The Minimal Overlap Rule: Restrictions on Mergers for Creditors' Consensus[*]

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January 19, 2012

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Abstract

This paper proposes a notion of partial Additivity in bankruptcy, μ-Additivity. We show that this property, together with Anonymity and Continuity, identify the Minimal Overlap rule, introduced by O'Neill (1982).

1 Introduction

There is empirical evidence pointing out that mergers continue to be a highly popular form of corporative development (Weitzel and McCarthy, 2011). To this regard, let us mention, as a recent example, that Spain is expected to suffer from a drastic reduction in the number of regional saving banks: from 45 in 2009 to 20 after the restructuring process that the financial sector is going through. In the financial literature, the avoidance of bankruptcy has been suggested as a plausible motive, among many others, for merging (see, for instance, Shrieves and Stevens 1979, and Billingsley et al. 1988). Therefore the ongoing global financial crisis, which explains the considerably increase in the number of firms going bankrupt, could somewhat support the current merger wave.

In this context our goal is to find out sharing rules for bankruptcy situations that are not affected by business alliances, that is, neutral distributive methods against mergers. In other words, and taking into account that creditors of a bankrupted firm can neither avoid nor promote any merger process, we try to find distribution rules that do not cause externalities. Just to illustrate it, let us consider a creditor lending some funds to two firms, say A and B. After a merger course, involving both firms, the new corporation C bankrupts. When reimbursing this corporation's creditors, two procedures can be proposed. The first one considers the possibility of partially reimbursing C's creditors, whereas the second one lies in paying the debts that A and B had with their creditors before the merger process. What neutrality requires is that both ways of proceeding yield the same outcome. So there will be no disagreement about how to proceed and, in this sense, creditors' consensus will be reached.

To deal with this question which is quite simple but, as this paper will point out, not trivial, we formalize general rationing problems by means of the so-called bankruptcy problems. This is a simple and robust economic model for describing situations where the demand of some endowment over-

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comes its supply. Moreover, we use the axiomatic method, introduced by Young (1988). The goal of this approach is to identify sharing rules with sets of properties that represent 'Equity Principles', helping to understand the nature and applicability of the different solutions. The study by Thomson (2003) provides a nice and complete overview of the main bankruptcy rules following this research line.

In this framework, neutrality against mergers is called Additivity and, as far as we know, the paper by Bergantiños and Méndez-Naya (2001) is the only one which explores this property. Their conclusion on this matter can be summarized as follows: There is no bankruptcy rule satisfying Additivity but, considering a very restrictive family of bankruptcy problems, the Ibn Ezra's rule, due to Rabad (12th Century), is the only one which conciliates Additivity and Equal Treatment of Equals.

The previous impossibility result shows that Additivity is a very demanding axiom.[1] This fact is not surprising since bankruptcy rules should allocate the debtor's assets according to essential characteristics of creditors' claims, and addition can substantially distort them. In general, there are two ways to overcome these findings. One of them involves restricting the domain keeping intact the required property. The other one is the opposite, based on weakening the property considering the whole domain. The former method is the one followed by Bergantiños and Méndez-Naya (2001). But, as far as we know, the latter one has not been analyzed for bankruptcy problems yet, although it has provided interesting results in other contexts (see, for instance, Peters (1986) who introduces a restricted Additivity for bargaining problems).

Taking into account the previous ideas, this paper presents a weak notion of Additivity, that we call μ-Additivity. Our proposal says that neutrality against mergers should only be required for problems with a similar internal

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[1]Section 3 discusses the rationale of such a fact from both economic and mathematical viewpoints.

structure.

The concept of similarity that we introduce concentrates on crucial char-acteristics of bankruptcy problems that on the one hand gather its whole essence and on the other hand are invulnerable to addition. Specifically, we say that two bankruptcy problems have a similar internal structure when, in both of them, every creditor's claim has a threefold fixed position with regard to: (i) any other creditor's claim, (ii) the debtor's assets and (iii) the magnitude of the whole problem.

The interpretation of the above conditions (i) and (ii) is straightforward. We analyze condition (iii) by dichotomously classifying the creditors on the basis of the certainty of being partially paid off, that is, paying attention to the part of the debtor's assets that each creditor has guaranteed. From our point of view, this aspect stands for the creditors' position against the severity of the bankruptcy in a 'natural' way. Furthermore the idea of establishing reasonable bounds on awards is not new, quite the contrary since it has underlied the theoretical bankruptcy problems from its beginnings to present day.[2]

Our starting point is to take as reference the security level of a creditor, that is, one n-th part of her feasible claim. This lower bound on awards was introduced for bankruptcy problems by Moreno-Ternero and Villar (2004) as a weakening of a limit previously proposed by Moulin (2002). Broadly speaking, we say that a creditor is 'privileged' when she is absolutely sure of getting an amount strictly greater than her security level. Otherwise we classify her as 'non-privileged'. Moreover, we consider that total certainty of a sharing out arises when whatever minimally fair division rule is applied such an outcome is achieved. Finally, we assume that the property of Order Preservation, which demands the respect of the ordering of claims, represents the minimal requirement of 'Equity'.

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[2]Jiménez-Gómez and Marco (2008) summarize the analysis of the concept of guarantee in the bankruptcy literature and propose a new method to establish lower bounds on awards.

Summarizing, what we demand to the problems that are going to be added up, besides the fact that every creditor's claim has the same relative position respect to both the rest of the claims and the debtor's assets, is that every creditor belongs to the same category according to the previous classification.

Our main result establishes that the Minimal Overlap rule, introduced by O'Neill (1982), is the only rule for which μ-Additivity is compatible with Anonymity and Continuity, two properties which have been widely justified in the literature for bankruptcy problems.

The rest of the paper is organized as follows: Section 2 introduces the model and the main definitions related to bankruptcy problems; Section 3 discusses the notion of general Additivity from a bankruptcy perspective and argues its lack of reasonableness in this context; Section 4 presents our proposal of partial Additivity and provides our main result: an axiomatization of the Minimal Overlap rule based on μ-Additivity. Technical proofs are provided in the Appendices.

2   The Framework and Main Definitions

Let us consider an individual, the debtor, who borrowed money. Let $\mathcal{N} = \{1, \ldots, i, \ldots, n\}$ denote the set of her creditors. This set is fixed throughout the paper. Let $E \geq 0$ denote the valuation of the debtor's assets, which we call the estate. For any fixed creditor, say i, $c_i \geq 0$ denotes her loan, i.e. the claim she has on the estate. Vector $c = (c_1, \ldots, c_i, \ldots, c_n)$ summarizes creditors' claims. We say that the debtor goes bankrupt if her assets are insufficient to reimburse her debts.

A bankruptcy problem, or simply **a problem**, is a vector $(E, c) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ such that

$$E \leq \sum_{i=1}^{n} c_i. \tag{1}$$

Condition (1) reflects that reimbursing all debts is not possible. Let $\mathcal{B}$ denote the family of all problems.

A solution provides a way of selecting, for each problem, a division between the creditors of the available amount satisfying natural lower and upper bound requirements.

Definition 1 A bankruptcy rule, or simply **a rule**, is a function $\varphi : \mathcal{B} \to \mathbb{R}_+^n$ such that for each problem $(E, c) \in \mathcal{B}$,

(a) $\sum_{i=1}^n \varphi_i(E, c) = E$ and

(b) $0 \leq \varphi_i(E, c) \leq c_i$ for each creditor i.

The main goal of the axiomatic method for the analysis of bankruptcy problems is to identify bankruptcy rules with sets of properties that represent 'Equity Principles'. In this sense, and taking into account that in our setting there are no priority classes, neither absolute nor relative, we consider that a bankruptcy rule is **minimally fair** whenever it respects the ordering of claims. This axiom, called Order Preservation, was introduced by Aumann and Maschler (1985) as a minimal requirement of fairness and it has been understood as such by many authors.

A rule φ satisfies **Order Preservation** if for each $(E, c) \in \mathcal{B}$ and each $\{i, j\} \subseteq \mathcal{N}$, whenever $c_i \geq c_j$, then

$$\varphi_i(E, c) \geq \varphi_j(E, c) \text{ and } c_i - \varphi_i(E, c) \geq c_j - \varphi_j(E, c).$$

Let Φ^* denote the set of minimally fair rules. Let us note that, when there are no priority classes, all the rules proposed in the literature belong to Φ^*.

Next we introduce those two rules that will be most useful in our analysis. The first one, to be called Ibn Ezra's rule, is a 'semi-solution' in the sense that it is not defined for every problem. The second one, known as the Minimal

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Overlap rule, was proposed by O'Neill (1982) as a possible extension of Ibn Ezra's rule to the whole class of problems.

Henceforth, for expository and technical convenience, we assume without loss of generality, that creditors' claims are increasingly ordered, i.e. $c_i \leq c_j$ whenever $i < j$. Let us remark that if this is not the case, there is a permutation[3] π such that $\pi(c)$ is increasingly ordered. Hence, we can compute $\varphi(E, c) = \pi^{-1}[\varphi(E, \pi(c))]$.

As mentioned above, Ibn Ezra's rule is only defined for a restricted class of bankruptcy problems, those having at least one 'super-creditor', i.e. an individual whose claim is not lower than the estate. Let $\mathcal{B}_S$ denote this class,

$$\mathcal{B}_S \equiv \{(E, c) \in \mathcal{B} \colon E \leq c_n\}.$$

Definition 2 Ibn Ezra's rule is the function $\varphi^{IE} \colon \mathcal{B}_S \to \mathbb{R}^n_+$ that associates to each problem $(E, c) \in \mathcal{B}_S$ and each creditor $i \in \mathcal{N}$, the amount

$$\varphi_i^{IE}(E, c) = \sum_{j=1}^{i} \frac{\min\{E, c_j\} - \min\{E, c_{j-1}\}}{n - j + 1}, \tag{2}$$

where $c_0 = 0$.

Chun and Thomson (2005) propose a formal description for the Minimal Overlap rule. What these authors suggest is to proceed as follows. Let $(E, c) \in \mathcal{B}$. Then, what each creditor $i \in \mathcal{N}$ recovers is:

(a) If $(E, c) \in \mathcal{B}_S$,

$$\varphi_i^{MO}(E, c) = \varphi_i^{IE}(E, c) \text{ and}$$

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[3]A permutation is a bijection applying $\mathcal{N}$ to itself. In this paper, and abusing notation, $\pi(c)$ will denote the claim vector obtained by applying permutation π to its components, i.e. the i-th component of $\pi(c)$ is c_j whenever $j = \pi(i)$. Similar reasoning considerations apply for $\pi[\varphi(E, c)]$.

(b) If $(E, c) \notin \mathcal{B}_S$, then there is a unique $t^* \geq 0$ such that

$$t^* = E - \sum_{i=1}^{n} \max \{c_i - t^*, 0\}$$

and in such a case,

$$\varphi_i^{MO}(E, c) = \varphi_i^{IE}(E - t^*, c) + \max \{c_i - t^*, 0\}.$$

Recently, Alcalde et al. (2008) found an alternative expression for the Minimal Overlap rule. They showed that this rule can be understood as a decomposition involving Ibn Ezra's rule and the Constrained Equal Losses rule. Before presenting their result we introduce the Constrained Equal Losses rule, which chooses the awards vector at which losses from the claims vector are the same for all creditors subject to no-one receiving a negative amount.

The **Constrained Equal Losses rule** is the function, $\varphi^{CEL} \colon \mathcal{B} \to \mathbb{R}_+^n$ that associates to each problem $(E, c) \in \mathcal{B}$ and each creditor $i \in \mathcal{N}$,

$$\varphi_i^{CEL}(E, c) = \max \{0, c_i - \beta\},$$

where β is such that $\sum_{i \in \mathcal{N}} \max\{0, c_i - \beta\} = E$.

Definition 3 The **Minimal Overlap rule** is the function $\varphi^{MO} \colon \mathcal{B} \to \mathbb{R}_+^n$ that associates to each problem $(E, c) \in \mathcal{B}$ and each creditor $i \in \mathcal{N}$, the amount

$$\varphi_i^{MO}(E, c) = \varphi_i^{IE}(\min \{E, c_n\}, c) + \varphi_i^{CEL}(E^r, c^r),$$

where $E^r = \max \{E - c_n, 0\}$, and $c^r = c - \varphi^{IE}(\min \{E, c_n\}, c)$.

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3 Additivity and Bankruptcy Rules

The aim of this section is to discuss the notion of Additivity in the framework of bankruptcy problems. Firstly we present the basis of this property from the axiomatic approach. Let us consider the following example. A creditor, say i, loans some quantity to two firms, say F and G, which go bankrupt. Let c_i^F and c_i^G denote these quantities. After a merger course, firm H emerges as the fusion of F and G and, initially H's assets are insufficient to reimburse its debts. If we denote by E^F and E^G the valuations of firms F and G respectively; and their respective debts vectors are denoted by c^F and c^G, we have that[4]

(a) $E^H = E^F + E^G$, and

(b) $c^H = c^F + c^G$,

what creditor i would expect to obtain at the division process, for any bankruptcy rule, say φ, is

$$\varphi_i\left(E^H, c^H\right) \geq \varphi_i\left(E^F, c^F\right) + \varphi_i\left(E^G, c^G\right).$$

Otherwise, creditor i could claim that she has been 'penalized' due to the merger process, and she would not have the possibility to object against such a decision made by the two firms. If such an argument is extended to all the creditors we have that the above inequality must become an equality. This is the essence of the Additivity notion.

Definition 4 A bankruptcy rule φ satisfies **Additivity** if for each pair of problems in $\mathcal{B}$, (E^1, c^1), and (E^2, c^2), we have that

$$\varphi\left(E^1, c^1\right) + \varphi\left(E^2, c^2\right) = \varphi\left(E^1 + E^2, c^1 + c^2\right).$$

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[4]We are implicitly assuming that the set of creditors, $\mathcal{N}$, is the same for both firms and that there are no intra-group debts, i.e. F is not a G's creditor or debtor.

Secondly, the requirement of Additivity of rules comes from the interpretation of bankruptcy problems as TU-games. This formulation, suggested by O'Neill (1982), associates to each problem (E, c) the TU-game $(\mathcal{N}, V)$, where the characteristic function $V : 2^{\mathcal{N}} \to \mathbb{R}_+$ assigns to each non-empty coalition $S \subseteq \mathcal{N}$ the amount

$$V(S) = \max\left\{E - \sum_{i \notin S} c_i, 0\right\}. \tag{3}$$

Following this approach, both TU-games solutions and properties can be extended to bankruptcy problems. On the one hand, Curiel et al. (1987) showed that not all bankruptcy rules can be interpreted as solutions of bankruptcy games. They provided a necessary and sufficient condition, known as Claims Truncation Invariance, for having this correspondence. This property demands a solution to depend only on the feasible claims and debtor's assets.

A rule φ satisfies **Claim Truncation Invariance** if for each problem $(E, c) \in \mathcal{B}$, $\varphi(E, c) = \varphi(E, c^E)$, where is the feasible claims vector; i.e. for each creditor i,

$$c^E = \min\{c_i, E\}$$

On the other hand, when concentrating on properties for TU-games solutions reflecting 'Equity' of the distributive process, Additivity is one of the most extensively imposed requirements. As it is well known Shapley (1953), in his seminal paper, pointed out the Additivity of the value he proposed. In fact, the fulfillment of this property has been demanded in a huge family of allocation problems analyzed from a cooperative perspective. As an example, Moretti and Patrone (2008) refer to the Shapley value application to cost allocations, social networks, water issues, biology, reliability theory and belief formation.

Despite the previous mainstays of Additivity, Bergantiños and Méndez-Naya (2001) pointed out, by means of an example, that this property is overly demanding for bankruptcy rules. Actually they showed that no rule is additive.

Next, we deal with some reasons that justify this finding. The first one lies in the relationship between bankruptcy problems and TU-games. Specifically, the next example shows that Additivity of bankruptcy problems might not induce Additivity of the respective TU-games.

Example 5 Let us consider the three-agent problems $(E, c) = (9, (8, 8, 8))$ and $(E', c') = (31, (4, 12, 22))$. Therefore, the aggregate problem is $(E'', c'') = (E + E', c + c') = (40, (12, 20, 30))$. Let V (resp. V', V'') denote the characteristic function relative to (E, c) (resp. (E', c'), (E'', c'')). By equation (3) we have that

S	$V(S)$	$V'(S)$	$V''(S)$	$V(S) + V'(S)$
$\{1\}$	0	0	0	0
$\{2\}$	0	5	**0**	**5**
$\{3\}$	0	15	**8**	**15**
$\{1, 2\}$	1	9	10	10
$\{1, 3\}$	1	19	20	20
$\{2, 3\}$	1	27	28	28
$\{1, 2, 3\}$	9	31	40	40

Therefore, the TU-game induced by adding the two problems differs from the addition of the TU-games they induce.

The second reason which explains that Additivity is a strong requirement, comes from an economic perspective. Let us consider a company that is the result of a merger process involving some firms. When the company as a

whole goes bankrupt, the **bankruptcy degree**[5] of all the firms that configure the company is not usually similar. This will justify the fact that not all the creditors should be rationed considering the company's financial situation but that of the firms receiving their credits. Therefore, what this argument suggests is that Additivity, as in Definition 4, might not be a reasonable property for bankruptcy rules, except that the problems to be added share, at least, some similarity.

4 μ-Additivity and the Minimal Overlap Rule

This section describes reasonable conditions, reflecting similarity in the internal structure of problems, under which Additivity is both justified from an economic point of view and consistent with the interpretation of problems as TU-games. The main idea of our requirements starts from considering that the complete essence of a problem can be described, in addition to the relative position of each creditor's claim in regard to both the rest of claims and the debtor's assets, by means of each creditor' claim position against the magnitude of the whole problem.

From our point of view, the severity of a problem for each creditor can be adequately represented through her certainty of getting a minimal share of the debtor's assets. To analyze this aspect we take as reference the security level of each creditor. This notion, defined as one n-th part of the feasible claim, was introduced by Moreno-Ternero and Villar (2004) as a fair lower bound on awards.

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[5]The notion of bankruptcy degree is introduced by Alcalde et al. (2012) as follows. Given a bankruptcy problem (E, c) with $c \neq 0$, we can define its bankruptcy degree as the expression

$$\mathcal{D}(E, c) = 1 - \frac{E}{\sum_{i=1}^{n} c_i}.$$

Definition 6 Given a problem, $B = (E, c)$, the **security levels vector**, $\sigma(E, c)$, provides to each creditor $i \in \mathcal{N}$,

$$\sigma_i(E, c) = \frac{1}{n} \min\{c_i, E\}.$$

Now we establish a partition of the set of creditors on the basis of receiving, with complete certainty, preferential treatment regarding their security levels. Specifically, we say that a creditor is privileged if for any structure of creditors' claims demanding less than she does and, whatever minimally fair rule is applied, her security level either is exceeded or if it is exactly achieved, any other creditor's security level is less than hers. Otherwise we say that such a creditor is non-privileged.

Definition 7 Given a problem $B = (E, c)$, we say that creditor i is **privileged in B** if, and only if, for each $c' \in \mathbb{R}_+^n$, such that $c'_j \leq c_i$ for each $j < i$ and $c'_k = c_k$ for each $k \geq i$, and for each minimally fair bankruptcy rule, $\varphi \in \Phi^*$,

(a) $\varphi_i(E, c') > \sigma_i(E, c)$ or

(b) $\varphi_i(E, c') = \sigma_i(E, c)$ and $\sigma_i(E, c) > \sigma_j(E, c)$ for each $j \in \mathcal{N}$, $j \neq i$.

To illustrate the notion of a privileged creditor, we provide an example, relegated to Appendix B, in which the partition of the set of creditors according to such a qualification is determined for two different problems.

The following results provide alternative definitions of the concept of a privileged creditor and highly intuitive interpretations of its meaning. The first one pays attention to the original data of the problem whereas the second one is stated in terms of the associated TU-game.

Next proposition can be expressed as follows: creditor i is privileged whenever there is some partial reimbursement of the debts such that in the residual problem the feasible part of agent i's claim is greater than any of her 'rivals'.

Proposition 8 Given a problem $B = (E, c)$, creditor i is privileged in B if, and only if, there is $\tau \in \mathbb{R}_+^n$ such that

(a) $B' = (E', c') = (E - \sum_{i=1}^n \tau_i, c - \tau) \in \mathcal{B}$, and

(b) $\min \{c_i', E'\} > \min \{c_j', E'\}$ for all $j \in \mathcal{N}$, $j \neq i$.

Proof. See Appendix C.

Our second result comes directly from the previous one. It says that creditor i is privileged whenever there is some partial reimbursement of the debts such that in the residual bankruptcy game agent i's marginal contributions exceed those of any of her rivals.

To introduce it formally, we proceed as follows. Given a TU-game, V, a coalition $S \subseteq \mathcal{N}$ and an agent $i \notin S$, let $\mathcal{M}_i^V (S)$ denote agent i's marginal contribution to S, i.e.

$$\mathcal{M}_i^V (S) = V (S \cup \{i\}) - V (S).$$

Corollary 9 Given a problem $B = (E, c)$, creditor i is privileged in B if, and only if, there is $\tau \in \mathbb{R}_+^n$ such that

(a) $B' = (E', c') = (E - \sum_{h=1}^n \tau_i, c - \tau) \in \mathcal{B}$,

(b) for each $j \neq i$, and each $S \subseteq \mathcal{N} \setminus \{i, j\}$, $\mathcal{M}_i^{V_{B'}} (S) \geq \mathcal{M}_j^{V_{B'}} (S)$ and

(c) for each $j \neq i$, there is $T_j \subseteq \mathcal{N} \setminus \{i, j\}$ such that $\mathcal{M}_i^{V_{B'}} (T_j) > \mathcal{M}_j^{V_{B'}} (T_j)$.

Now we introduce a notion of partial Additivity which lies in allowing adding up problems in which creditors belong to the same category according to the previous dichotomously classification: privileged and non-privileged.

Axiom 10 A rule φ satisfies μ-**Additivity** if

$$\varphi (E, c) + \varphi (E', c') = \varphi (E + E', c + c')$$

for any two problems (E, c) and (E', c') such that

(a) $(c_i - c_j)(c'_i - c'_j) \geq 0$ for each i and j in $\mathcal{N}$,

(b) $(E - c_i)(E - c'_i) \geq 0$ for each i in $\mathcal{N}$ and

(c) Each creditor is privileged in (E, c) if, and only if, she is privileged in (E', c').

Axiom 10 suggests that Additivity should be preserved in problems sharing their essential features from each creditor's point of view: rivals' claims, available amount to divide and severity of the problem. That is:

(a) In both problems the creditors' claims should be ordered in the same way: if i's claim is greater than j's claim in a problem, it should not be the case that j's claim is greater than i's claim in the other one.

(b) In both problems each creditor's claim should have the same position related to the estate: if some creditor's claim is lower than the estate in a problem, it should not be the case that, for the other problem, her claim exceeds the estate.

(c) In both problems the set of privileged creditors is the same: if a creditor is privileged in a problem, it should not be the case that she is non-privileged in the other one.

In order to present our main result we need to introduce two standard axioms. The first one states that the identity of the creditors should no matter. The second one says that small changes in the data of the problem should not lead to large changes in the chosen awards vector.

Axiom 11 A rule φ satisfies **Anonymity** if for each problem (E, c) and any permutation π,[6]

$$\pi\left[\varphi\left(E, c\right)\right] = \varphi\left(E, \pi\left(c\right)\right).$$

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[6]To introduce this axiom we must not to assume that c is increasingly ordered.

Axiom 12 A rule φ satisfies **Continuity** if for each sequence of problems $\{(E^\nu, c^\nu)\}_{\nu \in \mathbb{N}}$ such that

$$\lim_{\nu \to \infty} (E^\nu, c^\nu) = (E, c) \in \mathcal{B}, \ \lim_{\nu \to \infty} \varphi(E^\nu, c^\nu) = \varphi(E, c).$$

At this point it is worth noting that μ-Additivity together with Continuity ensure that a rule is also a solution for bankruptcy games.

Proposition 13 Let φ a rule satisfying μ-Additivity and Continuity. Then φ satisfies Claims Truncation Invariance.

Proof. See Appendix D.

We can now establish our main result.

Theorem 14 The **Minimal Overlap rule** is the only rule satisfying **Anonymity**, **Continuity** and μ-**Additivity**.

Proof. See Appendix E.

Next we show that Theorem 14 requires each and every one of the axioms. The Constrained Equal Losses rule is both continuous and anonymous but fails satisfy μ-Additivity. Any Weighted Minimal Overlap rule (see Alcalde et al. 2008) is both continuous and μ-additive, but does not satisfy Anonymity. Finally, let us consider the rule φ^c that recommends for any problem such that $E \leq c_n$ the Ibn Ezra's proposal and otherwise the following modification of the Constrained Equal Losses rule,

$$\varphi_i^c(E, c) = \begin{cases} 0 & \text{if } i \notin \mathcal{P}(E, c) \\ \varphi_i^{CEL}\left(E, \left(\{0\}_{j \notin \mathcal{P}(E,c)}, \{c\}_{j \in \mathcal{P}(E,c)}\right)\right) & \text{if } i \in \mathcal{P}(E, c) \end{cases},$$

where $\mathcal{P}(E, c)$ denotes the set of privileged creditors for (E, c). This rule is anonymous and μ-additive but not continuous.

To sum up, this paper introduces reasonable economic conditions, on the whole class of bankruptcy problems, to analyze the possibility of preserving

shareholder's wealth when bankrupted firms merge. Specifically, it finds out that if we are looking for neutral solution when adding bankruptcy problems that have fixed creditor's position, in a broad sense, the Minimal Overlap rule should be selected. Therefore this analysis goes into the nature of this solution in depth and helps to understand its applicability.

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APPENDICES

A   General Remark

Next we note how to check if a creditor $i \in \mathcal{N}$ is privileged (Definition 7) when facing a problem $B = (E, c)$. This reasoning will be used in Appendices B and C.

Let us introduce the **Constrained Equal Awards rule**, which will be useful in the sequel. This rule is the function $\varphi^{CEA} \colon \mathcal{B} \to \mathbb{R}^n_+$ that associates to each problem $(E, c) \in \mathcal{B}$ and each creditor $i \in \mathcal{N}$, the amount

$$\varphi_i^{CEA}(E, c) = \min\{c_i, \alpha\},$$

where α is such that $\sum_{i \in \mathcal{N}} \min\{c_i, \alpha\} = E$.

Remark 15 Given a problem $B = (E, c)$, to prove that a creditor $i \in \mathcal{N}$ is privileged, i.e. that some of the conditions, (a) or (b), in Definition 7 is satisfied, it will be enough to check them in the least favorable situation for such a creditor with regard to both the rest of the claims and the rule that prevails.

Let $c^{W(i)}$ and $\varphi^{W(i)}$ respectively denote the claims vector and the minimally fair bankruptcy rule corresponding to the worst situation for creditor i. Clearly, $c^{W(i)}$ is such that $c_j^{W(i)} = c_i$ for each $j < i$, and $c_k^{W(i)} = c_k$ for each $k \geq i$. And $\varphi^{W(i)}$ is obtained by applying the following two step rule: first the Constrained Equal Losses rule, φ^{CEL}, is applied until no creditor claims more than creditor i does (see Section 2 for a formal definition). Afterwards the remainder, if any, is divided according to the Constrained Equal Awards rule, φ^{CEA}.

B Classification of Creditors: Privileged and Non-Privileged

To illustrate the notion of privileged creditor, we present, taking into account Remark 15, the partition of the set of creditors according to such a qualification for two different problems. In the first one, all the claims are lower than the estate whereas in the second problem, there is some creditor who claims more than the debtor's assets.

Example 16 Let $B^1 = (E^1, c^1) = (120, (10, 20, 50, 80))$. By definition, the security levels vector is

$$\sigma \left(B^1 \right) = \left(\frac{5}{2}, 5, \frac{25}{2}, 20 \right),$$

and the claims vectors $c^{W(i)}$ are

$$c^{W(1)} = (10, 20, 50, 80), \quad c^{W(2)} = (20, 20, 50, 80),$$
$$c^{W(3)} = (50, 50, 50, 80), \quad c^{W(4)} = (80, 80, 80, 80).$$

Therefore,

(1) creditor 1 is non-privileged since

$$\varphi_1^{W(1)} \left(E^1, c^{W(1)} \right) = \varphi_1^{CEL} \left(E^1, c^{W(1)} \right) = 0 < \sigma_1(B^1) = \frac{5}{2};$$

(2) creditor 2 is privileged since

$$\varphi_2^{W(2)} \left(E^1, c^{W(2)} \right) = \varphi_2^{CEL} \left(90, c^{W(2)} \right) + \varphi_2^{CEA} \left(30, (20, 20, 20, 20) \right) =$$

$$0 + \tfrac{30}{4} > \sigma_2 \left(B^1 \right) = 5;$$

(3) creditor 3 is also privileged since

$$\varphi_3^{W(3)}\left(E^1, c^{W(3)}\right) = \varphi_3^{CEL}\left(30, c^{W(3)}\right) + \varphi_3^{CEA}\left(90, (50, 50, 50, 50)\right) =$$

$$0 + \frac{90}{4} > \sigma_3\left(B^1\right) = \frac{25}{2}; \text{ and}$$

(4) finally, creditor 4 is privileged since

$$\varphi_4^{W(4)}\left(E^1, c^{W(4)}\right) = \varphi_4^{CEA}\left(120, (80, 80, 80, 80)\right) = \frac{120}{4} > \sigma_4\left(B^1\right) = 20.$$

Example 17 Let $B^2 = \left(E^2, c^2\right) = \left(70, (30, 30, 60, 90)\right)$. Now, the security levels vector is

$$\sigma\left(B^2\right) = \left(\frac{15}{2}, \frac{15}{2}, 15, \frac{35}{2}\right)$$

and the agents' 'worst claim vectors' are

$$c^{W(1)} = (30, 30, 60, 90), \quad c^{W(2)} = (30, 30, 60, 90),$$
$$c^{W(3)} = (60, 60, 60, 90), \quad c^{W(4)} = (90, 90, 90, 90).$$

Therefore, we have that

(1) creditors 1 and 2 are non-privileged since for $i = 1, 2$,

$$\varphi_i^{W(i)}\left(E^2, c^{W(i)}\right) = \varphi_i^{CEL}\left(E^2, c^{W(1)}\right) = 0 < \sigma_i\left(B^2\right) = \frac{15}{2};$$

(2) creditor 3 is also non-privileged since

$$\varphi_3^{W(3)}\left(E^2, c^{W(3)}\right) = \varphi_3^{CEL}\left(30, c^{W(3)}\right) + \varphi_3^{CEA}\left(40, (70, 70, 70, 70)\right) =$$

$$0 + \frac{40}{4} = 10 < \sigma_3\left(B^2\right) = 15; \text{ and}$$

(3) finally, creditor 4 is privileged since

$$\varphi_4^{W(4)}\left(E^2, c^{W(4)}\right) = \varphi_4^{CEA}\left(70, (90, 90, 90, 90)\right) = \tfrac{70}{4} = \sigma_4\left(B^2\right), \text{ and}$$

$$\sigma_i\left(B^2\right) < \sigma_4\left(B^2\right) \text{ for } i = 1, 2, 3.$$

C Proof of Proposition 8

Firstly we show that Definition 7 implies conditions (a) and (b) of Proposition 8.

Let $i \in \mathcal{N}$ be a privileged creditor in $(E, c) \in \mathcal{B}$, then for each $c' \in \mathbb{R}_+^n$, such that $c'_j \leq c_i$ for each $j < i$ and $c'_k = c_k$ for each $k \geq i$, and for each minimally fair rule, $\varphi \in \Phi^*$, we have one of the two following cases.

Case 1 $\varphi_i\left(E, c'\right) > \sigma_i\left(E, c\right)$.

Taking into account Remark 15, the above inequality should be met for $\varphi^{W(i)}$. It implies that once the Constrained Equal Losses rule, φ^{CEL}, has been applied until no creditor claims more than creditor i does, the remainder is greater than c_i. Therefore

$$E - \sum_{j>i}\left(c_j - c_i\right) > c_i.$$

Let us define

$$\alpha \equiv E - c_i - \sum_{j>i}\left(c_j - c_i\right), \text{ and}$$

let $S \subset \mathcal{N}$ be the subset of creditors, whose cardinality is denoted by s, such that $k \in S$ if and only if $k \neq i$ and $c_k \geq c_i$, and let $\beta = \tfrac{1}{s}\min\{c_i, \alpha\}$. Now we define $\tau \in \mathbb{R}_+^n$ such that

$$\tau_k = \left(c_k - c_i\right) + \beta \text{ for all } k \in S, \text{ and } \tau_j = 0 \text{ for all } j \in \mathcal{N}\backslash S.$$

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Thus

$$B' = (E', c') = \left(E - \sum_{i=1}^{n} \tau_i, c - \tau \right)$$

is a problem in $\mathcal{B}$ and for all $j \in \mathcal{N}$, $j \neq i$, $\min \{c_i', E'\} > \min \{c_j', E'\}$.

Case 2 $\varphi_i(E, c') = \sigma_i(E, c)$, and $\sigma_i(E, c) > \sigma_j(E, c)$ for all $j \in \mathcal{N}$, $j \neq i$.

By Definition 6, the previous inequality is $\frac{1}{n} \min \{c_i, E\} > \frac{1}{n} \min \{c_j, E\}$ for all $j \in \mathcal{N}$, $j \neq i$. Therefore, for $\tau \in \mathbb{R}_+^n$ such that $\tau_i = 0$ for all $i \in \mathcal{N}$, conditions (a) and (b) of Proposition 8 are obviously satisfied.

Secondly we show that conditions (a) and (b) of Proposition 8 imply Definition 7.

Let us suppose that there is $\tau \in \mathbb{R}_+^n$ such that

(a) $B' = (E', c') = (E - \sum_{i=1}^{n} \tau_i, c - \tau) \in \mathcal{B}$ and

(b) $\min \{c_i', E'\} > \min \{c_j', E'\}$ for all $j \in \mathcal{N}$, $j \neq i$.

Let us note that the previous inequality implies that for all $j \in \mathcal{N}$, $j \neq i$, either $E' \geq c_i' > c_j'$ or $c_i' > E' > c_j'$. This means that there is a partial reimbursement of debts, τ, such that in the residual problem, (E', c'), $c_i' > c_j'$ for all $j \in \mathcal{N}$. Let us look for the minimum partial reimbursement of debts, τ^*, for which

$$(E^*, c^*) = \left(E - \sum_{i=1}^{n} \tau_i^*, c - \tau^* \right) \in \mathcal{B}.$$

Let us consider the following two cases.

Case 1 $c_i = c_n > c_{n-1}$.

Then $\tau^* = 0$ and, by Remark 15, $\varphi^{W(i)} = \varphi^{CEA}$. If $E > c_i$,

$$\varphi_i^{W(i)}\left(E, c^{W(i)} \right) = \frac{E}{n} > \sigma_i(E, c) = \frac{c_i}{n}$$

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and condition (a) in Definition 7 is satisfied. Otherwise, i.e. if $c_i > E > c_{n-1}$,

$$\varphi_i^{W(i)}\left(E, c^{W(i)}\right) = \frac{c_i}{n} = \sigma_i\left(E, c\right) < \sigma_j\left(E, c\right) = \frac{c_j}{n}$$

for all $j \in \mathcal{N}$, $j \neq i$, which implies condition (b) in Definition 7.

Case 2 There is $j \in \mathcal{N}$, $j \neq i$ such that $c_j \geq c_i$.

Then it is possible to partially reimburse debts in such a way that: (i) no creditor has a new claim greater than creditor i does and (ii) what is left of estate is greater than c_i. Otherwise conditions (a) and (b) of Proposition 8 would not be fulfilled. Therefore, the previous statement is equivalent to

$$E - \left[\sum_{i \in \mathcal{N}} \varphi_i^{CEL}\left(\sum_{j \in \mathcal{N}} \tau_j^*, c^{W(i)}\right)\right] > c_i,$$

where $\tau^* \in \mathbb{R}_+^n$ is such that $\tau_j^* = \max\left\{c_j - c_i, 0\right\}$ for all $j \in \mathcal{N}$.

From the previous inequality,

$$\varphi_i^{CEA}\left(E - \sum_{i \in \mathcal{N}} \varphi_i^{CEL}\left(\sum_{j \in \mathcal{N}} \tau_j^*, c^{W(i)}\right), c^{W(i)}\right) > \frac{c_i}{n} = \sigma_i\left(E, c\right).$$

Now, taking into account Remark 15

$$\varphi_i^{W(i)}\left(E, c^{W(i)}\right) > \sigma_i\left(E, c\right),$$

which implies condition (a) in Definition 7. ∎

D   Proof of Proposition 13

Let φ be a rule satisfying μ-Additivity and Continuity. Let (E, c) be a problem where $0 < E < c_n = \max_{i \in \mathcal{N}}\left\{c_i\right\}$; and let $S \subset \mathcal{N}$ be the subset of agents claiming zero. Let us consider the following two cases.

Case 1 $E \leq c_{n-1}$.

By μ-Additivity,

$$
\varphi\left(E, c\right) = \varphi\left(E - \frac{1}{r}, \left((0)_{i \in S}, \left(c_i^E - \frac{1}{r}\right)_{i \in \mathcal{N} \setminus S}\right)\right) +
$$
$$
+ \varphi\left(\frac{1}{r}, \left((0)_{i \in S}, \left(c_i - c_i^E + \frac{1}{r}\right)_{i \in \mathcal{N} \setminus S}\right)\right)
$$

for all $r \in \mathbb{N}$ such that

$$
\frac{1}{r} < \min\left\{\min_{i \in \mathcal{N} \setminus S}\left\{c_i\right\}, c_n - E\right\}.
$$

Now, we let r go to infinity in the previous equation and obtain

$$
\lim_{r \to \infty} \varphi\left(E, c\right) = \lim_{r \to \infty} \varphi\left(E - \frac{1}{r}, \left((0)_{i \in S}, \left(c_i^E - \frac{1}{r}\right)_{i \in \mathcal{N} \setminus S}\right)\right) +
$$
$$
+ \lim_{r \to \infty} \varphi\left(\frac{1}{r}, \left((0)_{i \in S}, \left(c_i - c_i^E + \frac{1}{r}\right)_{i \in \mathcal{N} \setminus S}\right)\right).
$$

Taking into account that φ is continuous,

$$
\varphi\left(E, c\right) = \varphi\left(E, c^E\right) + \varphi\left(0, c - c^E\right) = \varphi\left(E, c^E\right).
$$

Case 2 $E > c_{n-1}$.

By μ-Additivity,

$$
\varphi\left(E, c\right) = \varphi\left(E - \frac{1}{r}, \left((0)_{i \in S}, \left(c_i^E - \frac{1}{r}\right)_{i \in \mathcal{N} \setminus S \cup \{n\}}, c_n^E - \frac{1}{2r}\right)\right) +
$$
$$
+ \varphi\left(\frac{1}{r}, \left((0)_{i \in S}, \left(c_i - c_i^E + \frac{1}{r}\right)_{i \in \mathcal{N} \setminus S \cup \{n\}}, c_n - c_n^E + \frac{1}{2r}\right)\right)
$$

for all $r \in \mathbb{N}$ such that

$$\frac{1}{r} < \min \left\{ \min_{i \in \mathcal{N} \setminus S} \{c_i\}, \max_{\{i \in \mathcal{N} \,:\, E < c_i < c_n\}} \{c_i - E\}, 2\,(c_n - c_{n-1}) \right\}.$$

Now, we let r go to infinity in the previous equation and obtain

$$
\begin{aligned}
\lim_{r \to \infty} \varphi\,(E, c) &= \lim_{r \to \infty} \left[\varphi \left(E - \frac{1}{r}, \left((0)_{i \in S}, \left(c_i^E - \frac{1}{r} \right)_{i \in \mathcal{N} \setminus S \cup \{n\}}, c_n^E - \frac{1}{2r} \right) \right) + \right. \\
&\quad \left. + \varphi \left(\frac{1}{r}, \left((0)_{i \in S}, \left(c_i - c_i^E + \frac{1}{r} \right)_{i \in \mathcal{N} \setminus S \cup \{n\}}, c_n - c_n^E + \frac{1}{2r} \right) \right) \right].
\end{aligned}
$$

Taking into account that φ is continuous,

$$\varphi\,(E, c) = \varphi\left(E, c^E\right) + \varphi\left(0, c - c^E\right) = \varphi\left(E, c^E\right).$$

$\blacksquare$

E   Proof of Theorem 14

Firstly, it is straightforward to verify that the Minimal Overlap rule satisfies Anonymity, Continuity and μ-Additivity.

Now, let φ be a rule satisfying these axioms. Given a problem $(E, c) \in \mathcal{B}$, let us consider the following two cases:

Case 1 $E \leq c_n$.

By Proposition 13,

$$\varphi\,(E, c) = \varphi\left(E, c^E\right).$$

Let us denote $P^1 = c_1^E$; $P^i = c_i^E - c_{i-1}^E$ for $1 < i \leq n$; and $c^{P^i} = \left((0)_{j<i}, (P^i)_{j \geq i} \right)$ for each $i \in \mathcal{N}$.

Now, let us consider the following two subcases.

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Subcase 1. $c_n^E = c_{n-1}^E$.

μ-Additivity implies that

$$\varphi\left(E, c^E\right) = \sum_{i \in \mathcal{N}} \varphi\left(P^i, c^{P^i}\right).$$

By Anonymity and Proposition 13,

$$\varphi_j\left(P^i, c^{P^i}\right) = \begin{cases} 0 & \text{if } j < i \\ \frac{P^i}{n-i+1} & \text{if } j \geq i \end{cases},$$

i.e.

$$\varphi_j\left(P^i, c^{P^i}\right) = \begin{cases} 0 & \text{if } j < i \\ \frac{c_i^E - c_{i-1}^E}{n-i+1} & \text{if } j \geq i \end{cases}$$

with $c_0 = 0$.[7] Thus, for each agent h

$$\varphi_h\left(E, c^E\right) = \sum_{i \in \mathcal{N}} \varphi_h\left(P^i, c^{P^i}\right).$$

Since, for any $j > h$ we have that $c_h^{P^j} = 0$,

$$\begin{aligned} \varphi_h\left(E, c^E\right) &= \sum_{i=1}^{h} \varphi_h\left(P^i, c^{P^i}\right) = \sum_{i=1}^{h} \frac{c_i^E - c_{i-1}^E}{n-i+1} = \\ &= \sum_{i=1}^{h} \frac{\min\{c_i, E\} - \min\{c_{i-1}, E\}}{n-i+1} = \varphi_h^{MO}\left(E, c\right). \end{aligned}$$

Subcase 2. $c_n^E \neq c_{n-1}^E$.

Let $q(j)$ denote the cardinality of the set $\{i \leq j : P^i \neq 0\}$. μ-Additivity

_____

[7]Throughout this proof, and for notational convenience, we will consider $c_0 = c_0^E = 0$.

implies that

$$\varphi\left(E, c^E\right) = \varphi\left(P^1 + \tfrac{1}{r}, \left(\left(c_i^{P^1}\right)_{i<n}, c_n^{P^1} + \tfrac{1}{r}\right)\right) +$$

$$+ \sum_{\{1<i<n:\, P^i \neq 0\}} \varphi\left(P^i - \frac{1}{r\left[q\left(n\right) - 1\right]}, \left(\max\left\{0, c_j^{P^i} - \frac{1}{r\left[q\left(j\right) - 1\right]}\right\}\right)_{j \in \mathcal{N}}\right) +$$

$$+ \varphi\left(P^n - \frac{1}{r[q(n)-1]}, \left(0, \left(\min\left\{c_i^{p^i}, \tfrac{1}{r}\right\}\right)_{1<i<n}, c_n^{P^n} - \frac{1}{r[q(n)-1]}\right)\right),$$

where $r \in \mathbb{N}$ is such that

$$\frac{1}{r} < \min\left\{\left(1 - \frac{1}{q\left(n\right)}\right) P^n, \min_{\{i:\, P^i \neq 0\}} \left\{P^i\right\}\right\}.$$

Now, we let r go to infinity in the previous equation and obtain

$$\lim_{r \to \infty} \varphi\left(E, c^E\right) = \lim_{r \to \infty} \left[\varphi\left(P^1 + \frac{1}{r}, \left(\left(c_i^{P^1}\right)_{i<n}, c_n^{P^1} + \frac{1}{r}\right)\right) +\right.$$

$$+ \sum_{\{1<i<n:\, P^i \neq 0\}} \varphi\left(P^i - \frac{1}{r\left[q\left(n\right) - 1\right]}, \left(\max\left\{0, c_j^{P^i} - \frac{1}{r\left[q\left(j\right) - 1\right]}\right\}\right)_{j \in \mathcal{N}}\right) +$$

$$\left. + \varphi\left(P^n - \frac{1}{r[q(n)-1]}, \left(0, \left(\min\left\{c_i^{p^i}, \tfrac{1}{r}\right\}\right)_{1<i<n}, c_n^{P^n} - \frac{1}{r[q(n)-1]}\right)\right)\right].$$

By Continuity,

$$\varphi\left(E, c^E\right) = \sum_{i \in \mathcal{N}} \varphi\left(P^i, c^{P^i}\right).$$

By using the reasoning of the Subcase (a) above, we obtain that for each agent h,

$$\varphi_h\left(E, c^E\right) = \varphi_h^{MO}\left(E, c\right).$$

Case 2 $E > c_n$.

In such a case, there is a unique t, $0 \leq t < c_n$, such that

$$\sum_{i \in \mathcal{N}} \max \{0, c_i - t\} = E - t.$$

Let k be the unique agent such that $c_k - t > 0$, and $c_{k-1} - t \leq 0$. Note that this implies that each agent j, with $j \geq k$, is privileged in (E, c). Then, for each $r \in \mathbb{N}$ such that

$$\frac{1}{r} < \min \{E - t, (n - k - 1)(c_k - t)\}.$$

By μ-Additivity,

$$
\begin{aligned}
\varphi(E, c) = \ & \varphi\left(t + \tfrac{1}{r}, \left((c_i)_{i<k}, \left(t + \tfrac{1}{r(n-k+1)}\right)_{i \geq k}\right)\right) \\
& + \varphi\left(E - t - \tfrac{1}{r}, \left((0)_{i<k}, \left(c_i - t - \tfrac{1}{r(n-k+1)}\right)_{i \geq k}\right)\right).
\end{aligned}
\tag{4}
$$

Now, we let r go to infinity in equation (4) above and obtain

$$
\begin{aligned}
\lim_{r \to \infty} \varphi(E, c) = \ & \lim_{r \to \infty} \left[\varphi\left(t + \tfrac{1}{r}, \left((c_i)_{i<k}, \left(t + \tfrac{1}{r(n-k+1)}\right)_{i \geq k}\right)\right) \right. \\
& \left. + \varphi\left(E - t - \tfrac{1}{r}, \left((0)_{i<k}, \left(c_i - t - \tfrac{1}{r(n-k+1)}\right)_{i \geq k}\right)\right) \right].
\end{aligned}
$$

By Continuity,

$$\varphi(E, c) = \varphi\left(t, (\min \{c_i, t\})_{i \in \mathcal{N}}\right) + \varphi\left(E - t, (\max \{0, c_i - t\})_{i \in \mathcal{N}}\right).$$

Observe that the problem $\left(t, (\min \{c_i, t\})_{i \in \mathcal{N}}\right)$ was analyzed in Case 1

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above. Therefore, for each agent h,

$$\varphi_h\left(t, (\min\{c_i, t\})_{i \in \mathcal{N}}\right) = \sum_{i=1}^{h} \frac{\min\{c_i, t\} - \min\{c_{i-1}, t\}}{n - i + 1}. \tag{5}$$

Moreover, note that for agent h,

$$\max\{0, c_h - t\} = \begin{cases} 0 & \text{if } h < k \\ c_h - t & \text{if } h \geq k \end{cases}.$$

Since by construction

$$\sum_{i=1}^{n} \max\{0, c_i - t\} = E - t,$$

we conclude that for each agent h,

$$\varphi_h\left(E - t, (\max\{0, c_i - t\})_{i \in \mathcal{N}}\right) = \max\{0, c_h - t\}. \tag{6}$$

Finally, by combining equations (4), (5), and (6), we have that, for each agent h,

$$\varphi_h(E, c) = \sum_{i=1}^{h} \frac{\min\{c_i, t\} - \min\{c_{i-1}, t\}}{n - i + 1} + \max\{0, c_h - t\} = \varphi_h^{MO}(E, c).$$

∎