On the angular momentum of a system of quantum particles

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September 2, 2013

Abstract

The properties of angular momentum and its connection to magnetic momentum are explored, based on a reconsideration of the Stern-Gerlach experiment and gauge invariance. A possible way to solve the so called spin crisis is proposed. The separation of angular momentum of a quantum system of particles into orbital angular momentum plus intrinsic angular momentum is reconsidered, within the limits of the Schrödinger theory. A proof is given that, for systems of more than two particles, unless all of them have the same mass, the possibility of having eigenvalues of the form $(n + 1/2)\hbar$ is not excluded.
PACS: 03.65.Ta angular momentum, magnetic moment, spin crisis

1 introduction

We have no means to study the states of angular momentum of material quantum systems but through their interaction with *external* electromagnetic fields. The simplest experimental arrangement is that of Stern-Gerlach, were a collimated beam of silver atoms passes through a region where there is a strong non-homogeneous magnetic field. The force acting on the magnetic moment of individual atoms produced the famous two state separation which, years after the original experiment, will be taken as part of the experimental evidence in support of the theory of electronic *spin*. In this paper we get back in time to make a critical analysis of the Stern-Gerlach Experiment.

As we will prove, there have been some misconceptions, involving the relation between spin and magnetic moment, in particular, that have perhaps misleaded Theoretical Physics in the task of building a mathematically consistent theoretical framework to explain those experimental results, as well as in the formulation of a relativistic quantum theory and a quantum electrodynamics that's free of unwelcome mathematical inconsistencies.

In our first section, we prove that it is not true that

$$\hat{\mu} \propto \hat{\mathbf{J}},$$

where

$$[\hat{J}_i, \hat{J}_j] \propto i\epsilon_{ijk}\hat{J}_k,$$

for the components μ_i of the magnetic moment of a quantum system, in the presence of an electromagnetic field. We find in this fact an explanation of the observed anomalies in the magnetic moment of protons and neutrons and of the so called *proton spin crisis* [8, 9].

In our second section we prove that, within the limits of Schrödinger theory, and for isolated systems of three or more particles, it is not true that the projection of the orbital angular momentum and the internal angular momentum—along a given direction in space—have to be, in general, integer multiples of $\hbar$.

2 Angular Momentum and Electrodynamics

Let's start with a refresher of the way angular momentum gets to play a role in electrodynamics. Consider a system of particles restricted to move, for whatever reason, inside a bounded region Ω of space, where there is an external magnetic field. The force on the i^{th} particle is $\mathbf{f}_i = q_i \mathbf{v}_i \wedge \mathbf{B}(\mathbf{r}_i)$. Let $\mathbf{r}$ be a fixed point inside Ω, then, on Taylor's theorem we can approximate

$$\mathbf{f}_i = q_i \mathbf{v}_i(t) \wedge \left(\mathbf{B}(\mathbf{r}) + \frac{\partial \mathbf{B}}{\partial \mathbf{r}} \cdot (\mathbf{r}_i(t) - \mathbf{r}) \right).$$

So the instantaneous net force is made of two terms:

$$\mathbf{F} = \sum_{i=1}^{n} \left[q_i \mathbf{v}_i(t) \wedge \left(\mathbf{B}(\mathbf{r}) - \left(\frac{\partial \mathbf{B}}{\partial \mathbf{r}} \cdot \mathbf{r} \right) \right) + q_i \mathbf{v}_i(t) \wedge \left(\frac{\partial \mathbf{B}}{\partial \mathbf{r}} \cdot \mathbf{r}_i(t) \right) \right]$$

.

The average of the first in a period of time τ that's *long enough* to minimize statistical fluctuations, must be zero: otherwise the system will drift. So, we are left with an <u>AVERAGE FORCE</u>:

$$\begin{aligned} \overline{\mathbf{F}} &= \frac{1}{\tau} \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \left(\sum_{i=1}^{n} \left[q_i \mathbf{v}_i(z) \wedge \left(\frac{\partial \mathbf{B}}{\partial \mathbf{r}} \cdot \mathbf{r}_i(z) \right) \right] dz \right) \\ &= \frac{1}{2\tau} \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \left[\sum_{i=1}^{n} (q_i \mathbf{r}_i \wedge \mathbf{v}_i)\, dz \right] \cdot \frac{\partial \mathbf{B}}{\partial \mathbf{r}} \end{aligned}$$

(For the steps omitted see [3].) If all the particles have the same charge/mass ratio q/m, then

$$\overline{\mathbf{F}} = \frac{q}{2m\tau} \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \left[\sum_{i=1}^{n} (m_i \mathbf{r}_i \wedge \mathbf{v}_i)\, dz \right] \cdot \frac{\partial \mathbf{B}}{\partial \mathbf{r}} = \frac{q}{2m} \overline{\mathbf{L}} \cdot \frac{\partial \mathbf{B}}{\partial \mathbf{r}} \qquad (1)$$

Where $\overline{\mathbf{L}}$ is the <u>AVERAGE ANGULAR KINETIC MOMENTUM</u>.
Remarks:

1. We have assumed the system of charges to be restricted to move inside a fixed region Ω where there is an external constant magnetic field.
2. We are considering here an angular kinetic momentum, which is not necessarily the same physical magnitude as the canonical angular momentum, which is precisely the case in the presence of an external magnetic field.
3. We have averaged the instantaneous force in a period of time which is *long enough* to minimize statistical fluctuations. That's the reason that we have obtained a force that is the gradient of a potential, despite the fact that it is impossible for the magnetic force to do any work at all.

It is precisely for those reasons that the attempt to portrait Schrödinger's theory as failing to explain the result of the Stern-Gerlach experiment as in [2, 4] is unsound: because the atoms in the Stern-Gerlach experiment are in motion; there is an external magnetic field and, therefore, the operator of kinetic angular momentum is not $-i\hbar\mathbf{r}\wedge\vec{\nabla}$; and the average kinetic angular momentum is not the same physical magnitude as the instantaneous kinetic angular momentum. We have explained our second remark before in [7], based on a conspicuous remark by Weyl[1]:

Let's consider the Schrödinger's equation for an elementary particle with mass m and charge e in the presence of an *external* electromagnetic field:

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{(-i\hbar\vec{\nabla} - e\mathbf{A})^2}{2m}\Psi + eV\Psi, \tag{2}$$

where V and $\mathbf{A}$ are the electrodynamic potentials.

As it's well known from electrodynamics, the configuration of the electromagnetic field will not change in the gauge transformation

$$(V, \mathbf{A}) \to \left(V' = V + \frac{\partial\lambda}{\partial t}, \mathbf{A}' = \mathbf{A} + \vec{\nabla}\lambda\right)$$

It is not difficult to prove that if Ψ is a solution of (2) then $\Psi e^{-i\frac{e\lambda}{\hbar}}$ is a solution of

$$i\hbar\frac{\partial\Psi'}{\partial t} = \frac{(-i\hbar\vec{\nabla} - e\mathbf{A}')^2}{2m}\Psi' + eV'\Psi',$$

corresponding to the same physical state, because the corresponding density and current of probability are the same. However:

$$-i\hbar\int\Psi'^{\star}\left(\mathbf{r}\wedge\frac{\partial}{\partial\mathbf{r}}\right)\Psi' = -i\hbar\int\Psi^{\star}\left(\mathbf{r}\wedge\frac{\partial}{\partial\mathbf{r}}\right)\Psi - e\int\Psi^{*}\Psi\,\mathbf{r}\wedge\frac{\partial\lambda}{\partial\mathbf{r}}.$$

In other words: the expected value of the operator $-i\hbar\mathbf{r}\wedge\vec{\nabla}$ depends on the calibration of the electrodynamic potentials and, as a consequence, it cannot be the operator of an observable physical magnitude. Equation (2) gives us the clue to get around this problem. The operator of kinetic momentum $\mathbf{p} = m\mathbf{v}$ must be $-i\hbar\vec{\nabla} - e\mathbf{A}$ and, as a consequence, the operator of kinetic angular momentum must be this

$$\hat{\mathbf{L}} = -\mathbf{r} \wedge (i\hbar\vec{\nabla} + e\mathbf{A}). \tag{3}$$

We come to the same conclussion if we use Ehrenfest theorem, because

$$m\hat{\mathbf{v}} = \frac{i}{h}\left[\hat{\mathcal{H}}, \mathbf{r}\right] = -i\hbar\vec{\nabla} - e\mathbf{A}$$

As we have shown before [7] this substitution resolves the mathematical difficulties but, at the same time, it disrupts our understanding of magnetic momentum, which is based on the commutation relations of the components of the canonical angular momentum operator

$$[\hat{L}_i, \hat{L}_j] \propto i\epsilon_{ijk}\hat{L}_k.$$

It is simply not true that

$$[\hat{\mu}_i, \hat{\mu}_j] \propto i\epsilon_{ijk}\hat{\mu}_k$$

for the components of magnetic moment $\hat{\vec{\mu}}$ in the presence of an electromagnetic field: The configuration of the electromagnetic field determines the eigenvalues of the kinetic angular momentum—as it is clear from (3)—and, correspondingly, it determines also the eigenvalues of the magnetic moment, that plays a central role in atomic and nuclear physics, and in our understanding of the magnetic properties of condensed matter as well. The problem is that we will always have an electromagnetic field where there is a magnetic moment. In general, it is not true that

$$\hat{\vec{\mu}} \propto \hat{\mathbf{J}},$$

where

$$[\hat{J}_i, \hat{J}_j] \propto i\epsilon_{ijk}\hat{J}_k.$$

As an example, most of the *rest mass* of a proton or a neutron is supposed to come from the kinetic energy of the corresponding *quarks*; this suggests high speeds and high intensities of the electromagnetic field inside those particles and, as a consequence, the term $e\mathbf{A}$ might very well be the most important in (3) for protons and neutrons, explaining the huge anomalies observed in their magnetic moment, and suggesting a way to resolve the so called *proton spin crisis* [8, 9].

There are some other misconceptions involving the concept of angular momentum which we will address.

3 Intrinsic Angular Momentum of an Isolated System of Particles in Schrödinger's Theory

To motivate the discussion that follows, let's consider a system of classical particles with masses m_i and charges q_i. We will use the symbols $\mathbf{r}_i$ and $\mathbf{v}_i$ for the position and the velocity of the i^{th} particle, respectively; for the corresponding variables associated to the center of mass—the system *as a whole*—we will use M, Q, $\mathbf{r}$ and $\mathbf{v}$:

$$M = \sum m_i,\ Q = \sum q_i,\ \mathbf{r} = \frac{\sum m_i \cdot \mathbf{r}_i}{M},\ \text{and } \mathbf{v} = \frac{\sum m_i \cdot \mathbf{v}_i}{M}.$$

The total angular momentum of the system $\mathbf{L} = \sum \mathbf{l}^{(i)}$ is:

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \wedge \mathbf{v}_i = M\mathbf{r} \wedge \mathbf{v} + \sum_i m_i \vec{r}_i \wedge \vec{v}_i,$$

where $\vec{r}_i = \mathbf{r}_i - \mathbf{r}$ and $\vec{v}_i = \mathbf{v}_i - \mathbf{v}$ are the position vectors and velocities of the particles, in the system of reference where the center of mass is at rest.

The term

$$\mathbf{L}_o = M\mathbf{r} \wedge \mathbf{v} \tag{4}$$

is the **orbital angular momentum** and

$$\mathbf{L}_s = \sum_i m_i \vec{r}_i \wedge \vec{v}_i \tag{5}$$

is the **internal angular momentum**.

For a system made of two particles it is common to introduce the auxiliary vector

$$\vec{\rho} = \vec{r}_2 - \vec{r}_1 = \mathbf{r}_2 - \mathbf{r}_1. \tag{6}$$

Considering that $m_1 \cdot \vec{r}_1 + m_2 \cdot \vec{r}_2 = 0$ we have

$$\vec{r}_1 = -\frac{m_2}{m_1} \cdot \vec{r}_2$$

which, by virtue of (Eq. 6), implies that

$$\vec{r}_2 = \frac{m_1}{M} \cdot \vec{\rho},$$

and, in a similar fashion

$$\vec{r}_1 = -\frac{m_2}{M} \cdot \vec{\rho}.$$

From those equations we can prove that

$$\mathbf{L}_s = \mu \cdot \vec{\rho} \times \dot{\vec{\rho}}, \tag{7}$$

where $\mu = \frac{m_1 \cdot m_2}{M}$, which is a well known result in classical mechanics, where the solution of a two-body problem (if the potential energy depends only on the distance between the two particles) is reduced to the solution of a single body problem in a central field—whilst the center of mass moves like a free particle—by means of the transformation

$$\mathbf{r} = \frac{m_1 \cdot \mathbf{r}_1 + m_2 \cdot \mathbf{r}_2}{M},\ \vec{\rho} = \mathbf{r}_2 - \mathbf{r}_1.$$

The use of those coordinates has a similar effect in quantum mechanics, where a Schrödinger equation of the form

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m_1} \Delta_{\mathbf{r}_1} \Psi - \frac{\hbar^2}{2m_2} \Delta_{\mathbf{r}_2} \Psi + V(|\mathbf{r}_2 - \mathbf{r}_1|)$$

is transformed into the separable equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2M}\Delta_{\mathbf{r}}\Psi - \frac{\hbar^2}{2\mu}\Delta_{\vec{\rho}}\Psi + V(\vec{\rho})\Psi.$$

The Hamiltonian takes this form in correspondence with the classical decomposition of the kinetic energy:

$$K = \frac{\mathbf{P_r}^2}{2M} + \frac{\mathbf{P}_{\vec{\rho}}}{2\mu}.$$

The corresponding terms in the Hamiltonian are

$$-\frac{\hbar^2}{2M}\Delta_{\mathbf{r}} \text{ and } -\frac{\hbar^2}{2\mu}\Delta_{\vec{\rho}}.$$

There is an analogous decomposition of the total angular momentum:

$$\hat{\vec{L}} = -i\hbar\mathbf{r}\wedge\frac{\partial}{\partial\mathbf{r}} - i\hbar\vec{\rho}\wedge\frac{\partial}{\partial\vec{\rho}} \tag{8}$$

Based on the principle of correspondence, we assume that the first term is the operator of the orbital angular momentum and the second is the operator of internal, or intrinsic, angular momentum of the system, considered as a whole. We can do a little better and prove that the last operator actually corresponds to the total angular momentum:

$$-i\hbar\mathbf{r}_1\wedge\frac{\partial}{\partial\mathbf{r}_1} - i\hbar\mathbf{r}_2\wedge\frac{\partial}{\partial\mathbf{r}_2}$$

We show how to do this, though it might be obvious, because the existence of a mathematical proof is relevant for the work to come. To start we have

$$\mathbf{r}_1 = \mathbf{r} - \frac{m_2}{M}\vec{\rho} \text{ and } \mathbf{r}_2 = \mathbf{r} + \frac{m_1}{M}\vec{\rho}$$

In consequence

$$-i\hbar\mathbf{r}\wedge\frac{\partial}{\partial\mathbf{r}} - i\hbar\vec{\rho}\wedge\frac{\partial}{\partial\vec{\rho}}$$

$$= -i\hbar\frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}\wedge\left(\frac{\partial}{\partial\mathbf{r}_1} + \frac{\partial}{\partial\mathbf{r}_2}\right) - i\hbar\left(\mathbf{r}_2 - \mathbf{r}_1\right)\wedge\left(-\frac{m_2}{M}\frac{\partial}{\partial\mathbf{r}_1} + \frac{m_1}{M}\frac{\partial}{\partial\mathbf{r}_2}\right)$$

$$= -i\hbar\mathbf{r}_1\wedge\frac{\partial}{\partial\mathbf{r}_1} - i\hbar\mathbf{r}_2\wedge\frac{\partial}{\partial\mathbf{r}_2}.$$

Two sets of spherical coordinates can be used to represent the components of $\mathbf{r}$ and $\vec{\rho}$. Therefore, in this case of a system of two particles, because of (8), the allowed values of the projection of either, orbital or internal angular momentum, along any direction in space, are integer multiples of $\hbar$, and the same is true for the total angular momentum.

Let's consider, in general, a system of n particles, introducing the new variables

$$\vec{\rho}_i = \sum_{i=1}^{n}\alpha_{ij}\mathbf{r}_j,$$

where

$$\mathbf{r}_i = \sum_{i=1}^{n} \beta_{ij} \vec{\rho}_j.$$

In other words, we suppose the matrix $(\alpha_{ij})_{n\times n}$ to be invertible and, furthermore:

$$(\alpha_{ij})_{n\times n}^{-1} = (\beta_{ij})_{n\times n}$$

in such manner that:

$$\sum_{k=1}^{n} \alpha_{ik}\beta_{kj} = \sum_{k=1}^{n} \beta_{ik}\alpha_{kj} = \delta_{ij}. \tag{9}$$

As a consequence, we have

$$\frac{\partial}{\partial \mathbf{r}_i} = \sum_{k=1}^{n} \frac{\partial}{\partial \vec{\rho}_k} \frac{\partial \vec{\rho}_k}{\partial \mathbf{r}_i} = \sum_{k=1}^{n} \alpha_{ki} \frac{\partial}{\partial \vec{\rho}_k}$$

and

$$\begin{aligned} -i\hbar \sum_{i=1}^{n} \mathbf{r}_i \wedge \frac{\partial}{\partial \mathbf{r}_i} &= -i\hbar \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \alpha_{ki}\beta_{ij}\vec{\rho}_j \wedge \frac{\partial}{\partial \vec{\rho}_k} = -i\hbar \sum_{j=1}^{n}\sum_{k=1}^{n} \delta_{jk}\vec{\rho}_j \wedge \frac{\partial}{\partial \vec{\rho}_k} \\ &= -i\hbar \sum_{j=1}^{n} \vec{\rho}_j \wedge \frac{\partial}{\partial \vec{\rho}_j} \end{aligned} \tag{10}$$

This is interesting for us, because it rises the question if it is possible to write n linear combinations of the position vectors $\mathbf{r}_i$:

$$\vec{\rho}_i = \sum_{j=1}^{n} \alpha_{ij}\mathbf{r}_j,$$

in such manner that the first of them is the position vector of the center of mass:

$$\vec{\rho}_1 = \frac{\sum_{j=1}^{n} m_j \mathbf{r}_j}{M}$$

the Schrödinger's equation in the new variables takes the form

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2M}\Delta_{\vec{\rho}_1}\Psi + \hat{\mathcal{H}}\left(\vec{\rho}_2, ..., \vec{\rho}_n, \frac{\partial}{\partial \vec{\rho}_2}, \cdots, \frac{\partial}{\partial \vec{\rho}_n}\right)\Psi \tag{11}$$

and

$$\left[\hat{\mathcal{H}}, -i\hbar \sum_{j=2}^{n} \vec{\rho}_j \wedge \frac{\partial}{\partial \vec{\rho}_j}\right] = \hat{\vec{0}}.$$

If we can prove that this problem has a solution, we will be justified in our assumption that Schrödinger's theory predicts that the allowed values of the projection of the internal angular momentum along an arbitrary spatial direction are integer multiples of Planck's constant.Though this can be true for the total angular momentum it is not necessarily true for its components: the orbital angular momentum and the intrinsic angular momentum.

3.1 The Case of Three Particles

To simplify our discussion we consider a system made out of three particles. Then

$$M\vec{\rho}_1 = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 \tag{12}$$
$$\vec{\rho}_2 = \alpha_{21}\mathbf{r}_1 + \alpha_{22}\mathbf{r}_2 + \alpha_{23}\mathbf{r}_3$$
$$\vec{\rho}_3 = \alpha_{31}\mathbf{r}_1 + \alpha_{32}\mathbf{r}_2 + \alpha_{33}\mathbf{r}_3$$

The inverse of this relation, as follows from an almost trivial application of Cramer's rule, is given by:

$$\mathbf{r}_1 = \frac{M\left(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}\right)\vec{\rho}_1 + \left(m_3\alpha_{32} - m_2\alpha_{33}\right)\vec{\rho}_2 + \left(m_2\alpha_{23} - m_3\alpha_{22}\right)\vec{\rho}_3}{m_1\left(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}\right) + m_2\left(\alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33}\right) + m_3\left(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}\right)}$$

$$\mathbf{r}_2 = \frac{M\left(\alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33}\right)\vec{\rho}_1 + \left(m_1\alpha_{33} - m_3\alpha_{31}\right)\vec{\rho}_2 + \left(m_3\alpha_{21} - m_1\alpha_{23}\right)\vec{\rho}_3}{m_1\left(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}\right) + m_2\left(\alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33}\right) + m_3\left(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}\right)}$$

$$\mathbf{r}_3 = \frac{M\left(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}\right)\vec{\rho}_1 + \left(m_2\alpha_{31} - m_1\alpha_{32}\right)\vec{\rho}_2 + \left(m_1\alpha_{22} - m_2\alpha_{21}\right)\vec{\rho}_3}{m_1\left(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}\right) + m_2\left(\alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33}\right) + m_3\left(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}\right)}$$

The transformation (12) has to be invertible; in consequence we require:

$$\begin{vmatrix} m_1 & m_2 & m_3 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{34} & \alpha_{33} \end{vmatrix} = M \tag{13}$$

(The value of this determinant can be fixed at will, with the only condition that it is not zero.)

For an isolated system, the forces can only depend on the differences $\mathbf{r}_i - \mathbf{r}_j$, and not on the position of the center of mass, therefore, we must have that

$$\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} = \alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33} = \alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31} \tag{14}$$

Let's consider now the operator of kinetic energy:

$$\hat{K} = -\frac{\hbar^2}{2M}\left[\left(\frac{\partial}{\partial\mathbf{r}_1}\right)^2 + \left(\frac{\partial}{\partial\mathbf{r}_2}\right)^2 + \left(\frac{\partial}{\partial\mathbf{r}_3}\right)^2\right]$$

From equations (12) we get

$$\frac{\partial}{\partial\mathbf{r}_i} = \frac{m_i}{M}\frac{\partial}{\partial\vec{\rho}_1} + \alpha_{2i}\frac{\partial}{\partial\vec{\rho}_2} + \alpha_{3i}\frac{\partial}{\partial\vec{\rho}_i} +$$

Therefore, to achieve the separation (11) we must have:

$$\sum_{i=1}^{3} m_i\alpha_{2i} = \sum_{i=1}^{3} m_i\alpha_{3i} = 0$$

which means that the formal vector $\vec{\alpha}_1 = (m_1, m_2, m_3)$ must be orthogonal to the formal vectors $\vec{\alpha}_2 = (\alpha_{21}, \alpha_{22}, \alpha_{23})$ and $\vec{\alpha}_3 = (\alpha_{31}, \alpha_{32}, \alpha_{33})$ and, therefore, it is parallel, or anti-parallel to $\vec{\alpha}_2 \wedge \vec{\alpha}_3$. However, according to (14) the components of $\vec{\alpha}_2 \wedge \vec{\alpha}_3$ are identical and, as a consequence, the Schrödinger equation will be separable in the form (11), by a linear transformation of the form (12) if and only if the masses are identical, in the case of three particles.

Notice that the quantities in (14) are the minor determinants D_{11}, D_{12}, and D_{13} of the matrix:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \equiv \begin{pmatrix} m_1 & m_1 & m_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

and that a similar condition will be necessary for an arbitrary number of particles, as well as the orthogonality conditions: the Schrödinger equation will be separable in the form (11), by a linear transformation of the form (12) if and only if the masses are identical. This is a very strong condition which is not even true for protons and neutrons, if we consider the experimental facts in support of the theory that they are made of *up* and *down quarks*[5, 6], with possibly different masses[10]. (Disregarding the fact, apparently, most of the *rest mass* of *nucleons* comes from the kinetic energy of the corresponding quarks and the energy of the *gluon field*.)

The fact that we cannot separate Schrödinger's equation in the form (11) doesn't mean that we cannot decompose the total angular momentum as the sum of an ORBITAL ANGULAR MOMENTUM (of the center of mass) and an INTRINSIC ANGULAR MOMENTUM. We can do that:

$$\hat{\mathbf{L}}_{\text{orbital}} = -i\hbar \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{M} \wedge \left[\sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{r}_i} \right]$$

and

$$\hat{\mathbf{L}}_{\text{intrinsic}} = -i\hbar \sum_{i=1}^{n} \mathbf{r}_i \wedge \left(\frac{\partial}{\partial \mathbf{r}_i} \right) - \hat{\mathbf{L}}_{\text{orbital}}.$$

We can prove that each of those operators satisfies the well known commutation relations. What we cannot guarantee is that the eigenvalues of the projection of the orbital angular momentum or the intrinsic angular momentum along an arbitrary spatial direction will be multiples of $\hbar$: they can be multiples of $\hbar/2$, as follows from the commutation relations. Some additional complications appear when we consider charged particles as we have pointed out before [7].

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