SURREAL TIME AND ULTRATASKS

HAIDAR AL-DHALIMY
Department of Philosophy, University of Minnesota
and
CHARLES J. GEYER
School of Statistics, University of Minnesota

Abstract. This paper suggests that time could have a much richer mathematical structure than that of the real numbers. Clark & Read (1984) argue that a hypertask (uncountably many tasks done in a finite length of time) cannot be performed. Assuming that time takes values in the real numbers, we give a trivial proof of this. If we instead take the surreal numbers as a model of time, then not only are hypertasks possible but so is an ultratask (a sequence which includes one task done for each ordinal number—thus a proper class of them). We argue that the surreal numbers are in some respects a better model of the temporal continuum than the real numbers as defined in mainstream mathematics, and that surreal time and hypertasks are mathematically possible.

§1. Introduction.

To perform a *hypertask* is to complete an uncountable sequence of tasks in a finite length of time. Each task is taken to require a nonzero interval of time, with the intervals being pairwise disjoint. Clark & Read (1984) claim to "show that no hypertask can be performed" (p. 387). What they in fact show is the impossibility of a situation in which both (i) a hypertask is performed *and* (ii) time has the structure of the real number system as conceived of in mainstream mathematics—or, as we will say, time is $\mathbb{R}$-*like*. However, they make the argument overly complicated.

THEOREM 1.1. *It is not possible, in* $\mathbb{R}$-*like time, to perform a hypertask.*

Proof. Let t be the finite amount of time it takes to accomplish all of the tasks. For each integer n, at most nt of them can take time greater than $1/n$. All tasks take some nonzero time. Thus, the set of all tasks is a countable union of finite sets, which is countable. □

This kind of argument is familiar from many areas of mathematics. Clark & Read cite Cantor without stating how simple the argument really is.

Theorem 1.1 relies on the properties of the real numbers. If one uses the surreal numbers (Conway, 2001, first edition 1976) instead, then it is possible to perform not only a hypertask, but also what we will call an *ultratask*,[1] which includes the doing of one task for each ordinal number.[2] Ultratasks are a kind of hypertask, since there are uncountably many ordinals. We will call time that is structured like the surreal numbers *surreal time*.

There are many different notions of possibility: there are various kinds of *logical* possibility (corresponding to different logics), various kinds of *mathematical* possibility

Received: May 13, 2016.

[1] Ultratasks have no relation to ultrafilters or ultraproducts. Szabó (2010) defines "ultratask" differently.

[2] Thanks to Roy Cook for the idea of an ultratask.

doi:10.1017/S1755020316000289

(classical ones, constructive ones, intuitionistic ones, and others), and there are *conceptual*, *epistemic*, *nomic*, *physical*, and *metaphysical* notions of possibility. Our main claim is that surreal time and hypertasks are mathematically possible, in the sense of being compatible with the truths of *classical* mathematics.[3] This kind of mathematical possibility will be indicated by the subscript *cm*.[4]

THEOREM 1.2. *It is possible$_{cm}$, in surreal time, to perform an ultratask.*

Proof. For each ordinal α, start the α-th task at time $\alpha/(\alpha+1)$. □

Readers who are not familiar with the surreal numbers may not understand this proof. We explain the surreal numbers in §2 and §4, and expand and discuss the proof in §3.

§2. Surreal numbers. The surreal number system **No** (Conway, 2001) is a totally ordered Field with a capital "F" (Conway, 2001, chap. 1); Conway capitalizes names for mathematical structures whose domains are proper classes. **No** includes the real number system $\mathbb{R}$ as a subfield, but is much larger. **No** includes (as a subclass) the proper class **On** of all ordinals. The real numbers and ordinal numbers, considered as elements of **No**, have all of the properties of the real numbers and ordinal numbers from ordinary mathematics (Conway, 2001, chap. 2). **No** is also a real-closed Field (Conway, 2001, chap. 4), meaning **No** has all the first-order properties that $\mathbb{R}$ has. Hence, if α is a nonzero surreal number, then so is $1/\alpha$, and if α is a non-negative surreal number, then so is $\sqrt{\alpha}$. Any arithmetic and algebraic operations that take real numbers to real numbers also take surreal numbers to surreal numbers. In particular, if α is a nonzero ordinal, then $\alpha - 1$, $1/\alpha$, and $\sqrt{\alpha}$ are surreal numbers.

The proof of Theorem 1.1 would not work if time can take values in the surreal numbers. It would go wrong because it assumes non-first-order properties of $\mathbb{R}$ that **No** does not share. In particular, it assumes that the sequence $\langle 1/n \rangle_{n \in \mathbb{N}}$ converges to zero—although this holds in the real numbers, in the surreal numbers this sequence does not converge to zero. There are many infinitesimal surreal numbers smaller than $1/n$ for all integers n, for example, $1/\alpha$ for any infinite ordinal α.

Formally, it is not true in **No** that for every $\varepsilon > 0$ there is a natural number N such that $1/n < \varepsilon$ for all natural numbers $n \geq N$. It is not true whenever ε is infinitesimal—this, in fact, is the meaning of the word 'infinitesimal'.

§3. Ultratasks. Recall that an *ultratask* includes the doing of one task T_α for each ordinal α. If time takes values in the surreal numbers, then there is an easy way to do an ultratask in one unit of time.[5] Define the function f by

$$f(x) = \frac{x}{x+1}, \qquad x \geq 0. \tag{1}$$

This function makes sense for surreal numbers x just like for real numbers x. We could replace f by any other algebraic function taking non-negative argument values that is strictly increasing and bounded above by a real number. Start T_α at time $f(\alpha)$.

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[3] It suffices here to include NBG with Global Choice among "classical" mathematics.

[4] We take it that whatever is possible$_{cm}$ is also logically possible (in the sense of classical logic), but not vice versa.

[5] In fact, an ultratask—or even a separate ultratask for each ordinal—could be performed in an arbitrarily short positive interval of surreal time.

The length of time between the start of T_α and the start of $T_{\alpha+1}$ is

$$\frac{\alpha+1}{\alpha+2}-\frac{\alpha}{\alpha+1}=\frac{(\alpha+1)^2-\alpha(\alpha+2)}{(\alpha+1)(\alpha+2)}=\frac{1}{(\alpha+1)(\alpha+2)}. \tag{2}$$

The operations here are the Field operations for **No**, which do not agree with the usual ordinal arithmetic used in set theory (Conway, 2001, p. 28). The addition and multiplication operations for **No** when applied to ordinals are called the *natural* or *Hessenberg* or *Hessenberg–Conway* sum and product in set theory. There are no subtraction or division operations defined for ordinals except the Conway ones, which are the Field operations for **No**.

If α is an infinite ordinal, then the length of time (2) is infinitesimal, but it is a strictly positive surreal number. If one likes rests (staccato performance, Clark & Read, 1984, their §3), then one may (optionally) take part of this interval to do T_α and the remainder of the interval to rest. As we shall see in §6, there may be rests even if one does not have rests of the kind just described.

§4. Dedekind cuts. Is the theory in §3 philosophically satisfying? Is it philosophically legitimate to consider surreal time? Does any philosophical principle dictate that time is *necessarily* $\mathbb{R}$-like, in which case, by Theorem 1.1, no hypertask would be doable in any possible world in which there is time?

The real numbers can be constructed as Dedekind cuts of the rational numbers. Dedekind cuts were indeed a great idea (Reck, 2012), but if they are such a great idea, why stop at the rationals? What about Dedekind cuts of the real numbers? Why aren't they a great idea too? We raise this issue because it leads to a construction of the surreal numbers. We quote Conway (2001, pp. 3–4):

> Let us see how those who were good at constructing numbers have approached this problem in the past.
>
> *Dedekind* (and before him the author—thought to be Eudoxus—of the fifth book of Euclid) constructed the real numbers from the rationals. His method was to divide the rationals into two sets L and R in such a way that no number of L was greater than any number of R, and use this "section" to define a new number $\{L|R\}$ in the case that neither L nor R had an extremal point.
>
> His method produces a logically sound collection of real numbers (if we ignore some objections on the grounds of ineffectivity, etc.), but has been criticised on several counts. Perhaps the most important is that the rationals are supposed to have been already constructed in some other way, and yet are "reconstructed" as certain real numbers. The distinction between the "old" and "new" rationals seems artificial but essential.
>
> *Cantor* constructed the infinite ordinal numbers. Supposing the integers 1, 2, 3, ... given, he observed that their *order-type* ω was a new (and infinite) number greater than all of them. Then the order-type of $\{1, 2, 3, \ldots, \omega\}$ is a still greater number $\omega+1$, and so on, and on, and on. The similar objections to Cantor's procedure have already been met by von Neumann, who observes that it is unnecessary to suppose 1, 2, 3, ... given, and that it is natural to start at 0 rather than 1. He also takes each ordinal as the *set* (rather than the order-type) of all previous ones. Thus for von Neumann, 0 is the empty set, 1 is the set $\{0\}$, 2 is the set $\{0, 1\}, \ldots, \omega$ is the set $\{0, 1, 2, \ldots\}$, and so on.

> In this chapter we shall show that these two methods are part of a simpler and more general one by which we can construct a very large Class **No** of "Surreal Numbers," which includes the real numbers and the ordinal numbers, as well as others ....
>
> If L, R are any two sets of [surreal] numbers, and no member of L is $\geq$ any member of R, then there is a number $\{L|R\}$. All [surreal] numbers are constructed this way.

Conway's construction requires a definition of $\geq$ as it applies to surreal numbers constructed this way, and he (Conway, 2001, p. 4) gives an inductive definition that does the job, helping to make the surreal numbers a real-closed ordered Field.

In order to construct a surreal number $\{L|R\}$ in this way, one must already have some numbers to go in the sets L and R. So how does the process get started? When L and R are both empty, we get the number $\{|\} = 0$. Then we can make the new numbers $\{0|\} = 1$ and $\{|0\} = -1$. Then we can make the new numbers $\{0, 1|\} = 2$, $\{0|1\} = \frac{1}{2}$ and $\{-1|0\} = -\frac{1}{2}$, and $\{| -1, 0\} = -2$. At each stage, we can make some more numbers. The process never stops.

For each ordinal α, Conway (2001) denotes the set of numbers "born" on "day" α by N_α, where the "days" are what we were calling stages above. Thus,

$$
\begin{aligned}
N_0 &= \{0\}, \\
N_1 &= \{-1, 1\}, \\
N_2 &= \left\{-2, -\frac{1}{2}, \frac{1}{2}, 2\right\},
\end{aligned}
$$

and so forth. Each N_α contains the ordinal α, but also many more numbers. The union of all the N_α for finite ordinals α contains all of the dyadic rational numbers (rational numbers whose denominator is a power of 2). Then on day ω, that being the first transfinite ordinal, are born all the real numbers (as Dedekind cuts of the dyadic rationals) and also some infinite and infinitesimal numbers, including

$$
\begin{aligned}
\omega &= \{0, 1, 2, \ldots |\}, \\
-\omega &= \{| \ldots, -2, -1, 0\}, \\
\frac{1}{\omega} &= \left\{0 \,\middle|\, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}.
\end{aligned}
$$

And the process keeps on going. On day $\omega + 1$ are born

$$
\begin{aligned}
\omega + 1 &= \{\omega|\}, \\
\omega - 1 &= \{0, 1, 2, \ldots |\omega\},
\end{aligned}
$$

and all of the Dedekind cuts of the real numbers. For example, if

$$
\begin{aligned}
L &= \left\{x \in \mathbb{R} \colon x \leq \sqrt{2}\right\}, \\
R &= \left\{x \in \mathbb{R} \colon x > \sqrt{2}\right\},
\end{aligned}
$$

then

$$
\{L|R\} = \sqrt{2} + \frac{1}{\omega},
$$

and this is not a real number. And the process keeps on going, never stopping.

§5. The continuum. Who is to say whether the real numbers, as conceived by Dedekind and modern mainstream mathematics, agree with conceptions of the continuum of Euclid, Archimedes, Newton, Leibniz, and Euler?[6] Is it really the case that there is nothing between $\sqrt{2}$ and the real numbers greater than it? Or should we prefer a notion of continuity that is more continuous than that?

One might object that the surreal numbers, just like the real numbers, are punctate. They are neither gunky (Lewis, 1991; Hellman & Shapiro, 2013), nor nonpunctate in the sense of Bell (2008, 2014). The real numbers and the surreal numbers, considered as bare sets, consist only of points (numbers). Of course both number systems have a lot of additional structure in that each real is a Dedekind cut and each surreal is defined by Conway's kind of Dedekind cut with any surreal numbers, not just rational numbers, in the left and right sets—we say "defined by" because different Conway–Dedekind cuts can define the same number (Conway, 2001, p. 5); for example $\{0, 1|\}$, $\{1|\}$, and $\{1|3\}$ all define the same number (2). And, of course, the real and the surreal number systems also have the arithmetic operations, the order, roots, and other mathematical structure.

There are more surreal numbers than real numbers, so many as to make a proper class. This prevents us from considering Dedekind cuts of the surreal numbers (which Conway, 2001 does discuss, pp. 37–38, and we discuss, §7 below) as surreal numbers; the definition requires that L and R in $\{L|R\}$ be sets, so they cannot be proper classes. Thus, this construction fills up the continuum with numbers as much as can be done without running into the Burali-Forti paradox.

The real numbers are defined the way they are for reasons of mathematical convenience. In calculus, it is important that we have $1/n \to 0$ as $n \to \infty$ (and many similar results). So the real numbers are defined to make it so (and to make Theorem 1.1 correct). But in contexts where calculus plays no essential role, that gives no philosophical support for the notion that a continuum is structured like the real number system.

The surreal numbers are not well suited for calculus (or real and functional analysis) because they make the usual operations of calculus problematic (last two sections of chap. 4 and the Epilogue of Conway, 2001, but see also Costin, Ehrlich, & Friedman, preprint, and Rubinstein-Salzedo & Swaminathan, 2014). But there are several senses in which the surreal number system is maximal.[7] The surreal numbers form the largest possible real-closed ordered Field (Conway, 2001, theorems 28 and 29).[8] So they form the largest mathematical structure that has the arithmetic properties of the real numbers.[9] They are as continuous as they can possibly be and still be an ordered Field. In the terminology of Ehrlich (2012), the real numbers are the largest *Archimedean* arithmetic continuum (the Archimedean property being $1/n \to 0$ as $n \to \infty$, so there are no infinitesimals), but the surreal numbers are the *absolute* arithmetic continuum. If we think the field properties of numbers are important—and it is hard to imagine precise measurement of time using a continuum that does not have numbers or does not have arithmetic—and we want our temporal continuum to be as continuous as possible, then that would suggest that surreal time is not only possible in this context but preferable to $\mathbb{R}$-like time.

It is not clear that performing a supertask (a countably infinite sequence of tasks completed in a finite length of time) is compatible with the physical laws of the

[6] Six conceptions of the continuum are discussed in Feferman (2009).

[7] On a view like that of Zermelo (1930), the system of surreal numbers is maximal only *relative to* a model of the background set theory. See our footnote 11.

[8] This is provable, for example, in NBG with Global Choice. See Ehrlich (2012).

[9] This is so at least with respect to consistent theories that prove the aforementioned theorem.

actual world.[10] So talk even of supertasks might only be about in-principle philosophical idealization. And in that realm, it is not clear to us that the real numbers are preferable to the surreal numbers as a model of the temporal continuum.

§6. Rests. A rest is a positive duration of time during which no task is being started, performed, or finished. In surreal time, sometimes it is *required*, rather than optional, that there be a rest. As we shall see, there will be a rest right after any limit-ordinal-length sequence of tasks, even if we do not try to leave a rest, and even if we try *not* to leave one. Let us consider a few concrete examples.

Suppose that we are doing an ultratask as described in §3 with f given by (1) or as described in the text following that equation. For $0 < t < 1$, how much work has been accomplished by time t? Because the ordinals are well ordered (Conway, 2001, theorem 15), there is a unique least ordinal β satisfying $f(\beta) \geq t$. If $f(\beta) = t$, then we are just starting T_β at time t. If $f(\beta) > t$, then we have not yet started T_β at time t, in which case there are two subcases. If β is a successor ordinal, then $\beta - 1$ is also an ordinal, and $f(\beta - 1) < t < f(\beta)$, so at time t we are in the process of doing $T_{\beta-1}$ or in the optional rest (if we choose to leave one) between doing $T_{\beta-1}$ and T_β. If β is a limit ordinal, then $f(\alpha + 1) < t < f(\beta)$ for all $\alpha < \beta$, so we have already done every T_α for $\alpha < \beta$ and at time t are in the rest before doing T_β (regardless of whether we left a rest after each task).

The limit ordinal case is a bit counterintuitive: there will be a rest, no matter what. If α is an ordinal, β is a limit ordinal, and $\alpha < \beta$, then $\alpha + n < \beta$ for any integer n. Hence, by the Field properties of **No**, $\alpha < \beta - n$. Thus, every ordinal $\alpha < \beta$ is also less than $\beta - n$. Hence, there is a rest between times $f(\beta - n)$ and $f(\beta)$, so long as n is finite and β is a limit ordinal.

There are even longer rests before some special limit ordinals. If $\beta = \omega^\alpha$ for some ordinal α (this is the usual ordinal exponentiation from set theory; ω^x for any surreal x is defined in chap. 3 of Conway, 2001), then $n\gamma < \beta$ for all integers n and all ordinals $\gamma < \beta$ (van den Dries & Ehrlich, 2001, Corollary 3.1). Hence, there is a rest between times $f(\beta/n)$ and $f(\beta)$, so long as n is finite and $\beta = \omega^\alpha$ for some ordinal α.

If $\beta = \omega^{\omega^\alpha}$ for some ordinal α (again the usual ordinal exponentiation), then $\gamma^n < \beta$ for all integers n and all ordinals $\gamma < \beta$ (van den Dries & Ehrlich, 2001, Corollary 4.4). Hence, there is a rest between times $f\left(\sqrt[n]{\beta}\right)$ and $f(\beta)$, so long as n is finite and $\beta = \omega^{\omega^\alpha}$ for some ordinal α.

THEOREM 6.1. *In surreal time, there is a rest required right before each task that is indexed by a limit ordinal.*

Proof. For each ordinal α, denote the tasks by T_α, and suppose we start T_α at time s_α and finish T_α at time t_α. Let λ be a limit ordinal, and define

$$L = \{ t_\alpha : \alpha < \lambda \}.$$

Then every element of L is strictly less than s_λ, because for $\alpha < \lambda$,

$$t_\alpha \leq s_{\alpha+1} < t_{\alpha+1} < s_\lambda.$$

Thus $x = \{L | s_\lambda\}$ is a surreal number and for all $\alpha < \lambda$,

$$t_\alpha < x < s_\lambda$$

[10] For references to philosophical discussions of the physical (im)possibility of supertasks, see Laraudogoitia (2013).

(Conway, 2001, theorem 2). Since all tasks T_α for $\alpha < \lambda$ are done before time x, there is a rest that includes all times between x and s_λ. □

Thus, in surreal time, a supertask requires at least one rest, a hypertask requires at least a countably infinite number of rests, and an ultratask requires proper-class-many rests. Given a surreal $\varepsilon > 0$, the rest required right before task T_λ can be made shorter than ε by a suitable choice of the starting time s_λ. But because there is no earliest time that can be chosen for s_λ, any choice of s_λ leaves a rest right before T_λ. As we shall see in §7, there is another kind of rest that we have not described.

§7. Gaps. Conway (2001, pp. 37–38) defines a *gap* in the surreal numbers to be a mathematical object of the form $\{L|R\}$ that divides the surreals into two classes L and R, with every member of L less than every member of R. Unless either L or R is empty, both are proper classes. Conway (2001, p. 38) says

> Just as we speak of an *infinity* of objects when the collection of them is not finite, it seems natural to speak of a *University* of objects when the Collection is a Proper Class. But the collection of all gaps is not even a Proper Class, being an illegal object in most set theories. Informally, we may call it an IMPROPER CLASS, and speak of there being an IMPROPRIETY of gaps! There are very many gaps indeed. But we committed no impropriety in our discussion of them, which could all be formalized in such a way that at no point did the argument refer to more than one gap at a time.

Perhaps arguments mentioning gaps could be formalized so that they did not refer to gaps at all, but just had a much more long-winded discussion where we replace each reference to a gap by an explicit reference to the numbers to the left and the numbers to the right (in much the same way that ZFC “discusses” proper classes by long-winded descriptions that proper class terms succinctly capture e.g., in NBG).

The hyperreal number systems of nonstandard analysis (NSA) are real-closed fields too. If a real-closed field or Field has gaps (in which neither L nor R is empty) that are not right next to a number, then how can we consider it a continuum? We regard this question as privileging conventional mathematics and the real numbers. It is true that in conventional mathematics, the Dedekind completeness of $\mathbb{R}$ (every nonempty bounded subset has a least upper bound and a greatest lower bound) is key to calculus. But, as NSA shows, this is not the only way to go. NSA does calculus too, and an important part of the way it works is that the set of infinitesimals does not have a least upper bound or a greatest lower bound. So Dedekind completeness of the field is essential for calculus (or physics) only if one stays in mainstream mathematics. Not otherwise. Whether we should call a non-Archimedean real-closed field or Field with such gaps a “continuum” does not bear on whether it is possible for time to be structured like such a field or Field.

Now we will describe another kind of situation in which a rest is required in surreal time. Consider a bounded strictly increasing **On**-length sequence $\langle t_\alpha \rangle_{\alpha \in \mathbf{On}}$ of surreal numbers. This “converges” to a gap constructed as follows. Define

$$L = \{ x \in \mathbf{No} \colon \exists \alpha \in \mathbf{On}\ x \leq t_\alpha \}$$

(we are using NBG, since this is a proper class), and define $R = \mathbf{No} \setminus L$. Neither L nor R is empty, since L contains all of the t_α and R contains the upper bound of the sequence.

The sense in which $\langle t_\alpha \rangle_{\alpha \in \mathbf{On}}$ converges to this gap is that for every $l \in L$ and every $r \in R$, we have $l < t_\alpha < r$ for all sufficiently large α.

There are three kinds of gaps: those that are just to the right of a number, in which $L = \{ y \in \mathbf{No} \colon y \leq x \}$ for some surreal x, those that are just to the left of a number, in which $R = \{ y \in \mathbf{No} \colon y \geq x \}$ for some surreal x, and all other gaps. A strictly increasing **On**-length sequence of surreals cannot converge to a gap just to the right of a number, so only the latter two kinds matter.

Here are two examples that illustrate the two kinds of possibilities, where $\langle t_\alpha \rangle_{\alpha \in \mathbf{On}}$ is the **On**-length sequence of stopping times for an ultratask.

The stopping times of the ultratask discussed in §3 with $t_\alpha = (\alpha + 1)/(\alpha + 2)$ converge to the gap just to the left of 1. We know this because 1 is an upper bound on the sequence (so it goes in R). And for any surreal $\varepsilon > 0$ there is an $x \in L$ such that $x > 1 - \varepsilon$, in fact, a $t_\alpha > 1 - \varepsilon$ because otherwise we would have $\alpha/(\alpha + 1) \leq 1 - \varepsilon$ for all α, which is the same as $1/\varepsilon > \alpha$ for all α, and that is false because $\beta = \{ 1/\varepsilon | \}$ is an ordinal greater than $1/\varepsilon$, any surreal number of the form $\{L|\}$ being an ordinal (Conway, 2001, p. 27).

Now consider a different ultratask with stopping times $t_\alpha = \omega^{-1/(\alpha+1)}$, this denoting the $(\alpha + 1)$-th root of $1/\omega$. See pp. 31–32 in Conway (2001) for the meaning of this notation. Numbers of the form t_α are called "the largest infinitesimal numbers" on p. 214 in Conway (2001). The point is that every infinitesimal surreal is less than some t_α, which can be proven using the "normal form" of surreal numbers explained on pp. 32–33 in Conway (2001). If we define L to be the proper class of surreals that are negative or infinitesimal, and R to be the proper class of noninfinitesimal positive surreals, then $\{L|R\}$ is a gap, which Conway (2001, p. 37) denotes by $1/\infty$, and this **On**-length sequence of stopping times converges to this gap.

The duration of this ultratask does not correspond to any surreal length of time. Since gaps are not surreal numbers, they do not correspond to moments in surreal time, and do not measure lengths of surreal time. But in a manner of speaking, we might say that this ultratask lasts longer than any surreal infinitesimal duration, but shorter than any surreal noninfinitesimal duration. Every task in the ultratask is finished before any moment in R, but only after every moment in L is it the case that all of the tasks have been finished. We might say that this ultratask finishes before every moment in R and after every moment in L, even though no surreal number answers to that description.

Now consider what happens if an ultratask u is a part of some more complicated ultratask that has an earliest task T following all the tasks in u. There has to be a time when T starts, which comes after all the stopping times of u. In the case where the stopping times of u converge to a gap just to the left of a number, that number can be the starting time of T, and there is no rest at the end of u. In the case where the stopping times of u converge to a gap $\{L|R\}$ that is not just to the left of a number, any number that can be the starting time of T (that is, any number in R) leaves a rest at the end of u, because R has no least member.

In the latter case, there is necessarily a rest at the end of u, but there is no minimum length that the rest must have. If we start task T at time t, then there is an $s \in R$ such that $s < t$ and the interval between s and t is a rest. But we could have chosen an earlier $t \in R$, and that would make the rest shorter. The point is that no matter which t we choose, there will be a rest.

§8. ε-subfields of No. There are proper-class-many real-closed ordered subfields of **No**. In the terminology of van den Dries & Ehrlich (2001), these are **No**(λ) when λ is an ordinal satisfying $\omega^\lambda = \lambda$ (such ordinals are called *ε-numbers*). Each of these subfields (we will call them *ε-subfields*) of **No** is an elementary substructure of **No** and an elementary extension of $\mathbb{R}$, and is a field with a lowercase "f" (a set rather than a proper class) that contains all surreal numbers born before day λ, and thus all ordinals less than λ.

For each infinite ordinal α, define an *α-task* to be an α-length sequence of tasks (one for each ordinal less than α) done, in order, in a finite length of time. Then for each ε-number λ, time that is structured like **No**(λ) (or like **No**(δ) for any ε-number $\delta > \lambda$) allows λ-tasks, but not ultratasks. Thus, for each uncountable ordinal α, there are proper-class-many ways that time could be—in the sense of possibly$_{cm}$—that would allow α-tasks, with each of these continua (structured like an ε-subfield of **No**) having only set-many moments.[11]

§9. Discussion. Time that is structured like **No** (or its ε-subfields) is of interest independently of the possibility of hypertasks. But the idea of surreal time takes some getting used to. In the more than 300 years since Newton and Leibniz developed calculus and the nearly 150 years since Cauchy, Dedekind, Cantor, Weierstrass, and others made the calculus rigorous, we have come to accept as intuitive many concepts and arguments from the calculus that were formerly highly mysterious or paradoxical (Berkeley's ghosts of departed quantities, Zeno's paradoxes), in particular, $1/n \to 0$ as $n \to \infty$, which is key to our Theorem 1.1 above. Now we are asking readers to reconsider concepts from calculus—are they really intuitive?—and to consider Conway's surreal numbers, which violate many notions from the Newton–Leibniz–Dedekind-etc. orthodoxy.

We will consider four main kinds of arguments against the possibility of surreal time and hypertasks. The first kind of argument is from the premise, implicit in Clark & Read (1984), that time is necessarily $\mathbb{R}$-like. By Theorem 1.1, it follows that hypertasks are not possible. Filling in the reasoning: if time is necessarily Archimedean, then infinitesimal lengths of time are not possible, and so tasks cannot take infinitesimal durations.

One such argument involves *metaphysical* possibility. Since the first premise is a *de re* claim, we will use the subscript *mp* there to indicate metaphysical possibility.

P_1: Time is necessarily$_{mp}$ $\mathbb{R}$-like.
P_2: Hypertasks in $\mathbb{R}$-like time are not metaphysically possible (by Theorem 1.1).
$\therefore$ Hypertasks are not metaphysically possible.

From Theorem 1.1, it follows that *hypertasks in $\mathbb{R}$-like time are impossible$_{cm}$*. If whatever is impossible$_{cm}$ is metaphysically impossible, then P_2 follows from Theorem 1.1. But we do not know a good reason for thinking that P_1 is true. Even from the assumption that time is actually (i.e., in the actual world) $\mathbb{R}$-like, P_1 does not easily follow. Time in the *actual* world being $\mathbb{R}$-like—which may not be empirically accessible—is compatible with it being the case that in some metaphysically possible world, time is not structured like $\mathbb{R}$.

The second kind of argument takes it as a premise that completing a task in an infinitesimal duration is not metaphysically possible, from which it follows that hypertasks are not metaphysically possible. We do not know of a good argument for that premise. And that premise is consistent with it being possible$_{cm}$ to complete a task in an infinitesimal

[11] If the *set* vs. *proper class* distinction is not absolute but instead relative to a model of the 2nd-order axioms of a set theory, as in Zermelo (1930), then so too are the notions of *all ordinals*, *the Field of surreals*, *surreal time*, *ultratask*, and *hypertask*. It would then be difficult to maintain that *some but not all* real-closed ordered fields (or Fields) are structures that some possible temporal continuum instantiates (or similarly that hypertasks *but not ultratasks* are possible). One needs a principled basis on which to say whether a mathematical structure is instantiated by some *possible* temporal continuum.

duration. In the surreal number system, infinitesimals are just numbers like any other numbers, so there seems no reason to discriminate against them. If durations of zero-length and positive noninfinitesimal lengths are metaphysically possible, then why not recognize durations whose lengths are between those two as metaphysically possible as well?

The third kind of argument is from the premise that required rests are not metaphysically possible to the conclusion that hypertasks are not metaphysically possible. To insist that any sequence of tasks that is doable be doable without a required rest is tantamount to insisting that each task be counted with a successor ordinal. And, the first infinite ordinal ω being a limit ordinal, this is tantamount to insisting that each task be counted with a finite ordinal. In surreal time, a rest is required after even an ω-task. Thus, if required rests are not metaphysically possible, then in surreal time, even ω-tasks are not metaphysically possible.

It is unclear why one should think that required rests are not metaphysically possible. We can describe a situation even in $\mathbb{R}$-like time where there must be a rest before a task T. Suppose that we may begin T at any time $t > 0$, but not at any time $t \leq 0$. Thus, there is no earliest moment at which we may begin T. Since there is no least real number greater than 0, any time t that we choose to begin T leaves a rest during the half-open interval $[0, t)$. This example does not even involve infinitely many tasks. If required rests are metaphysically impossible, then so is this type of situation.

The fourth kind of argument proceeds from considerations about how many moments there could be. One such argument is the following:

P_3: It is not metaphysically possible for there to be proper-class-many moments.
$\therefore$ Surreal time is not metaphysically possible.

Denying the metaphysical possibility of proper-class-many moments comes awfully close to denying the metaphysical possibility of surreal time. And it is not easy to argue for P_3.

The argument from P_3 casts no doubt on the metaphysical possibility of $\mathbf{No}(\lambda)$-like time, with λ an ε-number, since $\mathbf{No}(\lambda)$-like time has only set-many (in particular, $2^{|\lambda|}$) moments.[12] If for some fixed uncountable cardinal κ, one assumes that it is not metaphysically possible for there to be κ-many moments, then one can argue that when $2^{|\lambda|} \geq \kappa$, $\mathbf{No}(\lambda)$-like time is not metaphysically possible. But each of these arguments (one for each choice of κ) nearly begs the question. And any choice of such a κ seems to place an arbitrary limitation on the sizes of metaphysically possible temporal continua.

Another argument from the same premise is the following:

P_3: It is not metaphysically possible for there to be proper-class-many moments.
$\therefore$ Ultratasks are not metaphysically possible.

This inference fails if it is metaphysically possible for there to be time without there being any moments—e.g., with a pointless temporal continuum consisting only of positive-length durations, as in Hellman & Shapiro (2013). For each ordinal α, an α-task could be done in a finite-length interval of moment-less time so long as the interval contains an α-length sequence of pairwise non-overlapping subintervals. Perhaps there is a pointless analog of the system of surreal numbers, along with other pointless continua that contain infinitesimal intervals. If it is metaphysically possible for time to be structured like a pointless analog of **No**, or like some pointless continuum in which a finite-length interval contains an **On**-length sequence of pairwise non-overlapping subintervals, then a lack of moments does not preclude the metaphysical possibility of ultratasks.

[12] The cardinality of ε-subfields of **No** is discussed in Alling (1985).

In each of these four kinds of arguments, the conclusion is nearly built into the premise(s). While none of these arguments assumes exactly what it is trying to prove, each comes so close as to make hardly any difference.

Showing that surreal time and hypertasks are mathematically possible prepares the way for arguments aiming to establish that they are metaphysically possible. Surreal time and hypertasks should be taken at least as seriously as many highly abstract and implausible concepts that philosophers take seriously.

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DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF MINNESOTA
831 HELLER HALL
271 19TH AVENUE SOUTH
MINNEAPOLIS, MN 55455, USA
E-mail: haidar@umn.edu

SCHOOL OF STATISTICS
UNIVERSITY OF MINNESOTA
313 FORD HALL
224 CHURCH STREET SE
MINNEAPOLIS, MN 55455, USA
URL: users.stat.umn.edu/∼geyer
E-mail: geyer@umn.edu