Hugo Albuquerque
Adam Přenosil
Umberto Rivieccio

An Algebraic View of Super-Belnap Logics

Abstract. The Belnap–Dunn logic (also known as First Degree Entailment, or FDE) is a well-known and well-studied four-valued logic, but until recently little has been known about its extensions, i.e. stronger logics in the same language, called super-Belnap logics here. We give an overview of several results on these logics which have been proved in recent works by Přenosil and Rivieccio. We present Hilbert-style axiomatizations, describe reduced matrix models, and give a description of the lattice of super-Belnap logics and its connections with graph theory. We adopt the point of view of Abstract Algebraic Logic, exploring applications of the general theory of algebraization of logics to the super-Belnap family. In this respect we establish a number of new results, including a description of the algebraic counterparts, Leibniz filters, and strong versions of super-Belnap logics, as well as the classification of these logics within the Leibniz and Frege hierarchies.

Keywords: Super-Belnap logics, Four-valued logic, Paraconsistent logic, Belnap–Dunn logic, FDE, Logic of Paradox, Kleene logic, Exactly True logic, De Morgan algebras, Abstract Algebraic Logic, Leibniz filters, Strong versions of logics.

1. Introduction

Since its introduction 40 years ago, four-valued Belnap–Dunn logic, also known as First Degree Entailment, has inspired a remarkable amount of research. Several papers in this extensive literature deal with the algebraic semantics of the Belnap–Dunn logic (denoted by $\mathcal{B}$ throughout the paper), highlighting in particular its connection with De Morgan algebras. However, only very recently has a systematic investigation of the family of *extensions* of $\mathcal{B}$ been attempted. Here by "extensions" we mean logics in the same language as $\mathcal{B}$ (i.e. conjunction, disjunction, and negation, perhaps with truth constants) which are stronger than $\mathcal{B}$. We call these *super-Belnap logics*, by analogy with the well-known family of super-intutionistic logics.

Special Issue: 40 years of FDE
Edited by **Hitoshi Omori and Heinrich Wansing**

Studia Logica (2017) 105: 1051–1086
DOI: 10.1007/s11225-017-9739-7
© Springer Science+Business Media B.V. 2017

As in the case of super-intuitionistic logics, classical logic turns out to be the strongest non-trivial extension of $\mathcal{B}$.

The family of super-Belnap logics includes three prominent systems that have been introduced with rather disparate motivations: $\mathcal{B}$ itself and its two three-valued cousins, the Logic of Paradox $\mathcal{LP}$ and strong Kleene logic $\mathcal{K}$. The logic $\mathcal{B}$ was first studied by Dunn [19,20] as the so-called first-degree fragment of Anderson and Belnap's [5] logic of entailment. Belnap [9,10] then provided a computer scientific motivation for this logic. Specifically, it was suggested by Belnap as a simple framework which a computer could use to handle information which is potentially both inconsistent and incomplete in a non-trivial way. The Logic of Paradox was introduced by Asenjo [6] as a calculus of antinomies and later by Priest [37] with a view to applying to the solution of various semantics paradoxes, such as the Liar Paradox. This logic has been commonly adopted by adherents of dialetheism, the view that some sentences may be both true and false. Finally, Kleene's original motivation for introducing what is now the semantics of strong Kleene logic in [31,32] was to handle partial functions, i.e. functions which are potentially undefined (Kleene himself did not introduce any notion of logical consequence based on his three-valued algebra). Later on, it was famously used by Kripke [33] in his theory of truth. Decidedly less attention was paid to the logic $\mathcal{K}^{\leq} = \mathcal{LP} \cap \mathcal{K}$ called Kleene's logic of order by Font [23] and Kalman implication by Makinson [34]. In one of the few papers dealing with $\mathcal{K}^{\leq}$, Dunn [21] proved that it is precisely the first-degree fragment of the relevance logic R-mingle.

Until recently, the study of super-Belnap logics has been constrained to the examples mentioned above. From a certain point of view, this is quite understandable: they all enjoy a strikingly simple semantics which comes with a transparent philosophical or computational motivation. On the other hand, the emphasis in contemporary logic on the systematic study of families of logics rather than isolated examples, embodied e.g. in the paradigm of Abstract Algebraic Logic (AAL) adopted here, makes it natural to view these logics as part a wider common framework. The wealth of results involving the lattices of superintuitionistic and modal logics suggests that this is a direction worth pursuing. In fact, not only does this perspective suggest new questions and new logics to study, but trying to pinpoint their location within a wider context sheds light even on the familiar examples above.

The impetus for a more systematic study of super-Belnap logics came with the recent discovery of the Exactly True Logic, denoted $\mathcal{ETL}$ here, by Pietz and Rivieccio [36]. A bilattice expansion was also independently introduced by Marcos [35]. The logic $\mathcal{ETL}$ uses the same four-valued algebra of

truth values as the Belnap–Dunn logic but has a different set of designated values: as the name suggests, $\mathcal{ETL}$ preserves the truth-and-non-falsity of the premises, whereas $\mathcal{B}$ is only concerned with preserving their truth (or equivalently, their non-falsity). This naturally suggested the question: are there any other extensions of $\mathcal{B}$ which have so far been overlooked? Rivieccio [44] answered in the positive: perhaps surprisingly, given the paucity of known examples, it turns out that there are infinitely many. In fact, Přenosil [38] showed that $\mathcal{B}$ has a continuum of finitary extensions.

This recent research suggests that the realm of super-Belnap logics is a large and unfamiliar place. The aim of the present paper is to bring the reader up to date with what is currently known about it. Compared to better-known classes such as modal and super-intuitionistic logics, the main peculiarity of the super-Belnap family lies in the weakness of the link between algebra and logic. In technical terms, this is due to the fact that non-classical super-Belnap logics are non-protoalgebraic. The standard methods of algebraization (developed by Blok, Pigozzi, and others) are thererefore not immediately applicable, and other techniques have to be used. This also means that super-Belnap logics form a good testing ground for some recently developed tools of AAL, such as the theory of Leibniz filters and strong versions of propositional logics [2].

Even though a systematic study of super-Belnap logics is a relatively new enterprise, individual super-Belnap logics have been studied from the point of view of AAL already since the 1990s. A Hilbert-style axiomatization of the Belnap–Dunn logic was obtained independently by Pynko [39] and Font [23]. The latter paper contains a comprehensive study of the Belnap–Dunn logic from the perspective of Abstract Algebraic Logic. A Hilbert-style axiomatization of the Logic of Paradox is given in [40]; Pynko also described in [42] the lattice of extensions of $\mathcal{LP}$, proving in particular that it is finite.

In the papers mentioned in the previous paragraph, and many others, the signature of the Belnap–Dunn logic is taken to consist of the conjunction ($\wedge$), the disjunction ($\vee$), and the negation ($\neg$). As explained below (Section 3), we find it convenient to extend this language with the truth constants 1 and 0 representing truth and falsity. The vast majority of the propositions that we will prove do not depend on this choice, and likewise Font's and Pynko's results may easily be seen to hold in both languages.

In this paper we present an overview of the main results of Rivieccio [44] and Přenosil [38], particularly as regards axiomatizations of super-Belnap logics, their reduced models, and the description of the lattice of super-Belnap logics. We then expand on these results in various directions, including a description of the algebraic counterparts, Leibniz filters, and strong

versions of selected logics, as well as the classification of super-Belnap logics in the Leibniz and Frege hierarchies.

The paper is organized as follows. In the following preliminary section, we review the fundamental notions of AAL which will be needed throughout the paper, as well as the basic theory of De Morgan algebras. The notions of strong versions and explosive extensions of logics are also introduced in Section 2. In Section 3 we present an overview of the known facts about super-Belnap logics, due mainly to Pynko [39], Font [23], and Rivieccio [44]. These include the semantic and syntactic characterizations of the best-known systems and a description of their reduced models. We also prove some facts which were originally stated in [44] without proof, such as the completeness theorems for Kleene's logics $\mathcal{K}$ and $\mathcal{K}^{\leq}$. The structure of the lattice of super-Belnap logics, including its connections to graph theory, is then described in Section 4, which draws on [38]. The results presented in Sections 5 and 6 are new except where we state otherwise in the text. In Section 5 we classify all super-Belnap logics within the Leibniz and Frege hierarchies of AAL, describe the algebraic counterparts of different super-Belnap logics, and provide algebraic semantics for the truth-equational logics in this family. Finally, Section 6 studies the implications of the theory of Leibniz filters and strong versions of logics in the super-Belnap setting, including the interactions between strong versions and explosive extensions of logics.

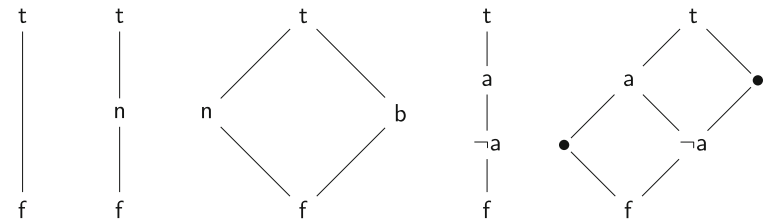
2. Preliminaries

2.1. Universal Algebra

We shall assume familiarity with the basic notions of Universal Algebra. For details we refer the reader e.g. to [15]. We only review some basic notation here: $\mathbb{V}(\mathsf{K})$ denotes the variety generated by a class of algebras K, $\mathbb{Q}(\mathsf{K})$ denotes the quasivariety generated by K, $\mathbb{I}(\mathsf{K})$ denotes the class of isomorphic images of algebras in K, and $\mathbb{P}_{\mathrm{S}}(\mathsf{K})$ denotes the class of subdirect products of algebras in K.

2.2. De Morgan Algebras

We now introduce the classes of algebras which form the algebraic counterparts of the Belnap–Dunn logic. Depending on whether we include the truth constants or not, these are the classes of De Morgan algebras and De Morgan lattices. *De Morgan lattices* are distributive lattices having an order-reversing involution, usually called the De Morgan negation, denoted



$\boldsymbol{B_2}$ $\boldsymbol{K_3}$ $\boldsymbol{DM_4}$ $\boldsymbol{K_4}$ $\boldsymbol{K_3} \times \boldsymbol{B_2}$

Figure 1. Some important De Morgan algebras

$\neg$. *De Morgan algebras* are De Morgan lattices with a top element 1 and a bottom element 0 which are part of the signature. Both of these classes are locally finite varieties.

There are exactly three subdirectly irreducible De Morgan algebras [30], namely the algebras $\boldsymbol{B}_2, \boldsymbol{K}_3$, and $\boldsymbol{DM}_4$ depicted in Figure 1. The negation $\neg$ is in all cases given by reflection with respect to the horizontal axis of symmetry. In particular, the algebras $\boldsymbol{K_3}$ and $\boldsymbol{DM_4}$ contain respectively one and two fixpoints of $\neg$. The algebra $\boldsymbol{K_4}$ will be needed in Theorem 2.1. Note that $\boldsymbol{B_2} \leq \boldsymbol{K_3} \leq \boldsymbol{DM_4}$.

Consequently, there are exactly three non-trivial varieties of De Morgan algebras, namely the whole variety DMA, the subvariety of Kleene algebras KA defined by the inequality $x \wedge \neg x \leq y \vee \neg y$, and the subvariety of Boolean algebras BA defined by $x \wedge \neg x \leq y$. The corresponding varieties of De Morgan lattices will be denoted by DML, KL, and BL.

Pynko [41] proved that there are only finitely many quasivarieties of De Morgan *lattices*: the quasivariety of *non-idempotent* De Morgan lattices NIDML defined by the quasiequation $x = \neg x \Rightarrow x = y$, the quasivariety of non-idempotent Kleene lattices $\mathsf{NIKL} = \mathsf{KL} \cap \mathsf{NIDML}$, the quasivariety $\mathsf{NIDML} \cup \mathsf{KL}$, the quasivariety of *regular* Kleene lattices RKL defined by the quasiequation $\neg x \leq x \ \& \ x \wedge \neg y \leq \neg x \vee y \Rightarrow \neg y \leq y$, and of course the varieties DML, KL, BL, as well as the trivial variety.

Moving to the setting of De Morgan algebras, the corresponding classes will be denoted NIDMA, NIKA, and RKA. Although there is a continuum of quasivarieties of Kleene algebras, as shown by Adams and Dziobiak [1], Pynko's results about the De Morgan lattices generating the above quasivarieties still hold, as observed already by Gaitán and Perea [27]. The following theorem is thus essentially due to Pynko [41].

Theorem 2.1. (i) $\mathsf{BA} = \mathbb{Q}(\boldsymbol{B_2})$.

(ii) $\mathsf{RKA} = \mathbb{Q}(\boldsymbol{K_4})$.

(iii) $\mathsf{NIKA} = \mathbb{Q}(\boldsymbol{K_3} \times \boldsymbol{B_2})$.

(iv) $\mathsf{KA} = \mathbb{Q}(\boldsymbol{K_3})$.

(v) $\mathsf{NIDMA} = \mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{B_2})$.

(vi) $\mathsf{NIDMA} \cup \mathsf{KA} = \mathbb{Q}(\boldsymbol{K_3}, \boldsymbol{DM_4} \times \boldsymbol{B_2})$.

(vii) $\mathsf{DMA} = \mathbb{Q}(\boldsymbol{DM_4})$.

2.3. Sentential Logics

We shall adopt the standard terminology and notation employed in Abstract Algebraic Logic (AAL). All the concepts not covered in this preliminary section can be found in the textbook [24].

Let $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ be a (sentential) logic in an (algebraic) language $\mathcal{L}$. A logic $\mathcal{S}' = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}'} \rangle$ in the same language $\mathcal{L}$ is an *extension of* $\mathcal{S}$ if $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{S}'}$. In this case we write $\mathcal{S} \leq \mathcal{S}'$ or $\mathcal{S}' \geq \mathcal{S}$. The extensions of $\mathcal{S}$ ordered by inclusion form a lattice which we denote $\mathrm{Ext}(\mathcal{S})$. The sublattice of finitary extensions of $\mathcal{S}$ will be denoted $\mathrm{Ext}_{\omega}(\mathcal{S})$. We shall also make use of the notation $[\mathcal{S}, \mathcal{S}']$ to denote the set of all logics extending $\mathcal{S}$ and extended by $\mathcal{S}'$. Given two logics $\mathcal{S}$ and $\mathcal{S}'$, we say that $\mathcal{S}'$ is *axiomatized* (*relative to* $\mathcal{S}$) by a set of logical rules $\Gamma_i \vdash \varphi_i$, with $i \in I$, if it is the least logic (extending $\mathcal{S}$) such that $\Gamma_i \vdash_{\mathcal{S}'} \varphi_i$. If a logic $\mathcal{S}$ is axiomatized by the rules $\Gamma_i \vdash \varphi_i$, with $i \in I$, and a logic $\mathcal{S}'$ is axiomatized by $\Delta_j \vdash \varphi_j$, with $j \in J$, then their join $\mathcal{S} \vee \mathcal{S}'$ is the logic axiomatized by the rules $\Gamma_i \vdash \varphi_i$ and $\Delta_i \vdash \psi_i$ for $i \in I$ and $j \in J$.

Given an algebra $\boldsymbol{A}$, an $\mathcal{S}$*-filter of* $\boldsymbol{A}$ is a subset $F \subseteq A$ such that for every homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$ and every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}$, if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $h(\Gamma) \subseteq F$, then $h(\varphi) \in F$. We shall denote the set of all $\mathcal{S}$-filters of $\boldsymbol{A}$ by $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a closure system, there is a least $\mathcal{S}$-filter of $\boldsymbol{A}$ containing a given $X \subseteq A$, which will be denoted by $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X)$.

A *(logical) matrix* is a pair $\mathcal{M} = \langle \boldsymbol{A}, F \rangle$, where $\boldsymbol{A}$ is an algebra and $F \subseteq A$. We say that $\mathcal{M}$ is *trivial* if $F = A$. Every class M of logical matrices induces a logic, which we denote by $\mathrm{Log}(\mathsf{M})$. Given a logic $\mathcal{S}$, a matrix $\mathcal{M}$ is a *model of* $\mathcal{S}$ if $\mathcal{S} \leq \mathrm{Log}(\mathcal{M})$. The notation $\mathrm{Log}(\langle \boldsymbol{A}, F \rangle)$ will usually be simplified to $\mathrm{Log}\langle \boldsymbol{A}, F \rangle$, and $\mathrm{Log}(\{\mathcal{M}, \mathcal{M}'\})$ to $\mathrm{Log}\{\mathcal{M}, \mathcal{M}'\}$. The class of all models of a logic $\mathcal{S}$ will be denoted by $\mathsf{Mod}(\mathcal{S})$. Classical references for the theory of sentential logics and logical matrices are [45,46].

Let $\boldsymbol{A}$ be an arbitrary algebra. A congruence $\theta \in \mathrm{Co}\boldsymbol{A}$ is *compatible* with a subset $F \subseteq A$ if it does not identify elements in F with elements outside F. The set of all congruences on $\boldsymbol{A}$ compatible with a given $F \subseteq A$ forms a complete sublattice of the lattice $\mathrm{Co}\boldsymbol{A}$. Its least element is the

identity congruence on $\boldsymbol{A}$, which we denote by $id_{\boldsymbol{A}}$. Its largest element is known as the *Leibniz congruence of* F, denoted $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. A trivial but useful observation is that $\boldsymbol{\Omega}^{\boldsymbol{A}}(\varnothing) = \boldsymbol{\Omega}^{\boldsymbol{A}}(A) = A \times A$.

We shall be interested in two further distinguished relations. The first (for $F \subseteq A$) is called the *Suszko congruence of* F *(relative to* $\mathcal{S}$*)*, defined as the largest congruence of $\boldsymbol{A}$ compatible with every $\mathcal{S}$-filter containing F. Equivalently,

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \coloneqq \bigcap \big\{ \boldsymbol{\Omega}^{\boldsymbol{A}}(G) : G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\ , F \subseteq G \big\}.$$

The second is called the *Frege relation of* F *(relative to* $\mathcal{S}$*)* and it is defined by

$$\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \coloneqq \big\{ \langle a, b\rangle \in A \times A : \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, a) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, b) \big\}\,.$$

Unlike the congruences $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ and $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, the equivalence relation $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ need not be compatible with the algebraic operations of $\boldsymbol{A}$.

The relation $\boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}(\varnothing)$ on the algebra of formulas is called the *interderivability relation*. Traditionally, $\langle \varphi, \psi\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}(\varnothing)$ is denoted by $\varphi \dashv\vdash_{\mathcal{S}} \psi$. A logic $\mathcal{S}$ whose interderivability relation is a congruence on $\boldsymbol{Fm}$ is called *selfextensional*. If this property lifts to arbitrary algebras, that is, if $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\varnothing)$ is a congruence on every algebra $\boldsymbol{A}$, then $\mathcal{S}$ is called *fully selfextensional*. One can generalize these definitions to arbitrary $\mathcal{S}$-theories and $\mathcal{S}$-filters as follows. If the Frege relation $\boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}(T)$ is a congruence on $\boldsymbol{Fm}$ for every $T \in \mathcal{T}h\mathcal{S}$, then $\mathcal{S}$ is *Fregean*; and if $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is a congruence on $\boldsymbol{A}$ for every algebra $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, then $\mathcal{S}$ is *fully Fregean*. These four classes of logics constitute the so-called *Frege hierarchy* in AAL.

A matrix $\langle \boldsymbol{A}, F\rangle$ is *Leibniz-reduced* when $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}}$, and it is *Suszko-reduced* when $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}}$. The classes of all reduced matrices according to these two criteria will be denoted respectively by

$$\mathsf{Mod}^{*}(\mathcal{S}) \coloneqq \big\{ \langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\mathcal{S}) : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}} \big\},$$
$$\mathsf{Mod}^{\mathrm{Su}}(\mathcal{S}) \coloneqq \big\{ \langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\mathcal{S}) : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}} \big\}.$$

These classes of matrices determine two classes of algebras that are considered in AAL as the natural algebraic counterparts of a logic:

$$\mathsf{Alg}^{*}(\mathcal{S}) \coloneqq \big\{ \boldsymbol{A} : \exists\ F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}} \big\},$$
$$\mathsf{Alg}(\mathcal{S}) \coloneqq \big\{ \boldsymbol{A} : \exists\ F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}} \big\}.$$

It is well known that $\mathsf{Alg}(\mathcal{S}) = \mathbb{IP}_{\mathrm{S}}\mathsf{Alg}^{*}(\mathcal{S})$ [26, Theorem 2.23]. As a consequence, the inclusion $\mathsf{Alg}^{*}(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{S})$ holds for arbitrary logics. The

converse inclusion holds for a broad class of logics, including all so-called *protoalgebraic logics*.[1] This is for instance the case of classical propositional logic, where $\mathsf{Alg}^*(\mathcal{CL}) = \mathsf{Alg}(\mathcal{CL})$ is the variety of Boolean algebras.

An $\mathcal{L}$-*equation* is a pair of formulas, written as $\varphi \approx \psi$. The set of all $\mathcal{L}$-equations will be denoted Eq. Given $\varphi(x_1, \ldots, x_n) \in \mathrm{Fm}$ and $a_1, \ldots, a_n \in A$, we denote by $\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n)$ the interpretation on $\boldsymbol{A}$ of the formula φ under any homomorphism $h : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h(x_i) = a_i$ for $1 \leq i \leq n$. Given an algebra $\boldsymbol{A}$ and a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}$, we denote by $\boldsymbol{\tau}(\boldsymbol{A})$ the set of all elements $a \in A$ satisfying the equations in $\boldsymbol{\tau}(x)$, that is,

$$\boldsymbol{\tau}(\boldsymbol{A}) \coloneqq \{a \in A : \boldsymbol{A} \vDash \boldsymbol{\tau}(x)[\![a]\!]\}.$$

Moreover, given $F \subseteq A$, we write

$$\boldsymbol{\tau}^{\boldsymbol{A}}(F) \coloneqq \left\{\left\langle \delta^{\boldsymbol{A}}(a), \epsilon^{\boldsymbol{A}}(a)\right\rangle : \delta \approx \epsilon \in \boldsymbol{\tau}(x), a \in F\right\}.$$

Given a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}$ and a class of matrices M, we say that *truth is equationally definable in* M *by* $\boldsymbol{\tau}$, or that $\boldsymbol{\tau}$ *defines truth in* M, if for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ we have $F = \boldsymbol{\tau}(\boldsymbol{A})$.

Given a class K of $\mathcal{L}$-algebras, the *equational consequence relation relative to* K is the relation $\vDash_{\mathsf{K}} \subseteq \mathscr{P}(\mathrm{Eq}) \times \mathrm{Eq}$ defined by:

$$\Pi \vDash_{\mathsf{K}} \varphi \approx \psi \qquad \text{iff} \qquad \begin{array}{l} \forall \boldsymbol{A} \in \mathsf{K}\ \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\ \forall \delta \approx \epsilon \in \Pi \\ h(\delta) = h(\epsilon) \ \Rightarrow\ h(\varphi) \approx h(\psi) \end{array}$$

for every $\Gamma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}$. It is well known that if K is a quasivariety, then the consequence relation $\vDash_{\mathsf{K}}$ is finitary.

2.4. Truth-Equational and Assertional Logics

The main classification of logics in AAL is the so-called *Leibniz hierarchy*, which classifies a given logic according to the algebraic properties enjoyed by the Leibniz operator. For instance, *protoalgebraic logics* are characterized through the order-preserving property of the Leibniz operator, i.e. for every algebra $\boldsymbol{A}$ if $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ are such that $F \subseteq G$, then $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. In this work we will be (almost) exclusively dealing with non-protoalgebraic logics, which means that the classification of super-Belnap logics inside the Leibniz hierarchy will reduce to two classes, namely truth-equational and assertional logics.[2]

[1] We refer the reader to [18] for a detailed treatment of protoalgebraic logics.

[2] For an overview of the Leibniz hierarchy, we refer the reader to [24]. Truth-equational and assertional logics were introduced by Raftery [43] and have more recently been studied in [3,4].

Definition 2.2. A logic $\mathcal{S}$ is *truth-equational* if truth is equationally definable in the class $\mathsf{Mod}^*(\mathcal{S})$.

A set of equations witnessing the truth-equationality of a logic $\mathcal{S}$ is called a *set of defining equations for* $\mathcal{S}$. This class of logics rightfully belongs to the Leibniz hierarchy because truth-equationality can be characterized by a lattice-theoretical property of the Leibniz operator, namely being completely order-reflecting [43, Theorem 28].

Since $\mathsf{Mod}(\mathcal{S}') \subseteq \mathsf{Mod}(\mathcal{S})$ whenever $\mathcal{S}' \geq \mathcal{S}$, truth-equationality is preserved through extensions. We record this fact for future reference.

Proposition 2.3. *If $\mathcal{S}$ is a truth-equational logic and $\mathcal{S} \leq \mathcal{S}'$, then $\mathcal{S}'$ is also truth-equational with same set of defining equations.*

Given a logic $\mathcal{S}$, an algebra $\boldsymbol{A}$ and an $\mathcal{S}$-filter F on $\boldsymbol{A}$, the *Leibniz class of* F is the set

$$[\![F]\!]^* := \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\}.$$

We say that F is a *Leibniz filter of* $\boldsymbol{A}$ if it is the least element of its Leibniz class. We denote the set of all Leibniz filters of $\boldsymbol{A}$ by $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$, and the least element of the Leibniz class $[\![F]\!]^*$ (which always exists) by F^*. Leibniz filters (and particularly a stronger version of this notion, called Suszko filters) turn out to be deeply connected with truth-equational logics, as shown in [3].

Theorem 2.4. *If $\mathcal{S}$ is truth-equational, then for every algebra $\boldsymbol{A}$, every $\mathcal{S}$-filter of $\boldsymbol{A}$ is Leibniz.*

In order to disprove the truth-equationality of a given logic $\mathcal{S}$, it therefore suffices to find an algebra $\boldsymbol{A}$ and an $\mathcal{S}$-filter which is not a Leibniz filter of $\boldsymbol{A}$. This will be the strategy followed in Theorem 5.3.

Given a class of algebras K and a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}$, the $\boldsymbol{\tau}$-*assertional logic of* K, denoted $\mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$, is defined by

$$\Gamma \vdash_{\mathcal{S}(\mathsf{K},\boldsymbol{\tau})} \varphi \quad \Leftrightarrow \quad \boldsymbol{\tau}(\Gamma) \vDash_{\mathsf{K}} \boldsymbol{\tau}(\varphi).$$

A class K is called a $\boldsymbol{\tau}$-*algebraic semantics for* $\mathcal{S}$ if $\mathcal{S} = \mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$. The notion of an algebraic semantics of a logic was introduced in [11], and it was further investigated in [13]. A relevant observation used later on is the following: if K is a $\boldsymbol{\tau}$-algebraic semantics of a finitary logic $\mathcal{S}$, then so is $\mathbb{Q}(\mathsf{K})$ [13, Proposition 2.2]. One particular situation in which we will be dealing with algebraic semantics is encompassed by the next result [13, Theorem 2.3].

Proposition 2.5. *A logic $\mathcal{S}$ is complete w.r.t. a class of matrices M where truth is equationally definable by $\boldsymbol{\tau}$ if and only if the class of algebraic reducts of M is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$.*

Assertional logics, sometimes also called 1-assertional [12], are the particular case of $\boldsymbol{\tau}$-assertional logics where the set of equations is of the form $\boldsymbol{\tau}(x) = \{x \approx 1\}$ and 1 is a constant term for some class of algebras K.

Definition 2.6. A logic $\mathcal{S}$ is *assertional* if it is the $\{x \approx 1\}$-assertional logic of some class of algebras K, where 1 is a constant term in K.

The next proposition relates the notions of algebraic semantics, truth-equational logics, and assertional logics [43].

Proposition 2.7. (i) *If $\mathcal{S}$ is truth-equational with a set of defining equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}$, then $\mathsf{Alg}(\mathcal{S})$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$, that is, $\mathcal{S} = \big(\mathsf{Alg}(\mathcal{S}), \boldsymbol{\tau}\big)$.*

(ii) *If $\mathcal{S}$ is 1-assertional, then it is truth-equational with a set of defining equations $\boldsymbol{\tau}(x) = \{x \approx 1\}$.*

2.5. The Strong Version of a Sentential Logic

The notion of the strong version of a protoalgebraic logic was first introduced by Font and Jansana [25]. More recently the theory has been generalized to non-protoalgebraic logics in [2]. We recall here the results from [2] that will be needed to characterize the strong versions of the super-Belnap logics considered in Section 3.

Definition 2.8. The *strong version* of a logic $\mathcal{S}$, denoted by $\mathcal{S}^+$, is the logic induced by the class of matrices $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A}$ is an algebra and F is the least $\mathcal{S}$-filter of $\boldsymbol{A}\}$.

We could have equivalently defined the logic $\mathcal{S}^+$ as the logic induced by the class of matrices $\big\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A}$ is an algebra and $F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}\big\}$. Theorem 2.4 then implies that the strong version of a truth-equational logic $\mathcal{S}$ coincides with $\mathcal{S}$ itself. We shall therefore (in Section 6) be only interested in finding the strong version of non-truth-equational super-Belnap logics.

Two different logics may share the same strong version. One such particular case is the following:

Lemma 2.9. *If $\mathcal{S} \leq \mathcal{S}' \leq \mathcal{S}^+$, then $(\mathcal{S}')^+ = \mathcal{S}^+$.*

Given an arbitrary logic $\mathcal{S}$, we know *a priori* very little about the logic $\mathcal{S}^+$. However, the study of Leibniz $\mathcal{S}$-filters may give us some insight into $\mathcal{S}^+$. We next introduce a definability criterion over the Leibniz $\mathcal{S}$-filters, which will be used in Section 6 for determining the strong versions of the super-Belnap logics under study.

Definition 2.10. A logic $\mathcal{S}$ *has equationally definable Leibniz filters* if there is a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}$ such that for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$

$$F^* = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}.$$

In this case we say that *the Leibniz $\mathcal{S}$-filters are equationally definable by $\boldsymbol{\tau}$*.

Equational definability of Leibniz filters has a most useful characterization in practice [2, Theorem 3.4]:

Proposition 2.11. *For any logic $\mathcal{S}$ and any set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}$, the following are equivalent:*

(i) *The Leibniz $\mathcal{S}$-filters are equationally definable by $\boldsymbol{\tau}$,*

(ii) *For every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, $\boldsymbol{\tau}(\boldsymbol{A})$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}$.*

The main result which makes equational definability of Leibniz $\mathcal{S}$-filters useful for determining the logic $\mathcal{S}^+$ is the following [2, Theorem 3.5]:

Theorem 2.12. *If the Leibniz $\mathcal{S}$-filters are equationally definable by a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}$, then:*

(i) *$\mathcal{S}^+$ is the $\boldsymbol{\tau}$-assertional logic of $\mathsf{Alg}(\mathcal{S})$,*

(ii) *$\mathcal{S}^+$ is truth-equational with set of defining equations $\boldsymbol{\tau}$,*

(iii) *$\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ for every $\boldsymbol{A}$.*

2.6. Semilattice-Based Logics

A class of algebras K has *semilattice reductions* if there exists a binary term $\wedge$ such that for every $\boldsymbol{A} \in \mathsf{K}$, $\langle \boldsymbol{A}, \wedge^{\boldsymbol{A}} \rangle$ is a semilattice. For every $\boldsymbol{A} \in \mathsf{K}$ let $\leq^{\boldsymbol{A}}$ denote the partial order induced by $\wedge^{\boldsymbol{A}}$, that is, $a \leq^{\boldsymbol{A}} b \Leftrightarrow a \wedge^{\boldsymbol{A}} b = a$ for every $a, b \in A$. The *semilattice-based logic* of K is defined for every $\gamma_1, \ldots, \gamma_n, \varphi \in \mathrm{Fm}$ by

$$\begin{aligned} \varnothing \vdash^{\leq}_{\mathsf{K}} \varphi \quad \Leftrightarrow \quad & \forall \boldsymbol{A} \in \mathsf{K} \ \ \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}) \ \ \forall a \in \boldsymbol{A} \\ & a \leq^{\boldsymbol{A}} h(\varphi), \end{aligned}$$

and

$$\begin{aligned} \{\gamma_1, \ldots, \gamma_n\} \vdash^{\leq}_{\mathsf{K}} \varphi \quad \Leftrightarrow \quad & \forall \boldsymbol{A} \in \mathsf{K} \ \ \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}) \\ & h(\gamma_1) \wedge^{\boldsymbol{A}} \cdots \wedge^{\boldsymbol{A}} h(\gamma_n) \leq^{\boldsymbol{A}} h(\varphi). \end{aligned}$$

Given a logic $\mathcal{S}$, a binary term $\wedge$ is a *conjunction* for $\mathcal{S}$ if $x, y \vdash_{\mathcal{S}} x \wedge y$, $x \wedge y \vdash_{\mathcal{S}} x$ and $x \wedge y \vdash_{\mathcal{S}} y$. The following facts about semilattice-based logics will be used in the sequel. The proofs of (i)–(iii) can be found in [29], and those of (iv)–(v) will appear in [2].

THEOREM 2.13. *Each finitary selfextensional logic with a conjunction is semilattice-based. Conversely, let $\mathcal{S}$ be the semilattice-based logic of a class of algebras* K.

(i) *$\mathcal{S}$ is fully selfextensional and has a conjunction.*

(ii) *If $\mathcal{S}$ has theorems, then for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, the set of $\mathcal{S}$-filters of $\boldsymbol{A}$ coincides with the set of lattice filters of $\boldsymbol{A}$.*

(iii) $\mathsf{Alg}(\mathcal{S}) = \mathbb{V}(\mathsf{K})$.

(iv) *If $\mathcal{S}$ has theorems, then the Leibniz $\mathcal{S}$-filters are equationally definable by $\boldsymbol{\tau}(x) = \{x \approx 1\}$, where 1 is any $\mathcal{S}$-theorem.*

(v) *If $\mathcal{S}$ has theorems, then $\mathcal{S}^+$ is the 1-assertional logic of* $\mathsf{Alg}(\mathcal{S})$, *where 1 is any $\mathcal{S}$-theorem.*

2.7. Explosive Extensions

The notion of explosive extension of a finitary base logic $\mathcal{B}$ (which is not necessarily the Belnap–Dunn logic), introduced in [38], will be useful in the following. It will suffice to consider this notion in the special case where the language of $\mathcal{B}$ contains a formula $\bot$ such that $\bot \vdash_{\mathcal{B}} p$, where p is a propositional atom which does not occur in $\bot$. An *explosive extension* of $\mathcal{B}$ is a logic that extends $\mathcal{B}$ with a set of explosive rules, where an *explosive rule* is a rule of the form $\Gamma \vdash \bot$. We shall use the notation $\Gamma \vdash \varnothing$ for $\Gamma \vdash \bot$, to emphasize the symmetry between axiomatic rules $\varnothing \vdash \varphi$ and explosive rules $\Gamma \vdash \varnothing$. Observe that a non-trivial matrix validates the rule $\Gamma \vdash \varnothing$ if and only if no valuation on it designates all formulas in Γ.

We shall now list some basic facts about explosive extensions proved in [38]. The explosive extensions of $\mathcal{B}$ form a lattice which will be denoted $\operatorname{Exp}\operatorname{Ext}(\mathcal{B})$ in the following. The sublattice of finitary explosive extensions will be denoted $\operatorname{Exp}\operatorname{Ext}_\omega(\mathcal{B})$.

PROPOSITION 2.14. *Let $\mathcal{S} \in \operatorname{Ext}(\mathcal{B})$ and $\mathcal{S}_{exp} \in \operatorname{Exp}\operatorname{Ext}(\mathcal{B})$. Then we have* $\mathsf{Mod}(\mathcal{S} \cap \mathcal{S}_{exp}) = \mathsf{Mod}(\mathcal{S}) \cup \mathsf{Mod}(\mathcal{S}_{exp})$.

COROLLARY 2.15. $\mathsf{Alg}^*(\mathcal{S} \cap \mathcal{S}_{exp}) = \mathsf{Alg}^*(\mathcal{S}) \cup \mathsf{Alg}^*(\mathcal{S}_{exp})$.

COROLLARY 2.16. *The (finitary) explosive extensions of a base logic $\mathcal{B}$ form a distributive (algebraic) sublattice of* $\operatorname{Ext}(\mathcal{B})$.

Axiomatizing the intersection of an explosive extension of $\mathcal{B}$ with an arbitrary extension of turns out to be easy.

PROPOSITION 2.17. *If $\mathcal{S}$ is the extension of $\mathcal{B}$ by the rules $\Gamma_i \vdash \varphi_i$ for $i \in I$ and $\mathcal{S}_{exp}$ is the explosive extension of $\mathcal{B}$ by the rule $\Delta \vdash \varnothing$, then $\mathcal{S} \cap \mathcal{S}_{exp}$ is the extension of $\mathcal{B}$ by the rules $\Gamma_i, \Delta \vdash \varphi_i$ for $i \in I$.*

The *explosive part* of an extension $\mathcal{S}$ of $\mathcal{B}$ (relative to the base logic $\mathcal{B}$), denoted $\mathrm{Exp}_{\mathcal{B}}(\mathcal{S})$, is the largest explosive extension of $\mathcal{B}$ below $\mathcal{S}$. The operator $\mathrm{Exp}_{\mathcal{B}}$ is an interior operator on $\mathrm{Ext}(\mathcal{B})$.

PROPOSITION 2.18. $\Gamma \vdash_{\mathrm{Exp}_{\mathcal{B}}(\mathcal{S})} \varphi$ *if and only if either* $\Gamma \vdash_{\mathcal{B}} \varphi$ *or* $\Gamma \vdash_{\mathcal{S}} \varnothing$.

The operator $\mathrm{Exp}_{\mathcal{B}}$ is particularly useful for determining the logics defined by products of matrices, as shown by the following proposition.

PROPOSITION 2.19. *If $\mathcal{B} = \mathrm{Log}\langle \boldsymbol{A}, F\rangle$ and $\mathcal{S} = \mathrm{Log}\langle \boldsymbol{B}, G\rangle$ is an extension of $\mathcal{B}$, then $\mathrm{Exp}_{\mathcal{B}}\,\mathcal{S} = \mathrm{Log}\langle \boldsymbol{A}, F\rangle \times \langle \boldsymbol{B}, G\rangle$.*

3. An Overview of Super-Belnap Logics

In this section we introduce the Belnap–Dunn logic $\mathcal{B}$ and some of its notable extensions, both from a semantic and a syntactic point of view. We mainly draw on the papers [23,38,44] but some new results regarding reduced models of various super-Belnap logics are also proved here.

We take the signature of $\mathcal{B}$ to consist of the two binary connectives $\wedge$ and $\vee$, the unary operator $\neg$, and the two constants 1 and 0. Although $\mathcal{B}$ has usually been studied in the language $\{\wedge, \vee, \neg\}$, we find it convenient to include the truth constants in the language. Virtually none of the results presented below depend on this choice. Occasionally, as in the case of Theorem 5.1, the almost trivial logic in the language without the constants (axiomatized by the rule $p \vdash q$) has to be excluded from consideration.

One reason for considering the language with the truth constants is that the theory of Leibniz filters and strong versions becomes trivial in the absence of theorems. Moreover, the application of Theorem 2.13 requires that the logics in question have theorems. Finally, the choice is in part also a matter of convenience when proving certain results: the duality theory for De Morgan algebras is simpler than for De Morgan lattices.

We start by a semantic presentation of the most important super-Belnap logics. Most of the logics mentioned in this section may be introduced via the matrices depicted in Figure 2 or their products. The algebraic reducts of these matrices are precisely the De Morgan algebras $\boldsymbol{B_2}$, $\boldsymbol{K_3}$, and $\boldsymbol{DM_4}$. The Belnap–Dunn logic $\mathcal{B}$ is defined by the matrix $\langle \boldsymbol{DM_4}, \{\mathsf{b}, \mathsf{t}\}\rangle$. Recall that we write this as $\mathcal{B} = \mathrm{Log}\ \langle \boldsymbol{DM_4}, \{\mathsf{b}, \mathsf{t}\}\rangle$. Apart from classical logic

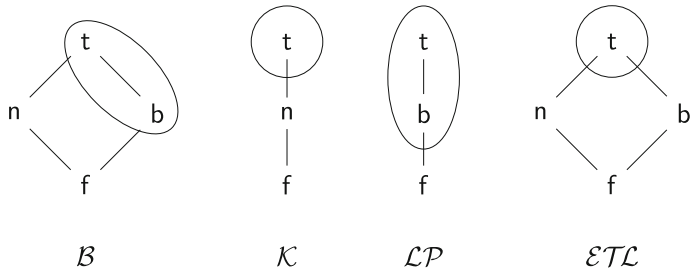


Figure 2. Logical matrices defining some basic super-Belnap logics

$\mathcal{CL} = \text{Log}\ \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle$, the oldest and best known extensions of $\mathcal{B}$ are the logics defined by the three-element submatrices of $\langle \boldsymbol{DM_4}, \{\mathsf{b}, \mathsf{t}\}\rangle$: strong three-valued Kleene logic $\mathcal{K} = \text{Log}\ \langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle$ and the Logic of Paradox $\mathcal{LP} = \text{Log}\ \langle \boldsymbol{K_3}, \{\mathsf{n}, \mathsf{t}\}\rangle$ (relabelling b as n for the sake of uniformity). These logics are linked by the following principle, whose easy proof we omit.

Proposition 3.1. *$\varphi \vdash_{\mathcal{K}} \psi$ if and only if $\neg\psi \vdash_{\mathcal{LP}} \neg\varphi$.*

Kleene's logic of order $\mathcal{K}^{\leq}$ is defined as $\mathcal{K} \cap \mathcal{LP}$, or in semantic terms as $\text{Log}\ \{\langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle, \langle \boldsymbol{K_3}, \{\mathsf{n}, \mathsf{t}\}\rangle\}$. This logic is briefly mentioned by Font [23] and Makinson [34], who calls it Kalman implication. It was also considered by Dunn [20,21], who proved that it is precisely the first-degree fragment of the logic R-mingle (in the same sense that $\mathcal{B}$ is the first-degree fragment of the relevant logic E). This logic, however, has received considerably less attention in the literature than $\mathcal{K}$ and $\mathcal{LP}$.

All three of the three-valued logics $\mathcal{K}$, $\mathcal{LP}$, and $\mathcal{K}^{\leq}$ are studied in Dunn's paper [22]. However, note that Dunn's axiomatization of these logics involves a "metarule" which allows us to infer $\varphi \vee \psi \vdash \chi$ from $\varphi \vdash \chi$ and $\psi \vdash \chi$. Dunn's syntactic descriptions of $\mathcal{K}$, $\mathcal{LP}$, and $\mathcal{K}^{\leq}$ are therefore not (Hilbert-style) axiomatizations in our sense of the word.

The super-Belnap logics introduced above turn out to be the best-behaved in the family. As proved in [38], they are the only non-trivial super-Belnap logics to satisfy the (weak) proof by cases property, and moreover $\mathcal{B}$, $\mathcal{K}^{\leq}$, and $\mathcal{CL}$ are the only ones that satisfy the contraposition property. Finally, $\mathcal{K}$ and $\mathcal{CL}$ are the only structurally complete super-Belnap logics.[3]

[3]Following Cintula and Noguera [16], we say that a super-Belnap logic $\mathcal{S}$ enjoys the *(weak) proof by cases property* if $\Gamma, \varphi \vee \psi \vdash_{\mathcal{S}} \chi$ whenever $\Gamma, \varphi \vdash_{\mathcal{S}} \chi$ and $\Gamma, \psi \vdash_{\mathcal{S}} \chi$ (for $\Gamma = \varnothing$). It enjoys the *contraposition property* if $\varphi \vdash_{\mathcal{S}} \psi$ implies $\neg\psi \vdash_{\mathcal{S}} \neg\varphi$.

A logic $\mathcal{S}$ is *structurally complete* if each admissible rule of $\mathcal{S}$ is in fact valid in $\mathcal{S}$, or more formally if $\Gamma \vdash_{\mathcal{S}} \varphi$ whenever $\varnothing \vdash_{\mathcal{S}} \sigma(\gamma)$ for all $\gamma \in \Gamma$ implies $\varnothing \vdash_{\mathcal{S}} \sigma(\varphi)$ for each

More recently, the Exactly True Logic $\mathcal{ETL} = \mathrm{Log}\,\langle \boldsymbol{DM_4}, \{\mathsf{t}\}\rangle$ has been introduced by Pietz and Rivieccio [36] and studied in more detail by Rivieccio [44], initiating the search for new super-Belnap logics.

The logics $\mathcal{B}$, $\mathcal{K}^{\leq}$, and $\mathcal{CL}$ are intimately connected with the three subvarieties of De Morgan algebras: they are precisely the semilattice-based logics of De Morgan, Kleene, and Boolean algebras, respectively.

Proposition 3.2. *The following are equivalent for a finite* Γ*:*

(i) $\Gamma \vdash_{\mathcal{B}} \varphi$,

(ii) *the inequality* $\bigwedge \Gamma \leq \varphi$ *holds in* DMA,

(iii) *the inequality* $\bigwedge \Gamma \leq \varphi$ *holds in the algebra* $\boldsymbol{DM_4}$.

Analogous equivalences hold for $\mathcal{K}^{\leq}$, KA, $\boldsymbol{K_3}$ *and* $\mathcal{CL}$, BA, $\boldsymbol{B_2}$.

The reduced models of $\mathcal{B}$ were described by Font [23]. In particular, his characterization entails that $\boldsymbol{B_2}$ and $\boldsymbol{K_3}$ are the only chains in $\mathsf{Alg}^*(\mathcal{B})$.

Theorem 3.3. [23] $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{B})$ *if and only if* $\boldsymbol{A}$ *is a De Morgan algebra,* F *is a lattice filter on* $\boldsymbol{A}$*, and for all* $a < b$ *there is some* $c \in \boldsymbol{A}$ *such that either (i)* $a \vee c \notin F$ *and* $b \vee c \in F$*, or (ii)* $\neg b \vee c \notin F$ *and* $\neg a \vee c \in F$.

Conversely, as a consequence of Proposition 3.2, each De Morgan algebra equipped with a lattice filter is a model (not necessarily reduced) of $\mathcal{B}$.

A finite Hilbert-style axiomatization of $\mathcal{B}$ was found independently by Font [23] and Pynko [39] in the 1990s. They only consider the language without the truth constants 1 and 0, but it is easy to show that adding the following rules yields a calculus for the whole language:

$$0 \vee p \vdash p \qquad\qquad \varnothing \vdash 1$$
$$\neg 1 \vee p \vdash p \qquad\qquad \varnothing \vdash \neg 0$$

All other logics mentioned in this paper will be axiomatized *relative to* $\mathcal{B}$.

Theorem 3.4. (i) $\mathcal{B} = \mathrm{Log}\,\langle \boldsymbol{DM_4}, \{\mathsf{b}, \mathsf{t}\}\rangle$.

(ii) $\mathcal{LP} = \mathrm{Log}\,\langle \boldsymbol{K_3}, \{\mathsf{n}, \mathsf{t}\}\rangle$ *is axiomatized by* $\varnothing \vdash p \vee \neg p$.

(iii) $\mathcal{K} = \mathrm{Log}\,\langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle$ *is axiomatized by* $(p \wedge \neg p) \vee q \vdash q$.

(iv) $\mathcal{ETL} = \mathrm{Log}\,\langle \boldsymbol{DM_4}, \{\mathsf{t}\}\rangle$ *is axiomatized by* $p, \neg p \vee q \vdash q$.

(v) $\mathcal{CL} = \mathrm{Log}\,\langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle$ *is axiomatized by* $\varnothing \vdash p \vee \neg p$ *and* $p, \neg p \vee q \vdash q$.

Footnote 3 continued
substitution σ. Equivalently, a logic $\mathcal{S}$ is structurally complete if each proper extension adds some new theorems $\mathcal{S}$.

(vi) $\mathcal{K}^{\leq} = \mathcal{LP} \cap \mathcal{K}$ *is axiomatized by* $(p \wedge \neg p) \vee r \vdash (q \vee \neg q) \vee r$.

Proof. Item (i) holds by definition, item (ii) was proved by Pynko [40], item (iv) was proved by Pietz and Rivieccio [36], and to prove (v) it suffices to take any standard Hilbert axiomatization of classical logic and prove all of its axioms using the rule disjunctive syllogism (Modus Ponens), the law of the excluded middle, and inferences valid in $\mathcal{B}$.

Item (iii) may be reduced to (ii): suppose that $\Gamma \vdash_{\mathcal{K}} \varphi$, where Γ is without loss of generality finite. Then $\gamma \vdash_{\mathcal{K}} \varphi$ for $\gamma = \bigwedge \Gamma$, hence $\neg\varphi \vdash_{\mathcal{LP}} \neg\gamma$ by Proposition 3.1. It follows from (ii) that there are formulas $\tau_1, \ldots, \tau_n$ of the form $\psi \vee \neg\psi$ such that $\neg\varphi, \tau_1, \ldots, \tau_n \vdash_{\mathcal{B}} \neg\gamma$, hence $\gamma \vdash_{\mathcal{B}} \varphi \vee \neg\tau_1 \vee \cdots \vee \neg\tau_n$ and $\Gamma \vdash_{\mathcal{B}} \varphi \vee \chi_1 \vee \cdots \vee \chi_n$ for some formulas $\chi_1, \ldots, \chi_n$ of the form $\psi \wedge \neg\psi$. In each logic which extends $\mathcal{B}$ by the rule $(p \wedge \neg p) \vee q \vdash q$, the rule $\Gamma \vdash \varphi$ is therefore valid. Conversely, the rule $(p \wedge \neg p) \vee q \vdash q$ is valid in $\mathcal{K}$.

To prove item (vi), suppose that $\langle \boldsymbol{A}, F \rangle$ is a reduced model of $\mathcal{B}$ which validates the rule $(p \wedge \neg p) \vee r \vdash (q \vee \neg q) \vee r$. In particular, $\boldsymbol{A}$ is a De Morgan algebra. Suppose for the sake of contradiction that $\boldsymbol{A}$ is not a Kleene algebra. Then there are $a, b \in \boldsymbol{A}$ such that $a \wedge \neg a \nleq b \vee \neg b$. By Theorem 3.3 there is some $c \in \boldsymbol{A}$ such that either $((a \wedge \neg a) \wedge (b \vee \neg b)) \vee c \notin F$ and $(a \wedge \neg a) \vee c \in F$ or $(a \vee \neg a) \vee c \notin F$ and $((a \vee \neg a) \wedge (b \wedge \neg b)) \vee c \in F$. The latter option contradicts the assumption that F is a lattice filter, therefore $(a \wedge \neg a) \vee c \in F$ and either $(a \wedge \neg a) \vee c \notin F$ or $(b \vee \neg b) \vee c \notin F$. The former option is again a contradiction, therefore $(a \wedge \neg a) \vee c \in F$ and $(b \vee \neg b) \vee c \notin F$. But this contradicts the validity of the rule $(p \wedge \neg p) \vee r \vdash (q \vee \neg q) \vee r$ in $\langle \boldsymbol{A}, F \rangle$. It follows that if $\Gamma \vdash \varphi$ is not valid is the extension of $\mathcal{B}$ by the rule $(p \wedge \neg p) \vee r \vdash (q \vee \neg q) \vee r$, then the inequality $\bigwedge \Gamma \leq \varphi$ fails in some Kleene algebra, hence $\Gamma \nvdash_{\mathcal{K}^{\leq}} \varphi$. Conversely, the rule $(p \wedge \neg p) \vee r \vdash (q \vee \neg q) \vee r$ is valid in $\mathcal{K}^{\leq}$. Therefore $\Gamma \vdash \varphi$ is valid in the extension of $\mathcal{B}$ by this rule if and only if $\Gamma \vdash_{\mathcal{K}^{\leq}} \varphi$. ■

Observe that the rule $(p \wedge \neg p) \vee q \vdash p$ is equivalent to the familiar rule of resolution $p \vee q, \neg q \vee r \vdash p \vee r$. Note also that the rule $p \wedge \neg p \vdash q \vee \neg q$ yields a (much) weaker logic than $\mathcal{K}^{\leq}$, namely the logic $\mathcal{LP} \cap \mathcal{ECQ}$. It follows from items (ii), (iii), and (iv) that $\mathcal{CL} = \mathcal{LP} \vee \mathcal{ETL} = \mathcal{LP} \vee \mathcal{K}$.

We now introduce several other families of super-Belnap logics. The logic $\mathcal{ECQ}$ extends $\mathcal{B}$ by the rule $p, \neg p \vdash q$. More generally, we may introduce an infinite chain of logics $\mathcal{ECQ}_n$, each axiomatized by the rule

$$(p_1 \wedge \neg p_1) \vee \cdots \vee (p_n \wedge \neg p_n) \vdash \varnothing.$$

The limit of this sequence is $\mathcal{ECQ}_\omega = \bigvee_{n \in \omega} \mathcal{ECQ}_n = \bigcup_{n \in \omega} \mathcal{ECQ}_n$. We may also consider logics obtained by adding these rules to $\mathcal{ETL}$ rather than $\mathcal{B}$.

This yields logics $\mathcal{ETL}_n = \mathcal{ETL} \vee \mathcal{ECQ}_n$ and $\mathcal{ETL}_\omega = \mathcal{ETL} \vee \mathcal{ECQ}_\omega$, which were introduced in [44] under the names $\mathcal{B}_n$ and $\mathcal{B}_\omega$.

Note that $\mathcal{ECQ}_1 = \mathcal{ECQ}$ and $\mathcal{ETL}_1 = \mathcal{ETL}$. The logics $\mathcal{ECQ}_n$ and $\mathcal{ETL}_n$ are shown to be pairwise distinct in [44], which implies that the logics $\mathcal{ECQ}_\omega$ and $\mathcal{ETL}_\omega$ are not finitely axiomatizable. On the other hand, it was shown in [44] and [38] that the logics $\mathcal{ETL}_\omega$ and $\mathcal{ECQ}_n$ may be defined by a finite matrix, unlike the logics $\mathcal{ECQ}_n$ and $\mathcal{ETL}_n$ for $n \geq 2$.

Finally, another infinite chain of logics may be defined by the rules

$$(p_1 \wedge \neg p_1) \vee \cdots \vee (p_n \wedge \neg p_n) \vee q, \neg q \vee r \vdash r.$$

The logic axiomatized by all of these rules for $n \geq 1$ will be denoted $\mathcal{K}^-$. It was proved in [38] that this logic is not finitely axiomatizable.

To provide a semantic characterization of the above logics, the theory of explosive extensions introduced in the previous section is useful. The following facts were proved in [38] and used to establish Theorem 3.6.

Proposition 3.5. (i) $\mathrm{Exp}_{\mathcal{B}}\, \mathcal{ETL}_n = \mathcal{ECQ}_n$.

(ii) $\mathrm{Exp}_{\mathcal{B}}\, \mathcal{CL} = \mathcal{ECQ}_\omega$.

(iii) $\mathrm{Exp}_{\mathcal{ETL}}\, \mathcal{CL} = \mathcal{ETL}_\omega$.

(iv) $\mathrm{Exp}_{\mathcal{LP}}\, \mathcal{CL} = \mathcal{LP} \vee \mathcal{ECQ}$.

(v) $\mathrm{Exp}_{\mathcal{K}^{\leq}}\, \mathcal{CL} = \mathcal{K}^{\leq} \vee \mathcal{ECQ}$.

Proposition 2.19 now immediately yields the following results.

Theorem 3.6. (i) $\mathcal{ECQ} = \mathrm{Log}\ \langle \boldsymbol{DM_4}, \{\mathsf{b}, \mathsf{t}\}\rangle \times \langle \boldsymbol{DM_4}, \{\mathsf{t}\}\rangle$.

(ii) $\mathcal{ECQ}_\omega = \mathrm{Log}\ \langle \boldsymbol{DM_4}, \{\mathsf{b}, \mathsf{t}\}\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle$.

(iii) $\mathcal{ETL}_\omega = \mathrm{Log}\ \langle \boldsymbol{DM_4}, \{\mathsf{t}\}\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle$.

(iv) $\mathcal{LP} \vee \mathcal{ECQ} = \mathrm{Log}\ \langle \boldsymbol{K_3}, \{\mathsf{n}, \mathsf{t}\}\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle$.

(v) $\mathcal{K}^{\leq} \vee \mathcal{ECQ} = \mathrm{Log}\ \{\langle \boldsymbol{K_3}, \{\mathsf{n}, \mathsf{t}\}\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle, \langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle\}$.

In the proof of item (v), we use the fact that $\mathcal{LP} \vee \mathcal{ECQ} = \mathcal{LP} \cup \mathcal{ECQ}_\omega$, hence $\mathcal{K} \cap (\mathcal{LP} \vee \mathcal{ECQ}) = \mathcal{K} \cap (\mathcal{LP} \cup \mathcal{ECQ}_\omega = \mathcal{K}^{\leq} \cup \mathcal{ECQ}_\omega = \mathcal{K}^{\leq} \vee \mathcal{ECQ}_\omega$.

We now describe the reduced models of various super-Belnap logics. Recall that NIKA denotes the quasivariety of non-idempotent Kleene algebras, that is, Kleene algebras where the negation has nofixpoints.

Theorem 3.7. *Let* $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{B})$ *be non-trivial. Then,*

(i) $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{ETL})$ *iff* $F = \{1\}$,

(ii) $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{K}^{\leq})$ *iff* $\boldsymbol{A} \in \mathsf{KA}$,

(iii) $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{K})$ *iff* $\boldsymbol{A} \in \mathsf{KA}$ *and* $F = \{1\}$,

(iv) $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{LP})$ *iff* $\boldsymbol{A} \in \mathsf{KA}$ *and* $F = \{a \in A : \neg a \leq a\}$,

(v) $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{ETL}_\omega)$ *iff* $\boldsymbol{A} \in \mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{K_3})$ *and* $F = \{1\}$,

(vi) $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{LP} \vee \mathcal{ECQ})$ *iff* $\boldsymbol{A} \in \mathsf{NIKA}$ *and* $F = \{a \in A : \neg a \leq a\}$.

Proof. (i) See [44, Theorem 10].

(ii) Assume $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{K}^{\leq})$. Since $\mathsf{Alg}^*(\mathcal{K}^{\leq}) \subseteq \mathsf{Alg}(\mathcal{K}^{\leq})$, by Proposition 5.7 we have $\boldsymbol{A} \in \mathsf{KA}$. Conversely, assuming $\boldsymbol{A} \in \mathsf{KA}$, we only need to check that $\langle \boldsymbol{A}, F\rangle$ is a model of the rule $(p \wedge \neg p) \vee r \vdash (q \vee \neg q) \vee r$, which axiomatizes $\mathcal{K}^{\leq}$ relative to $\mathcal{B}$. To see this, it is enough to observe that the inequality $(x \wedge \neg x) \vee z \leq (y \vee \neg y) \vee z$ is valid in KA, and that F is an upper set.

(iii) If $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{K})$, we use Proposition 5.7 to conclude that $\boldsymbol{A} \in \mathsf{KA}$ as in the preceding item. Moreover, since $\mathcal{ETL} \leq \mathcal{K}$, we have $F = \{1\}$ by item (i) above. Conversely, we need to show that if $\boldsymbol{A} \in \mathsf{KA}$, then the matrix $\langle \boldsymbol{A}, \{1\}\rangle$ satisfies the rule $(p \wedge \neg p) \vee q \vdash q$, which axiomatizes $\mathcal{K}$ relative to $\mathcal{B}$. To see this, it is enough to check that the quasiequation $(x \wedge \neg x) \vee y \approx 1 \Rightarrow y \approx 1$ is valid in $\boldsymbol{K_3}$ and therefore in every Kleene algebra.

(iv) Assume that $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{LP})$. Since $\mathsf{Alg}^*(\mathcal{LP}) \subseteq \mathsf{Alg}(\mathcal{LP})$, by Proposition 5.8 we have $\boldsymbol{A} \in \mathsf{KA}$. Since $p \vee \neg p$ is a theorem of $\mathcal{LP}$, we have that all elements $a \in A$ such that $a = a \vee \neg a$ (i.e. such that $\neg a \leq a$) must be in F. Now suppose there is $a \in F$ with $a \neq a \vee \neg a$ (i.e $a < a \vee \neg a \in F$). Since $\langle \boldsymbol{A}, F\rangle$ is reduced, there must be $c \in A$ such that $\neg a \vee c \in F$ and $\neg(a \vee \neg a) \vee c \notin F$. Applying the De Morgan laws and distributivity, we have $\neg(a \vee \neg a) \vee c = (a \wedge \neg a) \vee c = (a \vee c) \wedge (\neg a \vee c) \notin F$. Since $\neg a \vee c \in F$ and F is closed under $\wedge$, we conclude that $a \vee c \notin F$. But this is absurd, for the assumption that $a \in F$ implies that $a \vee c \in F$, since F is an upper set. Therefore any element $a \in F$ must be such that $\neg a \leq a$ as required. Conversely, assume $\boldsymbol{A} \in \mathsf{KA}$ and $F = \{a \in A : \neg a \leq a\}$. Then $\langle \boldsymbol{A}, F\rangle$ is certainly a model of $\mathcal{LP}$ because $\mathcal{LP}$ is axiomatized, relative to $\mathcal{B}$, by the axiom $p \vee \neg p$ and F contains the interpretation of $p \vee \neg p$ under every valuation.

(v) Follows easily from [44, Lemma 13].

(vi) Since $\mathcal{LP} \leq \mathcal{LP} \vee \mathcal{ECQ}$, assuming $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{LP} \vee \mathcal{ECQ})$, we know by item (iv) above that $\boldsymbol{A} \in \mathsf{KA}$ and $F = \{a \in A : \neg a \leq a\}$. If there were some $b \in \boldsymbol{A}$ such that $b = \neg b$, then we would have $b = b \wedge \neg b \in F$. Then, applying $p \wedge \neg p \vdash q$ (which is a rule of $\mathcal{ECQ}$, hence *a fortiori*

of $\mathcal{LP} \vee \mathcal{ECQ}$), we would obtain $F = A$, against our assumption that $\langle \boldsymbol{A}, F\rangle$ was non-trivial and reduced. Conversely, if $\boldsymbol{A} \in \mathsf{NIKA}$ and $F = \{a \in A : \neg a \leq a\}$, then $\langle \boldsymbol{A}, F\rangle$ is a model of $\mathcal{LP}$ by item (iv). Moreover, $\langle \boldsymbol{A}, F\rangle$ is a model of the rule $p, \neg p \vdash q$, which axiomatizes $\mathcal{ECQ}$ relative to $\mathcal{B}$. To see this, notice that $b \in F$ implies $\neg b \leq b$, so $\neg b \in F$ would imply $b = \neg b$, which cannot happen in a non-trivial algebra in NIKA.

■

The simple description of the reduced models of $\mathcal{ETL}$ holds even if we drop the constants 1 and 0 from the signature, i.e. the reduced models of $\mathcal{ETL}$ are bounded even if the language does not contain the truth constants.

4. The Lattice of Super-Belnap Logics

Having introduced various super-Belnap logics in the previous section, we now summarize what is known about the structure of the lattice of super-Belnap logics $\mathrm{Ext}(\mathcal{B})$. We also outline the connections between finite graphs and the reduced models of super-Belnap logics described in more detail in [38], where the results stated without proof in this section are proved.

Figure 3 depicts a part of the lattice $\mathrm{Ext}(\mathcal{B})$. Note that this figure is not a lattice diagram: although joins are faithfully represented by Figure 3, meets are not. Full lines in Figure 3 represent covers, whereas dashed lines merely represent inclusions.

Recall that a lattice is called modular if it satisfies the equality $(a \wedge b) \vee c = (a \vee c) \wedge b$ for $c \leq b$. In particular, every distributive lattice is modular.

THEOREM 4.1. *The lattice* $\mathrm{Ext}_\omega(\mathcal{B})$ *is non-modular, in particular we have* $(\mathcal{LP} \cap \mathcal{ETL}) \vee \mathcal{ECQ} < (\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{ETL}$.

The lattice of super-Belnap logics splits into several disjoint parts.

THEOREM 4.2. *Each super-Belnap logic* $\mathcal{S}$ *either has the same theorems as* $\mathcal{B}$ *or the same theorems as* $\mathcal{CL}$. *In the former case we have* $\mathcal{S} \leq \mathcal{K}$, *while in the latter case we have* $\mathcal{LP} \leq \mathcal{S}$.

THEOREM 4.3. *Each non-trivial proper extension of* $\mathcal{B}$ *belongs to one of the disjoint intervals* $[\mathcal{LP} \cap \mathcal{ECQ}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, *and* $[\mathcal{ETL}, \mathcal{CL}]$.

In particular, $\mathcal{CL}$ is the largest non-trivial extension of $\mathcal{B}$ and the logic $\mathcal{LP} \cap \mathcal{ECQ}$ is the smallest proper extension of $\mathcal{B}$. Recall that by Proposition 2.17 the logic $\mathcal{LP} \cap \mathcal{ECQ}$ is axiomatized by the rule $p, \neg p \vdash q \vee \neg q$.

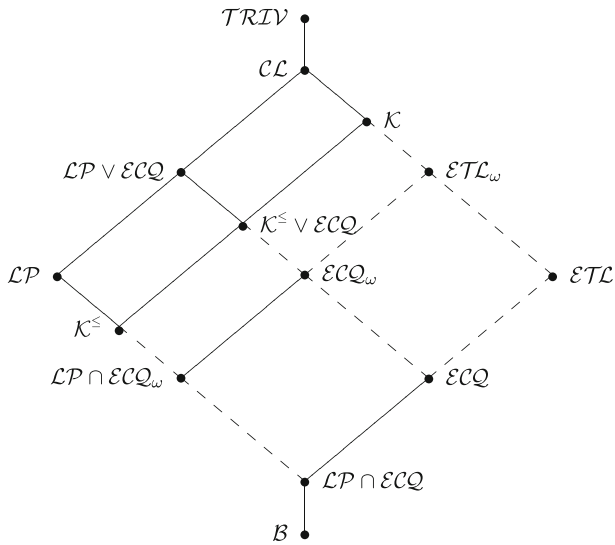


Figure 3. Super-Belnap logics under study

Corollary 4.4. *The only non-trivial proper extensions of $\mathcal{K}^{\leq}$ are the logics $\mathcal{K}^{\leq} \vee \mathcal{ECQ}$, $\mathcal{K}$, $\mathcal{LP}$, $\mathcal{LP} \vee \mathcal{ECQ}$, and $\mathcal{CL}$.*

In other words, all extensions of $\mathcal{K}^{\leq}$ are shown in Figure 3. This corollary in particular subsumes the description of $\mathrm{Ext}(\mathcal{LP})$ due to Pynko [42].[4]

Theorem 4.5. *Let $\mathcal{S}$ be a super-Belnap logic. Then either $\mathcal{LP} \cap \mathcal{ECQ}_2 \leq \mathcal{S}$ or $\mathcal{S} \leq \mathcal{ETL}$. Likewise, either $\mathcal{K}^{\leq} \leq \mathcal{S}$ or $\mathcal{S} \leq \mathcal{K}^{-}$.*

In particular, $\mathcal{ETL}_2$ is the smallest proper extension of $\mathcal{ETL}$ and $\mathcal{K}^{-}$ is the largest extension of $\mathcal{ETL}$ strictly below $\mathcal{K}$.

Some of the main results of [38] concern the connection between super-Belnap logics and graph theory. This connection takes a particularly simple form if we restrict our attention to the explosive extensions of $\mathcal{B}$.

Theorem 4.6. *The lattice $\mathrm{Exp}\,\mathrm{Ext}_\omega(\mathcal{B})$ is isomorphic to the order dual of the lattice of classes of finite graphs closed under homomorphisms.*

Classes of finite graphs closed under homomorphisms are precisely the upward closed sets of the so-called homomorphism order on finite graphs, a remarkable structure which has been the object of much mathematical attention, see in particular the monograph of Hell and Nešetřil [28].

[4]More precisely, Pynko described $\mathrm{Ext}(\mathcal{LP})$ for the language without 1 and 0.

The whole lattice $\mathrm{Ext}_\omega(\mathcal{B})$ may in fact be described as the order dual of a lattice of certain classes of triples (G, H, k) satisfying some suitable closure conditions, where G and H are finite graphs and $k \in \{0, 1\}$. The proof of this claim involves the Cornish–Fowler duality [17] restricted to finite De Morgan algebras. The core of this duality is the following construction: given a De Morgan frame, i.e. a poset $(W, \leq)$ equipped with an order-inverting involution δ, its *complex algebra* will be the lattice of all upsets of $(W, \leq)$ equipped with the operation $\neg U = W \setminus \delta[U]$. Finite De Morgan algebras are precisely the complex algebras of finite De Morgan frames.

The dual spaces of finite reduced models of $\mathcal{ETL}$ may essentially be viewed as graphs, where by graphs we mean finite symmetric (undirected) graphs which possibly contain reflexive vertices (loops), i.e. vertices which are self-adjacent. More precisely, given a graph $G = (X, R)$, let $X \sqcup \delta X$ be the set consisting of the vertices $u \in X$ and their copies $\delta u \in \delta X$, equipped with the natural involution δ. We then define a partial order on $X \sqcup \delta X$ such that $u \leq \delta v$ if and only if uRv, $u \leq v$ if and only if $u = v$, and $\delta u \not\leq v$ for all $u, v \in X$. The complex algebra of $(X \sqcup \delta X, \leq, \delta)$ will be denoted $\boldsymbol{\alpha}(G)$. We define two different filters on $\boldsymbol{\alpha}(G)$. The filter $F = \{1\}$ will be denoted $F_+(G)$, and the filter generated by the upset δX will be denoted $F_-(G)$.

Proposition 4.7. *The finite reduced models of $\mathcal{B}$ are exactly the matrices of the form $\langle \boldsymbol{\alpha}(G), F_+(G)\rangle \times \langle \boldsymbol{\alpha}(G), F_-(H)\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle^k$ for some graphs G and H and some $k \in \omega$. The finite reduced models of $\mathcal{ETL}$ are the matrices of the form $\langle \boldsymbol{\alpha}(G), F_+(G)\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle^k$ for some graph G and $k \in \omega$.*

Corollary 4.8. *Every finite reduced model of the logic $\mathcal{B}$ has the form $\langle \boldsymbol{A}, F\rangle \times \langle \boldsymbol{B}, G\rangle$ where $\langle \boldsymbol{A}, F\rangle$ is a reduced model of $\mathcal{B}$ but not a model of $\mathcal{ECQ}$ and $\langle \boldsymbol{B}, G\rangle$ is a reduced model of $\mathcal{ETL}$.*

With the help of Theorem 4.6 and Proposition 4.7, results from graph theory may be used to yield a better understanding of super-Belnap logics. For example, the following two results may be proved using the embeddability of the free countably generated semilattice into the homomorphism order on finite graphs and Erdős's theorem on the existence of graphs with an unbounded girth among graphs with a chromatic number at most n.

Proposition 4.9. *There is a non-finitary super-Belnap logic.*

Proposition 4.10. *For every $n \geq 2$, the logics $\mathcal{ECQ}_n$ and $\mathcal{ETL}_n$ are not complete with respect to any finite set of finite matrices.*

The lattice of classes of finite graphs under homomorphisms is known to have the cardinality of the continuum. It now follows from Theorem 4.6 that $|\mathrm{Ext}_\omega(\mathcal{B})| = 2^{\aleph_0}$. We may in fact make a more precise claim.

Proposition 4.11. $\operatorname{Exp}\operatorname{Ext}_\omega(\mathcal{B}) = \{\mathcal{B}\} \cup \operatorname{Exp}\operatorname{Ext}_\omega(\mathcal{ECQ})$.

Proposition 4.12. $\operatorname{Exp}\operatorname{Ext}_\omega(\mathcal{ECQ})$ *is isomorphic to* $\operatorname{Exp}\operatorname{Ext}_\omega(\mathcal{ETL})$ *via the maps* $\mathcal{S} \mapsto \mathcal{S} \vee \mathcal{ETL}$ *and* $\mathcal{S} \mapsto \operatorname{Exp}_{\mathcal{ECQ}}(\mathcal{S})$. $\operatorname{Exp}\operatorname{Ext}_\omega(\mathcal{ECQ})$ *is isomorphic to a sublattice of* $[\mathcal{B}, \mathcal{LP}]$ *via the maps* $\mathcal{S} \mapsto \mathcal{S} \cap \mathcal{LP}$ *and* $\mathcal{S} \mapsto \mathcal{S} \vee \mathcal{ECQ}$.

The theorem below now follows from these observations.

Theorem 4.13. *Each of the intervals* $[\mathcal{LP} \cap \mathcal{ECQ}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, *and* $[\mathcal{ETL}, \mathcal{CL}]$ *of the lattice* $\operatorname{Ext}_\omega(\mathcal{B})$ *has the cardinality of the continuum, as do the distributive sublattices* $\operatorname{Exp}\operatorname{Ext}_\omega(\mathcal{B})$ *and* $\operatorname{Exp}\operatorname{Ext}_\omega(\mathcal{ETL})$.

5. Algebraic Properties

In this section we shall place all super-Belnap logics within the Leibniz and Frege hierarchies. We study the classes $\mathsf{Alg}^*(\mathcal{S})$ and $\mathsf{Alg}(\mathcal{S})$ for the super-Belnap logics $\mathcal{S}$ in Figure 3, and prove that some well-known quasivarieties of De Morgan algebras in fact form an algebraic semantics for some of these logics.

5.1. Classification Within the Leibniz and Frege Hierarchies

We start by proving that no non-trivial super-Belnap logic apart from $\mathcal{CL}$ is protoalgebraic. A syntactic proof of this fact was given in [38], but we provide a simpler semantic proof.

Theorem 5.1. *The only non-trivial protoalgebraic super-Belnap logic is* $\mathcal{CL}$.

Proof. Let $\mathcal{S} \in \operatorname{Ext}(\mathcal{B})$ such that $\mathcal{S} \neq \mathcal{CL}$. Then either $\mathcal{S} \leq \mathcal{LP} \vee \mathcal{ECQ}$ or $\mathcal{S} \leq \mathcal{K}$ by Theorems 4.2 and 4.3 and Corollary 4.4. Since protoalgebraicity is preserved by extensions, it suffices to prove that neither of the logics $\mathcal{LP} \vee \mathcal{ECQ}$ and $\mathcal{K}$ is protoalgebraic. Recall that a logic $\mathcal{S}$ is protoalgebraic if and only if the Leibniz operator is monotone on $\mathcal{S}$-filters. To show that $\mathcal{LP} \vee \mathcal{ECQ}$ is not protoalgebraic, consider the algebra $\boldsymbol{K_8} = \boldsymbol{K_4} \times \boldsymbol{B_2}$ and the filters $F = \{\langle \mathsf{a}, 1\rangle, \langle 1, 1\rangle\}$ and $G = \{\langle \neg\mathsf{a}, 1\rangle, \langle \mathsf{a}, 1\rangle, \langle 1, 1\rangle\} \in \mathcal{F}i_{\mathcal{LP} \vee \mathcal{ECQ}}\boldsymbol{K_8}$. We have $F \subseteq G$ and $\langle\langle \mathsf{a}, 1\rangle, \langle 1, 1\rangle\rangle \in \boldsymbol{\Omega}^{\boldsymbol{K_8}}(F)$ but $\langle\langle \mathsf{a}, 1\rangle, \langle 1, 1\rangle\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{K_8}}(G)$. To show that $\mathcal{K}$ is not protoalgebraic, consider the algebra $\boldsymbol{K_4}$ and the filters $F = \{\mathsf{a}, 1\}$ and $G = \{\neg\mathsf{a}, \mathsf{a}, 1\}$. We have $F \subseteq G$ and $\langle \mathsf{a}, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{K_4}}(F)$ but $\langle \mathsf{a}, 1\rangle \notin \boldsymbol{\Omega}^{\boldsymbol{K_4}}(G)$. ∎

It is known that being protoalgebraic is equivalent to having the so-called parametrized local deduction-detachment theorem [18, Theorem 2.15], in

particular each logic which has the "classical" deduction-detachment theorem, as defined e.g. by Czelakowski in [18, Section 2.6], is protoalgebraic. The result of Beall et al. [8] that the logic $\mathcal{LP}$ does not have the deduction-detachment theorem can therefore be improved as follows.

COROLLARY 5.2. *The only non-trivial super-Belnap logic which enjoys the deduction-detachment theorem is* $\mathcal{CL}$.

As mentioned in Section 2, Theorem 5.1 reduces the characterization of super-Belnap logics inside the Leibniz hierarchy to two classes: truth-equational and assertional logics. As it turns out, such logics belong either to the interval $[\mathcal{LP}, \mathcal{CL}]$ or to the interval $[\mathcal{ETL}, \mathcal{CL}]$.

THEOREM 5.3. *A non-trivial super-Belnap logic* $\mathcal{S}$ *is truth-equational if and only if either* $\mathcal{S} \in [\mathcal{LP}, \mathcal{CL}]$ *or* $\mathcal{S} \in [\mathcal{ETL}, \mathcal{CL}]$. *Truth is defined in* $\mathsf{Mod}^*\mathcal{S}$ *by* $x \vee \neg x \approx x$ *in the former case, and by* $x \approx 1$ *in the latter case.*

PROOF. Suppose that $\mathcal{S} \in [\mathcal{LP}, \mathcal{CL}]$ or $\mathcal{S} \in [\mathcal{ETL}, \mathcal{CL}]$. Since truth-equationality is preserved by extensions, it suffices to show that $\mathcal{LP}$ and $\mathcal{ETL}$ are truth-equational. In the first case, let us fix $\boldsymbol{\tau}(x) := \{x \vee \neg x \approx x\}$ and let $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{LP})$. Since $\mathcal{B} \leq \mathcal{LP}$, we have $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^*(\mathcal{B})$. It follows by Theorem 3.6 (iv) that $F = \{a \in A : a \vee \neg a = a\} = \boldsymbol{\tau}(\boldsymbol{A})$. Therefore, the equation $\{x \vee \neg x \approx x\}$ defines truth in $\mathsf{Mod}^*(\mathcal{LP})$ by Theorem 3.6 (iv) and the equation $\{x \approx 1\}$ defines truth in $\mathsf{Mod}^*(\mathcal{ETL})$ by Theorem 3.6 (i).

As for the converse implication, the proof proceeds by contraposition. Suppose then that $\mathcal{S}$ is a non-trivial super-Belnap logic such that $\mathcal{S} \notin [\mathcal{LP}, \mathcal{CL}]$ and $\mathcal{S} \notin [\mathcal{ETL}, \mathcal{CL}]$. It follows by Theorem 4.5 that $\mathcal{S} \leq \mathcal{K}$ and $\mathcal{S} \leq \mathcal{LP} \vee \mathcal{ECQ}$. Therefore, $\mathcal{S} \leq \mathcal{K} \cap (\mathcal{LP} \vee \mathcal{ECQ}) = \mathcal{K}^{\leq} \vee \mathcal{ECQ}$. It now suffices to show that the logic $\mathcal{K}^{\leq} \vee \mathcal{ECQ}$ is not truth-equational. We know by Theorem 3.6 (v) that $\mathcal{K}^{\leq} \vee \mathcal{ECQ} = \big\{\langle \boldsymbol{K_3}, \{\mathsf{n}, \mathsf{t}\}\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle, \langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle\big\} = \big\{\langle \boldsymbol{K_3} \times \boldsymbol{B_2}, \{\mathsf{t}, \neg\mathsf{a}\}, \langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle\big\}$. Fix $F := \{\mathsf{a}, \mathsf{t}\}$. Then $F \in \mathcal{F}i_{\mathcal{K}^{\leq} \vee \mathcal{ECQ}}(\boldsymbol{K_3} \times \boldsymbol{B_2})$. Also, $\boldsymbol{K_3} \times \boldsymbol{B_2} \in \mathsf{KA} = \mathsf{Alg}(\mathcal{K}^{\leq} \vee \mathcal{ECQ})$. We claim that F is not a Leibniz filter of $\boldsymbol{A}$. First, it is easy to check that $\boldsymbol{\Omega}^{\boldsymbol{K_3} \times \boldsymbol{B_2}}(F) = id_{\boldsymbol{K_3} \times \boldsymbol{B_2}}$. Now, since $\boldsymbol{\Omega}^{\boldsymbol{K_3} \times \boldsymbol{B_2}}(F) = id_{\boldsymbol{K_3} \times \boldsymbol{B_2}}$, it follows that $[\![F]\!]^* = K_6$, and therefore $F^* = \bigcap \mathcal{F}i_{\mathcal{K}^{\leq} \vee \mathcal{ECQ}}(\boldsymbol{K_3} \times \boldsymbol{B_2})$. Now, $\{\mathsf{t}\} \in \mathcal{F}i_{\mathcal{K}^{\leq} \vee \mathcal{ECQ}}(\boldsymbol{K_3} \times \boldsymbol{B_2})$, because $\{\mathsf{t}\} \in \mathcal{F}i_{\mathcal{K}}(\boldsymbol{K_3} \times \boldsymbol{B_2})$ and $\mathcal{K}^{\leq} \vee \mathcal{ECQ} \leq \mathcal{K}$. Moreover, since $\mathcal{K}^{\leq} \leq \mathcal{K}^{\leq} \vee \mathcal{ECQ}$ and $\boldsymbol{K_3} \times \boldsymbol{B_2} \in \mathsf{Alg}(\mathcal{K}^{\leq})$, it follows that $\mathcal{F}i_{\mathcal{K}^{\leq} \vee \mathcal{ECQ}}(\boldsymbol{K_3} \times \boldsymbol{B_2}) \subseteq \mathcal{F}i_{\mathcal{K}^{\leq}}(\boldsymbol{K_3} \times \boldsymbol{B_2}) = \mathrm{Filt}(\boldsymbol{K_3} \times \boldsymbol{B_2})$, where $\mathrm{Filt}(\boldsymbol{K_3} \times \boldsymbol{B_2})$ denotes the set of lattice filters of $\boldsymbol{K_3} \times \boldsymbol{B_2}$. So, $\{\mathsf{t}\} = \bigcap \mathrm{Filt}(\boldsymbol{K_3} \times \boldsymbol{B_2}) \subseteq \bigcap \mathcal{F}i_{\mathcal{K}^{\leq} \vee \mathcal{ECQ}}(\boldsymbol{K_3} \times \boldsymbol{B_2})$. Then $\{\mathsf{t}\} = \bigcap \mathcal{F}i_{\mathcal{K}^{\leq} \vee \mathcal{ECQ}}(\boldsymbol{K_3} \times \boldsymbol{B_2})$, hence $F \neq F^*$, i.e. F is not a Leibniz $\mathcal{K}^{\leq} \vee \mathcal{ECQ}$-filter of $\boldsymbol{K_3} \times \boldsymbol{B_2}$. It finally follows by Theorem 2.4 that $\mathcal{K}^{\leq} \vee \mathcal{ECQ}$ is not truth-equational, as desired. ∎

THEOREM 5.4. *A non-trivial super-Belnap logic $\mathcal{S}$ is 1-assertional if and only if $\mathcal{S} \in [\mathcal{ETL}, \mathcal{CL}]$.*

PROOF. Let $\mathcal{S} \in [\mathcal{ETL}, \mathcal{CL}]$. Since 1-assertionality is preserved by extensions, and $\mathcal{ETL}$ is 1-assertional by Theorem 5.3 (i), it is clear that $\mathcal{S}$ is 1-assertional as well. Conversely, let $\mathcal{S}$ be a non-trivial 1-assertional super-Belnap logic. Then $\mathsf{Alg}(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{B}) = \mathsf{DMA} = \mathsf{Alg}(\mathcal{ETL})$. Since $\mathcal{S} = \big(\mathsf{Alg}(\mathcal{S}), \boldsymbol{\tau}_1\big)$ and $\mathcal{ETL} = \big(\mathsf{Alg}(\mathcal{ETL}), \boldsymbol{\tau}_1\big)$, with $\boldsymbol{\tau}_1(x) = \{x \approx 1\}$, it follows that $\mathcal{ETL} \leq \mathcal{S}$. Since $\mathcal{S}$ is non-trivial, we also have $\mathcal{S} \leq \mathcal{CL}$. ∎

Note that the only algebraic constants in DMA are 1 and 0, and no non-trivial super-Belnap logic is 0-assertional.

Summing up, the top right-most side of Figure 3 is composed exclusively by assertional (hence truth-equational) logics, while the top left-most side of Figure 3 is composed exclusively by truth-equational logics (but not assertional). Furthermore, all truth-equational and 1-assertional super-Belnap logics fall into one of these intervals.

The classification of super-Belnap logics within the Frege hierarchy is rather straightforward in the finitary case given the results seen in Section 2 about semilattice-based logics. A proof of the theorem which does not rely on Theorem 2.13 was also given in [38].

THEOREM 5.5. *The only (fully) selfextensional super-Belnap logics are $\mathcal{CL}$, $\mathcal{B}$ and $\mathcal{K}^{\leq}$. The logics $\mathcal{B}$ and $\mathcal{K}^{\leq}$ are non-Fregean.*

PROOF. The logics $\mathcal{B}$, $\mathcal{K}^{\leq}$, and $\mathcal{CL}$ are known to be (fully) selfextensional by virtue of their connection to the varieties of De Morgan, Kleene, and Boolean algebras, respectively, see Theorem 2.13 (i). It was proved by Font [23, Theorem 2.11] that the logic $\mathcal{B}$ is non-Fregean, and his counterexample applies equally well to $\mathcal{K}^{\leq}$, taking into account that $p, \neg p \nvdash_{\mathcal{K}^{\leq}} q$.

There are only three non-trivial varieties of De Morgan algebras, therefore by Theorem 2.13 (iii) each non-trivial finitary selfextensional super-Belnap logic is one of the logics $\mathcal{B}$, $\mathcal{K}^{\leq}$, $\mathcal{CL}$. Clearly, a logic $\mathcal{S}$ is selfextensional if and only if its finitary companion $\mathcal{S}_\omega$ is, where $\Gamma \vdash_{\mathcal{S}_\omega} \varphi$ if and only if $\Gamma' \vdash_{\mathcal{S}} \varphi$ for some finite $\Gamma' \subseteq \Gamma$. But by Theorem 4.3 each proper extension of $\mathcal{B}$ is an extension of $\mathcal{LP} \cap \mathcal{ECQ}$, therefore $\mathcal{B}$ cannot be its finitary companion. The same holds for $\mathcal{K}^{\leq}$ and $\mathcal{CL}$ by Corollary 4.4. ∎

5.2. The Class $\mathsf{Alg}(\mathcal{S})$ of Super-Belnap Logics

We shall now focus our attention on the classes of algebras $\mathsf{Alg}^*(\mathcal{S})$ and $\mathsf{Alg}(\mathcal{S})$ of a super-Belnap logic $\mathcal{S}$, with a particular interest in the super-

Belnap logics in Figure 3. As it turns out, apart from $\mathcal{CL}$, every super-Belnap logic $\mathcal{S}$ is such that $\mathsf{Alg}^*(\mathcal{S}) \subsetneq \mathsf{Alg}(\mathcal{S})$ (Proposition 5.12). This is a typical phenomenon of non-protoalgebraic logics, although there are of course examples of non-protoalgebraic logics $\mathcal{S}$ such that $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$.

Let us start with Belnap's logic $\mathcal{B}$ and Kleene's logic $\mathcal{K}$, whose algebraic counterparts were already determined in [23]. We improve on these two results as follows:

Proposition 5.6. *For every logic* $\mathcal{S} \in [\mathcal{B}, \mathcal{ETL}]$, $\mathsf{Alg}(\mathcal{S}) = \mathsf{DMA}$.

Proof. Since $\mathcal{B} \leq \mathcal{S} \leq \mathcal{ETL}$, it is clear that $\mathsf{Alg}(\mathcal{ETL}) \subseteq \mathsf{Alg}(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{B}) = \mathsf{DMA}$, where the equality follows by Theorem 2.13 (iii). On the other hand, observe that the subdirectly irreducible De Morgan algebras $\boldsymbol{DM_4}$, $\boldsymbol{K_3}$ and $\boldsymbol{B_2}$, all belong to $\mathsf{Alg}^*(\mathcal{ETL})$. Indeed, since the three algebras are simple, it follows that $\langle \boldsymbol{DM_4}, \{\mathsf{t}\}\rangle, \langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle, \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle \in \mathsf{Mod}^*(\mathcal{ETL})$. Applying a classical result of Birkhoff, we have $\mathsf{DMA} = \mathbb{IP}_{\mathrm{S}}\{\boldsymbol{DM_4}, \boldsymbol{K_3}, \boldsymbol{B_2}\} \subseteq \mathbb{IP}_{\mathrm{S}}\mathsf{Alg}^*(\mathcal{ETL}) = \mathsf{Alg}(\mathcal{ETL})$. ∎

Proposition 5.7. *For every logic* $\mathcal{S} \in [\mathcal{K}^{\leq}, \mathcal{K}]$, $\mathsf{Alg}(\mathcal{S}) = \mathsf{KA}$.

Proof. Since $\mathcal{K}^{\leq} \leq \mathcal{S} \leq \mathcal{K}$, it is clear that $\mathsf{Alg}(\mathcal{K}) \subseteq \mathsf{Alg}(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{K}^{\leq})$. On the one hand, it follows by Theorem 2.13 (iii) that $\mathsf{Alg}(\mathcal{K}^{\leq}) = \mathsf{KA}$. On the other hand, similarly to the proof of Proposition 5.6, observe that the subdirectly irreducible Kleene algebras $\boldsymbol{K_3}$ and $\boldsymbol{B_2}$ belong to $\mathsf{Alg}^*(\mathcal{K})$. So, $\mathsf{KA} = \mathbb{IP}_{\mathrm{S}}\{\boldsymbol{K_3}, \boldsymbol{B_2}\} \subseteq \mathbb{IP}_{\mathrm{S}}\mathsf{Alg}^*(\mathcal{K}) = \mathsf{Alg}(\mathcal{K})$. ∎

The algebraic counterpart of Priest's Logic of Paradox $\mathcal{LP}$ can be established in a similar way.

Proposition 5.8. $\mathsf{Alg}(\mathcal{LP}) = \mathsf{KA}$.

Proof. Since $\mathcal{K}^{\leq} \leq \mathcal{LP}$, the inclusion $\mathsf{Alg}(\mathcal{LP}) \subseteq \mathsf{Alg}(\mathcal{K}^{\leq}) = \mathsf{KA}$ holds. As for the converse inclusion, since the subdirectly irreducible Kleene algebras $\boldsymbol{K_3}$ and $\boldsymbol{B_2}$ belong to $\mathsf{Alg}^*(\mathcal{LP})$, it follows that $\mathsf{KA} = \mathbb{IP}_{\mathrm{S}}\{\boldsymbol{K_3}, \boldsymbol{B_2}\} \subseteq \mathbb{IP}_{\mathrm{S}}\mathsf{Alg}^*(\mathcal{LP}) = \mathsf{Alg}(\mathcal{LP})$. ∎

We next prove that the bottom two chains of logics in Figure 3 have the same algebraic counterpart, modulo n.

Proposition 5.9. *For every* $n \in \omega + 1$, $\mathsf{Alg}(\mathcal{LP} \cap \mathcal{ECQ}_n) = \mathsf{Alg}(\mathcal{ECQ}_n)$.

Proof. The inclusion $\mathsf{Alg}(\mathcal{ECQ}_n) \subseteq \mathsf{Alg}(\mathcal{LP} \cap \mathcal{ECQ}_n)$ is clear, because $\mathcal{LP} \cap \mathcal{ECQ}_n \leq \mathcal{ECQ}_n$. For the converse inclusion, notice that

$$\begin{aligned}\mathsf{Alg}^*(\mathcal{LP} \cap \mathcal{ECQ}_n) &= \mathsf{Alg}^*(\mathcal{LP}) \cup \mathsf{Alg}^*(\mathcal{ECQ}_n)\\ &\subseteq \mathsf{Alg}(\mathcal{LP}) \cup \mathsf{Alg}(\mathcal{ECQ}_n)\\ &= \mathsf{KA} \cup \mathsf{Alg}(\mathcal{ECQ}_n)\\ &= \mathsf{Alg}(\mathcal{ECQ}_n),\end{aligned}$$

where the first equality holds by Corollary 2.15 and because $\mathsf{KA} = \mathsf{Alg}(\mathcal{K}) \subseteq \mathsf{Alg}(\mathcal{ECQ}_n)$. Hence we have $\mathsf{Alg}(\mathcal{LP} \cap \mathcal{ECQ}_n) = \mathbb{IP}_{\mathrm{S}}\mathsf{Alg}^*(\mathcal{LP} \cap \mathcal{ECQ}_n) \subseteq \mathbb{IP}_{\mathrm{S}}\mathsf{Alg}(\mathcal{ECQ}_n) = \mathsf{Alg}(\mathcal{ECQ}_n)$. ■

We do not know whether for every $n \in \omega + 1$, $\mathsf{Alg}(\mathcal{ECQ}_n) = \mathsf{Alg}(\mathcal{ETL}_n)$. It is however easy to see that $\mathsf{Alg}(\mathcal{ETL}_m) \neq \mathsf{Alg}(\mathcal{ETL}_n)$ and $\mathsf{Alg}(\mathcal{ECQ}_m) \neq \mathsf{Alg}(\mathcal{ECQ}_n)$, for $m \neq n$.

Proposition 5.10. *For every $n \in \omega$,*

(i) $\mathsf{Alg}(\mathcal{ETL}_{n+1}) \subsetneq \mathsf{Alg}(\mathcal{ETL}_n)$,

(ii) $\mathsf{Alg}(\mathcal{ECQ}_{n+1}) \subsetneq \mathsf{Alg}(\mathcal{ECQ}_n)$.

Proof. (i) Let $n \in \omega$. Suppose for the sake of contradiction that $\mathsf{Alg}(\mathcal{ETL}_{n+1}) = \mathsf{Alg}(\mathcal{ETL}_n)$. Since $\mathcal{ETL}_{n+1}$ is the 1-assertional logic of $\mathsf{Alg}(\mathcal{ETL}_{n+1})$ and $\mathcal{ETL}_n$ is the 1-assertional logic of $\mathsf{Alg}(\mathcal{ETL}_n)$, it follows that $\mathcal{ETL}_{n+1} = \mathcal{ETL}_n$, which is absurd.

(ii) Let $n \in \omega$. Suppose for the sake of contradiction that $\mathsf{Alg}(\mathcal{ECQ}_{n+1}) = \mathsf{Alg}(\mathcal{ECQ}_n)$. We know by Theorems 2.12 and 6.8 (iv) that $\mathcal{ETL}_n = (\mathcal{ECQ}_n)^+$ is the 1-assertional logic of $\mathsf{Alg}(\mathcal{ECQ}_n)$ and $\mathcal{ETL}_{n+1} = (\mathcal{ECQ}_{n+1})^+$ is the 1-assertional logic of $\mathsf{Alg}(\mathcal{ECQ}_{n+1})$. So, it must be the case $\mathcal{ETL}_{n+1} = \mathcal{ETL}_n$, and we reach a contradiction again.

■

Other strict inclusions involving the classes $\mathsf{Alg}(\mathcal{S})$ of particular super-Belnap logics can be established. We next provide two such examples, which share an interesting feature in common, as we shall see.

Proposition 5.11. (i) $\mathsf{Alg}(\mathcal{ETL}_\omega) \subsetneq \mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{K_3})$.

(ii) $\mathsf{Alg}(\mathcal{LP} \vee \mathcal{ECQ}) \subsetneq \mathsf{NIKA}$.

Proof. (i) It follows by Theorem 3.7 (v) that $\mathsf{Alg}^*(\mathcal{ETL}_\omega) \subseteq \mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{K_3})$. So clearly, $\mathsf{Alg}(\mathcal{ETL}_\omega) = \mathbb{IP}_{\mathrm{S}}\mathsf{Alg}^*(\mathcal{ETL}_\omega) \subseteq \mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{K_3})$. To see that this inclusion is strict, let $\boldsymbol{A}$ be the subalgebra of $\boldsymbol{DM_4} \times \boldsymbol{K_3}$ with universe $A = \{0, 1, \mathsf{a}, \neg\mathsf{a}, \mathsf{b}, \neg\mathsf{b}\}$, following the notation of [44, Figure 1]

(here, $\boldsymbol{DM_4} \times \boldsymbol{K_3}$ is denoted by $\boldsymbol{D}_{12}$). Clearly, $\boldsymbol{A} \in \mathbb{S}(\boldsymbol{DM_4} \times \boldsymbol{K_3}) \subseteq \mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{K_3})$. We claim that $\boldsymbol{A} \notin \mathsf{Alg}(\mathcal{ETL}_\omega)$. It is easy to check that the only $\mathcal{ETL}_\omega$-filters on $\boldsymbol{A}$ are the sigleton $\{1\}$ and the universe A. Hence, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{ETL}_\omega}(\{1\}) = \boldsymbol{\Omega}^{\boldsymbol{A}}(\{1\}) \cap \boldsymbol{\Omega}^{\boldsymbol{A}}(A) = \boldsymbol{\Omega}^{\boldsymbol{A}}(\{1\})$. But, $\langle \mathsf{a}, \neg\mathsf{a}\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(\{1\})$. Hence, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{ETL}_\omega}(\{1\}) \neq id_{\boldsymbol{A}}$. Moreover, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{ETL}_\omega}(A) = A \times A \neq id_{\boldsymbol{A}}$. Thus, $\boldsymbol{A} \notin \mathsf{Alg}(\mathcal{ETL}_\omega)$.

(ii) We know by Theorem 3.7 (vi) that $\mathsf{Alg}^*(\mathcal{LP} \vee \mathcal{ECQ}) \subseteq \mathsf{NIKA}$, hence $\mathbb{IP}_{\mathrm{S}}\mathsf{Alg}^*(\mathcal{LP} \vee \mathcal{ECQ}) = \mathsf{Alg}(\mathcal{LP} \vee \mathcal{ECQ}) \subseteq \mathbb{IP}_{\mathrm{S}}\mathsf{NIKA} = \mathsf{NIKA}$. To see that the inclusion is strict, notice that $\boldsymbol{K_4} \in \mathsf{NIKA}$, however the system of $\mathcal{LP} \vee \mathcal{ECQ}$-filters on $\boldsymbol{K_4}$ is not reduced, i.e. $\boldsymbol{K_4} \notin \mathsf{Alg}(\mathcal{LP} \vee \mathcal{ECQ})$. ■

Notice that $\boldsymbol{DM_4} \times \boldsymbol{K_3} \in \mathsf{Alg}^*(\mathcal{ETL}_\omega) \subseteq \mathsf{Alg}(\mathcal{ETL}_\omega)$, and therefore the algebra $\boldsymbol{A}$ above witnesses that $\mathsf{Alg}(\mathcal{ETL}_\omega)$ is not closed under subalgebras. Likewise, $\boldsymbol{K_3} \times \boldsymbol{B_2} \in \mathsf{Alg}(\mathcal{LP} \vee \mathcal{ECQ})$, however $\boldsymbol{K_4}$ which is a subalgebra of $\boldsymbol{K_3} \times \boldsymbol{B_2}$ is not in $\mathsf{Alg}(\mathcal{LP} \vee \mathcal{ECQ})$. So neither $\mathsf{Alg}(\mathcal{ETL}_\omega)$ nor $\mathsf{Alg}(\mathcal{LP} \vee \mathcal{ECQ})$ are quasivarieties. Since moreover $\mathsf{Alg}^*(\mathcal{ETL}_\omega) \subsetneq \mathsf{Alg}(\mathcal{LP} \vee \mathcal{ECQ})$ and $\mathsf{Alg}^*(\mathcal{LP} \vee \mathcal{ECQ}) \subsetneq \mathsf{Alg}(\mathcal{ETL}_\omega)$, as we shall see in the next theorem, we are here in the presence of two examples of logics $\mathcal{S}$ such that $\mathsf{Alg}^*(\mathcal{S}) \subsetneq \mathsf{Alg}(\mathcal{S}) \subsetneq \mathbb{V}(\mathcal{S})$. The existence of such a logic was posed as an open problem in the first edition of [26]. To the authors' knowledge, the only examples in the literature are a (subintuitionistic) logic due to Bou [14], and an *ad hoc* example due to Babyonyshev [7].

Since all super-Belnap logics $\mathcal{S}$ under study are non-protoalgebraic, the class $\mathsf{Alg}^*(\mathcal{S})$ can be very difficult to characterize. In [23], it is proved that $\mathsf{Alg}^*(\mathcal{B}) \subsetneq \mathsf{Alg}(\mathcal{B})$. We now generalize this result to arbitrary super-Belnap logics.

Proposition 5.12. *For every non-trivial super-Belnap logic $\mathcal{S}$ except $\mathcal{CL}$ we have* $\mathsf{Alg}^*(\mathcal{S}) \subsetneq \mathsf{Alg}(\mathcal{S})$.

Proof. Let $\mathcal{S} \in \mathrm{Ext}(\mathcal{B})$ such that $\mathcal{S} \neq \mathcal{CL}$. Suppose for the sake of contradiction that $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$. We know that $(\mathcal{LP}, \mathcal{K})$ is a splitting pair of $\mathrm{Ext}(\mathcal{B})$, by Theorem 4.5. Then either $\mathcal{LP} \leq \mathcal{S}$ or $\mathcal{S} \leq \mathcal{K}$. In the latter case, $\mathsf{KA} = \mathsf{Alg}(\mathcal{K}) \subseteq \mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{B})$. In particular, $\boldsymbol{K_4} \in \mathsf{Alg}^*(\mathcal{B})$. We reach a contradiction, since the only chains in $\mathsf{Alg}^*(\mathcal{B})$ are $\boldsymbol{B_2}$ and $\boldsymbol{K_3}$, by [23, p. 16]. In the former case, since $\mathcal{LP}$ is truth-equational, it follows by Proposition 2.3 that $\mathcal{S}$ is truth-equational. It now follows by Proposition [3, Corollary 6.12] that $\mathcal{S}$ is protoalgebraic, which contradicts Proposition 5.1.

■

5.3. Algebraic Semantics of Truth-Equational Super-Belnap Logics

We finish our algebraic study of super-Belnap logics by looking at some distinguished algebraic semantics. Recall that if $\mathcal{S}$ is a truth-equational logic with defining equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}$, then the class $\mathsf{Alg}(\mathcal{S})$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$, by Proposition 2.7 (i). However, it turns out that some of our truth-equational super-Belnap logics also admit as algebraic semantics some quasivarieties of De Morgan algebras which are well studied in the literature, witnessed by very simple equational translations. In order to demonstrate this, let us fix the following two equational translations:

$$\boldsymbol{\tau}_0 \coloneqq \{x \vee \neg x \approx x\} \quad \text{and} \quad \boldsymbol{\tau}_1 \coloneqq \{x \approx 1\}.$$

THEOREM 5.13. (i) *The class* NIDMA *is a* $\boldsymbol{\tau}_1$*-algebraic semantics for the logic* $\mathcal{ETL}_\omega$.

(ii) *The class* NIKA *is a* $\boldsymbol{\tau}_0$*-algebraic semantics for the logic* $\mathcal{LP} \vee \mathcal{ECQ}$.

(iii) *The classes* KA *and* $\mathsf{NIDMA} \cup \mathsf{KA}$ *are* $\boldsymbol{\tau}_0$*-algebraic semantics for the logic* $\mathcal{LP}$.

(iv) *The classes* KA, RKA, NIKA, *and* $\mathsf{NIDMA} \cup \mathsf{KA}$, *are* $\boldsymbol{\tau}_1$*-algebraic semantics for the logic* $\mathcal{K}$.

(v) *The class* DMA *is a* $\boldsymbol{\tau}_1$*-algebraic semantics, as well as a* $\boldsymbol{\tau}_0$*-algebraic semantics, for the logic* $\mathcal{ETL}$.

PROOF. (i) Since $\mathrm{Log}\,\langle \boldsymbol{DM_4}, \{\mathsf{t}\}\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle$, by Theorem 3.6 (iii), and $\{\mathsf{t}\} = \boldsymbol{\tau}_1(\boldsymbol{DM_4} \times \boldsymbol{B_2})$, it follows by Proposition 2.5 that $\mathcal{ETL}_\omega$ is the $\boldsymbol{\tau}_1$-assertional logic of $\{\boldsymbol{DM_4} \times \boldsymbol{B_2}\}$, hence of $\mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{B_2})$ as well. The result now follows because $\mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{B_2}) = \mathsf{NIDMA}$, by Theorem 2.1.

(ii) Since $\mathcal{LP} \vee \mathcal{ECQ} = \mathrm{Log}\,\langle \boldsymbol{K_3}, \{\mathsf{n}, \mathsf{t}\}\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle = \mathrm{Log}\langle \boldsymbol{K_3} \times \boldsymbol{B_2}, \{\neg\mathsf{a}, \mathsf{t}\}\rangle$, by Theorem 3.6 (iv), and $\{\mathsf{a}, \mathsf{t}\} = \boldsymbol{\tau}_0(\boldsymbol{K_3} \times \boldsymbol{B_2})$, it follows by Proposition 2.5 that $\mathcal{LP} \vee \mathcal{ECQ}$ is the $\boldsymbol{\tau}_0$-assertional logic of $\{\boldsymbol{K_3} \times \boldsymbol{B_2}\}$, hence of $\mathbb{Q}(\boldsymbol{K_3} \times \boldsymbol{B_2})$ as well. The result now follows because $\mathbb{Q}(\boldsymbol{K_3} \times \boldsymbol{B_2}) = \mathsf{NIKA}$, by Theorem 2.1.

(iii) Since $\mathcal{LP} = \mathrm{Log}\langle \boldsymbol{K_3}, \{\mathsf{n}, \mathsf{t}\}\rangle$, by Theorem 3.4 (ii), and $\{\mathsf{n}, \mathsf{t}\} = \boldsymbol{\tau}(\boldsymbol{K_3})$, it follows by Proposition 2.5 that $\mathcal{LP}$ is the $\boldsymbol{\tau}_0$-assertional logic of $\{\boldsymbol{K_3}\}$, hence of $\mathbb{Q}(\boldsymbol{K_3})$ as well. Having in mind that $\mathbb{Q}(\boldsymbol{K_3}) = \mathsf{KA}$, the first part of the result follows. Next, since $\mathcal{LP} \leq \mathcal{LP} \vee \mathcal{ECQ}$, it follows that $\mathcal{LP} = \mathrm{Log}\left\{\langle \boldsymbol{K_3}, \{\mathsf{n}, \mathsf{t}\}\rangle, \langle \boldsymbol{K_3}, \{\mathsf{n}, \mathsf{t}\}\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle\right\} = \mathrm{Log}\left\{\langle \boldsymbol{K_3}, \{\mathsf{n}, \mathsf{t}\}\rangle, \langle \boldsymbol{K_3} \times \boldsymbol{B_2}, \{\neg\mathsf{a}, \mathsf{t}\}\rangle\right\}$. Now, since $\{\mathsf{n}, \mathsf{t}\} = \boldsymbol{\tau}_0(\boldsymbol{K_3})$ and $\{\mathsf{a}, \mathsf{t}\} = \boldsymbol{\tau}_0(\boldsymbol{K_3} \times \boldsymbol{B_2})$, $\mathcal{LP}$ is the $\boldsymbol{\tau}_0$-assertional logic of $\{\boldsymbol{K_3}, \boldsymbol{K_3} \times \boldsymbol{B_2}\}$. Since $\mathbb{Q}(\boldsymbol{K_3}, \boldsymbol{K_3} \times$

$\boldsymbol{B_2}) = \mathsf{NIDMA} \cup \mathsf{KA}$, by Theorem 2.1, the second part of the result also follows.

(iv) Using Theorem 3.4 it is not difficult to see that $\mathcal{K} = \mathrm{Log}\langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle = \mathrm{Log}\langle \boldsymbol{K_4}, \{\mathsf{t}\}\rangle = \mathrm{Log}\langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle$. Since moreover $\{\mathsf{t}\} = \boldsymbol{\tau}_1(\boldsymbol{K_3}) = \boldsymbol{\tau}_1(\boldsymbol{K_4}) = \boldsymbol{\tau}_1(\boldsymbol{K_3} \times \boldsymbol{B_2})$, the result follows as before, having in mind that $\mathbb{Q}(\boldsymbol{K_4}) = \mathsf{RKA}$ and $\mathbb{Q}(\boldsymbol{K_3} \times \boldsymbol{B_2}) = \mathsf{NIKA}$, by Theorem 2.1. Finally, since $\mathcal{K} = \mathrm{Log}\left\{\langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle \times \langle \boldsymbol{B_2}, \{\mathsf{t}\}\rangle, \langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle\right\} = \mathrm{Log}\left\{\langle \boldsymbol{K_3} \times \boldsymbol{B_2}, \{\mathsf{t}\}\rangle, \langle \boldsymbol{K_3}, \{\mathsf{t}\}\rangle\right\}$, $\mathcal{K}$ is also the $\boldsymbol{\tau}_1$-assertional logic of $\mathbb{Q}(\boldsymbol{K_3}, \boldsymbol{K_3} \times \boldsymbol{B_2}) = \mathsf{NIDMA} \cup \mathsf{KA}$, by Theorem 2.1.

(v) Given the previous items, just notice that $\mathcal{ETL} = \mathrm{Log}\langle \boldsymbol{DM_4}, \{\mathsf{t}\}\rangle$ and $\boldsymbol{\tau}_1(\boldsymbol{DM_4}) = \{\mathsf{t}\} = \boldsymbol{\tau}_0(\boldsymbol{DM_4})$. ■

Theorem 5.13 provides new examples of several phenomena that may occur when dealing with algebraic semantics. For instance, we see that the same class of algebras can be an algebraic semantics for two different logics (witnessed by different equational translations, of course); the same logic can have different algebraic semantics, witnessed by the same equational translation; and finally, two algebraic semantics for the same logic need not generate the same quasivariety.

Using the terminology of [12], items (iii) and (iv) above tell us that the quasivarieties KA and $\mathsf{NIDMA} \cup \mathsf{KA}$ are $\boldsymbol{\tau}_0$-assertionally equivalent, while the quasivarieties KA, RKA, NIKA, and $\mathsf{NIDMA} \cup \mathsf{KA}$, are $\boldsymbol{\tau}_1$-assertionally equivalent.

Another consequence of Theorem 5.13 is that $\mathsf{KA} \subsetneq \mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{K_3}) \subsetneq \mathsf{DMA}$. Indeed, since $\mathcal{ETL}_\omega = \mathrm{Log}\langle \boldsymbol{DM_4} \times \boldsymbol{K_3}, \{1\}\rangle$ and $\{1\} = \boldsymbol{\tau}_1(\boldsymbol{DM_4} \times \boldsymbol{K_3})$, $\mathcal{ETL}_\omega$ is the $\boldsymbol{\tau}_1$-assertional logic of $\{\boldsymbol{DM_4} \times \boldsymbol{K_3}\}$, hence of $\mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{K_3})$ as well. Therefore, if $\mathsf{KA} = \mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{K_3})$, then $\mathcal{ETL}_\omega$ would coincide with $\mathcal{K}$; and if $\mathsf{DMA} = \mathbb{Q}(\boldsymbol{DM_4} \times \boldsymbol{K_3})$, then $\mathcal{ETL}_\omega$ would coincide with $\mathcal{ETL}$.

6. Leibniz Filters and Strong Versions

In this last section, we examine the logics in Figure 3 from the point of view of AAL, using some of its recent developments on non-protoalgebraic logics. Namely, the strong version of a sentential logic and its Leibniz filters, a theory that is developed in full generality in [2], extending that of [25] for protoalgebraic logics.

6.1. The Strong Version of an Explosive Extension

We first undertake a small a detour in order to prove some general results concerning the strong version of an explosive extension. Throughout the present subsection, $\mathcal{B}$ denotes an arbitrary (base) logic, *not* necessarily the Belnap–Dunn logic, and $\mathcal{S}_{exp}$ denotes an arbitrary explosive extension of $\mathcal{B}$. In Section 6.2 we will instantiate the results here obtained for the explosive extensions $\mathcal{ECQ}_n$ and $\mathcal{ECQ}_\omega$ of $\mathcal{B}$.

Our strategy to determine the strong version of $\mathcal{S}_{exp}$ is to prove that the least $\mathcal{S}_{exp}$-filter and least $\mathcal{B}$-filter always coincide, for algebras in $\mathsf{Alg}(\mathcal{S}_{exp})$ (Corollary 6.2). The key observation to establish it is the next proposition.

Proposition 6.1. *If $\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\mathcal{S}_{exp})$ is a non-trivial model of $\mathcal{S}_{exp}$, then for every $\langle \boldsymbol{A}, G\rangle \in \mathsf{Mod}(\mathcal{B})$ with $G \subseteq F$, $\langle \boldsymbol{A}, G\rangle \in \mathsf{Mod}(\mathcal{S}_{exp})$ is a (non-trivial) model of $\mathcal{S}_{exp}$.*

Proof. If $\langle \boldsymbol{A}, G\rangle$ is a model of $\mathcal{B}$ but not of $\mathcal{S}_{exp}$, then some explosive rule $\Gamma \vdash \varnothing$ which is valid in $\mathcal{S}$ fails in $\langle \boldsymbol{A}, G\rangle$. There is therefore a valuation $v : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $v[\Gamma] \subseteq G$. Since $G \subseteq F$, we have $v[\Gamma] \subseteq F$. The rule $\Gamma \vdash \varnothing$ thus fails in the matrix $\langle \boldsymbol{A}, F\rangle$, hence $\langle \boldsymbol{A}, F\rangle$ is not a model of $\mathcal{S}_{exp}$. ■

Corollary 6.2. *Let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}_{exp})$. The least $\mathcal{S}_{exp}$-filter of $\boldsymbol{A}$ coincides with the least $\mathcal{B}$-filter of $\boldsymbol{A}$.*

Proof. Let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}_{exp})$. Taking $G = \bigcap \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$ in Proposition 6.1, it follows that either $\boldsymbol{A}$ has no non-trivial $\mathcal{S}$-filters, or the least $\mathcal{B}$-filter of $\boldsymbol{A}$ is an $\mathcal{S}$-filter, in which case $\bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$. In the latter case we are done. In the former case, we have $\bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A} = A$. So, $id_{\boldsymbol{A}} = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}_{exp}}\big(\bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A}\big) = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}_{exp}}(A) = A \times A$. But then it must be the case that $\boldsymbol{A}$ is a trivial algebra, say $A = \{a\}$. Hence $\bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A} = \{a\} = \bigcap \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$. ■

We are now ready to characterize the strong version of an explosive extension in terms of the strong version of its base logic (Theorem 6.4).

Lemma 6.3. $\mathcal{B}^+ \leq \mathcal{S}^+_{exp}$.

Proof. On the one hand, $\mathsf{Alg}(\mathcal{S}_{exp}) \subseteq \mathsf{Alg}(\mathcal{B})$, because $\mathcal{B} \leq \mathcal{S}_{exp}$. On the other hand, for every $\boldsymbol{A} = \mathsf{Alg}(\mathcal{S}_{exp})$ we have $\bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A} = \bigcap \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$ by Corollary 6.2. Having in mind that $\mathcal{B}^+$ is the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \bigcap \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}\rangle : \boldsymbol{A} \in \mathsf{Alg}(\mathcal{B})\}$ and that $\mathcal{S}^+_{exp}$ is the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A}\rangle : \boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}_{exp})\}$, the result follows. ■

Notice that the operator $(\ )^+ : \mathcal{S} \mapsto \mathcal{S}^+$ is not monotonic in general. For instance, $\mathcal{K}^{\leq} \leq \mathcal{LP}$, but as we shall see $(\mathcal{K}^{\leq})^+ = \mathcal{K} \not\leq \mathcal{LP} = \mathcal{LP}^+$.

THEOREM 6.4. $\mathcal{S}^+_{exp} = (\mathcal{B}^+ \vee \mathcal{S}_{exp})^+$.

PROOF. Let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{B}^+ \vee \mathcal{S}_{exp})$. Since $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{B}^+ \vee \mathcal{S}_{exp}) \subseteq \mathsf{Alg}(\mathcal{S}_{exp})$, and $\mathcal{S}_{exp}$ is an explosive extension of $\mathcal{B}$, it follows by Corollary 6.2 that $\bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A} = \bigcap \mathcal{F}i_{\mathcal{B}}\boldsymbol{A} = \bigcap \mathcal{F}i_{\mathcal{B}^+}\boldsymbol{A}$. So, $\bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A}$ is both an $\mathcal{S}_{exp}$-filter of $\boldsymbol{A}$ and a $\mathcal{B}^+$-filter of $\boldsymbol{A}$, hence $\bigcap \mathcal{F}i_{\mathcal{B}^+\vee\mathcal{S}_{exp}}\boldsymbol{A} = \bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A}$. Having in mind that $(\mathcal{B}^+ \vee \mathcal{S}_{exp})^+$ is the the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \bigcap \mathcal{F}i_{\mathcal{B}^+\vee\mathcal{S}_{exp}}\boldsymbol{A}\rangle : \boldsymbol{A} \in \mathsf{Alg}(\mathcal{B}^+\vee\mathcal{S}_{exp})\}$ and that $\mathcal{S}^+_{exp}$ is the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A}\rangle : \boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}_{exp})\}$, and moreover $\mathsf{Alg}(\mathcal{B}^+ \vee \mathcal{S}_{exp}) \subseteq \mathsf{Alg}(\mathcal{S}_{exp})$, it follows that $\mathcal{S}^+ \leq (\mathcal{B}^+ \vee \mathcal{S}_{exp})^+$. Together with Lemma 6.3, we get $\mathcal{B}^+ \vee \mathcal{S}_{exp} \leq \mathcal{S}^+_{exp} \leq (\mathcal{B}^+ \vee \mathcal{S}_{exp})^+$. The result now follows by Lemma 2.9, having in mind that in general $(\mathcal{S}^+)^+ = \mathcal{S}^+$. ■

By imposing equational definability over the Leibniz $\mathcal{B}$-filters, we can obtain a refined version of Theorem 6.4. To this end, let us first see that equational definability of Leibniz filters is preserved from the base logic to explosive extensions.

PROPOSITION 6.5. *Let $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}$. If the Leibniz $\mathcal{B}$-filters are equationally definable by $\boldsymbol{\tau}$, then the Leibniz $\mathcal{S}_{exp}$-filters are equationally definable by $\boldsymbol{\tau}$.*

PROOF. Assume that the Leibniz $\mathcal{B}$-filters are equationally definable by $\boldsymbol{\tau}$. Let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}_{exp})$. Since $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}_{exp}) \subseteq \mathsf{Alg}(\mathcal{B})$, it follows by assumption that $\bigcap \mathcal{F}i_{\mathcal{B}}\boldsymbol{A} = \boldsymbol{\tau}(\boldsymbol{A})$. Since $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}_{exp})$, it follows by Corollary 6.2 that $\bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A} = \bigcap \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$. Hence $\boldsymbol{\tau}(\boldsymbol{A}) = \bigcap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A}$, and by Proposition 2.11 the Leibniz $\mathcal{S}_{exp}$-filters are equationally definable by $\boldsymbol{\tau}$. ■

Putting together Theorem 6.4 and Proposition 6.5, we obtain:

THEOREM 6.6. *If $\mathcal{B}$ has equationally definable Leibniz filters, then $\mathcal{S}^+_{exp} = \mathcal{B}^+ \vee \mathcal{S}_{exp}$.*

PROOF. The inclusion $\mathcal{B}^+ \vee \mathcal{S}_{exp} \leq \mathcal{S}^+_{exp}$ hols in general, by Theorem 6.4. As for the converse inclusion, since the Leibniz $\mathcal{B}$-filters are equationally definable, it follows by Theorem 2.12 (iii) that for every algebra $\boldsymbol{A}$, $\mathcal{F}i_{\mathcal{B}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{B}}\boldsymbol{A}$. Hence $\mathcal{F}i_{\mathcal{B}^+\vee\mathcal{S}_{exp}}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{B}^+}\boldsymbol{A} \cap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{B}}\boldsymbol{A} \cap \mathcal{F}i_{\mathcal{S}_{exp}}\boldsymbol{A} \subseteq \mathcal{F}i^*_{\mathcal{S}_{exp}}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}^+_{exp}}\boldsymbol{A}$ for every algebra $\boldsymbol{A}$. Thus $\mathcal{S}^+_{exp} \leq \mathcal{B}^+ \vee \mathcal{S}_{exp}$. ■

6.2. Strong Versions of Super-Belnap Logics

In our setting of super-Belnap logics, matters are fairly well-behaved, due to the fact that all non-truth-equational logics considered in Figure 3 have their Leibniz filters equationally definable by the set of equations $\boldsymbol{\tau}_1(x) = \{x \approx 1\}$. We proceed to prove this fact.

THEOREM 6.7. *Fix* $\boldsymbol{\tau}_1(x) \coloneqq \{x \approx 1\} \subseteq \mathrm{Eq}$.

(i) *The Leibniz* $\mathcal{B}$*-filters are equationally definable by* $\boldsymbol{\tau}_1$.

(ii) *The Leibniz* $\mathcal{K}^{\leq}$*-filters are equationally definable by* $\boldsymbol{\tau}_1$.

(iii) *The Leibniz* $\mathcal{K}^{\leq} \vee \mathcal{ECQ}$*-filters are equationally definable by* $\boldsymbol{\tau}_1$.

(iv) *The Leibniz* $\mathcal{ECQ}_n$*-filters are equationally definable by* $\boldsymbol{\tau}_1$ *for* $n \in \omega + 1$.

(v) *The Leibniz* $\mathcal{LP} \cap \mathcal{ECQ}_n$*-filters are equationally definable by* $\boldsymbol{\tau}_1$ *for* $n \in \omega + 1$.

PROOF. (i) and (ii) Having in mind that $\mathcal{B}$ and $\mathcal{K}^{\leq}$ are the semilattice-based logics of DMA and KA, respectively, and since moreover 1 is a theorem of both logics, the results follow by Theorem 2.13 (iv).

(iii) Just notice that, for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{K}^{\leq} \vee \mathcal{ECQ}) = \mathsf{KA}$, $\{1\} = \bigcap \mathcal{F}i_{\mathcal{K}^{\leq}}\boldsymbol{A} \subseteq \bigcap \mathcal{F}i_{\mathcal{K}^{\leq} \vee \mathcal{ECQ}}\boldsymbol{A} \subseteq \bigcap \mathcal{F}i_{\mathcal{K}}\boldsymbol{A} = \{1\}$, using Theorem 2.13 (ii). So, $\mathcal{F}i_{\mathcal{K}^{\leq} \vee \mathcal{ECQ}}\boldsymbol{A} = \{1\}$. The result now follows by Proposition 2.11 (iv) follows by (i) and Proposition 6.5.

(v) follows by (iv) and Proposition 2.11, having in mind that $\mathsf{Alg}(\mathcal{LP} \cap \mathcal{ECQ}_n) = \mathsf{Alg}(\mathcal{ECQ}_n)$, by Corollary 5.9. ∎

Theorem 6.7 enables us to determine the strong version of all non-truth-equational super-Belnap logics under study. As for the truth-equational ones, recall that these coincide with their respective strong versions.

THEOREM 6.8. (i) $\mathcal{B}^+ = \mathcal{ETL}$.

(ii) ${\mathcal{K}^{\leq}}^+ = \mathcal{K}$.

(iii) $(\mathcal{K}^{\leq} \vee \mathcal{ECQ})^+ = \mathcal{K}$.

(iv) $\mathcal{ECQ}_n^+ = \mathcal{ETL}_n$, *for every* $n \in \omega + 1$.

(v) $(\mathcal{LP} \cap \mathcal{ECQ}_n)^+ = \mathcal{ETL}_n$, *for every* $n \in \omega + 1$.

PROOF. (i) and (ii) Having in mind Theorem 3.4 (iii) and (iv), the results follow by Theorem 2.13 (v).

(iii) follows by (ii) and Lemma 2.9.

(iv) Since $\mathcal{B}$ has it Leibniz filters equationally definable, it follows by Theorem 6.6 that, for every $n \in \omega + 1$, $(\mathcal{ECQ}_n)^+ = \mathcal{B}^+ \vee \mathcal{ECQ}_n = \mathcal{ETL} \vee \mathcal{ECQ}_n = \mathcal{ETL}_n$.

(v) Since $\mathcal{LP} \cap \mathcal{ECQ}_n$ has its Leibniz filters equationally definable by $\boldsymbol{\tau}_1$, it follows by Theorem 6.8 (i) that $(\mathcal{LP} \cap \mathcal{ECQ}_n)^+$ is the $\boldsymbol{\tau}$-assertional logic of $\mathsf{Alg}(\mathcal{LP} \cap \mathcal{ECQ}_n)$. But $\mathsf{Alg}(\mathcal{LP} \cap \mathcal{ECQ}_n) = \mathsf{Alg}(\mathcal{ECQ}_n)$, by Corollary 5.9, therefore $(\mathcal{LP} \cap \mathcal{ECQ}_n)^+ = \mathcal{ETL}_n$. ∎

In the light of Theorem 6.8 (i), an immediate consequence of Lemma 2.9 is that we are able to find the strong version of every logic in the interval $[\mathcal{B}, \mathcal{ETL}]$.

COROLLARY 6.9. *For every* $\mathcal{S} \in [\mathcal{B}, \mathcal{ETL}]$, $\mathcal{S}^+ = \mathcal{ETL}$.

Interestingly, all the strong versions considered here turn out to be truth-equational logics. This need not hold in general, as proved in [2].

Since each non-truth-equational super-Belnap logic $\mathcal{S}$ shown in Figure 3 has equationally definable Leibniz $\mathcal{S}$-filters, it follows by Theorem 6.8 (iii) that the Leibniz $\mathcal{S}$-filters coincide with the $\mathcal{S}^+$-filters on arbitrary algebras.

COROLLARY 6.10. *For every algebra* $\boldsymbol{A}$,

(i) $\mathcal{F}i^*_{\mathcal{B}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{ETL}}\boldsymbol{A}$,

(ii) $\mathcal{F}i^*_{\mathcal{K}^{\leq}}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{K}^{\leq}\vee\mathcal{ECQ}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{K}}\boldsymbol{A}$,

(iii) *For every* $n \in \omega + 1$, $\mathcal{F}i^*_{\mathcal{ECQ}_n}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{LP}\cap\mathcal{ECQ}_n}\boldsymbol{A} = \mathcal{F}i_{\mathcal{ETL}_n}\boldsymbol{A}$.

Acknowledgements. Adam Přenosil gratefully acknowledges the support of the Grant P202/12/G061 of the Czech Science Foundation. The research of Umberto Rivieccio has been supported by EU FP7 Marie Curie PIRSES-GA-2012-318986 project and by CNPq process 482809/2013-2 of the Brazilian National Council for Scientific and Technological Development.

References

[1] ADAMS, M. E., and W. DZIOBIAK, Lattices of quasivarieties of 3-element algebras, *Journal of Algebra* 166: 181–210, 1994.

[2] ALBUQUERQUE, H., J. M. FONT, and R. JANSANA, The strong version of a sentential logic, *Studia Logica*, 2017. doi:10.1007/s11225-017-9709-0.

[3] ALBUQUERQUE, H., J. M. FONT, and R. JANSANA, Compatibility operators in abstract algebraic logic, *The Journal of Symbolic Logic* 81(2): 417–462, 2016.

[4] ALBUQUERQUE, H., J. M. FONT, R. JANSANA, and T. MORASCHINI, Assertional logics, truth-equational logics and the hierarchies of AAL, in J. Czelakowski, (ed.), *Don Pigozzi on Abstract Algebraic Logic and Universal Algebra*. To appear.

[5] ANDERSON, A. R., and N. D. BELNAP, *Entailment: The Logic of Relevance and Necessity*, vol. 1, Princeton University Press, 1975.

[6] ASENJO, F. G., A calculus of antinomies, *Notre Dame Journal of Formal Logic* 7(1): 103–105, 1966.

[7] BABYONYSHEV, S. V., Fully Fregean logics, *Reports on Mathematical Logic* 37: 59–77, 2003.

[8] BEALL, J. C., T. FORSTER, and J. SELIGMAN, A note on freedom from detachment in the Logic of Paradox, *Notre Dame Journal of Formal Logic* 54(1): 15–20, 2013.

[9] Belnap, N. D., How a computer should think, in G. Ryle, (ed.), *Contemporary Aspects of Philosophy*, Oriel Press, Boston, 1977, pp. 30–56.

[10] Belnap, N. D., A useful four-valued logic, in J. M. Dunn, and G. Epstein, (eds.), *Modern uses of multiple-valued logic*, Reidel, Dordrecht. Episteme, Vol. 2, 1977, pp. 5–37.

[11] Blok, W. J., and D. Pigozzi, Algebraizable logics, *Memoirs of the American Mathematical Society* 77, 396, vi+78, 1989.

[12] Blok, W. J., and J. G. Raftery, Assertionally equivalent quasivarieties, *International Journal of Algebra and Computation* 18(4): 589–681, 2008.

[13] Blok, W. J., and J. Rebagliato, Algebraic semantics for deductive systems, *Studia Logica* 74(1–2): 153–180, 2003.

[14] Bou, F., *Implicación estricta y lógicas subintuicionistas*, Master's thesis, Universitat de Barcelona, 2001.

[15] Burris, S., and H. P. Sankappanavar, *A Course in Universal Algebra*, vol. 78 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 1981.

[16] Cintula, P., and C. Noguera, The proof by cases property and its variants in structural consequence relations, *Studia Logica* 101(4): 713–747, 2013.

[17] Cornish, W. H., and P. R. Fowler, Coproducts of De Morgan algebras, *Bulletin of the Australian Mathematical Society* 16: 1–13, 1977.

[18] Czelakowski, J., *Protoalgebraic logics*, vol. 10 of *Trends in Logic—Studia Logica Library*, Kluwer Academic Publishers, Dordrecht, 2001.

[19] Dunn, J. M., *The algebra of intensional logics*, Ph.D. thesis, University of Pittsburgh, Ann Arbor.

[20] Dunn, J. M., Intuitive semantics for first-degree entailments and 'coupled trees', *Philosophical Studies* 29(3): 149–168, 1976.

[21] Dunn, J. M., A Kripke-style semantics for R-mingle using a binary accessibility relation, *Studia Logica* 35: 163–172, 1976.

[22] Dunn, J. M., Partiality and its dual, *Studia Logica* 66(1): 5–40, 2000.

[23] Font, J. M., Belnap's four-valued logic and De Morgan lattices, *Logic Journal of the IGPL* 5(3): 1–29, 1997.

[24] Font, J. M., *Abstract Algebraic Logic – An Introductory Textbook*, vol. 60 of *Studies in Logic*, College Publications, London, 2016.

[25] Font, J. M., and R. Jansana, Leibniz filters and the strong version of a protoalgebraic logic, *Archive for Mathematical Logic* 40: 437–465, 2001.

[26] Font, J. M., and R. J., *A general algebraic semantics for sentential logics*, vol. 7 of *Lecture Notes in Logic*, second revised edition, Association for Symbolic Logic, 2009. Electronic version freely available through Project Euclid at http://projecteuclid.org/euclid.lnl/1235416965. First edition published by Springer-Verlag, 1996.

[27] Gaitán, H., and M. H. Perea, A non-finitely based quasi-variety of De Morgan algebras, *Studia Logica* 78: 237–248, 2004.

[28] Hell, P., and J. Nešetřil, *Graphs and homomorphisms*, vol. 28 of *Oxford Lecture Series in Mathematics and its Applications*, Oxford University Press, Oxford, 2004.

[29] Jansana, R., Selfextensional logics with a conjunction, *Studia Logica* 84: 63–104, 2006.

[30] Kalman, J. A., Lattices with involution, *Transactions of the American Mathematical Society* 87(2): 485–491, 1958.

[31] Kleene, S. C., On notation for ordinal numbers, *The Journal of Symbolic Logic* 3(4): 150–155, 1938.

[32] Kleene, S. C., *Introduction to Metamathematics*, vol. 1 of *Bibliotheca Mathematica*, North-Holland Publishing Co., Amsterdam, 1952.

[33] Kripke, S. A., Outline of a theory of truth, *Journal of Philosophy* 72(19): 690–716, 1975.

[34] Makinson, D. C., *Topics in Modern Logic*, Methuen, London, 1973.

[35] Marcos, J., The value of the two values, in M. E. Coniglio, and J.-Y. Béziau, (eds.), *Logic without Frontiers: Festschrift for Walter Alexandre Carnielli on the occasion of his 60th birthday*, College Publications, London, 2011, pp. 277–294.

[36] Pietz, A., and U. Rivieccio, Nothing but the truth, *Journal of Philosophical Logic* 42(1): 125–135, 2013.

[37] Priest, G., The Logic of Paradox, *Journal of Philosophical Logic* 8(1): 219–241, 1979.

[38] Přenosil, A., The lattice of super-Belnap logics, 201x. Submitted manuscript.

[39] Pynko, A. P., Characterizing Belnap's logic via De Morgan's laws, *Mathematical Logic Quarterly* 41(4): 442–454, 1995.

[40] Pynko, A. P., On Priest's Logic of Paradox, *Journal of Applied Non-Classical Logics* 5(2): 219–225, 1995.

[41] Pynko, A. P., Implicational classes of De Morgan lattices, *Discrete Mathematics* 205(1–3): 171–181, 1999.

[42] Pynko, A. P., Subprevarieties versus extensions. Application to the logic of paradox, *The Journal of Symbolic Logic* 65(2): 756–766, 2000.

[43] Raftery, J. G., The equational definability of truth predicates, *Reports on Mathematical Logic* 41: 95–149, 2006.

[44] Rivieccio, U., An infinity of super-Belnap logics, *Journal of Applied Non-Classical Logics* 22(4): 319–335, 2012.

[45] Wójcicki, R., *Lectures on propositional calculi*, Ossolineum Publishing Co., Wrocław, 1984.

[46] Wójcicki, R., *Theory of logical calculi: Basic theory of consequence operations*, vol. 199 of *Synthese Library*, Kluwer Academic Publishers Group, Dordrecht, 1988.

H. Albuquerque
Departament de Filosofia
Universitat de Barcelona
Montalegre 7
08001 Barcelona
Spain
hugo.albuquerque@ua.pt

A. Přenosil
Institute of Computer Science
Czech Academy of Sciences
Pod Vodárenskou věží 271/2
Prague
Czechia
adam.prenosil@gmail.com

U. Rivieccio
Department of Informatics and Applied Mathematics
Federal University of Rio Grande do Norte
Campus Universitário Lagoa Nova
Natal RN
Brazil
urivieccio@dimap.ufrn.br