Strengthening Consistency Results in Modal Logic

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A fundamental question asked in modal logic is whether a given theory is consistent. But consistent with what? A typical way to address this question identifies a choice of background knowledge axioms (say, S4, D, etc.) and then shows the assumptions codified by the theory in question to be consistent with those background axioms. But determining the specific choice and division of background axioms is, at least sometimes, little more than tradition. This paper introduces *generic theories* for propositional modal logic to address consistency results in a more robust way. As building blocks for background knowledge, generic theories provide a standard for categorical determinations of consistency. We argue that the results and methods of this paper help to elucidate problems in epistemology and enjoy sufficient scope and power to have purchase on problems bearing on modalities in judgement, inference, and decision making.

1 Introduction

Many treatments of epistemological paradoxes in modal logic proceed along the following lines. Begin with some enumeration of assumptions that are individually plausible but when taken together fail to be jointly consistent (or at any rate fail to stand to reason in some way). Thereupon proceed to propose a resolution to the emerging paradox that identifies one or more assumptions that may be comfortably discarded or weakened and that in the presence of the remaining assumptions circumvents the troubling inconsistency defining the paradox [11] (cf. Chow [8] and de Vos et al. [16]). Typical among such assumptions are logical standards expressed in the form of inference rules and axioms pertaining to knowledge and belief, such as axiom scheme **K** — that is to say, the distributive axiom scheme of the form $\mathrm{K}(\varphi \rightarrow \psi) \rightarrow (\mathrm{K}\varphi \rightarrow \mathrm{K}\psi)$.

The choice of precisely *which* assumptions to temper can, at times, have an element of arbitrariness to it, especially when the choice is made from among several independent alternatives underpinning distinct resolutions in the absence of clear criteria or compelling grounds for distinguishing among them. In the present paper, we introduce a criterion for addressing this predicament based on the *genericity* of what a resolution assumes.

As a standard for knowledge, a theory is generic when its factivity cannot be overturned however the questions it leaves open are answered and what is known accordingly grows. Generic theories enjoy various desirable properties which are common in formal epistemology — arbitrary unions of generic theories, for example, are generic. We present both positive and negative results turning on genericness, which cast light on the structure of popular logics for belief and knowledge.

The concept of generic theories, as introduced in [4] and [5] for quantified modal logic, emerged in response to Carlson's proof [7] of a conjecture due to Reinhardt [13]. Carlson's proof, despite its significance, was limited by its dependency on a somewhat arbitrary choice of background knowledge axioms. Carlson proof, subject to but small changes, is likewise valid for various other sets of background axioms. The present paper examines generic theories for propositional modal logic. In our concluding

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remarks we discuss the developments of this paper in connection with work done to generalize Carlson's consistency result.

The paper is organized as follows. In Section 2 we state preliminaries. In Section 3 we state a propositional version of the Knower Paradox: a certain theory, consisting of standard background knowledge axioms plus an axiom intended to be read as "This sentence is known to be false," is inconsistent. We discuss a possible resolution to the paradox: weaken the background knowledge axioms in order to render the theory consistent. In Section 4 we introduce generic and closed generic theories. In Section 5 we use generic and closed generic theories to state very generalized versions of the consistency result from Section 3. In Section 6 we state some negative results about genericness and closed genericness. Proofs of these negative results naturally lead to the construction of exotic models which satisfy certain standard knowledge axioms while failing certain other standard knowledge axioms. In Section 7 we conclude the paper with a high-level discussion. In Appendix A we give proofs of some of the claims made in the above sections.

2 Preliminaries

Throughout, we fix a nonempty set of symbols called *propositional atoms* and a symbol K which is not a propositional atom. The following logic is a propositional version of Carlson's so-called *base logic* [7] (cf. [2] and [1]).

Definition 1. The set of *formulas* is defined recursively as follows:

(i) Every propositional atom is a formula;

(ii) Whenever φ and ψ are formulas, so are $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, and $(\varphi \rightarrow \psi)$; and

(iii) Whenever φ is formula, so too is $\mathrm{K}(\varphi)$.

A formula is said to be *basic* if it is either a propositional atom or a formula of the form $\mathrm{K}(\varphi)$ for some formula φ. A set of formulas is called a *theory*. ◂

We adopt standard conventions for omitting parentheses. Parentheses omitted from conditional formulas are assumed to be right-nested; thus, for example, we write $\phi \rightarrow \psi \rightarrow \rho$ for $\phi \rightarrow (\psi \rightarrow \rho)$, and similarly for longer chains of implications.

Definition 2. A *model* is a function mapping each basic formula to a truth value in $\{\text{True}, \text{False}\}$. ◂

Thus, in contrast with classical treatments of semantics for modalities, a model assigns truth values not only to propositional atoms but also to formulas prefixed with K.

We may define a binary relation $\models$ from models to basic formulas in the usual way — that is, by stipulating that $\mathscr{M} \models \varphi$ just in case $\mathscr{M}$ assigns to φ the value True. The next definition extends this relation to all formulas. We adopt the standard convention to write $\mathscr{M} \not\models \varphi$ if it is not the case that $\mathscr{M} \models \varphi$.

Definition 3. Let $\mathscr{M}$ be a model. Define formula φ to be *true* in $\mathscr{M}$, $\mathscr{M} \models \varphi$, by recursion on φ:

(i) If φ is a basic formula, then $\mathscr{M} \models \varphi$ if and only if $\mathscr{M}$ assigns to φ the value True;

(ii) $\mathscr{M} \models \neg\varphi$ if and only if $\mathscr{M} \not\models \varphi$;

(iii) $\mathscr{M} \models \varphi \wedge \psi$ if and only if both $\mathscr{M} \models \varphi$ and $\mathscr{M} \models \psi$;

(iv) $\mathscr{M} \models \varphi \vee \psi$ if and only if either $\mathscr{M} \models \varphi$ or $\mathscr{M} \models \psi$; and

(v) $\mathscr{M} \models \varphi \rightarrow \psi$ if and only if either $\mathscr{M} \not\models \varphi$ or $\mathscr{M} \models \psi$.

Given a theory T, we write $\mathscr{M} \models T$ just in case $\mathscr{M} \models \varphi$ for every $\varphi \in T$. ◂

Entailment and validity are given standard treatment.

Definition 4. A theory T is said to *entail* a formula φ, written $T \models \varphi$, if for all models $\mathscr{M}$, $\mathscr{M} \models T$ implies $\mathscr{M} \models \varphi$. A formula φ is said to be *valid*, written $\models \varphi$, if $\emptyset \models \varphi$. ◂

Since modal formulas of the form $\mathrm{K}\varphi$ are treated like propositional atoms, it follows that if p is a propositional atom, then $\mathrm{K}p \vee \neg\mathrm{K}p$ is valid but $\mathrm{K}(p \vee \neg p)$ is not. Routine argument establishes compactness. A useful result is the following corollary of compactness.

Lemma 5. *Let T be a theory and φ be a formula. Then $T \models \varphi$ if and only if there is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n \in T$ for which $\models \varphi_1 \to \cdots \to \varphi_n \to \varphi$.*

Lemma 5 provides a basis for adopting the following proof-theoretic terminology in what follows.

Definition 6. A theory T is said to be *consistent* if there is a model $\mathscr{M}$ for which $\mathscr{M} \models T$. ◂

The following definition captures the familiar notion of closedness under the K operator.

Definition 7. A theory T is *closed* if $\left\{ \mathrm{K}\varphi \,:\, \varphi \in T \right\} \subseteq T$. ◂

Thus a theory T is closed just in case for every formula φ, if $\varphi \in T$, then $\mathrm{K}\varphi \in T$.

Definition 8. We adopt the following conventions for naming standard schemas:

V is the theory consisting of all formulas of the form $\mathrm{K}\varphi$ such that φ is valid (Definition 4).

K is the theory consisting of all formulas of the form $\mathrm{K}(\varphi \to \psi) \to (\mathrm{K}\varphi \to \mathrm{K}\psi)$.

T is the theory consisting of all formulas of the form $\mathrm{K}\varphi \to \varphi$.

KK (sometimes also called **4**) is the theory consisting of all formulas of the form $\mathrm{K}\varphi \to \mathrm{KK}\varphi$. ◂

We conclude this section with an observation about necessitation (proved in Appendix A).

Lemma 9. (Simulated Necessitation) *Let T be a closed theory. If T includes both* **V** *and* **K***, then for every formula φ : if $T \models \varphi$, then $T \models \mathrm{K}\varphi$.*

3 A Formalization of the Knower Paradox

We will use a propositional version of the well-known Knower Paradox [12] to illustrate the ideas of this paper. The paradox is usually formalized in first-order modal logic, where appeal to Gödel's Diagonal Lemma admits construction of the problematic sentence without having to assume it as an axiom. In our propositional version, we instead assume the problematic sentence axiomatically, allowing us to focus on the epistemological contents of the paradox without arithmetical distractions.

Theorem 10 (The Knower Paradox)**.** *Let p be some propositional atom. Let T_{KP} be the smallest closed theory which contains*:

(i) **V***,* **K***, and* **T**

(ii) $p \leftrightarrow \mathrm{K}\neg p$ *"This sentence is known to be false"*

Then the theory T_{KP} is inconsistent. ■

Proof. From schema **T** and axiom (ii), it follows that $T_{KP} \models \neg p$ and therefore $T_{KP} \models \mathrm{K}\neg p$ by Lemma 9, whence $T_{KP} \models p$ by axiom (ii). Hence, T_{KP} is inconsistent. □

The next theorem provides one way the theory in Theorem 10 may be weakened in order to restore consistency (and so constitutes a candidate for resolving the paradox, in the sense of Haack [11] or Chow [8]).

Theorem 11. *Let p be some propositional atom. Inductively, let $(T^-_{KP})_0$ be the smallest closed theory which contains*:

(i) **V** *and* **K**

(ii) $p \leftrightarrow \mathrm{K}\neg p$ *"This sentence is known to be false"*

In addition, let T^-_{KP} be the theory which contains:

(a) $(T^-_{KP})_0$.

(b) **T**.

Then theory T^-_{KP} is consistent. ■

Observe that the Knower Paradox (Theorem 10), so formalized, rests on the assumption that the knower know its own truthfulness. The key difference between T_{KP} and T^-_{KP} is that, while the schema $\mathrm{K}\varphi \rightarrow \varphi$ is included in both theories, only T_{KP} includes the schema $\mathrm{K}(\mathrm{K}\varphi \rightarrow \varphi)$. Some treatments[1] of the Knower Paradox do not explicitly include $\mathrm{K}(\mathrm{K}\varphi \rightarrow \varphi)$ as an assumption at all, instead including $\mathrm{K}\varphi \rightarrow \varphi$ and using a logic where the *rule of necessitation* holds—the rule permitting one to conclude $T \models \mathrm{K}\varphi$ from $T \models \varphi$. In such logics, if T contains the schema $\mathrm{K}\varphi \rightarrow \varphi$, then trivially $T \models \mathrm{K}\varphi \rightarrow \varphi$, so by necessitation, $T \models \mathrm{K}(\mathrm{K}\varphi \rightarrow \varphi)$. Thus, $\mathrm{K}(\mathrm{K}\varphi \rightarrow \varphi)$ sneaks in implicitly, in such logics.

The logic (Definition 1) studied in this paper does not presume the rule of necessitation. The rule of necessitation can be simulated in our logic by using Lemma 9, but only if the Lemma's conditions are met—which, in the case of T^-_{KP}, they are not, as T^-_{KP} is not closed. Thus, it becomes possible to weaken knowledge-of-factivity without weakening factivity itself. Theorem 11 shows that doing so is one possible resolution, in the sense of Haack [11] or Chow [8], to the paradox.[2] See [1, 15] for discussion about the weakening of knowledge-of-factivity. Note that this requires departing from Kripke semantics, as the rule of necessitation always holds in Kripke semantics.

Rather than prove Theorem 11 directly, we will (in Section 5) prove a pair of more general theorems, and Theorem 11 is a special case of either one of them. In order to state the more general theorems, we need to first introduce certain notions of genericity.

4 Generic and Closed Generic Theories

The following definition is a variant of Carlson's concept of a *knowing entity* [7].

Definition 12. Let T be a theory, and let S be a set of propositional atoms. Let $\mathscr{M}_{T,S}$ be the model defined by stipulating:

(i) For any propositional atom p: $\mathscr{M}_{T,S} \models p$ if and only if $p \in S$; and

(ii) For any formula of the form $\mathrm{K}\varphi$: $\mathscr{M}_{T,S} \models \mathrm{K}\varphi$ if and only if $T \models \varphi$. ◂

[1] See [9, 10] for an exception.

[2] The same technique has been used to resolve (in Haack's or Chow's sense) a version of the surprise exam paradox [1]; to resolve a version of Fitch's paradox [2]; and to construct a machine that knows its own code [3]. Aldini et al suggest [1] it might be possible to *simultaneously* resolve multiple paradoxes at once by dropping $\mathrm{K}(\mathrm{K}\varphi \rightarrow \varphi)$, i.e., the *union* of *multiple* paradoxically inconsistent theories might be consistent when so weakened.

The model $\mathscr{M}_{T,S}$ may be loosely interpreted to be that of an agent who knows exactly the consequences of theory T in a world in which all propositions from S are true. We will see that these models are useful for establishing consistency results.

The following definition strengthens the notion of consistency.

Definition 13. A theory T is said to be *generic* (resp. *closed generic*) if for each set S of propositional atoms and each theory (resp. *closed theory*) T': if $T' \supseteq T$, then $\mathscr{M}_{T',S} \models T$. ◂

A theory T is generic when T is known regardless of contingent facts S and however theoretical knowledge might grow in conjunction with them. Generic theories are theories that cannot be made false by the addition of more information.

We catalogue basic properties of genericity.

Proposition 14. *Genericity enjoys the following properties*:

(1) *Unions of generic theories are generic*;

(2) *Unions of closed generic theories are closed generic*;

(3) *Every generic theory is closed generic*;

(4) $\mathbf{V}$ *is generic*; and

(5) $\mathbf{K}$ *is generic.* ■

Proof. Properties (1)–(3) are readily verified.

(4) Let S be a set of propositional atoms and let $T' \supseteq \mathbf{V}$. Let $\varphi \in \mathbf{V}$, we must show $\mathscr{M}_{T',S} \models \varphi$. By definition of $\mathbf{V}$, φ is $\mathrm{K}\psi$ for some valid ψ. Since ψ is valid, $T' \models \psi$. Thus $\mathscr{M}_{T',S} \models \mathrm{K}\psi$, as desired.

(5) Let S be a set of propositional atoms and let $T' \supseteq \mathbf{K}$. Let $\varphi \in \mathbf{K}$, we must show $\mathscr{M}_{T',S} \models \varphi$. By definition of $\mathbf{K}$, φ is $\mathrm{K}(\psi \rightarrow \rho) \rightarrow (\mathrm{K}\psi \rightarrow \mathrm{K}\rho)$ for some ψ and ρ. Assume $\mathscr{M}_{T',S} \models \mathrm{K}(\psi \rightarrow \rho)$ and $\mathscr{M}_{T',S} \models \mathrm{K}\psi$. This means $T' \models \psi \rightarrow \rho$ and $T' \models \psi$. By modus ponens, $T' \models \rho$. So $\mathscr{M}_{T',S} \models \mathrm{K}\rho$, as desired. □

Lemma 15. *The theory* $\mathbf{V} \cup \mathbf{K} \cup \mathbf{KK}$ *is closed generic.* ■

Proof. Let $T = \mathbf{V} \cup \mathbf{K} \cup \mathbf{KK}$. Let S be a set of propositional atoms and let $T' \supseteq T$ be closed. Let $\varphi \in T$, we must show $\mathscr{M}_{T',S} \models \varphi$. Consider two cases:

Case 1 $\varphi \in \mathbf{V} \cup \mathbf{K}$. Then $\mathscr{M}_{T',S} \models \varphi$ because $\mathbf{V} \cup \mathbf{K}$ is generic by Proposition 14, parts (1), (4), and (5).

Case 2 $\varphi \in \mathbf{KK}$. Then φ is $\mathrm{K}\psi \rightarrow \mathrm{KK}\psi$ for some ψ. Assume $\mathscr{M}_{T',S} \models \mathrm{K}\psi$. This means $T' \models \psi$. Since T' contains $\mathbf{V}$ and $\mathbf{K}$ and is closed, we may simulate necessitation: Lemma 9 implies $T' \models \mathrm{K}\psi$. Thus $\mathscr{M}_{T',S} \models \mathrm{KK}\psi$, as desired. □

Lemma 16. *Let T_0 be a theory, and let T be the smallest closed theory including theory T_0. Suppose theory T_0 is (closed) generic. Then T is (closed) generic.* ■

Proof. Let S be a set of propositional atoms and let T' be a theory (resp. closed theory) such that $T' \supseteq T$. Let $\varphi \in T$, we must show $\mathscr{M}_{T',S} \models \varphi$. Consider two cases:

Case 1 $\varphi \in T_0$. Then $\mathscr{M}_{T',S} \models \varphi$ because T_0 is generic (resp. closed generic).

Case 2 $\varphi \notin T_0$. The only other way for φ to be in T (besides being in T_0) is by way of the closure of T. So φ is $K\psi$ for some $\psi \in T$. Since $T' \supseteq T$ and $\psi \in T$, we have $T' \models \psi$, which means $\mathscr{M}_{T',S} \models \mathrm{K}\psi$, as desired. □

Lemma 17. *Suppose T_0 is a generic (resp. closed generic) theory. Let $T = \{\varphi : T_0 \models \varphi\}$. Then T is generic (resp. closed generic).* ■

Proof. Let S be an arbitrary set of propositional atoms, and let T' be a theory (resp. closed theory) such that $T' \supseteq T$. We establish that $\mathscr{M}_{T',S} \models T$.

For each formula φ such that $T \models \varphi$, let $N(\varphi)$ be the smallest positive integer n for which there is a sequence $\varphi_1, \ldots, \varphi_n$, with $\varphi_n = \varphi$, such that for each $i = 1, \ldots, n$, either $\varphi_i \in T_0$ or there exist $j, k < i$ such that φ_k is $\varphi_j \rightarrow \varphi_i$. Such an $N(\varphi)$ exists by the deduction theorem.

We prove by induction on $N(\varphi)$ that for every φ such that $T_0 \models \varphi$, $\mathscr{M}_{T',S} \models \varphi$.

Basis Step $N(\varphi) = 1$ can clearly only hold if $\varphi \in T_0$. In that case, $\mathscr{M}_{T',S} \models \varphi$ because T_0 is generic (resp. closed generic).

Inductive Step $N(\varphi) > 1$. If $\varphi \in T_0$, we are done as in the Base Case, but assume not. Let $\varphi_1, \ldots, \varphi_n$ be a sequence of length $n = N(\varphi)$ with the above properties.

For each $i < n$, the subsequence $\varphi_1, \ldots, \varphi_i$ is a shorter sequence (with the above properties) for φ_i, showing $N(\varphi_i) < N(\varphi)$. Thus by induction, $(*)$ for each $i < n$, $\mathscr{M}_{T',S} \models \varphi_i$. Since $\varphi \notin T_0$, there must be $j, k < n$ such that φ_k is $\varphi_j \rightarrow \varphi_n$. By $*$, $\mathscr{M}_{T',S} \models \varphi_j$ and $\mathscr{M}_{T',S} \models \varphi_k$. So $\mathscr{M}_{T',S} \models \varphi_j \rightarrow \varphi_n$. By modus ponens, $\mathscr{M}_{T',S} \models \varphi_n$, as desired.

□

We conclude this section with a result throwing light on the relationship between generic theories and normal modal logics. The proof is immediate by combining Lemmas 14, 16, and 17.

Theorem 18. *Suppose T_0 is a (closed) generic theory. Let T be the* normal Kripke closure *of T_0, i.e., the smallest closed theory containing T_0,* **V**, **K**, *and with the property that T contains ϕ whenever $T \models \phi$. Then T is (closed) generic.* ■

5 Two Generalized Consistency Statements

In what follows, we state two theorems, each generalizing Theorem 11. One might be curious whether adding **KK** to the statement of Theorem 11 would make the paradox reappear. Certainly the paradox as formulated in Theorem 10 does not use **KK** in its proof. But what if there is some other form of the Knower's Paradox that makes use of **KK**, and what if in fact we only managed to achieve consistency because we neglected to include **KK** among the background axioms? We could state a separate version of Theorem 11 which includes **KK** and then prove that separate version, with a proof that is extremely similar to a proof of Theorem 11 itself, but then maybe there's still some further background axiom that we are still neglecting, and we would then have to state and prove yet a third version of the theorem. This process might go on forever, we might never exhaustively think of all the different background axioms that critics might insist upon.

Theorem 19. *Let p be a propositional atom, and let H be a generic theory. Let $(T_{KP})_0$ be the smallest closed theory containing:*

(i) H

(ii) $p \leftrightarrow \mathrm{K}\neg p$ *"This sentence is known to be false"*

In addition, let T_{KP} be the theory containing:

(a) $(T_{KP})_0$; and

(b) $\mathbf{T}$.

For any set S of propositional atoms, if $p \notin S$ then $\mathscr{M}_{(T_{KP})_0,S} \models T_{KP}$. In particular, T_{KP} is consistent. ■

We prove Theorem 19 in Appendix A. Observe that since theory $\mathbf{V}\cup\mathbf{K}$ is generic by Proposition 14, Theorem 11 is a special case of Theorem 19.

Now modify Theorem 11 by replacing $\mathbf{V}\cup\mathbf{K}$ with $\mathbf{V}\cup\mathbf{K}\cup\mathbf{KK}$. We could not do that using Theorem 19 unless we first established that $\mathbf{V}\cup\mathbf{K}\cup\mathbf{KK}$ was generic (in fact, in the next section, we will show that $\mathbf{V}\cup\mathbf{K}\cup\mathbf{KK}$ is *not* generic). We do know that $\mathbf{V}\cup\mathbf{K}\cup\mathbf{KK}$ is closed generic (Lemma 15), so we would be done if we had a version of Theorem 19 involving closed generic theories.

Theorem 20. *Same as Theorem 19 but with "generic" replaced by "closed generic."* ■

A proof similar to the one for Theorem 19 establishes Theorem 20.

6 Negative Results about Genericness

We have established theory $\mathbf{V}\cup\mathbf{K}\cup\mathbf{KK}$ to be closed generic. Are these results preserved if one or more of the arguments to the union is dropped? For example, is theory $\mathbf{V}\cup\mathbf{KK}$ closed generic? Or the theory $\mathbf{K}\cup\mathbf{KK}$? What about the theory $\mathbf{KK}$ alone? Similarly, can we strengthen closed genericity of $\mathbf{V}\cup\mathbf{K}\cup\mathbf{KK}$ to full genericity? We show each of these questions has one and the same answer: No.

Theorem 21. *The theory $\mathbf{V}\cup\mathbf{K}\cup\mathbf{KK}$ fails to be generic.* ■

Proof. Let $T = \mathbf{V}\cup\mathbf{K}\cup\mathbf{KK}$. Let p be some propositional atom and let $T' = T\cup\{p\}$. We show that $\mathscr{M}_{T',\emptyset} \not\models \mathrm{K}p \rightarrow \mathrm{KK}p$, whereby $\mathscr{M}_{T',\emptyset} \not\models \mathbf{KK}$ and so $\mathscr{M}_{T',\emptyset} \not\models T$, showing T is not generic. Clearly $T' \models p$, so $\mathscr{M}_{T',\emptyset} \models \mathrm{K}p$. What remains to show is that $\mathscr{M}_{T',\emptyset} \not\models \mathrm{KK}p$ — that is, $T' \not\models \mathrm{K}p$.

To this end, inductively define models $\mathscr{N}_1$ and $\mathscr{N}_2$ simultaneously by stipulating $\mathscr{N}_1 \models q$ and $\mathscr{N}_2 \not\models q$ for each propositional atom q and requiring that $\mathscr{N}_1$ and $\mathscr{N}_2$ interpret formulas $\mathrm{K}\varphi$ in the following way:

$\mathscr{N}_2 \models \mathrm{K}\varphi$ if and only if $\mathscr{N}_2 \models \varphi$; and

$\mathscr{N}_1 \models \mathrm{K}\varphi$ if and only if $\mathscr{N}_2 \models \varphi$.

Since $\mathscr{N}_2 \not\models p$, $\mathscr{N}_1 \not\models \mathrm{K}p$. Thus, to show that $T' \not\models \mathrm{K}p$, and so conclude the proof, it suffices to show $\mathscr{N}_1 \models T'$.

Let $\varphi \in T'$. Consider four cases:

Case 1 $\varphi \in \mathbf{V}$. Then φ is $\mathrm{K}\varphi_0$ for some valid φ_0. Since φ_0 is valid, $\mathscr{N}_2 \models \varphi_0$, so $\mathscr{N}_1 \models \mathrm{K}\varphi_0$.

Case 2 $\varphi \in \mathbf{K}$. Then φ has the form $\mathrm{K}(\psi \rightarrow \rho) \rightarrow (\mathrm{K}\psi \rightarrow \mathrm{K}\rho)$. Assume $\mathscr{N}_1 \models \mathrm{K}(\psi \rightarrow \rho)$ and $\mathscr{N}_1 \models \mathrm{K}\psi$. Then $\mathscr{N}_2 \models \psi \rightarrow \rho$ and $\mathscr{N}_2 \models \psi$. By modus ponens, $\mathscr{N}_2 \models \rho$. Thus $\mathscr{N}_1 \models \mathrm{K}\rho$, as desired.

Case 3 $\varphi \in \mathbf{KK}$. Then φ has the form $\mathrm{K}\psi \rightarrow \mathrm{KK}\psi$. Assume $\mathscr{N}_1 \models \mathrm{K}\psi$. Then $\mathscr{N}_2 \models \psi$, so $\mathscr{N}_2 \models \mathrm{K}\psi$, whence $\mathscr{N}_1 \models \mathrm{KK}\psi$, as desired.

Case 4 φ is p. Then $\mathcal{N}_1 \models \varphi$ by construction. $\square$

Proposition 14 can be used to establish an immediate corollary of Theorem 21.

Corollary 22. *The theories* $\mathbf{V} \cup \mathbf{KK}$ *and* $\mathbf{K} \cup \mathbf{KK}$ *fail to be generic.* ■

The following corollary follows from Theorem 21 and Lemma 15.

Corollary 23. *Not every closed generic theory is generic.* ■

Theorem 24. *If* $\mathbf{V} \cup \mathbf{KK}$ *is closed generic, then there is at most one propositional atom.*

The preceding theorem, like the one stated next, is proven in Appendix A.

Theorem 25. *The theory* $\mathbf{K} \cup \mathbf{KK}$ *is not closed generic.* ■

The following corollary follows by Proposition 14.

Corollary 26. *The theory* $\mathbf{KK}$ *is not closed generic.* ■

The proof of Theorem 21 illustrates a technique common to all proofs appearing in Appendix A for the results stated in this section — each argument proceeds by constructing pathological models. Investigating negative results about genericness and closed genericness using this technique locates sharp edges at the boundaries of modal logic: we are led to consider models where common assumptions no longer hold, such as models where $\mathbf{K}$ fails or where $\mathbf{V}$ fails.

We have applied the theory of genericity to the Knower Paradox. In the proofs of the following theorems, we will reverse the direction of application, applying the Knower Paradox to the theory of genericity, rather than vice versa.

Theorem 27. *The theory* $\mathbf{T}$ *is not closed generic. In fact, no superset of* $\mathbf{T}$ *is closed generic.* ■

Proof. Assume $\mathbf{T}^+ \supseteq \mathbf{T}$ is closed generic. By Lemma 14, $H = \mathbf{V} \cup \mathbf{K} \cup \mathbf{T}^+$ is closed generic. Let T_{KP} be as in Theorem 20. By Theorem 20, T_{KP} is consistent. But it is easy to see that T_{KP} is at least as strong as the theory of the same name from Theorem 10 (the Knower Paradox), which is *inconsistent*. Absurd. $\square$

In particular, $S4$ is not closed generic (and thus not generic), and the same goes for $S5$. The following theorem implies that the same also goes for $KD45$.

Theorem 28. *Let* $\mathbf{5}$ *be the schema consisting of all formulas of the form* $\neg\mathrm{K}\phi \rightarrow \mathrm{K}\neg\mathrm{K}\phi$*. No superset of* $\mathbf{5}$ *is closed generic.* ■

Proof. Similar to Theorem 27 by reformulating the Knower's Paradox using $\mathbf{5}$ instead of $\mathbf{T}$. $\square$

7 Discussion

There are different forms of genericity, two of which we have examined above: generic theories and closed generic theories. These forms are particularly nice because of closure under union (Proposition 14 parts 1–2) and because they are simple enough that we can prove some results about them.

In future work, we intend to use closed generic theories to generalize Carlson's consistency result [7] (this is almost already done in [5], but not quite, because the latter paper relies on an axiom called *assigned validity* to avoid some tricky nuances, whereas Carlson does not).

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A Proofs

Lemma 9. (Simulated Necessitation) *Let T be a closed theory. If T includes both* **V** *and* **K**, *then for every formula φ : if $T \models \varphi$, then $T \models \mathrm{K}\varphi$.*

Proof of Lemma 9. By Lemma 5, there are $\varphi_1, \ldots, \varphi_n \in T$ such that $\varphi_1 \to \cdots \to \varphi_n \to \varphi$ is valid. By **V**,

$$T \models \mathrm{K}(\varphi_1 \to \cdots \to \varphi_n \to \varphi).$$

By repeated applications of **K**,

$$T \models \mathrm{K}(\varphi_1 \to \cdots \to \varphi_n \to \varphi) \to \mathrm{K}\varphi_1 \to \cdots \to \mathrm{K}\varphi_n \to \mathrm{K}\varphi.$$

Since T contains each φ_i, the closure of T ensures T contains each $\mathrm{K}\varphi_i$. Thus $T \models \mathrm{K}\varphi$. □

Theorem 19. *Let p be a propositional atom, and let H be a generic theory. Let $(T_{KP})_0$ be the smallest closed theory containing:*

(i) H

(ii) $p \leftrightarrow \mathrm{K}\neg p$ *"This sentence is known to be false"*

In addition, let T_{KP} be the theory containing:

(a) $(T_{KP})_0$; and

(b) **T**.

For any set S of propositional atoms, if $p \notin S$ then $\mathscr{M}_{(T_{KP})_0,S} \models T_{KP}$. In particular, T_{KP} is consistent. ■

Proof of Theorem 19. Let $\varphi \in T_{KP}$, we must show $\mathscr{M}_{(T_{KP})_0,S} \models \varphi$. Consider four cases:

Case 1 $\varphi \in H$. Then $\mathscr{M}_{(T_{KP})_0,S} \models \varphi$ because $(T_{KP})_0 \supseteq H$ and H is generic.

Case 2 φ is $p \leftrightarrow \mathrm{K}\neg p$. Since $p \notin S$, $\mathscr{M}_{(T_{KP})_0,S} \not\models p$, thus it suffices to show $\mathscr{M}_{(T_{KP})_0,S} \not\models \mathrm{K}(\neg p)$. Let S' be a set of propositional atoms with $p \in S'$, and let T_∞ be the set of all formulas.

We claim $\mathscr{M}_{T_\infty,S'} \models (T_{KP})_0$. To see this, let $\psi \in (T_{KP})_0$, we must show $\mathscr{M}_{T_\infty,S'} \models \psi$. Three subcases are to be considered:

Subcase 1 $\psi \in H$. Then $\mathscr{M}_{T_\infty,S'} \models \psi$ because H is generic and $T_\infty \supseteq H$.

Subcase 2 ψ is $p \leftrightarrow \mathrm{K}\neg p$. Since $p \in S'$, $\mathscr{M}_{T_\infty,S'} \models p$. And since T_∞ contains all formulas, $T_\infty \models \neg p$, thus $\mathscr{M}_{T_\infty,S'} \models \mathrm{K}\neg p$. So $\mathscr{M}_{T_\infty,S'} \models \psi$.

Subcase 3 ψ is $\mathrm{K}\rho$ for some ρ such that $\rho \in (T_{KP})_0$. Since T_∞ contains all formulas, $T_\infty \models \rho$, so $\mathscr{M}_{T_\infty,S'} \models \mathrm{K}\rho$.

This shows $\mathscr{M}_{T_\infty,S'} \models (T_{KP})_0$. Now since $\mathscr{M}_{T_\infty,S'} \models (T_{KP})_0$ and $\mathscr{M}_{T_\infty,S'} \models p$, this shows $(T_{KP})_0 \not\models \neg p$. Thus $\mathscr{M}_{(T_{KP})_0,S} \not\models \mathrm{K}(\neg p)$, as desired.

Case 3 φ is $\mathrm{K}\psi$ for some ψ such that $\psi \in (T_{KP})_0$. Since $(T_{KP})_0 \models \psi$, by definition $\mathscr{M}_{(T_{KP})_0,S} \models \mathrm{K}\psi$.

Case 4 $\varphi \in T_{KP} \backslash (T_{KP})_0$. Then φ is an instance of $\mathbf{T}$, i.e., φ is $\mathrm{K}\psi \rightarrow \psi$ for some ψ. Assume $\mathscr{M}_{(T_{KP})_0,S} \models \mathrm{K}\psi$. Then $(T_{KP})_0 \models \psi$. By Cases 1–3, $\mathscr{M}_{(T_{KP})_0,S} \models (T_{KP})_0$. Thus $\mathscr{M}_{(T_{KP})_0,S} \models \psi$.

□

Definition 29. Given a formula φ, define $\mathrm{K}^n\varphi$ by recursion on $n \in \mathbb{N}$ by $\mathrm{K}^0\varphi = \varphi$ and $\mathrm{K}^{n+1}\varphi = \mathrm{K}\mathrm{K}^n\varphi$. ◂

Theorem 24. *If $\mathbf{V} \cup \mathbf{KK}$ is closed generic, then there is at most one propositional atom.*

Proof of Theorem 24. Let $T = \mathbf{V} \cup \mathbf{KK}$. Assume there exist distinct propositional atoms p and q. Let T' be the theory which contains:

- $K^n\varphi$ for all $n \in \mathbb{N}$ and all $\varphi \in \mathbf{V}$.
- $K^n\varphi$ for all $n \in \mathbb{N}$ and all $\varphi \in \mathbf{KK}$.
- $K^n(p \rightarrow q)$ for all $n \in \mathbb{N}$.
- $K^n p$ for all $n \in \mathbb{N}$.

Clearly T' is closed and $T' \supseteq T$. We will show $\mathscr{M}_{T',\emptyset} \not\models Kq \rightarrow KKq$, so $\mathscr{M}_{T',\emptyset} \not\models T$, so T is not closed generic. Since T' contains p and $p \rightarrow q$, by modus ponens $T' \models q$, so $\mathscr{M}_{T',\emptyset} \models Kq$. It remains only to show $\mathscr{M}_{T',\emptyset} \not\models KKq$, i.e., that $T' \not\models Kq$.

Define models $\mathscr{N}_1$ and $\mathscr{N}_2$ inductively so that:

- For every propositional atom a, $\mathscr{N}_1 \models a$.
- For every propositional atom a, $\mathscr{N}_2 \not\models a$.
- For every formula φ, $\mathscr{N}_2 \models K\varphi$ iff $\mathscr{N}_2 \models \varphi$.
- For every formula φ, $\mathscr{N}_1 \models K\varphi$ iff $\mathscr{N}_2 \models \varphi$ or φ is $K^n p$ for some $n \in \mathbb{N}$.

Since q is distinct from p and $\mathscr{N}_2 \not\models q$, we have $\mathscr{N}_1 \not\models Kq$. So to show $T' \not\models Kq$ (and thus finish the proof), it suffices to show $\mathscr{N}_1 \models T'$. Let $\varphi \in T'$.

Case 1: φ is $K^n\psi$ for some $n \in \mathbb{N}$ and some $\psi \in \mathbf{V}$. Then φ is $K^{n+1}\psi_0$ for some valid ψ_0. Since ψ_0 is valid, $\mathscr{N}_2 \models \psi_0$, and it follows that $\mathscr{N}_1 \models K^{n+1}\psi_0$.

Case 2: φ is $K^n\psi$ for some $n \in \mathbb{N}$ and some $\psi \in \mathbf{KK}$. Then φ is $K^n(K\rho \rightarrow KK\rho)$ for some ρ. To show $\mathscr{N}_1 \models \varphi$, it suffices to show $\mathscr{N}_2 \models K\rho \rightarrow KK\rho$. Assume $\mathscr{N}_2 \models K\rho$, then by definition $\mathscr{N}_2 \models KK\rho$, as desired.

Case 3: φ is $K^n(p \rightarrow q)$ for some $n \in \mathbb{N}$. Since $\mathscr{N}_2 \not\models p$, we have $\mathscr{N}_2 \models p \rightarrow q$, thus $\mathscr{N}_1 \models K^n(p \rightarrow q)$.

Case 4: φ is p. Then $\mathscr{N}_1 \models \varphi$ by definition.

Case 5: φ is $K^n p$ for some $n > 0$. Then $\mathscr{N}_1 \models \varphi$ by definition. □

Proof of Theorem 25. Let $T = \mathbf{K} \cup \mathbf{KK}$. Let T' be the theory consisting of:

- $K^n\varphi$ for all $n \in \mathbb{N}$ and all $\varphi \in \mathbf{K}$.
- $K^n\varphi$ for all $n \in \mathbb{N}$ and all $\varphi \in \mathbf{KK}$.

Clearly T' is closed and $T' \supseteq T$. Let p be a propositional atom. We will show $\mathscr{M}_{T',\emptyset} \not\models K(p \vee \neg p) \rightarrow KK(p \vee \neg p)$, showing $\mathscr{M}_{T',\emptyset} \not\models T$ and thus proving T is not closed generic. Clearly $T' \models p \vee \neg p$, thus $\mathscr{M}_{T',\emptyset} \models K(p \vee \neg p)$. It remains to show $\mathscr{M}_{T',\emptyset} \not\models KK(p \vee \neg p)$, i.e., that $T' \not\models K(p \vee \neg p)$.

Call a formula *bad* if it is either $p \vee \neg p$ or is of the form $\varphi_1 \rightarrow \cdots \rightarrow \varphi_n \rightarrow (p \vee \neg p)$. Let $\mathscr{N}$ be the model such that:

- For every propositional atom a, $\mathscr{N} \models a$.
- For every formula φ, $\mathscr{N} \models K\varphi$ iff φ is not bad.

Since $p \vee \neg p$ is bad, we have $\mathscr{N} \not\models K(p \vee \neg p)$. Thus to show $T' \not\models K(p \vee \neg p)$ (and thus finish the proof), it suffices to show $\mathscr{N} \models T'$. Let $\varphi \in T'$.

Case 1: $\varphi \in \mathbf{K}$. Then φ has the form $K(\psi \rightarrow \rho) \rightarrow K\psi \rightarrow K\rho$. Assume $\mathscr{N} \models K(\psi \rightarrow \rho)$ and $\mathscr{N} \models K\psi$. Then $\psi \rightarrow \rho$ is not bad. This implies ρ is not bad, thus $\mathscr{N} \models K\rho$, as desired.

Case 2: $\varphi \in \mathbf{KK}$. Then φ has the form $K\psi \rightarrow KK\psi$. Clearly $K\psi$ is not bad, thus $\mathscr{N} \models KK\psi$, thus $\mathscr{N} \models \varphi$.

Case 3: φ is of the form $K(K(\psi \rightarrow \rho) \rightarrow K\psi \rightarrow K\rho)$. Clearly $K(\psi \rightarrow \rho) \rightarrow K\psi \rightarrow K\rho$ is not bad, so $\mathscr{N} \models \varphi$.

Case 4: φ is of the form $K(K\psi \rightarrow KK\psi)$. Clearly $K\psi \rightarrow KK\psi$ is not bad, so $\mathscr{N} \models \varphi$.

Case 5: φ is $K^n\psi$ for some $\psi \in \mathbf{K} \cup \mathbf{KK}$ and some $n \geq 2$. Then φ has the form $KK\rho$ for some ρ. Clearly $K\rho$ is not bad, thus $\mathscr{N} \models KK\rho$. $\square$

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