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The Algebraic Creativity in The Neutrosophic Square Matrices

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Abstract: The objective of this paper is to study algebraic properties of neutrosophic matrices, where a necessary and sufficient condition for the invertibility of a square neutrosophic matrix is presented by defining the neutrosophic determinant. On the other hand, this work introduces the concept of neutrosophic Eigen values and vectors with an easy algorithm to compute them. Also, this article finds a necessary and sufficient condition for the diagonalization of a neutrosophic matrix.

Keywords: Neutrosophic matrix, neutrosophic Eigen value, neutrosophic determinant, neutrosophic inverse, diagonalization of neutrosophic matrices

1. Introduction

Neutrosophy is a general form of logic founded by Smarandache to deal with indeterminacy in all fields of knowledge science. We find many applications in, decision making [2,3,23], optimization theory [1], topology [7], medical studies [26,27], energy studies [25], and number theory [16],

Recently, there is an increasing interesting in algebraic applications of neutrosophy such as neutrosophic modules [11,17], spaces [4,18], rings [14,16], and their generalizations [5,6,19].

After the emergence of the neutrosophic logic at 1995 there were a lot of applications to handle the indeterminacy notion. It is common for anyone to say that an unknown data is indeterminate than saying it is not exist as well in mathematics. Because when that the unknown data is not exist to a common mind it means that this data is absent does not exist. However, indeterminacy is suitable, for we can say to any layman, "We cannot determine what you ask for", but we cannot say, "your inquiry is not exist". Therefore, when we are in a moderate position as we cannot perceive $\emptyset$ for un known data, so we felt it is appropriate under these circumstances to introduce the notion of indeterminacy I where $I^2 = I$. Using this indeterminacy, we construct some notion regarding neutrosophic matrices, which can be used in neutrosophic models. Researchers have already defined the concept of neutrosophic matrices and have used them in Neutrosophic Cognitive Maps model and in the Neutrosophic Relational Equations models, which are analogous to Fuzzy Cognitive Map and Fuzzy Relational Equations models respectively.

In [21], Kandasamy et al, proposed for the first time the notion of bi-matrices. Also, a minimal study of their properties can be found in [8,12,13].

In this essay and for the first time sheds the light on the notion of determinant of a neutrosophic matrix, and we find the form of its inverse and illustrate them with examples. Also, we introduce easy algorithms to find Eigen values and vectors for neutrosophic matrices, with a direct application into the problem of diagonalization.

Neutrosophic matrices are useful in the study of indeterminacy and they have many important properties in algebra, from this point of view we introduce this work.

All matrices through this paper are defined over a neutrosophic field $F(I)$.

2. Preliminaries

Definition 2.1 [24]**:** Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $\{x, (\mu_A(x), \delta_A(x), \gamma_A(x)) : x \in X\}$, where $\mu_A(x)$, $\delta_A(x)$ and $\gamma_A(x)$ represent the degree of membership, the degree of indeterminacy, and the degree of non-membership respectively of each element $x \in X$ to the set A .

Definition 2.2 [10]: Let K be a field, the neutrosophic file generated by $\langle K \cup I \rangle$ which is denoted by $K(I) = \langle K \cup I \rangle$.

Definition 2.3 [9]: Classical neutrosophic number has the form $a + bI$ where a, b are real or complex numbers and I is the indeterminacy such that $0 \cdot I = 0$ and $I^2 = I$ which results that $I^n = I$ for all positive integers n.

Definition 2.4 (Neutrosophic matrix) [16]. Let $M_{m\times n} = \{(a_{ij}) : a_{ij} \in K(I)\}$, where $K(I)$ is a neutrosophic field. We call to be the neutrosophic matrix.

3. Main discussion

Definition 3.1:

Let $M = A + BI$ a neutrosophic $n$square matrix, where A and B are two n squares matrices, then M is called an invertible neutrosophic n square matrix, if and only if there exists an n square matrix $S = S_1 + S_2 I$, where S_1 and S_2 are two n square matrices such that

$S \cdot M = M \cdot S = U_{n\times n}$, where $U_{n\times n}$ denotes the $n \times n$ identity matrix.

Definition 3.2:

Let $M = A + BI$ be a neutrosophic n square matrix. The determinant of M is defined as

$detM = detA + I[det(A + B) - detA]$.

Theorem 3.3:

Let $M = A + BI$ a neutrosophic square $n \times n$ matrix, where A , B are two squares $n \times n$ matrices, then M is invertible if and only if$A$ and $A + B$ are invertible matrices and

$M^{-1} = A^{-1} + I[(A + B)^{-1} - A^{-1}]$.

Proof:

If A and $A + B$ are invertible matrices, then $(A + B)^{-1}$, A^{-1} are existed, and

$M^{-1} = A^{-1} + I[(A + B)^{-1} - A^{-1}]$ exists too. Now to prove $M^{-1}$is the inverse of M,

$$MM^{-1} = (A + BI) \cdot (A^{-1} + I[(A + B)^{-1} - A^{-1}])$$

$$= AA^{-1} + I[A(A + B)^{-1} - AA^{-1} + B \cdot A^{-1} + B(A + B)^{-1} - BA^{-1}]$$

$$= U_{n\times n} + I[(A + B)(A + B)^{-1} - U_{n\times n}]$$

$$= U_{n\times n} + I[U_{n\times n} - U_{n\times n}] = U_{n\times n} = M^{-1}M.$$

conversely, we suppose that M is invertible, thus there is a matrix $S = S_1 + S_2 I$, with the property

$M \cdot S = S \cdot M = U_{n\times n}$.

$MS = (A + BI)(S_1 + S_2 I) = AS_1 + I[(A + B)(S_1 + S_2) - AS_1] = U_{n\times n} + 0_{n\times n}$=$SM$. Hence, we get:

(a)$S_1 A = AS_1 = U_{n\times n}, thus\ A\ is\ invertible\ and\ A^{-1} = S_1$.

(b)$(A + B)(S_1 + S_2) - AS_1 = (S_1 + S_2)(A + B) - S_1 A = O_{n\times n}$, thus,

$(S_1 + S_2)(A + B) = (A + B)(S_1 + S_2) = AS_1 = U_{n\times n}$. This implies that $(A + B)$ is invertible.

Theorem 3.4:

M is invertible matrix if and only if $detM \neq 0$.

Proof:

From **Theorem 3.3** we find that M is invertible matrix if and only if $A+B, A$ are two invertible matrices, hence $det[A+B] \neq 0, detA \neq 0$ which means

$detM = detA + I[\det(A+B) - detA] \neq 0$.

Example 3.5:

Consider the following neutrosophic matrix

$M = A + BI = \begin{pmatrix} 1 & -1+I \\ I & 2+I \end{pmatrix}$. Where $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(a)$detA = 2, A + B = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}, \det(A+B) = 3, detM = 2 + I[3-2] = 2 + I \neq 0, hence\ M\ is\ invertible.$

(b) We have $A^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, (A+B)^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}, thus\ M^{-1} = (A^{-1}) + I[(A+B)^{-1} - A^{-1}]$

$= \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} + I \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} - \frac{1}{2}I \\ -\frac{1}{3}I & \frac{1}{2} - \frac{1}{6}I \end{pmatrix}.$

(c) We can compute $MM^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = U_{2\times2}$.

Theorem 3.6:

Let $M = A + BI$ be a neutrosophic n square matrix, were A and B are two n square matrices, then

3.6.1) $M^r = A^r + I[(A+B)^r - A^r]$.

3.6.2)$M\ is\ nilpotent\ if\ and\ only\ if\ A, A+B\ are\ nilpotent.$

3.6.3)$M\ isi\ dempotent\ if\ and\ only\ if\ A, A+B\ are\ idempotent.$

Proof:

(3.6.1) By using mathematical induction, it easy to see $P(r=1)$ is true.

Suppose$P(k)$, then we must prove $P(k+1)$ is true like the following

$M^{k+1} = M^k \cdot M = (A^k + I[(A+B)^k - A^k]) \cdot (A + IB)$

$= A^{k+1} + I[(A^k \cdot B + (A+B)^k \cdot A + (A+B)^k \cdot B - A^k \cdot A - A^k \cdot B)]$

$= A^{k+1} + I[(A+B)^k \cdot (A+B) - A^{k+1}]$

$= A^{k+1} + I[(A+B)^{k+1} - A^{k+1}]$.

(2) $M$is nilpotent $if\ and\ only\ if\ \exists\ r \in N^{+};\ M^{r}=0$, this is equivalent to

$A^{r}+I[(A+B)^{r}-A^{r}]=0$, thus

$A^{r}=(A+B)^{r}=0$. Which is equivalent to

$A, A+B$ are nilpotent.

(3) The proof is similar to (2).

Theorem 3.7:

Let $M=A+BI$ and $N=C+DI$ be two neutrosophic n square matrices, then

(3.7.1) $det(M\cdot N)=detM\cdot detN$.

(3.7.2) $\det(M^{-1})=(detM)^{-1}$.

(3.7.3) $detM$ =1 if and only if $detA=det(A+B)=1$.

Proof:

(a) $M\cdot N=A\cdot C+I[B\cdot C+B\cdot D+A\cdot D]$

$=A\cdot C+I[(A+B)(C+D)-A\cdot C]$.

$\det(M\cdot N)=\det(A\cdot C)+I\big[\det\big((A+B)(C+D)\big)-\det(A\cdot C)\big]$,

$=\det A\cdot\det C+I[\det(A+B)\cdot\det(C+D)-\det(A\cdot C)]$,

$=\det A\cdot\det C+I[\det(A+B)\cdot\det(C+D)-\det A\cdot\det C]$,

$=(\det A+I[\det(A+B)-\det A])\cdot(\det C+I[\det(C+D)-\det C])$,

$=\det M\cdot\det N$.

(b) We have

$\det(MM^{-1})=\det(U_{n\times n})=1, thus\ detM.\det(M^{-1})=1, so\ that\ \det(M^{-1})=(detM)^{-1}$.

(c) $detM=1$ is equivalent to $\det A+I[\det(A+B)-\det A]=1$, thus it is equivalent to

$detA=det(A+B)=1$.

Remark: The result in the section (c) can be generalized easily to the following fact:

$detM=detA$ if and only if $detA=det(A+B)$.

Definition 3.8:

Let $M=A+BI$ be a neutrosophic n square matrix, where A and B are two n squarematrices. M is satisfying the orthogonality property if and only if $M\cdot M^{T}=U_{n\times n}$.

Theorem 3.9:

Let $M = A + BI$ a neutrosophic n square matrix, then

(a) $M\ is$ orthogonal $if\ and\ only\ if\ A, B$ are two orthogonal matrices .

(b) $If\ M\ is$ orthogonal, then $detM \in \{1, -1, -1 + 2I, 1 - 2I\}$.

Proof:

(a) $M\ is$ orthogonal neutrosophic matrix $if\ and\ only\ if\ M^T = M^{-1}$, this is equivalent to

$A^T + B^T I = A^{-1} + I[(A + B)^{-1} - A^{-1}]$, thus

$A^{-1} = A^T, (A + B)^{-1} - A^{-1} = B^T$. This is equivalent to

$A^{-1} = A^T and\ (A + B)^{-1} = B^T + A^{-1} = B^T + A^T = (A + B)^T$. Thus the proof is complete.

(b) If M is orthogonal, we get that $det(M \cdot M^T) = det(U_{n\times n}) = 1$. This implies

$detM \cdot detM^T = 1,$

$(\det M)^2 = 1$, hence

$detM \in \{1, -1, -1 + 2I, 1 - 2I\}$.

Definition 3.10:

Let $M = A + BI$ be a square neutrosophic matrix, we say that M is diagonalizable if and only if there is an invertible neutrosophic matrix $S = C + DI$ such that $S^{-1}MS = D$. Where D is a diagonal neutrosophic matrix$\left(\ i.e.\ d_{ij} = 0\ \ \forall i \neq j, and\ d_{ij} \neq 0\ \ \forall i = j\right)$.

Theorem 3.11:

Let $M = A + BI$ be any square neutrosophic matrix. Then M is diagonalizable if and only if $A, A + B$ are diagonalizable.

Proof:

Consider a diagonalizable neutrosophic matrix M, then there exists an invertible matrix S such that $S^{-1}MS = K(k_{ij})$(3.11,1).

Now, to compute the entries elements k_{ij} , solve (3.1.11) as follows:

$\left[C^{-1} + I[(C + D)^{-1} - C^{-1}]\right](A + BI)(C + DI) = \left[C^{-1} + I[(C + D)^{-1} - C^{-1}]\right]\left[AC + I[(A + B)(C + D) - AC]\right] = C^{-1}AC + I[(C + D)^{-1}(A + B)(C + D) - C^{-1}AC] = D_1 + (D_2 - D_1)I = K$. Where K is a diagonal matrix, thus D_1, D_2 are diagonal, and $A, A + B$ are diagonalizable. Conversely, assume that $A, A + B$ are diagonalizable, then there are C, D, where $C^{-1}AC = D_1, D^{-1}(A + B)D = D_2$. Put $S = C + (D - C)I$.

Now we compute $S^{-1}MS = \left[C^{-1} + I[D^{-1} - C^{-1}]\right](A + BI)(C + (D - C)I)$

$= \left[C^{-1} + I[D^{-1} - C^{-1}]\right]\left[AC + I[(A + B)(D) - AC]\right] = C^{-1}AC + I[D^{-1}(A + B)D - C^{-1}AC]$

$= D_1 + (D_2 - D_1)I = K$. Thus,M is diagonalizable, that is because D_1, D_2 are diagonal matrices.

Remark 3.12:

If C is the diagonalization matrix of A, and D is the diagonalization matrix of $A + B$, then

$S = C + (D - C)I$is the diagonalization matrix of $M = A + BI$.

Example 3.13:

Consider the neutrosophic matrix defined in Example 3.5, we have:

(a) A is a diagonalizable matrix. Its diagonalization matrix is $C = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, the corresponding diagonal matrix is $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, we can see that $C^{-1}AC = D_1$. Also, the diagonalization matrix of $A + B$ is $D = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$, the corresponding diagonal matrix is $D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. It is easy to check that $D^{-1}(A + B)D = D_2$.

(b) Since $A, A + B$ are diagonalizable, then M is diagonalizable. The neutrosophic diagonalization matrix of M is $S = C + (D - C)I = \begin{pmatrix} 1 & 1 - I \\ -\frac{1}{2}I & -1 + 2I \end{pmatrix}$. The corresponding diagonal matrix is

$L = D_1 + I[D_2 - D_1] = \begin{pmatrix} 1 & 0 \\ 0 & 2 + I \end{pmatrix}$.

(c) It is easy to see that $S^{-1} = C^{-1} + I[D^{-1} - C^{-1}] = \begin{pmatrix} 1 & 1 - I \\ \frac{1}{2}I & -1 + 2I \end{pmatrix}$.

(d) We can compute $S^{-1}MS = \begin{pmatrix} 1 & 1 - I \\ \frac{1}{2}I & -1 + 2I \end{pmatrix}\begin{pmatrix} 1 & -1 + I \\ I & 2 + I \end{pmatrix}\begin{pmatrix} 1 & 1 - I \\ -\frac{1}{2}I & -1 + 2I \end{pmatrix}$ $= \begin{pmatrix} 1 & 0 \\ 0 & 2 + I \end{pmatrix} = L$.

Definition 3.14:

Let $M = A + BI$ be a $n$square neutrosophic matrix over the neutrosophic field $F(I)$, we say that $Z = X + YI$ is a neutrosophic Eigen vector if and only if $MZ = (a + bI)Z$. The neutrosophic number $a + bI$ is called the Eigen value of the eigen vector Z.

Theorem 3.15:

Let $M = A + BI$ be a n square neutrosophic matrix, then $a + bI$ is an eigen value of M if and only if a is an eigen value of A, and $a + b$ is an eigen value of $A + B$. As well as, the eigen vector of $M$is $Z = X + YI$ if and only if X is the corresponding eigen vector of A, and $X + Y$ is the corresponding eigen vector of $A + B$.

Proof:

We suppose that $Z = X + YI$ is an eigen vector of M with the corresponding eigen value $a + bI$, hence $MZ = (a + bI)Z$, this implies

$(A + BI)(X + YI) = (a + bI)(X + YI), thus\ AX + I[(A + B)(X + Y) - AX] = aX + I[(a + b)(X + Y) - aX]$. We get:

$AX = aX, (A + B)(X + Y) = (a + b)(X + Y)$, so that X is an eigen vector of A, $X + Y$ is an eigen vector of $A + B$. The corresponding eigen value of X is a, and the corresponding eigen value of $X + Y$ is $a + b$.

For the converse, we assume that X is an eigen vector of A with a as the corresponding eigen value, and $X + Y$ is an eigen vector of $A + B$ with $a + b$ as the corresponding eigen value, so that we get $AX = aX, (A + B)(X + Y) = (a + b)(X + Y)$.

Let us compute

$MZ = (A + BI)(X + YI) = AX + I[(A + B)(X + Y) - AX]$

$= aX + I[(a + b)(X + Y) - aX] = (a + bI)(X + YI) = (a + bI)Z$. Thus $Z = X + YI$ is an eigen vector of M with $a + bI$ as a neutrosophic eigen value.

Theorem 3.16:

The eigen values of a neutrosophic matrix $M = A + BI$ can be computed by solving the neutrosophic equation $det(M - (a + bI)\ U_{n\times n}) = 0$.

Proof:

We have $det(M - (a + bI)U_{n\times n}) = \det([A - aU_{n\times n}] + I[B - bU_{n\times n}])$

$= \det([A - aU_{n\times n}] + I[\det((A + B) - (a + b)U_{n\times n}) - \det[A - aU_{n\times n}]]$. Thus, the equation $det(M - (a + bI)\ U_{n\times n}) = 0$ is equivalent to

$\det([A - aU_{n\times n}] = 0\ (3{,}16{,}1), and\ [\det((A + B) - (a + b)U_{n\times n}) - \det[A - aU_{n\times n}] = 0$ (3.16.2).

From equation (3,16,1), we get a as eigen value of A, and from (3.16.2) we get

$[\det((A + B) - (a + b)U_{n\times n}) = \det[A - aU_{n\times n}] = 0$, thus$a + b$ is an eigen value of $A + B$.

Example 3.17:

Consider M the neutrosophic matrix defined in Example 3.5, we have

(a) The eigen values of the matrix A are {1,2}, and {1,3} for the matrix $A+B$. This implies that the eigen values of the neutrosophic matrix M are

$\{1+(3-1)I, 1+(1-1)I, 2+(3-2)I, 2+(1-2)I\} = \{1+2I, 1, 2+I, 2-I\}$.

(b) If we solved the equation $\det(M-(a+bI)U_{n\times n})$=0 has been solved, the same values will be gotten.

(c) The eigen vectors of A are $\{(1,0),(1,-1)\}$,the eigen vectors of $A+B$ are $\{(1,-1/2),(0,1)\}$. Thus, the neutrosophic eigen vectors of M are

$$\left\{(1,0)+I[(0,1)-(1,0)], (1,0)+I\left[\left(1,-\frac{1}{2}\right)-(1,0)\right], (1,-1)+I[(0,1)-(1,-1)], (1,-1)+\right.$$

$$\left. I\left[\left(1,-\frac{1}{2}\right)-(1,-1)\right]\right\} = \{(1,0)+I(-1,1), (1,0)+I(0,-1/2), (1,-1)+I(-1,2), (1,-1)+I(0,1/2)\} =$$

$$\{(1-I, I), (1, -1/2\, I), (1-I, -1+2I), (1, -1+1/2\, I)\}\ .$$

To determine the neutrosophic eigen vectors using Theorem 3.15. let X be an eigen vector of $A, and\ Y$ be an eigen vector of $A+B$, hence $X+[(Y)-X]I = X+(Y-X)I$ is an Eigen vector of $M = A+BI$.

Conclusion

In this article, we have determined necessary and sufficient conditions for the invertibility and diagonalization of neutrosophic matrices. Also, we have found an easy algorithm to compute the inverse of a neutrosophic matrix and its Eigen values and vectors.

As a future research direction, we aim to find the representation of neutrosophic matrices by linear transformations in neutrosophic vector spaces.

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