) infinitely divisible three-dimensional stuff, all of the parts of which are also three-dimensional (a kind of "atomless gunk" containing no indivisible parts[19]); and (b) physical points, lines, and surfaces that are responsible both for terminating and connecting an object's three-dimensional parts. (3) Finally, there are numerous anti-indivisibilists, headed up by William of Ockham,[20] who deny that extended objects possess any less-than-three-dimensional parts.

All three of these views remain live options throughout the 17th, 18th, and 19th centuries. Full-fledged indivisibilism survives in two forms. Some indivisibilists, like Christian Wolff, Berkeley, and Hume, build extended entities out of finite numbers of unextended elements.[21] Others, like Bolzano,[22] think that extended things are made by filling the infinitely many points of an extended region with infinitely many unextended simples. Moderate indivisibilism, ably defended in one form by Suarez at the very end of the 16th century, is still being defended by Brentano at the beginning of the 20th. Anti-indivisibilism is advocated by the likes of Malebranche and Descartes, but it receives its most vigorous modern defense when Euler battles the 18th century monadists.[23]

In the modern period, there is also considerable skepticism about whether the paradoxes about extended objects and their boundaries can be resolved at all. But if they cannot, there can be no extended substances; and then what is left besides absolutely unextended simple substances? Thus a sort of degenerate indivisibilism emerges in the quite different monadisms of Leibniz and Roger Joseph Boscovich: each, in his own way, rejects three-dimensional substances altogether in favor of extensionless simples.[24] The pre-critical Kant belongs in this category as well: extended substances are made out of point-sized physical monads, and these simples may be said to fill extended regions of space in virtue of repulsive forces which keep approaching objects at a distance.[25] Pierre Bayle heaps ridicule upon every possible theory of extended objects: the proponents of each succeed in refuting the others, so all are refuted; and thus three-dimensional bodies "can only exist ideally."[26] Although Bayle finds "very evident contradictions in the existence of extension", we should not conclude that he would join Boscovich and Leibniz in preferring unextended substances to extended ones; his arguments are probably just meant to confront the human mind with its limitations, "making it admit, in spite of itself, that there are things that exist though it is not capable of understanding them."[27]

An unstated but noteworthy assumption of the dispute throughout this period is that at most one of these alternative metaphysics of extended objects is possible. This is manifest in that (a) all parties argue that opposing views are incoherent or contradictory and therefore simply untenable, and (b) no one so much as considers the possibility of hybrid theories, according to which extended objects may differ from one another in the way indivisibles enter into their construction — no one stops to wonder, for example, whether some extended objects might have, and others lack, indivisible parts. As a matter of fact, I believe this presupposition of the medieval and modern debate is more or less correct; but its defense would involve some rather controversial premises,[28] and I shall reserve it for another occasion.

II. Full-fledged indivisibilism

Problems for indivisibilism

Some indivisibilists admit that extended regions are infinitely divisible, while others do not. In this section, I first briefly consider the position of the latter indivisibilists, who hold that space is "discrete". Then I shall suggest some difficulties that must be faced by the former, who are committed to the existence of the same infinite number of simples in every extended object. The overall aim is to show that indivisibilism — sometimes thought to be completely vindicated by 20th century dissolutions of Zenonian paradoxes — is not altogether free of problems. If one does not want to suppose that space has to be discrete, and if there are real difficulties for indivisibilism in continuous space, then one must take moderate indivisibilism and anti-indivisibilism more seriously.

Indivisibilism and discrete space

Wolff, Berkeley, and Hume are indivisibilists committed to discrete space. Wolff builds the three-dimensionally extended atoms of corpuscularian physics out of finitely many absolutely unextended "elements".[29] When such simples form an extended continuum, they are united in such a way that no additional simple could possibly be inserted between any one and the simples closest to it.[30] Whether or not Wolff himself realized this, Euler and Kant certainly saw that Wolff's picture was inconsistent with the continuous space of Euclidean geometry.[31] Berkeley and Hume, unlike Wolff, were constructing extended things out of finite collections of minima sensibilia. Given this phenomenalist bent, their relationship to the debates about the boundaries of (mind-independent) physical objects in a three-dimensional (mind-independent) space is somewhat ambiguous. But both realized an important fact about discrete space of any kind, phenomenal or physical: namely, that in discrete space many theorems of geometry can hold only approximately, at best.[32]

We need not take seriously, I should think, those theories of discrete space which suppose that the ultimate "space atoms" — and so the smallest atomic parts of extended physical objects — have some finite size and definite shape. For one thing, such theories seem hopelessly vitiated by "tile trouble"; they could not even approximate Euclidean space in the large, as Weyl's tile argument shows.[33] For another, there seems to me to be an incoherence in the very notion of a spatial region's having a definite size and shape — being, say, a square 1/1000th of an inch across — without its having a left and right half of smaller size — in this case, rectangular halves each 1/2000th of an inch across. A partless square region is no more possible than a round square region. Thus the only hope for a workable theory of discrete space would seem to be one which builds extended regions out of contiguous, but unextended, points. Perhaps, as some have argued, the hypothesis that space is so composed is a coherent one which can only be disproven by empirical results which are not yet available.[34] But only those indivisibilists who think this issue can be settled by a priori means can afford to ignore the hypothesis that space is not discrete.

Continuous space and infinite divisibility

Most philosophers and physicists suppose that space is in fact continuous. On this assumption, it becomes hard to see how anything extended could fail to be infinitely divisible. It is sometimes said that a literally partless entity could fill the whole of an infinitely divisible three-dimensionally extended region. But usually the thing alleged to fill space in this way is either a soul, or a physical substance which has a place in some supernatural way, or something that is not a substance at all. Ockham and others affirm that Christ's body is present in the Eucharist, not "as a whole in the whole and as a part in each part", but rather "as a whole in the whole and as a whole in each part".[35] Likewise a certain kind of "immanent realism" asserts that universals are spatiotemporally located, and wholly present wherever they are instantiated.[36]

Occasionally, however, this mode of spatial occupancy is contemplated for more mundane physical substances: Democritus's atoms come in various shapes and sizes, but may perhaps be "indivisible" in the sense of completely partless.[37] Bayle thinks that Epicurus' atoms are like this, at any rate,[38] and offers the following rebuttal:

> [F]or every extension, no matter how small it may be, has a right and a left side, an upper and a lower side. Therefore it is a collection of distinct bodies. I can deny concerning the right side what I affirm about the left side. These two sides are not in the same place. A body cannot be in two places both at the same time, and consequently every extension that occupies several parts of space contains several bodies.[39]

Note that someone who agrees with Bayle need not accept what Peter van Inwagen has called "the doctrine of arbitrary undetached parts" — roughly, the thesis that, for every "occupiable" region within the space occupied by a body, there is a part of the body.[40] If one held — as van Inwagen does — that every physical object is decomposable without remainder into parts that are physical simples; and also — as van Inwagen may or may not — that the simples are spatially simple, then Bayle's thesis would follow immediately:

every distinct pair of regions occupied by a single body would contain distinct parts of the body. And one could obviously hold this view while denying, with van Inwagen, that just any batch of these simples filling an occupiable region within a body constitutes a part of the body.

Bayle's convictions about extension seem quite compelling — although, as Euler points out, "[i]n speaking of the divisibility of body we must carefully distinguish what is in our power, from what is possible in itself." As long as we take "divisibility" to mean simply the having of distinct parts, then even an extended object "so hard that no force could break it" must be "as divisible in its own nature as the most brittle of the same magnitude."[41] Perhaps a miraculous substance or a universal may be partless yet spread throughout an extended region; but it is hard to imagine an extended solid — an object filling a precisely demarcated three-dimensional region of space — behaving in any way which would lead us to describe it as partless.[42]

But let us try anyway. Consider an extended simple that is supposed to fill an extended region. If it failed to have a part filling each subregion, then each subregion would be empty — and how could the whole region be filled, if it were just a sum of empty regions? If we agree that no subregion of the region occupied by an object could fail to be filled itself by a part of the thing, then the simple must be present "as a whole in the whole and as a whole in each part" — like Christ in the Eucharist, or a multiply-located universal in all its instances. But then no attempt to interact differentially with an object filling just a proper part of the region could be successful. When one sees the top half of the thing, one thereby sees the bottom half as well; when one touches the left side, one touches the right also. Perhaps this bizarre mode of space-filling is not absolutely impossible; but surely the more sensible kinds of extended objects ought not participate in it. At least the paradigmatic case of an extended body in continuous space must be a body that is divisible through-and-through.

Infinities of indivisibles

If extended objects in continuous space are infinitely divisible, then there are infinitely many planes cutting a cube, lines cutting a plane, and points cutting a line. Some of the medieval indivisibilists accepted this result, and realized that each extended object would have to contain infinitely many simple parts.[43] But indivisibilism remained a minority position, a non-starter in the minds of most scholastics and early moderns alike. In the 17th century, Pierre Bayle could write:

> [P]ersons of the slightest depth can comprehend with complete certainty, if they give the matter a little attention, that several nonentities of extension joined together will never make up an extension. Consult the first course of Scholastic philosophy that you come across; and you will find there the most convincing arguments in the world, supported by many geometrical demonstrations, against the existence of these points.[44]

Some of these arguments are so dependent upon the details of scholastic Aristotelianism as to be of only historical interest.[45] Others are based on the geometrical proofs that, for example, two concentric circles composed of indivisibles would contain the same number of indivisibles — something supposed to be impossible.[46] Perhaps the most influential of all such arguments, frequently cited by both medievals and moderns, is Aristotle's adoption of Zeno's maxim: a whole composed of even an infinite number of dimensionless entities must itself be dimensionless.[47]

Although the latter two objections were long thought to raise insurmountable difficulties for the supposition that any extended thing could be composed of unextended simple parts, they began to seem less serious after Cantor. In particular, Adolf Grünbaum has shown that Aristotle's Zenonian paradox of extension is significantly defused by Cantor's discovery of the distinction between denumerably and non-denumerably infinite numbers.[48] Thus Grünbaum finds no difficulty in agreeing with Bolzano that an extended physical object is quite literally a "linear continuum of points": "By a point of this body we then mean nothing more or less than an element of it possessing the formal properties prescribed for points by the postulates of geometry. And, on this interpretation, the ground is then cut from under the geometric parti pris against Cantor by the modern legatees of

Zeno."[49] Most pre-20th-century arguments against indivisibilism question "the consistency of conceiving of an extended continuum as an aggregate of unextended elements."[50] In the light of Grünbaum's work, these arguments seem rather unconvincing.

Paradoxes of the infinite

It appears, then, that indivisibilism in continuous space has a new lease on life; the notion that extended objects are made of infinities of point-sized parts is not as ridiculous as most medievals and moderns believed. Still, there are some serious problems facing this hypothesis. If Grünbaum is right, one can only avoid the Zenonian metrical paradox of extension by supposing that the number of points in a one-, two-, or three-dimensional region has the cardinality of the continuum.[51] But, on this assumption, the largest three-dimensional region contains the same number of points as the smallest line segment. Thus, if an extended body were nothing more than a continuous manifold of simple parts, one for each point in the region occupied by the body, then it should be possible for the same set of parts to be rearranged so as to form a body of <u>any</u> size you like.[52] A similar but perhaps even more unsettling result emerges from the Banach-Tarski theorem. Given the assumption that a sphere consists of simple parts — so that, even for a <u>nonmeasurable</u> set of points in the sphere there will be a corresponding part of the sphere which exactly fills that set of points, — it follows from this theorem that a sphere is divisible into just <u>four</u> parts which can be disassembled and reassembled by means of rigid motions so as to form two spheres of the same size as the original — or, indeed, so as to form <u>any number</u> of such spheres.[53]

These strange results become mere mathematical fictions, however, if we reject full-fledged indivisibilism. Cantor's conclusion and the Banach-Tarski theorem tell us what would happen if wholes decomposable into nonmeasurable sets of simples could be rearranged. But if extended objects are not identifiable without remainder with sets of simples (or with any other non-measurable sets of parts), then these geometrical proofs imply nothing about what could result from rearranging the real parts of such objects.

The force of these considerations should not be overestimated, however. First of all, in the case of the Banach-Tarski theorem, the paradoxical result is sometimes seen as casting doubt not upon the punctual make-up of bodies but upon the axiom of choice, which is essential to its proof. More importantly, all such paradoxes may readily be "taken neat": strange things happen whenever we confront infinities; paradoxical-sounding conclusions about the non-denumerably infinite number of simples in an extended body are only to be expected, and do not show that such objects are not wholly composed of simple parts. Bolzano, for example, accepted the following result with apparent equanimity:

> …[T]he very same set of substances, which at this particular moment fills this particular cubic foot, could be distributed another time throughout a space one millionfold larger, without any point in this larger space standing empty; and distributed yet another time throughout a space one millionfold smaller, without any point in this smaller space needing to accommodate two or more atoms.[54]

Whatever evidence such paradoxical-sounding results may provide for the thesis that extended objects are not made entirely of simple parts, it does not seem to me to be enough to settle the matter.

Problems of contact

There are, however, more serious problems confronting the view that extended objects are decomposable without remainder into simple parts, problems suggested by Brentano's objection to the indivisibilism of Bolzano. The account of continuous substance in Bolzano's Paradoxes of the Infinite, like the corresponding modern concept of a connected region, allows for the distinction between "closed" and "open" configurations:

> Now I define the limiting surface (Gränze) of a body as the aggregate of all the extreme (äusserst) ether-atoms which still belong to it. … A closer consideration further shows that many bodies are at certain places altogether devoid of limiting atoms; none of their atoms can be described as the extreme ones among those which still belong to it and would accompany it if it started to move.[55]

Clearly, any attempt to build extended objects out of simples by filling the points of a continuous space with point-sized parts will have difficulty resisting Bolzano's conclusion that objects may come with or without a final "skin" of simple parts, depending upon whether the simples fill a closed or open region.

Bolzano's theory is supposed to solve "the problem concerning the limiting surfaces of physical bodies: where exactly does one such body end and another begin?",[56] as well as answering "the question whether and when two bodies stand in immediate contact, or whether they are separated by an intervening space".[57] He defines "the contact of two bodies as taking place when the extreme atoms of the one,…together with certain atoms of the other, form a continuous extension".[58]

Brentano rejects Bolzano's theory, which he describes as the "monstrous doctrine" that there exist "bodies with and without surfaces", and that contact is "possible only between a body with a surface and another without." He claims that Bolzano's conception of continuous bodies "as sets of points runs counter to the concept of contact and thereby abolishes precisely what makes up the essence of the continuum."[59] I have argued at length elsewhere that Brentano's conviction is justified: if one maintains that three-dimensional objects are decomposable without remainder into sets of unextended parts, one must be able to provide an account of the relationship of contact which holds between distinct extended objects (for example, between the left and right halves of a single object); and the only options are implausible in the extreme. For it does not matter whether an indivisibilist supposes, with Bolzano, that some objects have and others lack final surfaces; or that all objects have surfaces; or that all objects lack surfaces. Accepting any of these alternatives will require either the postulation of repulsive forces which necessarily accompany certain shapes, or the admission of bizarre appearances and disappearances of simples, or the introduction of in principle undetectable differences between different sorts of contact. None of these results is particularly attractive; and, since the indivisibilist's rivals need not carry such strange metaphysical baggage, full-fledged indivisibilism begins to look needlessly, indeed perversely, complex.[60]

If these conclusions are correct, then it is very difficult to maintain that an extended object in a continuous space could be made entirely of simples. Thus discrete space appears to be the last refuge of the indivisibilist — at least of the indivisibilist who wants to say that there are extended objects.[61]

Of course one may always join the ranks of the "degenerate indivisibilists" mentioned above, those who accept simple substances but deny that there could be such a thing as a three-dimensionally extended object.[62] Indeed, in the case of Boscovich, it was precisely the seeming impossibility of contact between extended objects which led him to conclude that every apparently three-dimensionally extended object is really a swarm of disconnected, absolutely unextended atoms.[63]

Suppose the Brentanian arguments I have given elsewhere are correct; then, barring discrete space, indivisibilism about extended objects with sharp boundaries leads to the Boscovichian conclusion that there could be no such things.[64] Thus anyone who thinks that extended objects are possible, but also has doubts about the discreteness of space, is forced in the direction of moderate indivisibilism or anti-indivisibilism.

III. Moderate indivisibilism

Brentano and Suarez: doctrines held in common

The great defenders of moderate indivisibilism are Francisco Suarez and Franz Brentano. Both affirm all of the following: (i) Extended objects have indivisible parts, (ii) every extended object (including each of the infinitely many proper parts of a solid body) is surrounded by a "skin" of point-sized parts which constitutes its two-dimensional surface, (iii) distinct extended objects touch when two such indivisible boundaries coincide,[65] and (iv) the three-dimensionally extended parts of a thing are not made up out of indivisibles alone but also contain some "atomless gunk", a substance all of whose parts have proper parts.[66] Brentano was quite familiar with the work in which Suarez developed his views on boundaries,[67] and there are some striking similarities in their approaches.[68] I shall argue, however, that Brentano's theory is a definite step forward. But first let us see how Suarez's moderate indivisibilism is motivated.

Suarez rejects several versions of intermediate moderate indivisibilism: for example, the theory that extended objects have indivisibles of all three grades on their exteriors, but not between their interior parts; or that extended objects have two-dimensional surfaces on their exteriors only, and no indivisibles of lesser dimension anywhere.[69] His reasons for rejecting these views are compelling. First he asks, why should one think there are indivisible parts in addition to atomless gunk alone? The primary reasons come from "the powerful mathematical argument by which it is commonly proved that there are points, namely, that a perfectly spherical globe touches a perfect plane in a point."[70] Imagine a sphere descending towards a perfectly flat object. One cannot deny that the two will eventually come in contact; after all, if the sphere is heavy enough and falling towards the flat object, the sphere will carry it down with it. And when they touch, a part of the one must be in direct contact with a part of the other. But for any extended part of the sphere you pick, the whole of that part cannot be in direct contact with any part of the plane. Only a point-sized part of the sphere and of the surface could be such that all of the one is in contact with all of the other.[71] Now if indivisibles are introduced to explain how contact is possible between objects, we need point-sized parts all over the surface and throughout the inside of every extended object. The external parts of a solid may touch other solids at points, lines, and surfaces anywhere on their exteriors; and all the proper parts of a solid touch other parts in surfaces, lines, and points that lie within the object. Since every extended object is divisible ad infinitum, extended objects are filled through and through with zero-, one-, and two-dimensional parts.

Brentano vs. Suarez

So far, Brentano and Suarez are together. Both posit indivisibles throughout the interior of an extended object; both do so because they think that contact requires indivisibles and that extended objects are infinitely divisible into sets of proper parts in contact. Both are also somewhat discomfited by this result; for it forces them to recognize the existence of infinities of extended parts and indivisibles, and neither is very happy with an actual infinite.[72] Where they differ, however, is in the number of point-sized parts thought to occupy a point in the interior of a body. Suarez thinks that, typically, there is only one; while Brentano holds that there is one for each distinct part of the body which has a boundary running through the point in question — which is to say, infinitely many, given infinite divisibility. And it is here that Brentano's position represents an advance over previous versions of moderate indivisibilism.

If Suarez is right, and the non-overlapping but continuous parts of a solid meet at a plane that has only one simple for each point in the plane, then we need answers to a host of questions about what happens when a continuous body is broken up; and all the possible answers to these questions seem equally absurd.[73] Since every surface must have a final "skin" of simple parts surrounding it, and there is only one "skin" of simples lying in the internal plane at which breakage occurs, this internal plane of simples becomes the surface of either the one side, or the other, or neither. Suarez decides that it is best to suppose that the internal plane of simples is destroyed at breakage, and that two new surfaces come into being.[74] But in any case, an infinity of simples will have to appear from somewhere so that both sides can come in contact with other objects. Furthermore, one wants to know whether the continuous left and right halves of an object, once separated, can subsequently be at best merely contiguous with one another — that is, whether, after breakage, the two parts can touch only by virtue of having distinct "skins" in the same two-dimensional region, whereas before they touched by sharing a single "skin" of simples between them. If the original kind of continuity can never be restored, extended objects are like Humpty-Dumpty — they cannot be put back together again, at least not in quite the same way. To suppose, on the contrary, that the original type of continuity can be restored introduces its own bit of awkwardness. Since the appearance of simples at breakage is posited for purely a priori reasons, it cannot be supposed to be necessarily associated with any observable feature of extended bodies; likewise, then, for their disappearance. So Suarez's theory ends up implying an in principle undetectable difference between pairs of objects in contact: some merely share boundary parts and others have coincident but distinct boundary parts; and objects can go back and forth from one kind of contact to the other, for no empirical or metaphysical rhyme or reason.[75]

In light of the fact that Brentano's version avoids all such bizarre complications, they seem to me to weigh decisively against Suarez's version of moderate indivisibilism. For Brentano there is only one species of contact, whether or not the objects in contact be parts of a larger whole; contact always consists in the coincidence of zero-, one-, or two-dimensional boundaries. Thus, when an extended object is broken in half, none of the newly exposed parts are in danger of emerging "naked" — both halves already had a two-dimensional surface before the breakage, and they were in contact then in virtue of the two surfaces being coincident.

Thus Suarez's theory of extended objects may well deserve the (characteristically) rough treatment it receives at Bayle's hands:

> [S]ome School philosophers…suppose that nature has mixed some mathematical points in with the infinitely divisible parts to serve as connections between them and to make up the extremities of bodies. They believed by this they could also answer the objection about the penetrative contact of two surfaces, but this subterfuge is so absurd that it does not deserve to be refuted.[76]

Although Bayle would probably have given Brentano's version the same treatment, Brentano's theory seems to me to represent an important advance over the medieval tradition of moderate indivisibilism culminating in Suarez.

IV. Anti-indivisibilism

Ockham and the descending sphere

According to anti-indivisibilists like Ockham, three-dimensionally extended objects have no parts that are not themselves three-dimensionally extended; the ostensibly zero-dimensional tip of a cone, for instance, is a mere fiction or abstraction of some kind, as are the seemingly one-dimensional edges and two-dimensional square faces of a cube. Suarez, as we saw, rejects this view because he is unsatisfied with its account of contact; indeed, his main objections to anti-indivisibilism seem to depend upon the traditional sphere argument described above.[77] Ockham's response to the sphere argument goes like this:

> I, however, maintain that the spherical body does not touch the flat body primarily with a part that is such that each of its parts touches the flat body. Therefore, it does not touch it primarily with some part that is prior to all the other touching parts. Rather, any given touching part is still such that a half of it does not touch immediately, and a half of that half does not touch immediately, and so on ad infinitum.[78]

Suarez characterizes this response as the suggestion "that the sphere and plane have negative, but not positive contact; for, to the extent that they are not distant from one another, they are said to touch each other negatively, since there is no positive entity in which they touch."[79] Suarez replies:

> Real contact occurs in some entity which truly and formally exists in things; for the contact itself is real, and properly and formally exists in reality; therefore it occurs in some real entity which formally exists in the thing; and yet it occurs in an indivisible thing; therefore such an indivisible entity exists formally in the thing itself.[80]

But why not simply suppose, as Ockham clearly does, that the relation of "being in (at least partial) contact with" can hold between extended objects, but not in virtue of either one's having a part every part of which is in (at least partial) contact with some part of the other? According to the Ockhamist, even two objects whose surfaces mesh perfectly will not have any parts all of the proper parts of which are themselves in contact with parts of the other — since every part of the one that is right up against the other has some depth to it, and so has parts that are not themselves right up against the other. Indeed, it seems to me that Ockham has a very simple and unproblematic account of contact to offer: for two things to be in contact is for them to be distinct objects with no room for anything three-dimensional between them.[81]

Suarez realizes he is not really justified in attacking the Ockhamist's account of contact. After all, he must himself admit that the relationship between a two-dimensional line and one of its end points is a species of contact or unity which closely resembles the "negative contact" relation of the Ockhamist. For the end point must be in some sense united with the line it terminates, even though "there is no conceivable part which is the one and only part to which the point is joined, for no part taken as a whole is right next to the point; otherwise either that part would be just as indivisible as the point, or else the point would be as it were extended or present to a divisible space in the way that lines are." Suarez here admits that "this argument undermines to some extent the argument made above about the touching of the sphere in a point";[82] for if Suarez can posit a primitive relation of connection or unity which does not require the existence of any single part of the terminated line which is wholly connected with the terminating point — each candidate being, of course, further divisible into a shorter line terminated by the same point — then why cannot Ockham posit a similar sort of relationship of contact between extended objects, a relationship in which neither relatum has a single part which is the "one and only" part which "taken as a whole is right next to" a similar part of the other? To my mind, this tu quoque takes the teeth out of the sphere argument; for it shows that no moderate indivisibilism can justly deny Ockham the kind of contact relation he needs.

Whence points, lines, and surfaces?

But there is a more serious challenge to be faced: what is the anti-indivisibilist to make of the myriad uses to which indivisibles are put in geometry and physics? On the face of it, as Suarez is quick to point out, the view that there really are zero-, one-, and two-dimensional entities "is more consonant with the principles both of geometry and philosophy [including, presumably, natural philosophy], and makes it easier to give explanations of many effects, and to speak on many philosophical matters."[83]

This challenge is quite serious. Given even a modest relationism about space, it would seem that space should exhibit no topological distinctions not to be found within the topology of (actual or merely possible) spatial objects. Thus many anti-indivisibilists will reject not only physical but also spatial points, lines, and surfaces. But then the onus is upon the anti-indivisibilist to explain the tremendous success of geometrical methods in the physical domain. Geometry is full of points, lines, and surfaces; so how can physical space and the objects that fill it fail to have parts of the sort geometry describes? This burden was not truly discharged until Whitehead developed his "method of extensive abstraction" in the early years of this century.[84]

Whitehead's first exposition of the method in "La Théorie Relationniste de l'Espace" was, in its own way, very much in the older tradition of concern about the boundaries of physical objects. The paper includes a lengthy discussion of some traditional problems about contact,[85] and applies the method of extensive abstraction to three-dimensional physical objects at various distances from one another[86] rather than to a four-dimensional manifold of events, as in Whitehead's most well-known English expositions of the method.[87]

Whitehead's method was very well-received, quickly adopted by the likes of Bertrand Russell, Jean Nicod, C. D. Broad, and Alfred Tarski.[88] The general idea is now familiar enough: identify points, lines, and planes in a continuum with "abstractive sets" — sets containing infinitely many converging, nested, extended entities. When applied to regions of space, as in Broad's account, it is used to explain exactly how we are to think about the physicists "points" without having to suppose that space really contains point-sized regions:

> The first thing to notice is that it does not in the least matter to science what is the inner nature of a term, provided it will do the work that is required of it. If we can give a definition of points which will make them fulfil a certain pair of conditions, it will not matter though points in themselves should turn out to be entities of a very different kind from what we had supposed them to be. The two conditions are (i) that points must have to each other the kind of relations which geometry demands; and (ii) that points must have to finite areas and volumes such a relation that a reasonable sense can

> be given to the statement that such areas and volumes can be exhaustively analysed into sets of points.[89]

One can see how close Whitehead was to the problems of contact between solids by noting that, in his first implementation of the method, he applies it not to space itself (as in Broad's exposition) or to a four-dimensional manifold of events, but directly to three-dimensional physical objects at various distances from one another. His earliest paper offers the means for maintaining that, for example, the tip of a cone-shaped physical object is identifiable for all practical or theoretical purposes with an abstractive set of extended parts of the cone which form an infinite nested series honing in on the place where the cone ends — and likewise for every other inner or outer boundary of a part of the cone, or of any other extended body. During this period, his strict relationism about physical space made it hard to see how the method could be applied to regions of empty space; so he confines himself to constructing abstractive sets out of nothing but the parts of three-dimensional objects. "These correspond to 'occupied' points, lines, and surfaces"; the extension to "unoccupied space" awaits "a general theory of ideal points (which I hope to set forth in a later paper)."[90]

Can extensive abstraction really supply us with proper surrogates for the less-than-three-dimensional entities seemingly required for the geometrical treatment of physical space or extended objects? Recent years have seen increasing interest in the construction of point-free topologies, geometries, and mereologies, including quite elegant constructions of points by means of algebraic operations on Boolean algebras of regions. It is to this literature — which includes at least two of the papers in the present number of this journal — that we must look if we are to feel sure that a thorough-going anti-indivisibilism may be easily reconciled with the treatment of space in mathematical physics.[91] It would, however, be very surprising if it could not.

If the construction of spatial points out of extended spatial regions is successful, one may then deny that space contains less-than-three-dimensional parts without thereby jeopardizing any physical or metrical facts. This Whiteheadian treatment of space obviously yields the result that, if there are extended objects, then they have no simple parts — there being no simple regions of space in which to put them.

Not only does the method relate the points, lines, and planes of physical geometry to the simple-less three-dimensional objects of anti-indivisibilism; but it provides a nice supplement to the Ockhamistic account of contact stated above. The Ockhamist says that two discrete three-dimensional objects are in contact just in case they are separated by a less-than-three-dimensional region. Given a Whiteheadian approach to spatial points, lines, and surfaces, it follows that two distinct objects are in contact just in case there is an appropriate abstractive set of spatial regions (corresponding to a point, line, or surface) all of whose members contain parts of both objects.

Objections to a Whiteheadian anti-indivisibilism

All of the most serious criticisms levelled against Whitehead's project of building points, lines, and planes out of abstractive sets have turned upon the inadequacy of a phenomenal basis for the construction.[92] Whatever Whitehead's intentions may have been,[93] it should be clear that someone who is an anti-indivisibilist for metaphysical reasons — e.g., because she thinks the view eschews the bootless complexity and paradoxes of both kinds of indivisibilism, yet can still account for every possible metrical and topological fact about spatially located objects — should have license to appeal to the requisite degree of infinite divisibility for her regions of space or extended objects. If super-denumerable infinities of points are really required for a consistent metrical account of extension, for example, then the indivisibilist must posit super-denumerably many points in every extended region, and the anti-indivisibilist must posit super-denumerable infinities of non-equivalent abstractive sets.[94] But I cannot see why either should have more of a right to the required infinities.

One sometimes hears the following less familiar objection to anti-indivisibilism:[95]

Objector: If touching is really "separation" by no more than a point, then "bodies never touch, since something — if only a very fine something — is always in between."[96]

Furthermore, since all the extended parts of a thing are extended objects themselves, each complete decomposition of a supposedly solid cube into a set of discrete extended parts yields a set of parts separated from one another by "empty" points, lines, and planes. Extended objects not only do not touch, then, but they are full of holes and cracks, and therefore not solid space-fillers after all!

This objection is not hard to answer, but the answer is instructive:

Anti-indivisibilist: Your objection betrays a fundamental misunderstanding of my theory. On my view, what you tendentiously call "separation by a point" is not any kind of separation at all, it is contact. Does a sphere have a "hole" at its center simply because it has no part that occupies just that location? Does it have a hairline fracture running through it merely because it has no part exactly filling some two -dimensional, bisecting plain? The answer to both questions is obviously, "No": holes cannot be so small, nor cracks so fine.[97] According to us anti-indivisibilists, "non-three-dimensional gaps" are not gaps. Since extended objects do not fill space by filling in "gaps" of this sort, positing a hole or crack of such size is nonsensical.

Furthermore (the anti-indivisibilist continues), no measurable facts could be lost by my view. Even the friends of indivisible parts must admit that they add nothing to the height of a cone by supposing it to have an unextended simple at its "tip". The presence of the simple there does not make their cone any taller than the anti-indivisibilist's cone. Nor, then, would its absence make the cone any shorter. Consequently, we should not imagine that an unextended part is necessary to fill a "gap" between an open cone and an open sphere spinning on top of it in order to allow the two to touch. "Filling" such gaps has no effect on the size of an object, and this fact provides a fresh reason to think that the objector is mistaken. Non-three-dimensional differences are not differences; therefore non-three-dimensional "gaps" are not gaps.[98]

In fact, the anti-indivisibilist may well want to say that all her open extended objects exactly fill topologically closed regions of space. Consider a sphere *s* which fills an open region *R* and fills no distinct extended open region. The anti-indivisibilist says: "The idea of an object filling a closed region but not an open one, or vice versa, is fundamentally incoherent. So to say in this case that the sphere *s* does not fill the closure of *R*, just because it has no final skin of simple parts surrounding it, would be quite misleading. We should say instead that whenever an open region is filled by an extended object, it automatically follows that its closure is also filled by the very same object." One result of this policy would be that, for the anti-indivisibilist, no object could exactly fill an open region. Someone who thought that extended objects were made entirely of simples would no doubt object that, if *s* exactly fills the closure of *R*, then it must have a proper part that exactly fills the open region *R* and another proper part that exactly fills the set-theoretical difference between *R* and the closure of *R*. But this objection simply begs the question against the anti-indivisibilist. Furthermore, it cannot even be made by the anti-indivisibilist's most significant rival, the moderate indivisibilist. A moderate indivisibilist like Brentano or Suarez must, after all, admit that an extended object has three-dimensional parts made of atomless gunk; and each of these parts fills a region of space without itself including any less-than-three-dimensional parts to exactly fill its non-three-dimensional subregions.

As noted above, an anti-indivisibilist about extended objects may well want to be an anti-indivisibilist about space, too. In that case, she should maintain a similar thesis about regions: every extended region corresponds to two sets of "points" (i.e., abstractive sets), one that includes its boundary elements and one that does not. But she should not allow for a distinction between closed and open regions. Since the Whiteheadian treatment requires that regions have no parts so small as to make up the difference between an open and closed region, to admit this difference would require that there be two distinct regions with all the same regions as parts — and surely if *x* and *y* are regions with all the same regions as parts, then *x* and *y* are one and the same region.[99]

With extensive abstraction on board, anti-indivisibilism seems quite capable of explaining why it works to treat physical continua as if they consist of unextended elements; and it also has an obvious and elegant account of the nature of contact.

V. The status of the debate about indivisibles

An inconclusive conclusion

In recent years, historians of philosophy have begun in earnest to mine the wealth of medieval literature on indivisibles, continuity, and contiguity; and some are attending to these themes in the moderns as well. But only a few contemporary philosophers have addressed themselves directly to the problems of boundaries; only a few have been so bold as to suppose that the old questions are perfectly sensible and still deserving of answers.[100] In fact, serious and intense concern about the boundaries of solid bodies effectively ceased around the beginning of this century. The most obvious explanation for its sudden decline is a series of radical changes in physical theory. In conclusion, I offer (i) a sketch of some of the changes in science which were bound to make the older problems seem rather passé; and (ii) the assertion that the problems about indivisibles and extended objects which captivated philosophers a few hundred years ago have not become completely irrelevant in the light of modern physics.

Extended solids and the new physics

The time was when nearly everyone agreed that the furniture of our world includes numerous three-dimensionally extended substances, substances which are "solid" in the sense of completely filling a precisely determinate three-dimensional location. Philosophers naturally wondered about the nature of the edges and surfaces of such bodies, and about relationships of contact or continuity that might hold between two of them or among the parts of a single body. Newtonian atomism did not completely overturn this picture. If matter consists of three-dimensionally extended atoms, then questions about the boundaries of these objects will still arise. Even if one's physics says that two such atoms never actually touch, at least the left and right halves of a single atom must be touching — and this was enough, as can be seen from Boscovich's case, to raise all the important questions about how distinct extended bodies can be in contact.[101] Even the attempt to identify particles with perturbations in an underlying ether, a Cartesian move that was widely adopted in the 19th century, left intact all the important questions about the nature of boundaries — for the moveable portions of this pervasive fluid must come in and out of contact with one another. Indeed, ether theories showed a marked tendency to "go atomistic", so that even "fluid theories of matter" often ended up explaining physical phenomena in terms of extended particles with empty space between them.[102]

The 19th and 20th century, however, witnessed the gradual "dematerialization of matter".[103] More radical theories became prominent, theories which threatened to render the older questions about the boundaries of solid objects inapplicable to any actual physical entities. Faraday's Boscovichian dynamism treated the "ultimate atoms" of matter "as centres of force, and not as so many little bodies surrounded by forces…. In the latter view, these little particles have a definite form and a certain limited size; in the former view such is not the case…."[104] Faraday's conception altogether eliminated extended substances with definite boundaries; and it is hard to see how traditional questions about the boundaries of solids could be reformulated so as to be applicable to the "lines of force" emanating from each particle's point-sized center.

Related pressures came from Maxwell's electro-magnetic field theory. To many, there seemed something unseemly about the positing of both fields of force and particles as equally fundamental.[105] This conviction led to the development of theories according to which particles are something like "energy knots" propagating in fields.[106] Even more radically, some suggested that particles could be identified with "disturbances" in the fabric of a substantival space-time.[107] The boundaries of "energy knots" or "disturbances" need not, it would seem, be precise. And on either view particles are made out to be "modes" of an underlying something, be it field or space. Now space-time obviously is, and fields are most naturally taken to be, four-dimensional entities. And problems of contact are apt to seem less serious in this context. In the geometrodynamic case, serious questions about contact may still arise: for instance, we can take any connected three-dimensional, space-like part of space-time and ask, what is the relationship between the left and right halves of this chunk of space? However, the fact that the parts of a four-dimensional region of space-time cannot move around in any meaningful sense makes this sort of question less pressing: one can answer without much embarrassment that the chunk of space cannot really be split into disjoint halves which together comprise the whole space; for one must not forget the two-dimensional plane separating the halves. It is not so hard to be Bolzano if one's closed and open objects are "frozen in place".

Even though no concrete attempt to work out a field or geometrodynamic theory of matter has survived for long,[108] quantum phenomena have effectively brought into question the very notion of a precisely located material particle. Suppose quantum theory really implies that matter does not come in precisely located extended bits (a significant assumption, for at this juncture there is no settled opinion as to the kind of metaphysics quantum theory requires).[109] If particles are spread out like waves, and what boundaries they do have are fuzzy, then there need be no precise moment at which two particles are in contact — indeed, the whole notion of contact becomes problematic if the fields in question may interpenetrate or grow fuzzy around the edges.

There is little doubt, then, that some of the central questions about contact and the boundaries of solid extended objects do not have any analogues within modern conceptions of matter. But the significance of this fact should not be overestimated. For we can still ask ontological questions about the reality of indivisible parts in either a substantival space-time or a substantial field. And, more significantly, no one should feel comfortable suggesting that any of these amazing (and potentially transient) theories is <u>necessarily true</u>. Surely matter did not <u>have</u> to come in such strange, particle-wave packaging. And why should one suppose that the only kind of substances there could be are the parts of a substantival space-time or of a number of space-filling substantial fields? Why only such big, unified substances? Why not smaller, discrete, extended substances, of the traditional sort? And once the bare possibility of such objects is granted, all the old questions become at least intelligible again, and lead naturally to the kind of metaphysical arguments surveyed above.

Of course, given the widely held opinion that modern physics rules out discrete extended solids, one should not expect contemporary metaphysicians to take too great an interest in these matters — no more than one expects materialists who recognize the bare possibility of disembodied souls to spend much time debating whether or not such souls could be spatially located. However, I have tried to show elsewhere that there is considerably more at stake here than one might think. For even the recognition of the <u>possibility</u> of atomless gunk — rejected by full-fledged indivisibilists, but required by moderate indivisibilists and anti-indivisibilists alike — eliminates certain otherwise attractive options in the metaphysics of organisms and artifacts.[110] Since a great deal turns upon whether or not atomless gunk is really possible, the philosophical problems about the physical boundaries of (perhaps merely possible) precisely located solids are considerably more than just so many dusty pages in topology's prehistory.

NOTES

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* Special thanks are due to Alfred J. Freddoso, my tireless Suarez informant. The translations from Suarez that appear below are basically his; but if they contain any errors, blame me. I am also grateful to Leopold Stubenberg for information about Brentano's "Zur Lehre von Raum und Zeit"; and to Peter Roeper and Peter Forrest for helpful correspondence and the provision of unpublished materials.

[1] It is noteworthy that Bolzano and Cantor, situated at the very headwaters of modern topology, were familiar with the scholastic debates in which these questions arose (cf. Donald A. Steele's historical introduction to Bolzano's Paradoxes of the Infinite, trans. by Steele (London: Routledge and Kegan Paul, 1950), p. 33).

[2] R. L. Wilder, "Evolution of the Topological Concept of 'Connected'", American Mathematical Monthly, 85 (1978), pp. 720-26; quotation from p. 721.

[3] Bolzano, Paradoxes, §§38, 66, and 67. Franz Brentano's work on continuity resembles Bolzano's in blending mathematical and physical concerns. Brentano assumes that definitions of continuity like those found in Cantor, Dedekind, and Poincaré should be judged by their adequacy to our concepts of real continua, such as space, time, and extended bodies. Cf. Brentano, Philosophical Investigations on Space, Time and the Continuum, trans. by Barry Smith (London: Croom Helm, 1988), pp. 39-44 and 138-149 ("Addendum to the treatise on what is continuous" and "Nativistic, Empiricist and Anoetistic Theories of our Presentation of Space"). But Brentano's work, unlike Bolzano's, could hardly have had an impact upon the development of topology (cf. note 14 below).

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[4] The dimension theory built upon Poincaré's ideas by Brouwer, Menger, and Urysohn quickly became a central branch of the discipline. For a succinct account of its origins, cf. Witold Hurewicz and Henry Wallman, Dimension Theory (Princeton: Princeton University Press, 1948), ch. 1.

[5] The phrases occur in a passage from Gregory of Rimini's In secundum Sententiarum, quoted by Duhem in his Medieval Cosmology: Theories of Infinity, Place, Time, Void and the Plurality of Worlds, trans. by Roger Ariew (Chicago: University of Chicago Press, 1985), pp. 25-26. Poincaré's original suggestion is found in section two of "Pourquoi l'Espace à Trois Dimensions", Revue de Métaphysique et de Morale 20 (1912), pp. 482-504. For an interesting early metrical definition of one-, two-, and three-dimensional body, cf. Bolzano, Paradoxes, p. 134 (footnote to §40).

[6] Cf. Hurewicz and Wallman, Dimension Theory, pp. 4-5; and Karl Menger, "What is Dimension?", American Mathematical Monthly 50 (1943), pp. 2-7.

[7] I have in mind the many geometrical arguments showing a one-to-one correspondence between the points in geometrical or physical entities of quite different sizes. John Duns Scotus, for example, pointed out that, if lines and curves were composed of infinitely many indivisible points, two concentric circles would have to contain the same number of points; he also adapted an argument of Roger Bacon's, involving one-to-one correlations between, for instance, the indivisibles in the side of a square and the diagonal of the square. Scotus, and many following him, took these arguments to show that lines are not composed of points (cf. Duhem, Medieval Cosmology, pp. 18-35). Later considerations of such arguments may be found in Galileo, Isaac Barrow, and Berkeley, among many others (cf. Galileo, Dialogues Concerning Two New Sciences, trans. by Henry Crew and Alfonso de

Salvio (New York: McGraw-Hill, 1963), pp. 20-40; and the discussion and references in Robert Fogelin, "Hume and Berkeley on the Proofs of Infinite Divisibility", Philosophical Review 97 (1988), pp. 47-69). There are striking parallels between Suarez's and Bolzano's treatments of the same sort of geometrical arguments (cf. Francisco Suarez, Disputationes Metaphysicae (Salamanca, 1597), modern edition ed. by Carolo Berton as vols. 25 and 26 of Suarez, Opera Omnia: Nova Editio (Paris: 1866; reprinted in two volumes at Hildesheim: 1965), disputation 40, sect. 5, paras. 48-49; and Bolzano, Paradoxes, pp. 146-149 (§48)).

[8] Suarez says that all the lines lying on a surface can be regarded as a single connected line that runs back and forth, covering the whole of the surface. Cf. Suarez, Disputationes, paras. 48-49 (all citations of the Disputationes by paragraph refer to disputation 40, section 5) . Compare also Bolzano, Paradoxes, p. 147 (§48).

[9] Cf. Duhem, Medieval Cosmology, p. 24

[10] Euler describes the controversy of his day between the "monadists", led by Christian Wolff, who held that everything is made out of unextended atoms; and those, like Euler, who believed matter to be infinitely divisible and not composed of unextended parts. He says the dispute "forced its way into company of every description, that of the guard-room not excepted. There was scarcely a lady at court who did not take a decided part in favour of monads or against them. … The Royal Academy of Berlin took up the controversy, and being accustomed annually to propose a question for discussion, and to bestow a gold medal, of the value of fifty ducats, on the person who, in the judgment of the Academy, has given the most ingenious solution, the question respecting monads was selected for the year 1748." The prize, Euler tells us, went to a "Mr. Justi", an anti-monadist. "You may easily

imagine how violently this decision of the Academy must have irritated the partisans of monads, at the head of whom stood the celebrated Mr. Wolff" (Leonard Euler, Letters of Euler on different subjects in Natural Philosophy (a.k.a. "Letters to a German Princess"), trans. by Henry Hunter (New York: J. and J. Harper, 1833), vol. II, pp. 39-40).

[11] For interesting contemporary philosophical discussions of less-than-three-dimensional parts and problems of contact, cf. Anthony Quinton, "Matter and Space", Mind 73 (1964), pp. 332-52; David Sanford, "Volume and Solidity", Australasian Journal of Philosophy 45 (1967), pp. 329-40; Adolf Grünbaum, Philosophical Problems of Space and Time (New York: Alfred A. Knopf, 1963), ch. 6; David Kline and Carl A. Matheson, "The Logical Impossibility of Collision", Philosophy 62 (1987), pp. 509-515; and Chris Mortensen, "The Limits of Change", Australasian Journal of Philosophy 63 (1985), pp. 1-10. For brief passages bearing on these questions, cf. also Richard Cartwright, "Scattered Objects" in his Philosophical Essays (Cambridge: MIT Press, 1987), pp. 171-186, esp. 171-173; Chris Mortensen and Graham Nerlich, "Physical Topology", Journal of Philosophical Logic 7 (1978), pp. 209-223, esp. 221-222; and A. P. Hazen, "Slicing it Thin", Analysis 53 (1993), pp. 189-192, esp. 192. Although much of this work is important and very sophisticated, it is also surprisingly cut off from the tradition to be examined here. Kant and Brentano are the only participants in the older debate whose work on point-sized parts and boundaries has cast significant shadows into the second half of this century. For evidence of Kant's influence, cf. C. D. Broad, "Kant's Mathematical Antinomies", Proceedings of the Aristotelian Society 55 (1954-55), pp. 1-22, esp. 19-20; and James Van Cleve, "Reflections on Kant's Second Antinomy", Synthese 47 (1981), pp. 481-494, esp. 490-491. For Brentanian theories of boundaries and continuous bodies, cf. Roderick M. Chisholm, "Boundaries as Dependent Particulars", Gräzer Philosophische Studien 20 (1983), pp. 87-95; and H. Scott Hestevold, "Boundaries, Surfaces, and Continuous

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Wholes", Southern Journal of Philosophy 24 (1986), pp. 235-45. For a Brentanian approach to continuous space, cf. Barry Smith, "The Formal Ontology of Space: An Essay in Mereotopology", in L. Hahn, ed., The Philosophy of Roderick M. Chisholm (Library of Living Philosophers) (La Salle: Open Court, forthcoming).

[12] Its first appearance is in "La Théorie Relationniste de l'Espace", Revue de Métaphysique et de Morale 23 (1916), pp. 423-454 (read at Le Premier Congrès de Philosophie mathématique, Paris, April 8, 1914). A few excerpts from the paper are translated by Janet A. Fitzgerald in an appendix to her study, Alfred North Whitehead's Early Philosophy of Space and Time (Washington, D.C.: University Press of America, 1979), pp. 167-178. (For a complete translation of "La Théorie Relationniste", cf. Patrick J. Hurley, "Whitehead's 'Relational Theory of Space' — Text, Translation, and Commentary", Philosophy Research Archives 5 (1979).) This first presentation was followed by three English essays — published together in The Organisation of Thought, Educational and Scientific (London: Williams and Norgate, 1917), chs. 6, 7, and 8 — entitled "The Organisation of Thought" (first published in 1916), "The Anatomy of Some Scientific Ideas" (not previously published), and "Space, Time, and Relativity" (first published in 1915). These three chapters are reprinted in The Aims of Education and Other Essays (New York: MacMillan, 1929). For a discussion of Whitehead's views during these years, cf. Victor Lowe, Understanding Whitehead (Baltimore, Maryland: Johns Hopkins Press, 1966), pp. 178-186.

[13] Brentano's most sustained discussion of these matters dates from 1914 ("On what is continuous", Space, Time and the Continuum, pp. 1-38). Cf. also "The part-whole relation with reference to collectives, continua, and accidents" (1915), "On that which is relative to something" (1915), and "The third draft of the theory of categories" (1916), in Theory of

Categories, trans. by Roderick M. Chisholm and Norbert Guterman (The Hague: Martinus Nijhoff, 1981), pp. 54-57, 125-131, and 188-207, respectively; and "On Ens Rationis" (1917), in Psychology from an Empirical Standpoint, ed. Oskar Kraus, English edition ed. by Linda L. McAlister, trans. by Antos C. Rancurello, D. B. Terrell, and Linda L. McAlister (New York: Humanities Press, 1973), pp. 339-368.

[14] The earliest publication known to me in which Brentano discusses boundaries and continua is the posthumous "Zur Lehre von Raum und Zeit", Kant-Studien 25 (1920), pp. 1-23; but it contains only a comparison of the present instant (characterized as the boundary of non-existent future and past continua) with the utmost tip of a disappearing cone (which, at the last moment, is also the boundary of a non-existent continuum). A few more details slip out in 1924 at the back of Oskar Kraus's edition of the Psychologie (cf. Psychology from an Empirical Standpoint, pp. 353-358); a substantially clearer picture is available by 1933 with the publication of Kategorienlehre (cf. The Theory of Categories, pp. 54-57, 128-129, 157-158, and 200-202); but his most thorough treatments of the subject are found only in Philosophical Investigations on Space, Time and the Continuum, German edition first published in 1976.

[15] Cf. Whitehead, "La Théorie Relationniste", pp. 426-29; and Fitzgerald, Alfred North Whitehead's Early Philosophy of Space and Time, p. 170-74. He repeats the claim that certain central problems of contact are insoluble in "The Anatomy of Some Scientific Ideas", The Aims of Education, pp. 223-24.

[16] For some details, cf. Duhem, Medieval Cosmology, pp. 18-35; John E. Murdoch, "Infinity and Continuity", in Norman Kretzmann, Anthony Kenny, and Jan Pinborg, eds., The Cambridge History of Later Medieval Philosophy (Cambridge: Cambridge University

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Press, 1982), pp. 564-591; Murdoch, "William of Ockham and the Logic of Infinity and Continuity", in Norman Kretzmann, ed., Infinity and Continuity in Ancient and Medieval Thought (Ithaca: Cornell University Press, 1982), pp. 165-206; and Jack Zupko, "Nominalism Meets Indivisibilism", Medieval Philosophy and Theology 3 (1993), pp. 158-185.

[17] Murdoch, "Infinity and Continuity", pp. 575-577.

[18] For a discussion of Scotus's views, cf. Duhem, Medieval Cosmology, pp. 18-21. The details of Suarez's version of moderate indivisibilism are discussed in section III, below.

[19] David Lewis calls something "atomless gunk" if it is "an individual whose parts all have further proper parts" (Parts of Classes (Oxford: Basil Blackwell, 1991), p. 20).

[20] Ockham's anti-indivisibilist followers include Durandus de Sancto Porciano, Gregory of Rimini, Adam Wodeham, John Buridan, Albert of Saxony, Thomas Bradwardine, and William Heytesbury. Cf. Duhem, Medieval Cosmology, pp. 18-35; Zupko, "Nominalism Meets Indivisibilism"; and Murdoch, "Infinity and Continuity", pp. 574-75.

[21] For a description of Wolff's views, cf. Jean École, La métaphysique de Christian Wolff (Hildesheim: Georg Olms, 1990), III^ème partie, ch. 5, and IV^ème partie, ch. 3. Berkeley and Hume, unlike Wolff, carry on their discussion in terms of extended and simple sensibilia. For discussion of their views, cf. Fogelin, "Hume and Berkeley"; Phillip Cummins, "Bayle, Leibniz, Hume and Reid on Extension, Composites and Simples", History of Philosophy Quarterly 7 (1990), pp. 299-314; Donald L. M. Baxter, "Hume on Infinite

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Divisibility", History of Philosophy Quarterly 5 (1988), pp. 133-140; and Robert Gray, "Berkeley's Theory of Space", Journal of the History of Philosophy 16 (1978), pp. 415-434.

[22] Paradoxes of the Infinite, pp. 128-131, §38

[23] Cf. Nicolas Malebranche, The Search after Truth, trans. by Thomas M. Lennon and Paul J. Olscamp (Columbus, Ohio: Ohio State University Press, 1980; originally published in 1674), pp. 38-39 (bk. 1, ch. 8, § 2); René Descartes, Principles of Philosophy (1644), pp. 177-291 in vol. I of Descartes, The Philosophical Writings of Descartes, trans. by John Cottingham, Robert Stoothoff, and Dugald Murdoch (Cambridge: Cambridge University Press, 1988), pp. 231-232 (Part 2, §20); and Euler, Letters, vol. II, pp. 31-64 (Letters 7-17).

[24] Cf. Boscovich, A Theory of Natural Philosophy, trans. by J. M. Child (Cambridge: MIT Press, 1966). For references and caveats concerning Leibniz's views about extended substance, cf. Phillip Cummins, "Bayle, Leibniz, Hume and Reid", pp. 302-306; J. E. McGuire, "'Labyrinthus continui': Leibniz on Substance, Activity, and Matter", in Motion and Time, Space and Matter, ed. by Peter K. Machamer and Robert G. Turnbull (Ohio State University Press, 1976), pp. 290-326; and Glenn A. Hartz and J. A. Cover, "Space and Time in the Leibnizian Metaphysic", Noûs 22 (1988), pp. 493-519. According to McGuire and Hartz and Cover, the "mature" Leibniz held that physical bodies do not have the kind of continuous extension possessed by (purely "ideal") space; physical bodies, although in some sense phenomenally extended, are in fact aggregates composed of discrete monads.

[25] Cf. the Physical Monadology (trans. by Lewis White Beck in his Kant's Latin Writings: Translations, Commentaries, and Notes (New York: Peter Lang, 1986), pp. 110-134); and

also brief remarks in Dreams of a Spirit-Seer, 2nd ed., trans. Emanuel F. Goerwitz (London: New-Church Press, 1915), pp. 47-48. In Metaphysical Foundations of Natural Science, the critical Kant's matter still fills regions entirely by means of repulsive forces; but there is no longer talk about unextended "elements of matter" which exert these forces — the forces are like the ghosts of departed monads. Perhaps we can think of the monads as having beaten a hasty retreat into the noumenal realm. (Cf. Metaphysical Foundations, ch. 2, in Kant, Philosophy of Material Nature, trans. by James W. Ellington (Indianapolis: Hackett, 1985), pp. 40-94. For a helpful discussion of Kant's early views on physical simples and a compelling reading of the difficult Spirit-Seer, cf. Alison Laywine, Kant's Early Metaphysics and the Origins of the Critical Philosophy (North American Kant Society Studies in Philosophy, vol. 3) (Atascadero: Ridgeview, 1993).)

[26] Bayle, Historical and Critical Dictionary: Selections, trans. by Richard H. Popkin (Indianapolis: Bobbs-Merrill, 1965), p. 363 (Remark G in Bayle's entry on Zeno of Elea); for his rejection of moderate indivisibilism, cf. p. 370.

[27] Bayle, Dictionary, p. 372. (Here he is quoting, with approval, a passage from Nicole and Arnauld's treatment of infinite divisibility in the Port-Royal Logic, part IV, ch. 1.)

[28] These premises would include: a very weak version of relationism about space, the contingency of laws of nature, and the thesis that particulars have only categorical properties necessarily.

[29] Cf. École, La métaphysique de Christian Wolff, pp. 191-203, 238-39, and 242-45.

[30] École, La métaphysique de Christian Wolff, pp. 191-192.

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[31] Euler, Letters, vol. II, pp. 33-42 (letters 8, 9, and 10). Kant gives a geometrical argument in the Physical Monadology for the conclusion that a "physical line" must be infinitely divisible and cannot, therefore, be made out of "simple" and "primitive parts of matter" (Beck, Kant's Latin Writings, pp. 118-120).

[32] Cf. Fogelin, "Hume and Berkeley".

[33] Hermann Weyl, Philosophy of Mathematics and Natural Science (Princeton: Princeton University Press, 1949), p. 43. Cf. also Wesley C. Salmon, Space, Time, and Motion, 2nd ed. (Minneapolis: University of Minnesota Press, 1980), pp. 64-66.

[34] Cf. Peter Forrest, "Is Space-Time Discrete or Continuous? — An Empirical Question", Synthèse, forthcoming.

[35] Cf. Stump, "Theology and Physics in De sacramento altaris: Ockham's Theory of Indivisibles", in Kretzmann, ed., Infinity and Continuity in Ancient and Medieval Thought, pp. 207-230, esp. 211; for Aquinas's quite similar theory of the Eucharist, cf. Summa theologica 3, q. 76, a. 3.

[36] For a discussion of whether universals really have "multiple location in space" (and the contrast with physical objects which are said to always have "divided location in space" — i.e., to fill a region by having a part for each part of the region occupied), cf. Nicholas Wolterstorff, On Universals: An Essay in Ontology (Chicago: University of Chicago Press, 1970), pp. 223-34.

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[37] According to Simplicius, Democritus' atoms "have all sorts of forms and shapes, and differences in size"; and according to Aetius Doxographus, he believes "that the division stops at the indivisible substances and does not continue to infinity". Cf. Milton C. Nahm, Selections from Early Greek Philosophy, pp. 156 and 160.

[38] Perhaps wrongly; cf. Sorabji, Time, Creation, and the Continuum (Ithaca, New York: Cornell University Press, 1983), pp. 371-72.

[39] Bayle, Dictionary, p. 360.

[40] For more on this thesis, and the force of "occupiable", cf. van Inwagen, "The Doctrine of Arbitrary Undetached Parts", Pacific Philosophical Quarterly 62 (1981), pp. 123-37.

[41] Euler, Letters, vol. II, pp. 42-43.

[42] One might think that the "spread out" electrons posited by quantum theory are actually extended simples. It seems to me, however, that they could not really qualify as simple physical objects filling extended regions, for: (a) it is wave-like entities or "aspects" that are said to fill extended regions, and a wave is naturally thought of as filling a region by having different parts in different locations; and (b) in any case it is doubtful whether the region filled is a precisely determinate one, which puts a considerable distance between them and this older debate about the nature of solid objects with sharp boundaries (cf. section V, below).

[43] Cf. Murdoch, "Infinity and Continuity", pp. 575-76.

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[44] Bayle, Dictionary, pp. 359-360 (Remark G in Bayle's entry on Zeno of Elea). For some details, cf. John E. Murdoch, "Infinity and Continuity", pp. 575-84.

[45] Cf., for example, the arguments in Ockham's De sacramento altaris (recounted in Stump, "Theology and Physics in De sacramento altaris", pp. 218-221).

[46] Cf. Duhem, Medieval Cosmology, pp. 18-35.

[47] Aristotle, De Generatione et corruptione bk. I, ch.. 2, 316^a15-317^a17; and Metaphysics bk. III, ch. 4, 1001^b1-20. For typical endorsements, cf. Suarez, Disputationes, para. 35; and Arnauld and Nicole, The Port-Royal Logic, trans. by Thomas Spencer Baynes (Edinburgh and London: William Blackwood and Sons, 8th edition), p. 306 (part IV, ch. 1).

[48] Cf. Adolf Grünbaum, Philosophical Problems of Space and Time, ch. 6; and Grünbaum, "A Consistent Conception of the Extended Linear Continuum as an Aggregate of Unextended Elements", Philosophy of Science 19 (1952), pp. 288-306.

[49] Grünbaum, Modern Science and Zeno's Paradoxes (Middletown, Conn.: Wesleyan University Press, 1967), p. 133.

[50] Grünbaum, Modern Science and Zeno's Paradoxes, p. 115.

[51] Grünbaum has not, I think, shown that there could not possibly be other ways of avoiding this paradox; but if, with Russell, one supposes that a continuous physical space could consist of a merely denumerably infinite number of points, one must show exactly how the paradox is to be avoided. So far, it seems, no one has done this. For a brief discussion, cf.

Ben Rogers, "On Discrete Spaces", American Philosophical Quarterly 5 (1968), pp. 117-123, esp. 118-119.

[52] Brentano was sufficiently scandalized by the weaker result that, "if the line consisted in its points", then "the points of a smaller line can be set into mutual one-one correspondence with those of a larger line, those of one half line, for example, with those of the whole." Thus "the lines themselves would be brought into coincidence through the bringing of their points into coincidence piece by piece, which is impossible" (Space, Time and the Continuum, p. 146 ("Nativistic, empiricist and anoetistic theories of our presentation of space", §8)).

[53] For an informal but thorough discussion of the proof, cf. Robert M. French, "The Banach-Tarski Theorem", The Mathematical Intelligencer vol. 10, no. 4 (1988), pp. 21-28. For more formal treatments, cf. Thomas J. Jech, The Axiom of Choice (Amsterdam: North-Holland Pub. Co., 1973), pp. 3-6; and Stan Wagon, The Banach-Tarski Paradox (Cambridge: Cambridge University Press, 1985). The initial result appeared in S. Banach and A. Tarski, "Sur la decomposition des ensembles de points en parties respectivement congruents", Fundamenta Mathematicae 6 (1924), pp. 244-77.

[54] Bolzano, Paradoxes, p. 161, §59.

[55] Bolzano, Paradoxes, pp. 167-168, §66.

[56] Bolzano, Paradoxes, p. 167, §66.

[57] Bolzano, Paradoxes, p. 168, §67.

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[58] Bolzano, Paradoxes, p. 168, §67.

[59] Brentano, Space, Time, and the Continuum, pp. 147-148.

[60] Cf. Zimmerman, "Could Extended Objects Be Made Out of Simple Parts?", *Philosophy and Phenomenological Research* 56 (1996).

[61] If one were prepared (as I should not be) to suppose that there could be an ineliminable, non-semantical, and non-conceptual vagueness to the very structure of space, then one need not assume that the only alternatives are continuous space and discrete space. Cf. David H. Sanford, "Infinity and Vagueness", Philosophical Review 84 (1975), pp. 520-35.

[62] Leibniz denied this as well; but in Boscovich there is no hint of phenomenalism, and no rejection of the three-dimensional physical space in which extended objects ought to be located (cf. note 27, above).

[63] Cf. Boscovich, A Theory of Natural Philosophy, pp. 19-46 (§§1-90). Boscovich argues that no extended objects can come into contact with one another, since to do so would produce a discontinuous change in motion (his notion of an extended substance apparently including incompressibility), and discontinuous changes are impossible. Since "whatever occupies a distinct position is itself also a distinct thing", an extended body must always be divisible into distinct parts (p. 44; §§83-84). So an extended object would have to be made out of distinct extended parts in contact, something which is also impossible. For if these parts were in contact, then they could, at least in principle (perhaps only "due to the action of God surpassing the forces of Nature"), come apart; and then they would immediately get

themselves into similar trouble, bumping into one another to create an inadmissible discontinuous change (p. 44; §82). Therefore there are no extended things, only unextended atoms at various distances from one another. Boscovich has a couple of additional reasons for concluding that the ultimate atoms must be extensionless, one based on his principle of "the homogeneity of matter", the other appealing to the greater simplicity of his view which does not need to posit two different kinds of cohesion in nature — one for the proper parts of extended atoms and one for relations between distinct atoms (cf. pp. 43-44; §82).

[64] My Brentanian arguments do not imply that there could be no extended entities made of simples, since they ignore extended entities which lack precise spatial boundaries or which are not infinitely divisible into distinct proper parts. Cf. Zimmerman, "Could Extended Objects Be Made Out of Simple Parts?", section II.

[65] As we shall see, Suarez thought that closed objects could also touch by simply sharing an indivisible boundary element. I shall suggest momentarily that this alternative mode of touching is both superfluous and pernicious.

[66] More recently, Roderick Chisholm, developing some suggestions of Brentano's, formulates a theory of boundaries which clearly implies (i) and (iii), and is almost certainly meant to imply (ii). Although the implication may be unintended, Chisholm's theory can, I believe, be shown to require (iv) as well. Cf. Chisholm, "Boundaries as Dependent Particulars".

[67] Brentano mentions Suarez frequently, and cites the Disputationes Metaphysicae in particular at least twice (Space, Time, and the Continuum, pp. 112 and 52).

[68] Besides sharing the above four theses in common, Brentano and Suarez use similar terminology to describe the difference between inner and outer boundaries. Suarez points out that "a surface can continue a body in two directions" if it is inside a body; whereas if it is on the exterior, it terminates extended parts "in one direction" only. Unlike surfaces, points and lines "can terminate parts in infinitely many directions" — "in the middle of a body, we conceive of there being a coming together and continuation from all sides [in a particular line or point]; by contrast, in the case of those points or lines which are thought of as on the last surface of the body, one conceives of there being a coming together of the parts which meet [in the point or line], not from every direction, but only from a side, or from above or below" (Suarez, Disputationes, para. 16; this discussion occurs as a part of Suarez's exposition of a view he will refute (cf. paras. 18-20); but Suarez himself endorses the doctrines just described.)

Similarly, Brentano says that "every surface which divides a sphere in two halves is an inner boundary, the surface of the sphere an outer boundary"; the inner surface is connected with the rest of the continuum on both sides, but with the outer surface, this connection is "missing on one...side" (Space, Time, and the Continuum, pp. 10-11). In describing his quite non-Suarezian doctrine of the "plerosis" or "fullness" of a boundary, Brentano says that "a point located inside a physical thing serves as a limit in all directions, but a point on a surface or an edge or a vertex serves as a limit only in some directions" (Theory of Categories, pp. 60-61).

[69] Suarez considers three intermediate views, labeled the third, fourth, and fifth opinions (Disputationes, paras. 13-27). The third and fifth are versions of the view that extended objects have all three kinds of indivisibles on their surfaces, but not between their internal

parts; the fourth is the view that the only indivisibles are two-dimensional surfaces, which are only present on the exterior of an object.

[70] Suarez, Disputationes, para. 6.

[71] Suarez, Disputationes, para. 11. Note that, with Brentano and against Bolzano, Suarez insists that it would be crazy to suppose that two extended objects could not touch simply because of the presence or absence of indivisibles on their surfaces. Cf. section II, above; and Zimmerman, "Could Extended Objects Be Made Out of Simple Parts?", sections IV and V.

[72] Suarez actually seems to accept the reality of infinities of points with more equanimity than Brentano. Cf. Disputationes, para. 43; Brentano, Psychology from an Empirical Standpoint, pp. 351-58; and Brentano, The Theory of Categories, p. 55.

[73] Suarez's struggles with this problem are found in Disputationes, paras. 55-57. Compare also Chisholm's discussion of similar difficulties in "Boundaries as Dependent Particulars", pp. 88-89.

[74] Suarez, Disputationes, para. 56.

[75] These problems are discussed in more detail in Zimmerman, "Could Extended Objects Be Made Out of Simple Parts?", section VIII.

[76] Bayle, p. 370. Suarez is probably the primary "school philosopher" Bayle has in mind here.

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[77] For Suarez's final assessment of the debate between moderate indivisibilism and anti-indivisibilism, cf. Disputationes, para. 28.

[78] William of Ockham, Quodlibetal Questions, trans. in two volumes by Alfred J. Freddoso and Francis E. Kelley (New Haven: Yale University Press, 1991), p. 53 (first quodlibet, question 9, reply to argument 2).

[79] Suarez, Disputationes, para. 18.

[80] Suarez, Disputationes, para. 19. Bayle trades in the colorless sphere and plane for a freshly painted cannon ball marking up a table, and turns the example to a more skeptical use. He points out, in the spirit of Suarez, that "[t]he part of the ball that touches the table is a determinate body, and really distinct from the other parts of the ball that do not touch the table", and similarly for "the part of the table that is touched by the ball." He then seems to assume anti-indivisibilism; for he asserts that "[t]hese two parts that touch each other are each divisible to infinity in length, breadth, and depth. Therefore they mutually touch one another according to their depth, and consequently they penetrate one another." Scholastic philosophers, Bayle says, object to this conclusion "daily" in "public disputations"; but the "jargon of distinctions" they use — including, presumably, the distinction Suarez mentions between positive and negative contact — "are only suitable for preventing the disappointment the student's relatives might have had if they had seen him reduced to silence" (Bayle, p. 364). In this instance, I do not think that Bayle's clever barbs go very deep.

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[81] Cf. Eleonore Stump's discussion of Ockham on what it means to say that the parts of a line are joined at a point, in "Theology and Physics in De sacramento altaris", p. 227. Cf. also my definition of contact within an "open-objects metaphysics", in "Could Extended Objects Be Made Out of Unextended Parts?", section VI.

[82] Suarez, Disputationes, para. 67.

[83] Suarez, Disputationes, para. 28.

[84] For a survey of medieval anti-indivisibilist suggestions about how to understand talk of indivisibles in geometry and science, cf. Duhem, Medieval Cosmology, pp. 27-33.

[85] Whitehead, "La Théorie Relationniste", pp. 426-29.

[86] The method is also applied to "apparent objects" in two sorts of "apparent space" (Whitehead, "La Théorie Relationniste", pp. 429-433).

[87] Cf. Whitehead, An Enquiry Concerning the Principles of Natural Knowledge (Cambridge: Cambridge University Press, 1919), part 3; The Concept of Nature (Cambridge: Cambridge University Press, 1920), chs. 3, 4, and 5; and Process and Reality (New York: The Free Press, 1969; originally published in 1929), part 4.

[88] Cf. Russell, The Analysis of Matter (London: George Allen and Unwin, 1927), ch. 28; Nicod, Geometry and Induction (Berkeley: University of California Press, 1970), pp. 21-32 (part 1, ch. 4 of Nicod's Geometry in the Sensible World, first published in 1924); Broad, C. D. Broad, Scientific Thought (Paterson, New Jersey: Littlefield, Adams and Co., 1959;

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first published, 1923), ch. 1; and Tarski, "Foundations of the Geometry of Solids" in his Logic, Semantics, Metamathematics, trans. by J. H. Woodger (Oxford: Clarendon Press, 1956), pp. 24-29. Tarski mentions Whitehead and Nicod, but may also be suggesting that Lesniewski [ACCENT AIGU OVER 'S'] had at least been thinking of something like Whitehead's method independently: "Some years ago [presumably some years before 1927; cf. bibliographical note, Tarski, p. 24] Lesniewski [ACCENT AIGU OVER 'S'] suggested the problem of establishing the foundations of a geometry of solids, understanding by this term a system of geometry destitute of such geometrical figures as points, lines, and surfaces, and admitting as figures only solids…" (Tarski, p. 24).

For another early adoption of Whitehead's method, cf. Theodore De Laguna, "Point, Line, and Surface, as Sets of Solids", Journal of Philosophy 19 (1922), pp. 449-461.

[89] Broad, Scientific Thought, p. 39.

[90] Whitehead, "La Théorie Relationniste", p. 441 (my translation). A suggestion for the treatment of unoccupied ideal points appears in "The Anatomy of Some Scientific Ideas", The Aims of Education, pp. 219-220; but Whitehead soon gave up his old-fashioned relationism about space, in which space is "constituted by relations between objects", and objects are conceived as three-dimensional entities which may be at some distance from one another with nothing in between (cf. Whitehead, "La Théorie Relationniste", p. 429). Encouraged by modern physics, "which presupposes the play of an electromagnetic field throughout space and time", Whitehead repudiated empty space and posited "an ether of events" filling all of space-time (Whitehead, The Concept of Nature, p. 78). By this point in the development of his thought, the sense in which a physical object can be said to be

located in a particular region has become quite complex (cf., for example, Whitehead, An Enquiry Concerning the Principles of Natural Knowledge, pp. 95-99 and 165-189).

[91] Cf. Peter Forrest, "From Ontology to Topology in the Theory of Regions", and Peter Roeper, "Topology as a Theory of Regions", both in this issue. Cf. also Bowman L. Clarke, "Individuals and Points", Notre Dame Journal of Formal Logic 26 (1985), pp. 61-75; G. Gerla and R. Volpe, "Geometry Without Points", American Mathematical Monthly 29 (1985), pp. 707-711; and G. Gerla, "Pointless Metric Spaces", Journal of Symbolic Logic 55 (1990), pp. 207-219. The constructions of Clarke and Roeper are both built upon the same topological notion of "connection". For a criticism of Clarke's theory (one which does not seem to apply to the constructions of Roeper and Forrest), cf. Peter Simons, Parts: A Study in Ontology (Oxford: Clarendon Press, 1987), p. 98.

[92] Cf. Adolf Grünbaum, "Whitehead's Method of Extensive Abstraction", British Journal for the Philosophy of Science 4 (1953), pp. 215-226; Ernest Nagel, Sovereign Reason (Glencoe, Illinois: Free Press, 1954), ch. 9; and V. F. Lenzen, "Scientific Ideas and Experience", University of California Publications in Philosophy 8 (1926), pp. 175-189.

[93] For discussions of this question, and defenses against some objections to the method, cf. May, The Philosophy of Whitehead (London: Allen and Unwin, 1959), ch. VII; and Fitzgerald, Whitehead's Early Philosophy of Space and Time.

[94] Cf. Grünbaum, Philosophical Problems of Space and Time, ch. 6; and Grünbaum, "A Consistent Conception of the Extended Linear Continuum as an Aggregate of Unextended Elements".

[95] The objection was suggested to me by Larry Lombard, and is considered by Cartwright ("Scattered Objects", p. 172).

[96] Cartwright, "Scattered Objects", p. 172.

[97] This is Cartwright's response ("Scattered Objects", pp. 173-174).

[98] For a dissenting opinion, cf. Chris Mortensen, "The Limits of Change", pp. 2-3 (§3).

[99] Cf. Simons, Parts, p. 98; and Forrest, "From Ontology to Topology in the Theory of Regions".

[100] Cf., e.g., the authors cited in note 11, above.

[101] Cf. Boscovich's argument recounted in footnote 63, above.

[102] For a discussion of "fluid theories of matter", cf. Milic Capek [FUNNY MARKS ON 'C'S], The Philosophical Impact of Contemporary Physics (Princeton: D. Van Nostrand Co., 1961), pp. 106-117 and 280-87.

[103] Cf. N. R. Hanson, "The Dematerialization of Matter", in The Concept of Matter, ed. by Ernan McMullin (Notre Dame, Indiana: University of Notre Dame Press, 1963), pp. 549-561; and Milic Capek [FUNNY MARKS ON 'C'S], The Philosophical Impact of Contemporary Physics, chs. XIV-XVI.

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[104] Faraday, "Thoughts on ray vibrations", in Shmuel Sambursky, ed., Physical Thought from the Presocratics to the Quantum Physicists (New York: Pica Press, 1975), p. 415.

[105] Cf. Einstein's remark: "In a particular department of theoretical physics the continuous field thus appeared side by side with the material point as the representative of physical reality. This dualism remains even today, disturbing as it must be to every orderly mind" (Albert Einstein, "Maxwell's Influence on the Evolution of the Idea of Physical Reality", in Ideas and Opinions (New York: Bonanza Books, 1954), pp. 266-70; quotation from p. 268).

[106] In Hermann Weyl's proposed field-theory of matter, for example, an electron becomes an "energy-knot" which 'propagates itself in empty space in a manner no different from that in which a water-wave advances over the surface of the sea; there is no "one and the same substance" of which the electron is composed at all times.' Weyl's (empirically unsuccessful) theory implies that there is "no sharp line of demarcation between diffuse field-energy and that of electrons and atoms" (Weyl, Space-Time-Matter, trans. by Henry L. Brose from the 4th German ed. (New York: E. P. Dutton and Co., 1922; first published, 1920), pp. 202-3).

[107] In 1870, William Kingdon Clifford suggested a geometrodynamic theory of matter according to which the phenomena we associate with the motion of matter through space is really the propagation of variations in the curvature of space itself. Cf. "On the Space Theory of Matter" in Clifford's Mathematical Papers, ed. by Robert Tucker (London: Macmillan and Co., 1882). For a more recent proposal, cf. J. A. Wheeler, Geometrodynamics (New York: Academic Press, 1962).

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[108] J. A. Wheeler, for example, gave up his long-running geometrodynamic project in the early 70s (for a description of Wheeler's change of heart, cf. Adolf Grünbaum, "The Ontology of Empty Space in the Geometrodynamics of Clifford and Wheeler", in Patrick Suppes, ed., Space, Time and Geometry (Dordrecht: D. Reidel, 1973), pp. 268-95, esp. 268-69). For more recent suggestions about how a "geometrical theory of everything" might be worked out, cf. Robert Weingard, "Making Everything Out of Nothing", in Simon Saunders and Harvey R. Brown, eds., The Philosophy of Vacuum (Oxford: Clarendon Press, 1991), pp. 197-216.

[109] For one consideration of the metaphysical consequences of quantum theory, cf. Peter Forrest, Quantum Metaphysics (Oxford: Basil Blackwell, 1988). Forrest has suggested to me (in correspondence) that a number of approaches to quantum indeterminacy still face (in slightly altered form) the problems about precise boundaries raised here: "Quantum theoretic indeterminacy (fuzzyness) need not dissolve the problems with boundaries. For one way to get such indeterminacy is to think of the actual world as indeterminate between various quite determinate possible worlds. In that case the boundary problem seems to occur in these possible worlds." The "many-universes" interpretation of quantum physics, for example, eliminates vagueness of boundaries by consigning each (precisely defined) superposed state to a different "universe". For a discussion of the "many-universes" approach, cf. the interview with David Deutsch in P. C. W. Davies and J. R. Brown, *The Ghost in the Atom* (Cambridge: Cambridge University Press, 1986), pp. 83-105.

[110] Cf. Zimmerman, "Theories of Masses and Problems of Constitution", Philosophical Review 104 (1995), pp. 53-110. Cf. also Theodore Sider, "Van Inwagen and the Possibility of Gunk", Analysis 53 (1993), pp. 285-289. Sider deploys the possibility of

atomless gunk in an attack on the metaphysics of material objects in Peter van Inwagen's Material Beings (Ithaca, New York: Cornell University Press, 1990).