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Decoherence and the Classical Limit of Quantum Mechanics

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Contents

Chapter 1

Introduction

According to the general wisdom there should not be any problem with the classical limit of quantum mechanics. After all, in any textbook of quantum mechanics one can easily find a section where the solution to this problem is explained (see e.g. [37]). Usually, Ehrenfest theorem, WKB approximation or simply the observation that the canonical commutation relations become Poisson brackets are presented as providing a definite answer to the problem of understanding how the classical laws of Newtonian mechanics emerge from the more basic laws of quantum mechanics. There is also an enormous amount of mathematical work, called semiclassical analysis or, in more modern terms, microlocal analysis (see, e.g., [31]), in which the limit $\hbar \to 0$ of Schrödinger evolutions is rigorously studied[1]. Moreover, in the recent years, books and articles are appearing claiming that "decoherence" is the key for understanding the appearance of the classical world. There are two lines of thinking: those who believe that decoherence is necessary to explain the emergence of the classical world (see [23] and references therein) and those who think that it is not only necessary but also sufficient (see, e.g., [43], [4], [34]).

From the above overview, one might easily get the impression [33], that it is only a matter of putting all known results into order for obtaining a clear and

[1]For nonspecialists we remark that the heart of semiclassical analysis is the stationary phase method. For Schrödinger evolutions of the usual kind this method is well developed while, for Hamiltonians which are not quadratic in the momentum operator but are more general expressions, it has been generalized to what is now called microlocal analysis.

rigorous derivation of classical mechanics from quantum mechanics.

Is this true?

1.1 What Converges to What?

What is the relevant physical quantity whose convergence "in the classical limit" asserts in a satisfactory way that the classical world arises?

To came to grips with this issue, it is convenient to distinguish the various related problems associated them with classical limit of quantum mechanics and to group them into "easy" and "hard" problems. The easy problems include explaining how quantum probability distributions, say of position, in a certain regime, are governed by the classical laws; how the statistics of quantum observables converge to the "Liouville-statistics" [2] of corresponding classical observables; how the asymptotic of energy eigenstates for high quantum numbers converges to the classical values; how, in an appropriate limit, the non commutative algebra of observables converges to the classical commutative one, etc.

The "hard" problem consists in explaining how the world of familiar experience, made of bodies with definite positions and velocities, moving along definite trajectories according to Newtonian laws, arises from the basic principles of quantum mechanics.

Classical physics is about real objects, like apples falling from trees, whose motion is governed by Newtonian laws.

Whatever the mathematical or physical $\hbar$-small arguments prove, they cannot prove, by themselves, that an apple falls to earth along its Newtonian path. While mathematical expressions can look classical, there is nothing like an apple and nothing from which the apple can be derived. It is important to appreciate that the solutions to the easy problems cannot provide, by themselves, a solution to the hard problem.

In order to predict trajectories, one would need quantum trajectories con-

[2] That is, one proves the convergence of the quantum distribution at a given time to the classical distribution at that time, where the time dependence is such, that it allows propagation by a classical flow. We shall comment on the status of such assertions which hold true in very general circumstances in more detail in sections 8 and in appendix 8.3.

Figure 1.1: Newton and the apple.

verging to classical trajectories, i.e. to trajectories obeying Newtonian laws. But what are quantum trajectories?

Consider, for example, the standard explanation of why the center of mass of a body x follows a classical trajectory. If the initial wave function $\psi_0(x)$ is a narrow wave packet, it remains concentrated along the classical trajectory. This can be shown by the Ehrenfest theorem (that we shall recall in section 5.3), which states that the time evolution of the mean values of position and momentum are given by

$$\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}, \quad \frac{d\langle p\rangle}{dt} = -\langle \nabla V(x)\rangle, \tag{1.1}$$

where $\langle\rangle$ denotes the average with respect to an initial wave function ψ_0. If one supposes that initially the wave function of the position x of the center of mass is a narrow wave packet, we have

$$\langle \nabla V(x)\rangle \simeq \nabla V(\langle x\rangle), \tag{1.2}$$

then

$$m\frac{d^2\langle x\rangle}{dt} \simeq -\nabla V(\langle x\rangle). \tag{1.3}$$

Thus the (average) position of the center of mass evolves (approximately) according to Newtonian law.

What's wrong with the above explanation?

First of all, a wave packet typically spreads, so that (1.1) will hold only up to a certain time (depending on the mass of the body) after which the classical approximation (1.3) will break down.

Actually, it can be shown that for very massive bodies (such as planets) the wave packet will remain concentrated for times much longer than the life of the universe.[3] But consider those macroscopic bodies subjected to non linear interaction leading to chaotic behavior (think e.g. of an asteroid subjected to strong gravitational perturbation [4]). Then, even if at a certain time the spread of the wave function is very small on the macroscopic scale, due to the sensitive dependence to initial conditions (positive Lyapunov exponent), we have that in a very short time the wave function spreads out. Moreover, we expect the classical limit to hold even for not very massive objects (think, e.g., of a particle's beam in an accelerator). In these cases we don't even need to invoke chaos. In fact, in a scattering experiment, a very localized initial wave function becomes very quickly a well delocalized wave function. Thus we can conclude that interactions of one particle with a very generic potential generate very spread out wave functions.

Secondly, why should the wave function of the position of the center of mass of the body be initially narrow? Why should the wave function of a macroscopic body, a many particle system, factorize into a product wave function of the center of mass and the relative coordinates?

Third, from the foregoing considerations about the Ehrenfest theorem, it seems that the classical behavior is something deeply connected to the fact that

[3] In fact we can give some estimate for the growth of σ in time considering for simplicity a free Gaussian wave packet

$$\sigma_t = \sigma_0 \sqrt{1 + \left(\frac{\hbar t}{2m\sigma_0^2}\right)^2} = \sigma_0 \sqrt{1 + \left(\frac{t}{\tau}\right)^2}, \tag{1.4}$$

where $\tau = \frac{2m\sigma_0^2}{\hbar}$. The spread will become greater that the initial spread σ_0 for times $t \gg \tau$. Suppose that the initial wave function of a medium–weight planet of mass, for example, $M \simeq 10^{27}$ Kg is a wave packet. Simple calculations show that it will remain narrow for a time much longer than the life of the Universe (10^{10} years almost). In fact, for an initial spread of the order of $\sigma_0 = 10^{-10}$ m, we get $\tau \simeq 10^{44} s$, which correspond to almost 10^{37} years. So it seems that Ehrenfest theorem is valid for *enough* time to ensure classical limit for planets.

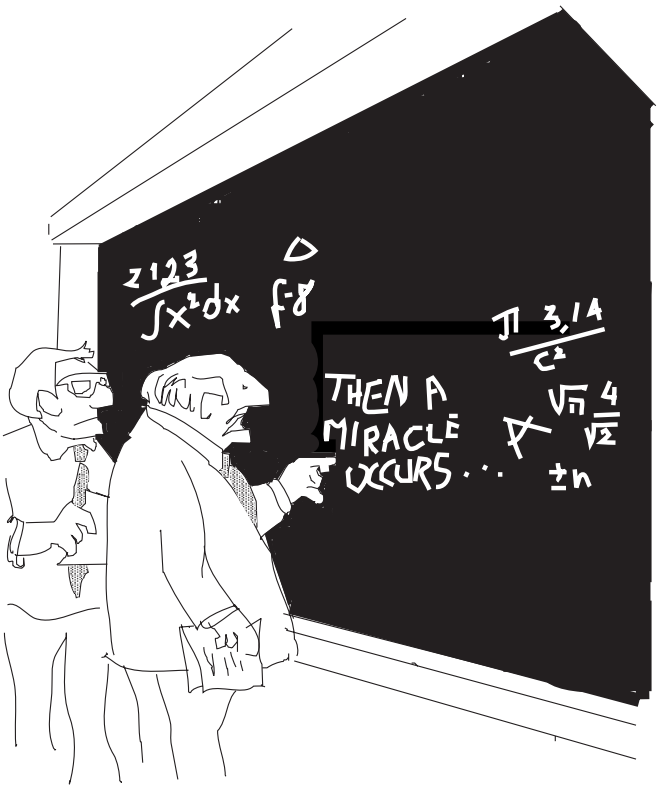

Figure 1.2: Then a miracle occurs

the wave function must be narrow at a certain time. But, for what we have just seen, Schrödinger's dynamics makes it a very transient feature, while we know from the world around us that the classical behavior is very stable. So, what's going on? How can we explain the trajectories of the classical world?

1.2 The Appeal to Decoherence

It has become quite common to believe that these objections are easily taken care of by recalling that no system is truly isolated but interacts with an environment which constantly "measures" the position of the particle and produces in this way a narrow wave packet (see [23], and references therein) an effect called nowadays *decoherence*.

Does taking into account the interaction with the environment really provide an answer to the problem of the emergence of the classical world without the

need of the collapse like it has been sometimes claimed? (Some examples of this belief can be found in [42], [4], [34] and [43].) The answer is no: in fact, by including the environment and considering the wave function evolving according to Schrödinger's equation as all there is in the theory, we simply shift the problem one level up. In fact the composite system formed by the subsystem of interest and by its environment is a closed system and Schrödinger's evolution of this system tends to produce spreading over configuration space. In this way the wave function of the composite system will typically evolve to a wave function supported by distinct and vastly different macroscopic configurations. This is the case of the Schrödinger cat in which Schrödinger's equation leads to the grotesque macroscopic superposition of a dead and an alive cat. Thus, decoherence is not sufficient to explain the emergence of the classical world from ordinary quantum mechanics: the collapse must be added to the usual axioms as an additional rule.

1.3 Beyond Standard Quantum Mechanics

However, as is well known and as Bell has often emphasized [3], Schrödinger's evolution correlated with the collapse is not a precise microscopic theory if the division between microscopic and macroscopic world (where the collapse has to take place) is not part of the theory. Moreover, coming back to our initial question (how can we derive classical mechanics, a theory formulated in terms of particles and their trajectories, in a framework in which they don't exist?), the following conclusion seems inevitable: quantum mechanics does not contain the means for describing the classical world in any approximate sense and one needs to go beyond quantum mechanics. There are only two possibilities for mending ordinary quantum mechanics: either the wave function is not all there is, or, Schrödinger's equation is wrong (for a proposal of modification of Schrödinger's equation, see [22]). In this thesis we'll formulate the problem of the classical limit within the framework of Bohmian mechanics, a theory in which the observer doesn't play any crucial role. It is a theory about reality, not about the result of measurements: it is about point particles which evolve according to a dynamical law generated by a function, the wave function ψ, which follows Schrödinger's equation (for the original article see [6] and for further development, see, e.g., [15],[13],[24], [12] and

[25]. For a complete review, see [11], unfortunately in german). Both Bohmian mechanics and classical mechanics are theories about the motion of particles. Without that, to answer the question of how the classical world can be part of the quantum world becomes a rather formidable task. Remember, in fact, that in quantum mechanics only the wave function exists: particles and trajectories do not. Therefore they must be deduced in some suitable sense in some suitable limit to obtain classical mechanics as a limit. The question *when have we got the classical limit?* in Bohmian terms becomes very simple:*when do the Bohmian trajectories converge to Newtonian ones?*

From the above discussion, one might now come to believe that (even in the framework of Bohmian mechanics) to solve the problem of the classical limit one just needs to rephrase what there is in the physics and mathematics literature to get the convergence of the Bohmian trajectories to classical ones. But this is still far from being true, and these other reasons we shall explain next.

1.4 Classical Limit as a Scaling Limit

Since $\hbar$ is known to be nonzero, the limit $\hbar \to 0$ must be understood as taking a scaling limit in which the relevant "action" of the problem is much bigger than $\hbar$. In other words, the classical limit should be understood in terms of an dimensionless parameter, that we shall call ϵ, which will be a combination of definite relevant physical quantities such that the classical laws emerge whenever ϵ is small. This small parameter is not directly given but must be extracted from the physics of the situation (see e.g. [29]). The common understanding is that $\epsilon = \frac{\lambda}{L}$, where λ is the "wave length of the particle" and L is some relevant "macroscopic length scale". This is very reminiscent of how geometric optics can be deduced from wave optics and thus seems reasonable, but to really pinpoint the λ and the L in a given physical situation is not an easy task. While in some situations it may be more or less clear what is meant by the "wave length of the particle" and the "macroscopic length scale" L, in general, however, the wave length depends on the wave function ψ, that is $\lambda = \lambda(\psi)$, and L is some length scale defined by the potential V, that is $L = L(V)$.

The dimensionless parameter ϵ may thus depend in general on a combination

of various quantities $x, y, z, ...$, which relate to the potential and to the wave function. Thus $\epsilon(x, y, z...) \to 0$ can be taken along different paths (different sequences) in the $x, y, z...$ space. Therefore, it turns out that the limit $\hbar \to 0$ is only a special case of the limit $\epsilon \to 0$: it is only one special sequence and it is not able to cover the complexity of the classical world as a whole. It is however natural to conjecture quite generally that classical physics applies on appropriate space and time scales whenever $\epsilon \to 0$, that is, we put forward that a classical limit on appropriate time and length scales (depending on ϵ) arises *uniformly* as long as $\frac{\lambda}{L} \to 0$. To show this is a mathematically hard problem, as we shall explain in section 3.2. How can one actually be sure that the extracted ϵ is small enough, i.e. that the physical situation is close to the *limit* $\epsilon = 0$? In truth ϵ is just some constant (or some function on the macroscopic scale) which is not equal to zero, but small, and the question is: is it small enough for classical laws to apply to the motion of the particles? This is a very hard problem, probably the hardest of them all.

We note by the way that this situation has an analogue in statistical physics. Consider for example in kinetic gas theory in which one for example "derives" Boltzmann's equation in some scaling limit, like the Boltzmann-Grad-limit for hard spheres [38]. The limit is of course only taken for definiteness, and it is hoped that the real situation is well approximated by the limit. In which sense it is so is again a very hard technical question, unsolved in general. Moreover, in realistic physical situations, the gas molecules do not only interact via elastic collisions (so that not only diameter, mean free path, density are the obvious physical scales) but also through potentials, which makes the findings of the right scales more involved.

We shall find that the crucial feature of the classical limit is the production of a "local plane wave", a wave which may be naturally partioned into wave packets each of which guides the Bohmian trajectory along a classical path. Thus for our question of having the Bohmian trajectory converge to a Newtonian one, it is sufficient to follow the evolution of the actual guiding packet, the evolution of which may be subjected to Ehrenfest's theorem. The λ will in some natural way be associated with such local plane waves and we suggest further that L can be extracted from a condition arising from Ehrenfest's theorem.

1.5 Organization of the Thesis

We shall start with a chapter introducing Bohmian mechanics, trying to explain what has been the reason to formulate such a theory and how it can be of help in solving in general the problems arising in standard quantum mechanics and in particular those associated with the classical limit.

In the third chapter we shall discuss what we mean by the classical limit and we shall analyze a very simple model of a system composed by several macroscopic bodies composed of N point–like particles in an external potential. Our basic conjecture is that we have classical behavior, on suitable macroscopic scales, provided that the dimensionless parameter ϵ defined as the ratio of the two relevant length scales of the situation, $\epsilon = \frac{\lambda}{L}$, is small, i.e. $\epsilon \ll 1$. The macroscopic scales on which the motion is classical are defined by λ and L. The first length scale λ, as we already anticipated, is roughly the mean de Broglie wave length derived by the mean kinetic energy or, more precisely, it can be defined as a function of the initial wave function of the particle. The other relevant scale L is the scale on which the potential varies that we will define according to an argument based upon the Ehrenfest theorem.

We shall show, in the fourth chapter, the emergence of the classical limit in the case of special families of initial wave functions and potentials. We will see how, in these cases, the dynamics tends to create a particular form of wave function, a local plane wave, whose formation we believe is crucial for the emergence of the classical world.

In the fifth chapter we will describe how this happens and we will arrive at the conclusion that on the macroscopic scales in which the motion is classical, provided that $\lambda \ll L$, the local plane wave forms. The fundamental feature of the local plane wave is that it can be thought of as a sum of non-interacting "virtual" wave packets with definite local wave length, such that only one of them is relevant for the dynamics of the trajectory of a given particle and we can forget the effects of the rest of the wave function.

As soon as we have a potential, caustics appear: they are associated with configurations for which the velocity field is mutivalued. In the sixth chapter we shall see what is the role of the external environment (the decoherence) in

suppressing interference produced by the presence of caustics.

Finally, in the seventh chapter, we shall analyze what happens if the approximation of motion in an external potential is abandoned. We will show that the general structure of the classical limit for a more realistic scenario in which the environment and the internal degrees of freedom of a macroscopic body are taken into account is again the emergence of local plane wave structure, as it has been established in the external potential approximation of chapter 3. This ensures that the classical world is something robust in structure, as we expected.

The last chapter is dedicated to comparing our work with what is actually present in the literature about the classical limit in ordinary quantum mechanics.

There are two appendices containing more technical material which would have distracted from the line of reasoning followed in the thesis but that can be useful to have at hand.

1.6 Acknowledgments

It is very difficult to really give merit to all the people who helped me during the period of my Ph.D. School, so I would like to be as short as I can and I hope that those I forget will not be upset with me. First of all, I am grateful to Prof. Nino Zanghí, my advisor, without whom this thesis would not exist. I am grateful also to Prof. Sheldon Goldstein, from Rutgers University, for his enlightening suggestions and to Prof. Detlef Dürr and to Dott. Stefan Teufel, from LMU, for all the helpful discussions we had in Munich and without which I would have been lost. Thanks al lot also to James Taylor for his very careful reading and for his precise an detailed comments. I also would like to say thank you to Prof. Renzo Collina, who shares the same office with Nino, to have so patiently accepted my obsessive presence in his office for such a long time. Most of all, I am grateful to Davide who convinced me so strongly to continue to study physics.

Chapter 2

A Precise Microscopic Theory: Bohmian Mechanics

Bohmian mechanics is a "quantum" theory with a clear ontology. To make clear what we mean by this, we shall proceed recalling what are the problems of quantum mechanics and we shall see how Bohmian mechanics can overcome them.

2.1 What is Quantum Mechanics About?

The basic problem of quantum mechanics is that it is not clear what is it about, as already stressed by Bell [3] and Goldstein [24]. Already Scrödinger pointed out, with his famous cat paradox [36], that the state of a system cannot be described only by the behaviour of the wave function: if the quantum phenomena are described by quantum mechanics (Schrödinger's equation of the wave function), the passage from the microscopic to the macroscopic world leads to paradoxical conclusions like, for example, a superposition between a dead and an alive cat.

It is useful to recall the Schrödinger's cat paradox. Bell [3] has rephrased this mental experiment in a less cruel way as follows: consider a cat in a perfectly isolated room. Together with the cat, the experimenter has put in the room a radioactive source and a complicated mechanism. If a radioactive nucleus decays, the mechanism opens a source of milk such that it fills a cup and the cat can drink. The room has no window so that what happens inside is completely obscure to the experimenter: she doesn't know if the cat is still hungry or if she enjoyed her

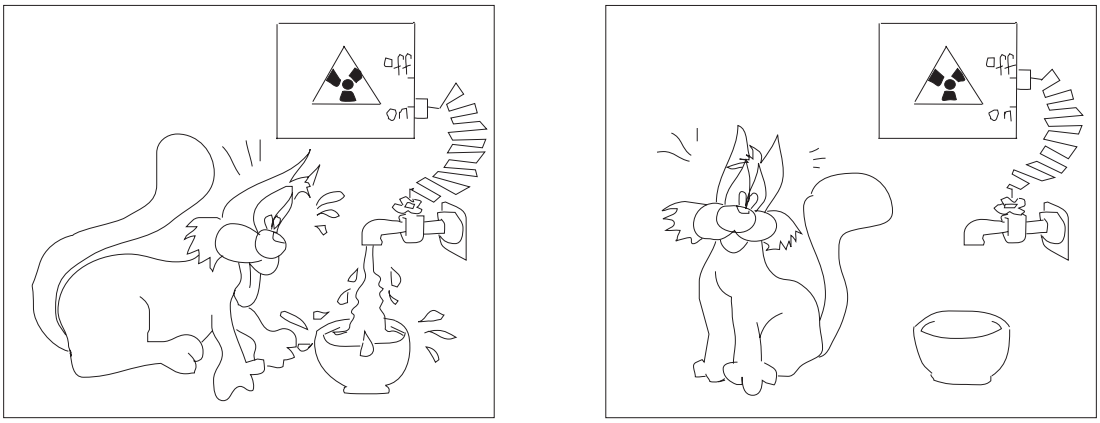

Figure 2.1: The Schrödinger's cat

meal. In this way the radioactive decay, a microscopic event, influences directly a macroscopic event, like the presence or not of some milk molecules in the stomach of the cat.

From the mathematical rules of quantum mechanics it follows that, given that the wave function of the radioactive nucleus is in a superposition of decayed–non decayed wave function, the cat is neither hungry nor filled up but she is a superposition of both states. From ordinary experience, we know that a macroscopic object cannot be in such a superposition of states with macroscopically disjoint supports, so somewhere, somehow, quantum mechanics gives the wrong answer. Note that, if the experimenter opens the door of the room, she finds out that the cat is always in one or in the other state. As a consequence of observation (measurement), the wave function has collapsed into one of the two possibilities.

From the Schrödinger's cat paradox we can conclude that it is not clear what is the role of the observer (and also who can be regarded as an observer, i.e., someone that is able to reduce the superposition wave function). Moreover, it is also obscure where to put the border between the microscopic world, in which superpositions can exist, and the macroscopic world, in which they cannot. From these considerations Bell has drawn the conclusion that we have only two possibilities. Either we add something to the wave function for the description of the state of the system or we modify Schrödinger's equation. Bohmian mechanics is a theory that follows the first direction.

2.2 The State of a System and The Dynamical Laws

It could be useful to recall that the first step in the construction of a physical theory is to establish what are the mathematical entities (points, fields, strings, membranes, and so on) with which one intends to describe the physical reality. These mathematical entities are what the theory is about and they are often called the ontology of the theory (a rather complicated word with a deep and simple physical meaning).

In Bohmian mechanics the world is described by point–like particles which follow trajectories determined by a law of motion. The evolution of the configurations of these particles is guided by the wave function which iteslf evolves according to Schrödinger's equation. In other words, in Bohmian mechanics the complete description of the state of an N point–like particle system is the couple (Ψ_t, Q_t), where $\Psi_t = \Psi_t(q)$ is the wave function of the system and $Q_t = (Q_1(t), ..., Q_N(t))$ is a point in the configuration space $\mathbb{R}^{3N}$. Each $Q_k(t)$ is the position of the k-th particle at time t in ordinary space $\mathbb{R}^3$. This is a very big difference with ordinary quantum mechanics in which the state of the same system is given only by the wave function and there is no position and no trajectory whatsoever.

We can think of Bohmian mechanics as a dynamical system and from this point of view we can compare it with classical mechanics. We all know that in Newtonian mechanics the dynamics of the point particles is determined by a *second* order differential equation

$$\ddot{Q}_t = \frac{1}{m} F(Q_t), \tag{2.1}$$

in which $F(Q)$ is a force field that in some cases can be derived from a potential ϕ as $F(Q) = -\nabla\phi$. In Bohmian mechanics the dynamics is given by a *first* order differential equation

$$\frac{dQ_t}{dt} = v^{\Psi}(Q_t), \tag{2.2}$$

where $v^{\Psi} = (v_1^{\Psi}, ..., v_N^{\Psi})$ is a velocity field on the configuration space. This field is generated by the wave function Ψ which itself evolves according to Schrödinger's

equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi\,, \tag{2.3}$$

where

$$H = -\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\nabla_k^2 + V$$

is the Hamiltonian.

The velocity field is determined by reasons of simplicity and symmetry (see, e.g., [15]):

$$v_k^{\Psi} = \frac{\hbar}{m_k}\mathrm{Im}\left[\frac{\nabla_k\Psi}{\Psi}\right]. \tag{2.4}$$

The factor $\frac{\hbar}{m}$ comes from the requirement of Galilei invariance, the imaginary part is a consequence of invariance for time reversal, the gradient is from rotational invariance and the fact that one has to divide for Ψ derives from the homogeneity of degree zero of the velocity field. If we have a magnetic field we should take ∇_k as the covariant derivative. If the wave function is a spinor, we should rewrite the velocity field as

$$v_k^{\Psi} = \frac{\hbar}{m_k}\mathrm{Im}\left[\frac{\Psi^*\nabla_k\Psi}{\Psi^*\Psi}\right], \tag{2.5}$$

when now in the numerator and denominator appears the scalar product in the spinor space. The global existence of Bohmian dynamics has been proven with full mathematical rigor in [7] where it has been shown that for a large class of Schrödinger Hamiltonians, including Coulomb with arbitrary charges and masses, and sufficiently regular initial datum ψ_0 of (2.3) the solution of (2.2) exists uniquely and globally in time for $|\psi_0|^2$-almost all initial configurations Q_0.

Equations (2.3) and (2.2) form a complete specification of the theory. What we have in Bohmian mechanics is a dynamical system for the dynamical variables (Ψ, Q). Without any other axiom, all the results obtained in the framework of non relativistic quantum mechanics follows from the analysis of the dynamics.

2.3 Bohmian Mechanics and Newtonian Mechanics

Note that the analogy we have done between Newtonian mechanics and Bohmian mechanics doesn't mean that Bohmian mechanics is a way of rephrasing quantum

mechanics in classical terms.

First of all, Bohmian mechanics is a first order theory while Newtonian mechanics is a second order one. This means that in Bohmian mechanics, given the position at a time t, we can compute the trajectory. In classical mechanics, on the contrary, we need both the velocity and the position to specify the motion. A consequence of this facts is that Bohmian mechanics trajectories cannot cross *in configuration space*, while there is no such restriction in Newtonian mechanics.

The only similarity between classical and Bohmian mechanics is that they are dynamical systems dealing with point particles. In the original work by Bohm [6] he tries to rewrite quantum mechanics as a second order theory. But this attempt may be misleading in the same sense as it would be rewriting Newton's equation as a third order equation. (See section 2.7 for further comments on that.) Note that the configurations Q of the particles composing the system are the primary objects of the theory, while the wave function is only a derivative concept.

2.4 Nonlocality and Hidden Variables

In the literature it is common to refer to Bohmian mechanics as a theory of *hidden variables*. This is a consequence of the famous EPR paper [19] in which Einstein, Podolsky and Rosen argued that quantum mechanics might be incomplete. Their proposal was to look for some non measurable variables (somehow hidden) to complete the theory.

It should be stressed that the problem faced by Einstein, Podolsky and Rosen in their paper was about the locality of quantum theory: they assumed implicitly that reality is local, i.e. action at a distance is impossible, and proposed a mental experiment (that we shall not recall here). Their conclusions were that, if reality is local, quantum mechanics is incomplete and there is need of some extra variables to take this into account. From the violation of Bell's inequality (see [3], [2]) it followed that their assumption was wrong: reality *is* non local and therefore from their reasoning we cannot conclude anything concerning the existence of hidden variables.

We should enphatize that the reason for introducing the configuration of the particles as an extra variable in quantum mechanics has nothing to do with

nonlocality. This has created and indeed still creates a lot of confusion in understanding which are the consequences of the violation of Bell's inequality –that reality is nonlocal and that any completion of quantum mechanics with *local* hidden variable is impossible. This is not the case of Bohmian mechanics, in which nonlocality follows directly from the fact that the wave function is a function in configuration space, not in ordinary space. This means that the velocity of each particle of a system composed of N particles, independently on how far are they. The degree of *action at distance* depends on the degree of entanglement.

It is interesting to note, as a side remark, that the true "hidden" variable is actually the wave function. In fact, it is not stressed sufficiently that it is indeed the wave function that cannot be measured. This can be seen as follows. A completely general experiment can be described by:

- a unitary map U transforming the initial state of system and apparatus $\psi_0(x) \otimes \phi_0(y)$ into a final state $\Psi(x, y) = U\left(\psi_0(x) \otimes \phi_0(y)\right)$ (x refers to the configurations of the system and y those of the apparatus);

- a pointer function $Z = F(y)$ representing the pointer orientation in terms of the microscopic configuratins y of the apparatus.

It can be shown that the probability distribution of Z must be a quadratic function of ψ_0[1] . (This is a direct consequence of quantum equilibrium and linearity of Schrödinger's equation, see [17]). If the wave function were measurable, the statistics of the pointer measuring the wave function would be formally given by

$$\mu^Z(d\psi) = \delta(\psi - \psi_0)d\psi, \tag{2.6}$$

which, however, is not a quadratic function of ψ_0 and thus the wave function is not measurable.

[1]Measured–valued quadratic forms on the Hilbert space of wave function are mathematically equivalent to positive operator–valued measures (POVM) and self–adjoint operators (which are, by the spectral theorem, in one to one corespondence with projector–valued measures) are therefore a particular class of POVM. In physical terms: general quantum observables are described by POVM and self-adjoint operators as observables are a very idealized notion coming from the special class of repeatable measurements [17].

2.5 The Quantum Equilibrium Hypothesis and the Experimental Predictions

Bohmian mechanics makes the same predictions as does non relativistic ordinary quantum mechanics for the results of any experiment, provided that we assume a random distribution for the configuration of the system and the apparatus at the beginning of the experiment given by $\rho(q,t) = |\Psi(q,t)|^2$. In fact, consider the quantum continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} J^{\Psi} = 0, \tag{2.7}$$

which is, by itself, a simple consequence of Schrödinger's equation. Here $J^{\Psi} = (J_1^{\Psi}, \ldots, J_N^{\Psi})$ is the quantum probability current

$$J_k^{\Psi} = \frac{\hbar}{m_k} \operatorname{Im} [\Psi^* \nabla_k \Psi] = |\Psi|^2 v_k^{\Psi}. \tag{2.8}$$

Equation (2.7) becomes the classical continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho\, v^{\Psi} = 0 \tag{2.9}$$

for the system $dQ_t/dt = v^{\Psi}$ and it governs the evolution of the probability density ρ under the motion defined by the guiding equation (2.2) for the particular choice $\rho = |\Psi|^2$. In other words, if the probability density for the configuration satisfies $\rho(q,t_0) = |\Psi(q,t_0)|^2$ at some time t_0, then the density to which this is carried by the motion (2.7) at any time t is also given by $\rho(q,t) = |\Psi(q,t)|^2$. This is an extremely important property of any Bohmian system. In fact it expresses a compatibility between the two equations of motion defining the dynamics, which we call the *equivariance* of $|\Psi|^2$.

The above assumption, which guarantees agreement between Bohmian mechanics and quantum mechanics regarding the results of any experiment, is what we call the *quantum equilibrium hypothesis*: when a system has a wave function Ψ, its configuration Q is random with probability distribution given by the measure (for a complete discussion of this, see [13])

$$I\!P^{\Psi}(q) \equiv \rho_{\Psi}(q)(dq) = |\Psi(q)|^2 dq. \tag{2.10}$$

It has sometimes been claimed that it is possible to experimentally discriminate between Bohmian mechanics and quantum mechanics. This claim is however totally unfounded: what we have just shown is that there *must* be experimental agreement as a consequence of quantum equilibrium.[2]

The experimental equivalence of Bohmian mechanics with quantum mechanics might appear, somehow, as a little frustrating fact. While, on the one hand, all the experimental evidence confirms Bohmian mechanics as well as quantum mechanics, on the other hand it would be easier if the experimental predictions were different. In fact, if there was a crucial experiment able to discriminate between the two theories, there would be something *objective* to establish which is the correct theory. We should make clear, however, that the experimental equivalence of Bohmian mechanics with quantum mechanics holds as long as the predictions of quantum mechanics are not ambiguous. There are in fact a variety of experimental issues that don't fit comfortably within the standard operator quantum formalism, such as dwell and tunneling times [30], escape times and escape positions [10], scattering theory [16], but are easily handled by Bohmian mechanics.

Actually, after the discussion of the previous sections, it should be clear that the comparison should not be made only on the level of experimental prediction but, on the contrary, the decision of what is the right theory should be taken on the deeper level of the ontology of the theory.

2.6 The Wave Function of a Subsystem and the Collapse

The existence of configurations in Bohmian mechanics allows for a natural and clear notion for the definition of the wave function of a subsystem. In fact, consider a complex system composed of a sub-system and by its environment. If

[2]Note that the quantum equilibrium hypothesis is a physical condition which might not be satisfied in our world. However, this would imply an experimental violation of the quantum mechanical predictions, which, given all the experimental evidence collected so far, is not the case. Indeed, all the experimental verifications of quantum mechanics give support to the quantum equilibrium hypothesis.

$Q_t = (X_t, Y_t)$, where X_t is the actual (i.e. what really is) configuration of the sub-system and Y_t the one of its environment, we can define the *conditional* wave function for the x-system at time t as follows

$$\psi_t(x) = \Psi_t(x, Y_t), \tag{2.11}$$

that is, the wave function of the whole universe (the biggest system of all) Ψ_t calculated in the actual configuration of the environment. Under appropriate conditions, on which we'll focus later (see chapter 7), $\psi_t(x)$ satisfies Schrödinger's equation in x. In this case it is indeed the *effective* wave function for the x-system, that is, the collapsed wave function that the ordinary quantum formalism assigns to the subsystem after a quantum measurement. In fact, suppose Ψ has the structure occurring in a measurement situation

$$\Psi_t(x, y) = \psi_t(x)\phi_t(y) + \Psi_t^{\perp}(x, y), \tag{2.12}$$

where $\phi_t(y)$ and $\Psi_t^{\perp}(x, y)$ (the part of Ψ_t which is not $\psi_t(x)\phi_t(y)$) have macroscopically disjoint y-supports. If Y_t belongs to the support of $\phi_t(y)$, $\psi_t(x)$ is the effective wave function of the x-system at time t. (For a clear exposition of this, see [15] or [13].)

Basically, from the above discussion the collapse of the wave function can be deduced from Bohmian mechanics without introducing any active role to the observer. Consider, again, the cat paradox in the original version, were the two superposing states are dead and alive cat. In Bohmian mechanics at any time t the cat is something real, she is dead or alive, independently of who is looking at her. Note that she *can* be in a superposition state because the wave function evolves according to Schrödinger's equation, but in Bohmian mechanics the state of the system is given by the couple (Ψ, Q) of the wave function and the configurations $Q = (q_1, ..., q_n)$ of all the particles composing the system (the cat). Thus, according to which support Q belongs to (to those of the wave function Ψ_{dead} describing the dead cat or to those of the wave function Ψ_{alive} describing the alive cat), the cat is actually dead or alive. Note that superpositions exist on all scales (from micro to macro) but don't influence at all the fact that the cat is this or that. At this point a question could arise: due to the presence of a superposition wave function, could it be possible that the cat, who at some time is dead, returns

alive? The cat has an actual configuration, belonging (in our example) to the support of Ψ_{dead}, and its evolution is guided by the wave function. There seems to be nothing to prevent Q from being guided to the support of Ψ_{alive}, making the dead–alive transition possible. Actually, this is very unlikeky to happen since the supports of the two wave functions are macroscopically distinguishable. By this we mean that the macroscopic variables, like, e.g., the temperature, assume different values in the two states, even if the microscopic quantities from which they have been derived might be similar. The temperature of a dead cat and of a live cat are, in general, different. Thus, if Q at some time belongs to the support of Ψ_{dead}, the effect of Ψ_{alive} is completely negligible: we can forget it for the dynamics of Q. The dead–alive transition could be possible if we could bring the two wave functions close to each other again. But the probability of having success in this would be even less probable than the fact that all the molecules of perfume we have sprayed in a room would come back spontaneously in the neighbourhood of the bottle: it is possible but extremely improbable.

2.7 Bohmian Mechanics and the Quantum Potential

To point out some interesting features of Bohmian mechanics, it can be useful to write the wave function Ψ in the polar form

$$\Psi = R\mathrm{e}^{\frac{i}{\hbar}S} \tag{2.13}$$

and then rewrite Schrödinger's equation in terms of these new variables. (This is what Bohm originally did in his 1952 paper [6].) In this way one obtains from (2.3) a pair of coupled equations: the continuity equation for R^2,

$$\frac{\partial R^2}{\partial t} + \mathrm{div}\left(\frac{\nabla_k S}{m}\right) R^2 = 0, \tag{2.14}$$

which suggests that $\rho = R^2$ can be interpreted as a probability density, and a modified Hamilton-Jacobi equation for S

$$\frac{\partial S}{\partial t} + \frac{(\nabla_k S)^2}{2m} + V - \sum_k \frac{\hbar^2}{2m_k}\frac{\nabla_k^2 R}{R} = 0 \tag{2.15}$$

Note that this equation differs from the usual classical Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{(\nabla_k S)^2}{2m} + V = 0 \tag{2.16}$$

only by the appearance of an extra term, the *quantum potential*

$$V_Q \equiv -\sum_k \frac{\hbar^2}{2m_k} \frac{\nabla_k^2 R}{R}. \tag{2.17}$$

This modified Hamilton-Jacobi equation can be used, together with the continuity equation for R, to define particle trajectories identifying the velocity with $v_k = \frac{\nabla_k S}{m}$. In this way the resulting motion is precisely what would have been obtained classically if the particles were subjected to the force generated by the quantum potential in addition to the usual forces.

As we have already anticipated, this rewriting of Schrödinger's equation through the polar variables (R, S) is somehow misleading. In fact, first of all, there is an increase in complexity: Schrödinger's equation is a linear equation while the modified Hamilton-Jacobi equation is highly nonlinear and still requires the continuity equation for its closure. Note that, since in Bohmian mechanics the dynamics is completely defined by Schrödinger's equation (2.3) and the guiding equation (2.2), there is no need of any further *axioms* involving the quantum potential and thus it should not be regarded as the most basic structure defining Bohmian mechanics.

Moreover, it is important to recall that Bohmian mechanics is not simply classical mechanics with an additional force term. In Bohmian mechanics the velocities are not independent of positions, as they are classically, but are constrained by the guiding equation (2.2). The correct way of regarding to Bohmian mechanics is as a first-order theory, in which the fundamental quantity is the velocity, which is specified directly and simply by the theory. In Bohmian mechanics the second-order (Newtonian) concepts of acceleration and force, work and energy play no fundamental role, and are emergent notions like the quantum potential. On the contrary, they are fundamental to the theory to which Bohmian mechanics converges in the classical limit, namely Newtonian mechanics. It might be objected that mass is also a second-order concept and it does play an important role in Bohmian mechanics. Note that the masses appear in

the basic equations only in the combination $m_k/\hbar \equiv \mu_k$. Thus equation (2.2) could more efficiently be written as

$$\frac{dQ_k}{dt} = \frac{1}{\mu_k}\mathrm{Im}\left[\frac{\psi^*\nabla_k\psi}{\psi^*\psi}\right]. \tag{2.18}$$

If we divide Schrödinger's equation by $\hbar$ we get

$$i\frac{\partial\psi}{\partial t} = -\sum_{k=1}^{N}\frac{1}{2\mu_k}\nabla_k^2\psi + \hat{V}\psi, \tag{2.19}$$

with $\hat{V} = V/\hbar$. Thus it seems more appropriate to regard the *naturalized* masses μ_k, rather than the original masses m_k, as the fundamental parameters of the theory. Note that if we naturalize also all other parameters, including also the electric charge $\hat{e} = e/\sqrt{\hbar}$, $\hbar$ disappears from this formulation. Planck's constant remains only in the equations $m_k = \hbar\mu_k$ and $e^2 = \hbar\hat{e}^2$ relating the masses and the charges in the microscopic units with those in the macroscopic scales [15].

In any case, regardless of whether or not we think of the quantum potential as fundamental, it can be useful. One sees from equation (2.15) that the quantum potential provides a rough measure of the deviation of Bohmian mechanics from its classical approximation. In fact the law of motion is

$$m\ddot{X}_t = -\nabla[V(X_t) + V_Q(X_t)]. \tag{2.20}$$

Equation (2.20) shows that all the *deviations* from classicality are embodied in the quantum force $F_Q = -\nabla V_Q$, so that, whenever F_Q is negligible, there is classical motion. In section 3.2 we'll see that our basic notion of deviation from classicality is simply a way to make clear and precise this concept.

Chapter 3

Classical Limit in Bohmian Mechanics

From the last remarks of the prevous chapter it follows that in the framework of Bohmian mechanics, the classical behavior of a body should emerge when the quantum potential is small. In this way it seems that the classical limit is something trivial: ensure the quantum potential is somehow small and then classical mechanics arises from Bohmian mechanics. What is not at all trivial is to understand what are the physical conditions corresponding to the smallness of the quantum potential. The goal of our work is indeed to show that there exists a well defined limit that, in turns, defines macroscopic scales in which the time evolution is approximately classical. With this we mean that the deviation from classicality (the quantum force) on those macroscopic scales in that limit is "almost" zero, as we shall make precise in section 3.2.

In the present chapter we shall discuss a simple model of a macroscopic body moving in an external potential and we shall see in which limit and on which macroscopic scales classical behavior should arise. In chapter 6 we shall study the general structure of the classical limit for more realistic models in which the approximation of motion in an external potential will be abandoned.

3.1 Motion in an External Potential

Consider a body composed of N particles, with positions $(x_1, ..., x_N)$, subjected only to internal forces, i.e. with Hamiltonian

$$H = \sum_i \frac{p_i^2}{2m_i} + \sum_{i>j} U(x_i, x_j).$$

The center of mass of the body is defined as

$$x = \frac{\sum_i m_i x_i}{\sum_i m_i} \tag{3.1}$$

and y_i is some suitable relative coordinate of the i-th particle. In the new coordinates $(x, y = (y_1, ..., y_{N-1}))$, the Hamiltonian factorizes as

$$H = H_{CM}^{(0)} + H_{rel},$$

where $H_{CM}^{(0)}$ is the kinetic energy of the center of mass. Note that the internal potential is only a function of the relative coordinates, $U(x, y_1, ..., y_{N-1}) = \sum_{i>j} W(y_i - y_j)$, and the total kinetic energy is given by the sum of the kinetic energy of the center of mass and the one of the relative coordinates, that is $KE = KE_{CM} + KE_{rel}$. (The first fact is true for any relative coordinate, not only for those with respect to the center of mass, while the second one is true only for those relative to the center of mass.) This means that the motion of the center of mass is not affected by internal forces, that is, it moves freely. Thus, if the total initial wave function of the total system factorizes as

$$\Psi_0(q) = \psi_0(x) \otimes \phi_0(y), \tag{3.2}$$

where $q = (x, y)$, ψ_0 is the wave function of the center of mass and ϕ_0 is the wave function of the internal degrees of freedom, then this product form will be preserved by the dynamics and the motion of the center of mass can be completely decoupled from the internal coordinates y.

If in addition to the internal force the particles interact are also subjected to an external potential, the potential energy in the Hamiltonian will be of the form

$$U = \sum_{i<j} U(x_i, x_j) + \sum_i V_i(x_i)\,.$$

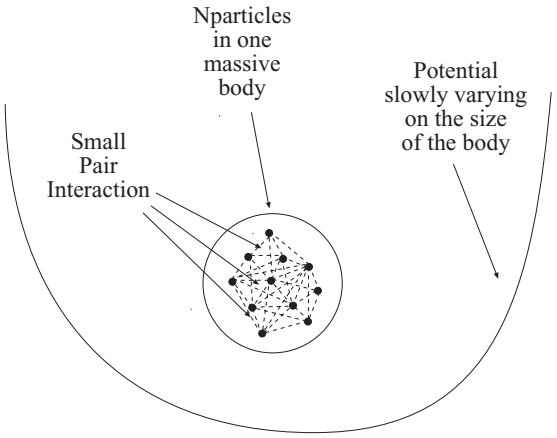

Figure 3.1: Motion in an external potential of N point–like particles composing a single massive body

Then under the change of variables $q = (x, y)$ the Hamiltonian assumes the form

$$H = \left(H^{(0)} + V\right)_{CM} + H_{rel} + H^I_{int}, \tag{3.3}$$

where $V_{CM} \equiv \sum_i V_i(x)$ and H^I_{int} is the interaction between the center of mass and the relative coordinates due to internal degrees of freedom. Thus the product form (3.2) is not preserved by the dynamics. If V_i are slowly varying on the size of the body, H^i_{int} can be treated as a small perturbation, and, in first approximation, neglected. Thus, if $\Psi = \psi(x) \otimes \phi(y)$, at some time, the time evolution of the center of mass decouples from that of the relative coordinates and we end up with a very simple one particle problem: the wave function ψ of the center of mass evolves according to one-particle Schrödinger's equation.

Note that we would have obtained the same conclusion considering several (M) macroscopic bodies composed of N particles subjected to a sufficiently slowly varying pair interaction U and to an external potential V, slowly varying in the size of each body. Assuming that the pair interaction is slowly varying is necessary to neglect any coupling between particles belonging to different bodies. An example of a system like this is the solar system.

Therefore, in this approximation, the product structure of the wave function is still preserved (assuming the product form at some "initial" time). In this way we end up with a very simple one body problem: the wave function $\psi(x)$ of the

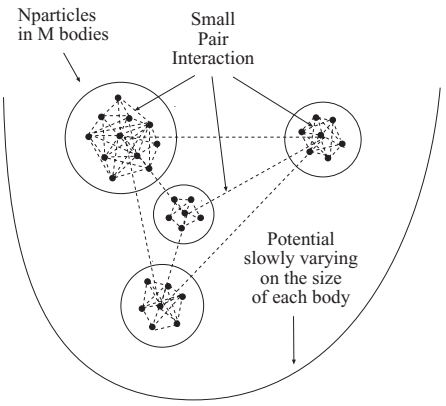

Figure 3.2: Motion in an external potential of N point–like particles composing M massive bodies

center of mass evolves according to

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = H\psi(x,t), \tag{3.4}$$

where

$$H = \left(H^{(0)} + V\right)_{CM}$$

is the Hamiltonian of the center of mass of the body and the time evolution of the center of mass of the body X_t is given by

$$\frac{dX_t}{dt} = \frac{\hbar}{m}\mathrm{Im}\left[\frac{\nabla\psi(X_t,t)}{\psi(X_t,t)}\right]. \tag{3.5}$$

We want to underline that this is an approximation which is allowed whenever the external potential V is slowly varying on the size of the body. We will see later in section 6 how all our analysis can be generalized when we don't neglect the interaction term and when the product structure of the wave function is no longer preserved.

3.2 Conjecture on Classicality

Usually physicists consider the limit $\hbar \to 0$, meaning by this $\hbar \ll A_0$, where A_0 is *some* characteristic action of the corresponding classical motion (see, e.g., [32],[37],[4]). In this regard one should observe that, while ψ doesn't have any

limit as $\hbar$ goes to zero, the couple (R, S) defined by the $\hbar$-dependent change of variables (2.13), does have a limit. Formally, this limit can be read by setting $\hbar \equiv 0$ in equations (2.14) and (2.15). In this way one has that the couple (R, S) becomes the pair (R^0, S^0), where R^0 satisfies the classical continuity equation and S^0 the classical Hamilton-Jacobi equation, i.e.

$$\frac{\partial R^{0\,2}}{\partial t} + \operatorname{div}\left(\frac{\nabla S^0}{m}\right) R^{0\,2} = 0, \tag{3.6}$$

$$\frac{\partial S^0}{\partial t} + \frac{(\nabla S^0)^2}{2m} + V = 0. \tag{3.7}$$

The condition $\hbar \ll A_0$ is often regarded as equivalent to another standard condition of classicality which involves the length scales of the motion (see, e.g., [29]): if the de Broglie wave length λ is small with respect to the characteristic dimension L determined by the scale of variation of the potential, the behavior of the system should be close to the classical behavior in the same potential. We regard this as the most natural condition of classicality since it relates in a completely transparent way a property of the state, namely its de Broglie wave length, and a property of the dynamics, namely the scale of variation of the potential. We believe that this condition should not depend too much on any detailed characterization of λ and L. In particular, the wave length λ could be taken as a function of the initial wave function ψ_0, i.e., $\lambda \equiv \lambda_{\psi_0}$. A rough estimate can be made in terms of the mean kinetic energy, that is[1]

$$\lambda_{\psi_0} = \frac{\hbar}{\sqrt{2mE_{kin,\psi_0}}} = \frac{\hbar}{\sqrt{2m\langle\psi_0, -\frac{\hbar^2}{2m}\nabla^2\psi_0\rangle}}{}^{2}. \tag{3.8}$$

We will see in section 5.4 that a suitable definition of L should be the following

$$L = \sqrt{\frac{V'}{V'''}}, \tag{3.9}$$

where V' and V''' are, respectively, the first and the third derivative of the potential V (see section 5.3 for more details).

[1] From now on, we shall denote by λ both the wave length and the wave length divided by 2π since the 2π-factor doesn't play any role in our analysis.

[2] Note that in formula (7) $\hbar$ and m cancel each other such that $\lambda = \frac{1}{\langle\psi_0, -\nabla^2\psi_0\rangle}$ but the h dependance is inside the wave function

But, regardless of the precise characterization of λ and L (a problem that we shall address later on), one should observe that they provide natural macroscopic scales for describing the motion. The macroscopic scales defined by λ and L are in fact

$$(x', t') = \left(\frac{x}{L}, \frac{t}{T}\right). \tag{3.10}$$

The time scale T is defined as $T = \frac{L}{v}$ and v is the speed defined by λ,

$$v = \frac{\hbar}{m\lambda}.$$

The scales L and T tell us what are the fundamental units of measure for the motion: L is the scale in which the potential varies and T is the time necessary for the particle to see its effects. We expect the Bohmian motion on these scales to look classical when the dimensionless parameter $\epsilon \equiv \frac{\lambda}{L}$ is getting smaller and smaller. This means in particular that we expect the quantum potential to become negligible (with respect to the kinetic energy) whenever $\epsilon \to 0$. Thus, it seems appropriate to formulate the emergence of classicality in terms of the *deviation from classicality*,

$$D = |\ddot{X}'_{t'} - \ddot{X}'^{0}_{t'}| = \frac{mL\lambda^2}{\hbar^2}\left|\nabla V_Q(X'_{t'}L, t'T)\right|, \tag{3.11}$$

expressing the quantum force $F_Q = -\nabla V_Q$, with V_Q defined by equation (2.17), on the macroscopic scales (3.10). $X'^{0}_{t'}$ and $X'_{t'}$ denote the classical and the Bohmian motion on the same scales[3]. (In the following, whenever there will be no ambiguity, we will replace the prime notation x' by x.)

Then a first rough notion of classical limit can be phrased as follows: $D \to 0$ on the macroscopic scales (3.10) whenever $\epsilon \to 0$. This condition involves however some subtleties which we shall now address.

First of all, note that the limits $\epsilon \to 0$ is much more general than merely requiring $\hbar \to 0$. In fact

$$\epsilon = \frac{\lambda}{L} = \frac{\hbar}{pL},$$

where $p = mv$ is the momentum. So, keeping fixed L and p, the limit $\hbar \to 0$ implies $\epsilon \to 0$. But there are several ways in which ϵ could go to zero according to

[3]Note that the classical trajectory is the one that passes through that point at that time, not the classical trajectory with same initial position and velocity as the Bohmian trajectory.

the combination of λ and L. In other words, the classical limit is a two-parameter limit, λ and L, and special cases like $\hbar \to 0$ cannot explain all the classical behavior. Note that the two parameters themselves live on infinite dimensional spaces since $\lambda = \lambda_{\psi_0}$, with ψ_0 varying in the Hilbert space of system's wave functions, and $L = L_V$, with V varying in the class of admissible one particle potentials (that is, potentials leading to a self-adjoint Hamiltonian).

Secondly, observe that demanding $D \to 0$ whenever $\epsilon \to 0$ is probably too strong a requirement. In fact, some special examples can be found in which, even if ϵ is small, D is not small. In appendix B we shall see, with the simple and completely controllable example of a free Gaussian wave packet, what is the behavior of the deviation from classicality D in the limit $\epsilon \to 0$ on the macroscopic scales as a function of all the parameters occurring in the problem. These parameters are in this example the mass m, the initial velocity u, the initial position spread σ_0 and the Planck's constant $\hbar$, then we have the macroscopic scales L_o and $T = \frac{mL_o\lambda}{\hbar}$. We'll see that there is no uniformity of the convergence. In fact there is a special combination of parameters for which D doesn't go to zero even if ϵ is small. This is connected to the fact that requiring that the deviation from classicality converges to zero for *any* initial condition is too much. In fact different initial conditions have different probabilities to happen. Therefore a conjecture on the emergence of the classical limit could be very generally stated as follows:

Let $\mathbb{P}^\psi$ be the probability distribution (2.10) on the space of initial conditions Ω, induced by the initial wave function ψ_0. D, given by (3.11), is a random variable on the probability space $(\Omega, \mathbb{P}^\psi)$. Then, for any $\delta > 0$, the probability of D being greater than δ goes to zero,

$$\mathbb{P}^\Psi(D > \delta) \to 0, \tag{3.12}$$

when the dimensionless parameter ϵ goes to zero,

$$\epsilon \equiv \epsilon(\psi_0, V) = \frac{\lambda_{\psi_0}}{L_V} \to 0,$$

uniformly in ψ_0 and V.

The above conjecture involves a notion of uniformity. By this we mean the following: let (V_n, ψ_n) be any sequence for which $\epsilon_n = \frac{\lambda_n}{L_n} \to 0$, where $\lambda_n = \lambda(\psi_n)$, e.g. with λ given by (7), and $L_n = L(V_n)$. Then the Bohmian motion on the macroscopic scales (3.10) defined by V_n and ψ_n is approximatively classical with deviation from classicality D tending to zero in probability as $n \to +\infty$. (We recall the definition of convergence in probability: let (X_n) be a sequence of random variables and let X be a random variable, we say that $X_n \to X$ in probability if for every $\eta > 0$,

$$P(|X_n - X| > \eta) \to 0$$

as $n \to +\infty$.) The conjecture can be equivalently formulated as follows: for any $\eta > 0$ and for any $\delta > 0$, there exists an $\epsilon_0 > 0$ such that the probability of D being greater than δ, $I\!P^\psi(D > \delta)$, is smaller than η, for any $\epsilon < \epsilon_0$.

Note that, though we have estimated λ by (7) and emphasized that it depends on ψ_0, λ in general could also depend on the initial condition x_0 and therefore be a random variable on $(\Omega, I\!P^\psi)$. Taking this into account requires a refinement of the above conjecture that we will address in section 5.6.

Moreover, note that if the potential is uniform or constant (free particle) one expects that its scale of variation is $L = +\infty$. Since for this latter case λ is always smaller than L, it is always possible to find a macroscopic scale in which the motion is classical. More precisely, if $\epsilon = 0$, our conjecture must be modified as follows: if $L = +\infty$, let L_o be *any* length scale chosen by the experimenter and T_o the corresponding time scale $T_o = \frac{L_o}{v}$. Then the motion is classical on the macroscopic scales $(\frac{x}{L_o}, \frac{t}{T_o})$, i.e. for any $\delta > 0$ the probability of D being greater than δ goes to zero, $I\!P^\Psi(D > \delta) \to 0$, when the dimensionless parameter ϵ goes to zero

$$\epsilon = \epsilon(\psi_0, L_o) = \frac{\lambda_{\psi_0}}{L_o} \to 0, \tag{3.13}$$

uniformity in ψ_0 and L_o.

We would like to underline that the above conjecture is really very hard to prove because it requires a lot of uniformity both in the initial wave function ψ_0 and in the potential V. Just to have an idea of the difficulty in proving it, one may think of the analogous problem in statistical mechanics, namely the

problem of studying the deviations from thermodynamic behavior of a large but finite system.

We should then observe that proving the conjecture is difficult but it is still not completely satisfactory from a physical point of view. In fact the conjecture states only that D depends on ϵ in such a way that $D \to 0$ as $\epsilon \to 0$ uniformly in all the other parameters appearing in the initial wave function and in the potential. The result that we would like to prove would involve estimating D not just showing that it goes to zero but how rapidly: like ϵ, like ϵ^2 and so on. Notice that this last result is the only one of practical value: given V and ψ_0 it provides an estimate for the deviation from classicality, while other results do not quite do this.

Finally, note that the conjecture (3.12) captures, of course, also the notion of classical limit for microscopic systems as an electron or an atom, whenever the approximation of motion in an external potential is appropriate.

To sum up, a mathematical derivation of the classical limit is a very difficult problem. One may hope to get a handle on it by separating it into a series of steps of growing difficulty:

1. Fix a particular sequence of initial wave functions and external potentials (ψ_0^λ, V^L) such that $\epsilon = \lambda/L \to 0$. For example, fix all the parameters but the scale of variation of the potential L which is going to infinity such that $\epsilon \simeq \frac{1}{L} \to 0$. Another example is to fix everything but λ which goes to zero such that $\epsilon \simeq \lambda \to 0$. Show that D goes to zero as ϵ gets small on the macroscopic scales (x', t').

2. Consider any sequence of initial wave function and external potential (ψ_0^λ, V^L) for which $\epsilon = \lambda/L \to 0$. Show that D is small with high probability as ϵ goes to zero on the macroscopic scales (x', t') uniformly in all the parameters (those characterizing the initial wave function ψ_0 and the macroscopic scales L and T).

3. For any sequence of initial wave function and external potential (ψ_0^λ, V^L) for which $\epsilon = \lambda/L \to 0$, show how the deviation from classicality D depends upon ϵ in the limit $\epsilon \to 0$.

In the next chapter we shall study two one-parameter families representing an example of the first step, that is two special cases of the limit $\epsilon \to 0$. The first is the usual limit $\hbar \to 0$, in which the mean de Broglie wave length λ of the initial wave function ψ_0 is going to zero, while the scale of variation of the potential is kept fixed. The second example is a limit connected to a slow variation of the potential. In this case, the initial wave function ψ_0 is fixed, while the scale of the variation of the potential L is going to infinity.

3.3 Closeness of Laws and Closeness of Trajectories

From a physical point of view, the main content of our conjecture is that, for $\epsilon \to 0$, the quantum laws of motion converge to the corresponding classical ones (on the appropriate scales described in the previous section). We have different possible notions of this kind of convergence. The most natural one is the convergence of modified Newton's equation to classical Newton's equation

$$m\ddot{x}_t = -\nabla \left[V(x_t) + V_Q(x_t)\right] \to m\ddot{x}_t = -\nabla V(x_t). \tag{3.14}$$

This happens when the quantum force $F_Q = -\nabla V_Q$ is going to zero. Equivalently, we may have convergence of the modified Hamilton-Jacobi equation to the classical Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + V_Q = 0 \to \tag{3.15}$$

$$\frac{\partial S^0}{\partial t} + \frac{(\nabla S^0)^2}{2m} + V = 0, \tag{3.16}$$

where S^0 is the classical action. This is a limit in which the quantum potential $V_Q = -\frac{\hbar^2}{m}\frac{\nabla R^2}{R}$ is going to zero. Finally, one may have the convergence of the quantum velocity field to the classical one

$$v = \frac{\nabla S}{m} \to v^0 = \frac{\nabla S^0}{m}. \tag{3.17}$$

Note that convergence of laws of motion *per se* doesn't imply convergence of the corresponding trajectories. But showing that two laws of motion are "close",

that is they are equal except for a small error, suggests that one can use classical or quantum equations indifferently.

It is just in the rules of physics to use approximate laws to describe the world. Think, for example, of the large variety of cases in which physicists trust the validity of their approximations, e.g. Newtonian gravity as an approximation of Einsteinian gravity or the use of reversible Newtonian laws whenever dissipative and viscous forces are negligible. In this regard, one should not forget that the rigorous mathematical justifications of the validity of these approximations might be in general a very difficult task (think, e.g., of the problem of showing that the solutions of Navier-Stokes equations converge to the solutions of Euler equations in the limit of small viscosity). The mathematical work gives us precise information on the time scales of validity of the approximation and this is really a relevant thing to know. But one should remember that mathematical sophistication should not be regarded as a substitute of the very physical reason for which we believe in these approximations.

Be that as it may, we are aware of the mathematical difficulties involved in a rigorous proof of the convergence of the Bohmian trajectories to the corresponding classical ones. In this regard we observe that convergence of trajectories from convergence of laws, at least *for sufficiently short times*, may follow from standard theorems of ordinary differential equation theory (see, e.g. [20]) concerning the dependence of solutions on initial values and other parameters.

While we take for granted that convergence of laws implies convergence of the corresponding trajectories, we observe that there are indeed different mathematical notions which express such a convergence and capture different physical notions of closeness of trajectories:

1. One may demand that, given the Bohmian trajectory on the macroscopic scales X_t^ϵ, there is a classical trajectory X_t^0 such that, in the limit $\epsilon \to 0$, the quantum trajectory X_t^ϵ is closed to X_t^0, that is

$$\lim_{\epsilon \to 0} \left| X_t^\epsilon - X_t^0 \right| = 0. \tag{3.18}$$

   Note that X_t^0 itself depends on ϵ, in the sense that it is the classical motion on the macroscopic scales defined by λ and L.

2. One could also ask for both the macroscopic quantum velocities and the quantum trajectories to be close to the classical ones, that is

$$\lim_{\epsilon\to 0}\left|X_t^\epsilon - X_t^0\right| = 0\ ,\ \lim_{\epsilon\to 0}\left|\dot{X}_t^\epsilon - \dot{X}_t^0\right| = 0. \tag{3.19}$$

3. One can also add the convergence of the macroscopic quantum accelerations to their classical counterparts

$$\lim_{\epsilon\to 0}\left|X_t^\epsilon - X_t^0\right| = 0\ ,\ \lim_{\epsilon\to 0}\left|\dot{X}_t^\epsilon - \dot{X}_t^0\right| = 0\ ,\ \lim_{\epsilon\to 0}\left|\ddot{X}_t^\epsilon - \ddot{X}_t^0\right| = 0. \tag{3.20}$$

4. We can even consider the closeness of higher order derivatives of the Bohmian trajectory to the corresponding derivatives of the classical trajectory

$$\lim_{\epsilon\to 0}\left|X_t^\epsilon - X_t^0\right| = 0\ ,\ \ldots\ ,\ \lim_{\epsilon\to 0}\left|X^{\epsilon}{}_t^{(n)} - X^{0}{}_t^{(n)}\right| = 0. \tag{3.21}$$

The first notion is the weakest one: quantum velocity and acceleration do not need to converge to their classical counterparts. On the contrary the other cases, requiring closeness also of higher order derivatives of quantum and classical trajectory, maybe are too stringent in their requirements. In fact, consider for example a turning point: it is unreasonable to expect the existence of a classical velocity and a classical acceleration close to the quantum ones while there is a classical trajectory close to the Bohmian trajectory, at least for a one dimensional motion.

Finally we note that, since the trajectories are random on the probability space $(\Omega, I\!P^\psi)$, one should also specify the kind of probabilistic convergence. For example, one may have (uniformly in t, $t \in (0,T)$ for some T), for Y denoting the trajectory or its time derivatives,

1. Pointwise convergence

$$\lim_{\epsilon\to 0}\left|Y_t^\epsilon - Y_t^0\right| = 0, \tag{3.22}$$

almost everywhere with respect to $I\!P^\Psi$ defined by equation (2.10).

2. L^2 convergence

$$\lim_{\epsilon\to 0}\left\|Y_t^\epsilon - Y_t^0\right\|_2 = 0, \tag{3.23}$$

that is, the convergence of the variance.

3. Convergence in Probability

$$\lim_{\epsilon \to 0} I\!P^{\Psi} \left(\left| Y_t^{\epsilon} - Y_t^0 \right| \right) = 0. \tag{3.24}$$

Note that our conjecture (3.12) characterize classicality in terms of convergence in probability for $Y = \ddot{X}$.

3.4 On the Hamilton-Jacobi Formulation

Consider the convergence of the modified Hamilton-Jacobi equation (2.15) to the classical Hamilton-Jacobi equation (3.7). Recall from classical mechanics [1] that the solution of the classical Hamilton-Jacobi equation is

$$S^0(x,t) = S_0^0(x_0(x,t)) + \int_0^t \mathcal{L}(\dot{x}(\tau), x(\tau), \tau) d\tau, \tag{3.25}$$

where

$$\mathcal{L}(\dot{x}(\tau), x(\tau), \tau) = \frac{\dot{x}^2}{2m} - V(x)$$

is the Lagrangian, S_0^0 is the initial classical action, and the initial and the final configurations are $x(0) = x_0(x,t)$ and $x(t) = x$. With this we mean that the function $x_0(x,t)$ is the initial condition which in the time t evolves into the configuration x. The momentum corresponding to the initial condition x_0 is $p_t = \frac{\partial S^0}{\partial x}$. The function $p_t(x)$ is not in general single-valued for all times. In fact, at a certain time t_c, it can happen that more than one classical trajectory arrives at the same final point x at time t_c. We shall call this time the "first caustic time" [32]. This time is an upper bound for the time interval in which we can expect convergence of Bohmian trajectories to classical trajectories. We shall explain how our analysis concerning the emergence of the classical laws from Bohmian mechanics can be extended to times bigger than t_c in chapter 6.

Chapter 4

Special Families

In this chapter we shall study two one-parameter families, which are examples of a fixed sequence $\epsilon = \frac{\lambda}{L}$. We shall show that, provided that $\epsilon \to 0$, the motion is classical on the macroscopic scales $(\frac{x}{L}, \frac{t}{T})$, where L is the scale of variation of the potential and T is the corresponding time scale, $T = \frac{L}{v} = \frac{mL\lambda}{\hbar}$.

4.1 Quasi Classical Wave Functions

Consider a family of wave functions depending on $\hbar$ of the *short wave* form

$$\psi_0^\hbar(x) = R_0(x)\mathrm{e}^{\frac{i}{\hbar}S_0(x)}, \tag{4.1}$$

where $R_0(x)$ and $S_0(x)$ are functions not depending on $\hbar$ and $R(x)$ is compactly supported. The limit $\hbar \to 0$ of this kind of wave function has been studied by many people (see, e.g., [32]) and corresponds to a mathematical trick to simulate the limit in which the mean de Broglie wave length $\lambda \equiv \lambda_{\psi_0}$ is tending to zero as $\hbar \to 0$. In fact, by a straightforward computation it follows that, for ψ_0 given by equation (4.1), the mean de Broglie wave length $\lambda_{\psi_0} \to 0$ as $\hbar \to 0$ [1]. So this limit, called short wave length limit, is a special case of the limit $\epsilon = \lambda/L \to 0$ in which the wave length is getting small and the external potential is fixed such that L is also fixed.

Observe that the limit $\hbar \to 0$ also simulates the limit of large masses (in which $m \to +\infty$) for which the potential rescales as $V = m\hat{V}$. In fact in this case m

[1] Note that this limit is equivalent to time dependent WKB [32].

and $\hbar$ play the same role in the Schrödinger's equation

$$i\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar}{2m}\nabla^2 + \frac{m}{\hbar}\hat{V}\right]\psi. \tag{4.2}$$

In the short wave length limit, to see the classical limit, one should go to the macroscopic coordinates given by $\left(\frac{x}{L}, \frac{t}{T}\right)$. Note that both L and T are constant so that there is no substantial difference between the microscopic and macroscopic scale. Therefore, there is no need to rescale the wave function to see the emergence of the classical behavior in this case.

The approximate solution of Schrödinger's equation in the short wave limit is given by [32]

$$\psi^{(0)}(x,t) = R^{(0)}(x,t)\mathrm{e}^{\frac{i}{\hbar}S^{(0)}(x,t)} + O(\hbar), \tag{4.3}$$

where $S^{(0)}(x,t) = S^{(0)}(x_0) + \int_0^t \mathcal{L}(\dot{x}(\tau), x(\tau), \tau)d\tau$ is the classical action. The initial position is $X_0^{(0)} = x_0$ and the initial velocity is given by $V_0^{(0)} = \frac{\nabla S^{(0)}(x_0,0)}{m}$.

The amplitude of the wave function is given by $R^{(0)}(x) = |dx/dx_0|^{-1/2}R_0(x_0)$. This is the evolution at time t of the initial amplitude $R_0(x)$ according to the classical laws, i.e. according to the classical continuity equation (2.14). Note that the shape of the initial wave function (4.1) is preserved by the dynamics.

Observe that in this case we have convergence to classical laws according to all the different notions we have discussed above in section 3.3. We see in fact that the velocity field, as $\hbar \to 0$, is the classical one

$$v^{(0)}(x,t) = \frac{1}{m}\nabla S^{(0)}(x,t) + O(\hbar^2), \tag{4.4}$$

given that $S^{(0)}$ is the classical action. Moreover, the quantum potential

$$V_Q^{(0)} = -\frac{\hbar^2}{m}\frac{\nabla^2 R^{(0)}}{R^{(0)}}$$

is zero because $R^{(0)}$ doesn't depend on $\hbar$, so that we have convergence of the modified Hamilton-Jacobi equation to the classical Hamilton-Jacobi equation. Also the quantum force is zero, just because it is the derivative of the quantum potential. Using the Hamilton-Jacobi equation, in fact, we can rewrite the equation for the velocity field given by equation (4.4) as a system of two coupled first order differential equations for position and velocity as follows

$$\begin{cases} m\dot{X}_t^{(0)} = \nabla S^{(0)}(X_t^{(0)}, t) + O(\hbar^2) \\ m\dot{V}_t^{(0)} = -\nabla V(X_t^{(0)}) + O(\hbar^2) \end{cases} \tag{4.5}$$

which corresponds to the second order equation

$$m\ddot{X}_t^{(0)} = -\nabla V(X_t^{(0)}) + O(\hbar^2). \tag{4.6}$$

As we have seen previously in section 3.3, the convergence of quantum laws to classical laws ensure the convergence of Bohmian trajectories to classical trajectories, in force of those theorems of ordinary differential equations theory about the convergence of an integral curve of a function to the integral curve of the limit function.

4.2 Slowly Varying Potentials

Another special case of the limit $\epsilon \to 0$ is given by the situation in which we have a slowly varying external potential. A slowly varying potential can be written as $V(x) = V(\frac{x}{L})$, where its scale of variation L is very big. Given that $\epsilon = \lambda/L$, we see that the slowly varying potential limit is a special limit corresponding to keeping fixed the initial wave function and letting $L \to +\infty$. To see the classical motion we should go on the macroscopic scales $\left(\frac{x}{L}, \frac{t}{T}\right)$, where $T = \frac{L}{v} = \frac{mL\lambda}{\hbar}$. Note that this limit is equivalent to a long time limit. In fact, if the potential is slowly varying, that is $L \to +\infty$, to see its effect the particle has to wait a time of order $T = \frac{mL\lambda}{\hbar} \to +\infty$. Note that both time and space rescalings are of the same order. Then it is useful, only for convenience of notation, not to define the macroscopic scales (as we have done previously) as $x' = \frac{x}{L}$ and $t' = \frac{t}{T}$. Instead we shall rescale space and time with the dimensionless parameter $\epsilon = \frac{\lambda}{L}$ such that the macroscopic coordinates are $x' = x\epsilon$, $t' = t\epsilon$ and the initial conditions become

$$\psi_0^\epsilon(x) = \epsilon^{-1/2}\psi_0(x/\epsilon), \quad x_0^\epsilon = x_0/\epsilon. \tag{4.7}$$

Under this transformation, it is easy to show that Bohmian equations become the usual equations of motion for $\psi^\epsilon(x,t)$ with $\hbar$ substituted by $\hbar\epsilon$

$$i\hbar\epsilon\frac{\partial\psi^\epsilon(x,t)}{\partial t} = -\frac{\hbar^2\epsilon^2}{2m}\nabla^2\psi^\epsilon(x,t) + V(x)\psi^\epsilon(x,t), \tag{4.8}$$

$$\frac{dX_t^\epsilon}{dt} = v^\epsilon(X_t^\epsilon, t) = \frac{\hbar\epsilon}{m}\mathrm{Im}\left[\frac{\nabla\psi^\epsilon(X_t^\epsilon,t)}{\psi^\epsilon(X_t^\epsilon,t)}\right]. \tag{4.9}$$

Note that in the limit $\epsilon \to 0$ the initial position tends to zero and the initial wave function converges to a delta function centered at the origin [2]

$$\begin{cases} x_0^\epsilon \to 0 \\ |\psi_0^\epsilon(x)|^2 \to \delta(0) \end{cases} \tag{4.10}$$

The solution $\psi^\epsilon(x,t)$ of the Schrödinger equation in the macroscopic coordinate can be expressed in terms of the propagator in which $\hbar$ is replaced by $\hbar\epsilon$ and the Fourier transform of the initial wave function. Let's recall that the propagator is the solution of the Schrödinger's equation for an initial wave function that is a δ-function. In terms of the rescaled propagator $K^\epsilon(x,t;x_0,0)$ and of the Fourier transform of the initial wave function $\hat{\psi^\epsilon}_0(k)$, the wave function at time t is given by

$$\psi^\epsilon(x,t) = \frac{1}{(2\pi\epsilon)^{d/2}} \int\int K^\epsilon(x,t;x_0,0) e^{i\frac{x_0\cdot k}{\epsilon}} \hat{\psi}_0(k) d^d x_0 d^d k, \tag{4.11}$$

where d is the dimension of the space. In general, the asymptotic form of the propagator in the limit $\epsilon \to 0$ is [32]

$$K^\epsilon(x,t;x_0,0) = \frac{1}{(2\pi i\hbar\epsilon)^{d/2}} \sqrt{C(x,x_0;t)} e^{\frac{i}{\hbar\epsilon}S^0(x,t;x_0,0)}[1+\hbar\epsilon z], \tag{4.12}$$

where $z = z(t,x_0,x,\epsilon)$ and $||z||_{L^2(\mathbf{R^d})} \leq c$ where c is a constant. This means that, taking the expression

$$K^\epsilon(x,t;x_0,0) = \frac{1}{(2\pi i\hbar\epsilon)^{d/2}} \sqrt{C(x,x_0;t)} e^{\frac{i}{\hbar\epsilon}S^0(x,t;x_0,0)}, \tag{4.13}$$

we are forgetting terms of order ϵ. S^0 is the classical action, X_t^0 is the classical path with boundary conditions $X_0^0 = x_0$, $X_t^0 = x$ and

$$C(x,x_0;t) = \left| -\nabla^2_{x,x_0} S^0(x,t;x_0,0) \right|.$$

This approximate expression of the propagator is valid when there is only one path joining x_0 to x in the same time t, that is when we have no caustics. Let's remember that a caustic is a point in which the velocity field is multivalued. In order to find the asymptotic $\epsilon \to 0$ of $\psi^\epsilon(x,t)$, we apply the method of stationary

[2] This is not a limitation, in fact we could have rescaled the wave function in a different way, i.e. $\psi_0 \to \psi_0^\epsilon(\epsilon(x - \frac{\bar{x}}{\epsilon}))$, and in this case the wave function would have been converging to a δ-function centered in $\bar{x}$.

phase: the main contribution to $\psi^\epsilon(x,t)$ comes from the x_0 and the k which make stationary the phase $\phi(x_0,k) = \frac{1}{\hbar}[S^0(x,t;x_0,0) + x_0 \cdot \hbar k]$. They are

$$x_0 = 0 \quad \text{and} \quad k_0(x,t) = -\frac{1}{\hbar}\,\nabla_{x_0} S^0(x,t;x_0,0)\Big|_{x_0=0}. \tag{4.14}$$

So we have

$$\psi^{(0)}(x,t) = \sqrt{C(x,0;t)}\left(\frac{i}{\hbar}\right)^{d/2} \hat{\psi}_0(k_0(x,t))\mathrm{e}^{\frac{i}{\hbar\epsilon}S^0(x,t;0,0)|} + O(\epsilon). \tag{4.15}$$

We can rewrite this as

$$\psi^{(0)}(x,t) = R^0(x,t)\mathrm{e}^{\frac{i}{\hbar\epsilon}S^0(x,t)} + O(\epsilon^2), \tag{4.16}$$

where

$$R^0(x,t) = \sqrt{C(x,0;t)}\left(\frac{i}{\hbar}\right)^{d/2} \hat{\psi}_0(k_0(x,t)) \tag{4.17}$$

and S^0 is the classical action. Always in the limit $\epsilon \to 0$, for any time $t > 0$, the velocity field becomes

$$v^{(0)}(x,t) = \frac{1}{m}\,\nabla_x S^0(x,t;x_0,0)\Big|_{x_0=0} + O(\epsilon^2). \tag{4.18}$$

Also in this case we have convergence of laws of motion. In fact, given that S^0 is the classical action, we have that (4.18) is the classical velocity field. Moreover, we have convergence of the modified Hamilton-Jacobi equation to the classical Hamilton-Jacobi equation. In fact, in the limit $\epsilon \to 0$, the quantum potential in the macroscopic scales is given by

$$V_Q^{(0)} = -\frac{\hbar^2\epsilon^2}{m}\frac{\nabla^2 R^0}{R^0}$$

where R^0 doesn't depend on ϵ, so that $V_Q \to 0$ as $\epsilon \to 0$. Just as before, we have convergence of the modified Newton's equation to Newton's equation since the quantum force is zero. Indeed, from equation (4.18) and from those theorems of ordinary differential equations theory about the convergence of an integral curve of a function to the integral curve of the limit function in the limit $\epsilon \to 0$, we can conclude that the Bohmian trajectories converge to the classical ones.

Note that this limit is, as the previous case of section 4.1, a limit in which $\epsilon \to 0$ but the wave function is rescaled in a different way. The effect on the motion equations is the same but the effect on the initial conditions is completely different.

4.3 Convergence of Probability Distributions

As a byproduct of our analysis, we obtain convergence of the quantum probability distributions to the corresponding classical ones.

Consider first convergence at the initial time $t = 0$. In the case of the familiy of quasi classical wave functions, in the limit $\hbar \to 0$, the initial position $X_0^{(0)}$ is distributed according to

$$\rho(x, 0) = |R_0(x)|^2, \tag{4.19}$$

that is the classical probability distribution, and the initial velocity $V_0^{(0)} = \frac{1}{m}\nabla S^0(X(0), 0)$ is distributed according to the classical probability distribution induced by the initial position $X_0^{(0)}$.

In the case of a slowly varying potential, the situation about the probability distributions of initial conditions is as follows. The initial conditions are

$$X_0^{(0)} = \lim_{t\to 0}\lim_{\epsilon\to 0} X_t^\epsilon = 0, \tag{4.20}$$

$$V_0^{(0)} = \lim_{t\to 0}\lim_{\epsilon\to 0} \frac{X_t^\epsilon}{t}. \tag{4.21}$$

The limit trajectory at time t $X_t^{(0)} = \lim_{\epsilon\to 0} X_t^\epsilon$ is distributed according to

$$\rho(x, t) = \frac{C(x, 0; t)}{\hbar^d}|\hat{\psi}_0(k_0(x, t))|^2, \tag{4.22}$$

where $k_0(x, t)$ is defined by equation (4.14). The initial velocity is a random variable whose probability distribution is the one induced by the probability distribution of X_t^ϵ given by equation (4.22) in the limit $t \to 0$. Note that

$$\lim_{t\to 0} C(x, 0; t) = \lim_{t\to 0} C_{free}(x, 0; t) = \left[\frac{m}{t}\right]^d \tag{4.23}$$

so that

$$\rho(v, 0) = \left(\frac{m}{\hbar}\right)^d \left|\hat{\psi}_0\left(\frac{mv}{\hbar}\right)\right|^2, \tag{4.24}$$

which is the classical distribution of the initial velocity if the classical probability distribution of position at time t is given by (4.22).

On the basis of the results concerning the convergence of trajectories in both cases of quasi classical wave functions and slowly varying potentials (see sections 4.1 ad 4.2), the time evolution of the probability distributions is given, in the limit, by the transport along the classical paths. We shall briefly recall the classical laws of transport in the next section.

4.4 Propagation of Classical Probability Distributions

Consider the differential equation

$$\frac{dQ_t}{dt} = v(Q_t), \tag{4.25}$$

where $v = v(q)$, $q \in \mathbb{R}^n$, is a field $v : \mathbb{R}^n \to \mathbb{R}^n$ (velocity field) and $Q_t \in \mathbb{R}^n$ is the state of the system at time t. The solution of the equation (4.25) is a function $f_t : \mathbb{R}^n \to \mathbb{R}^n$ (parametrized by $t \in \mathbb{R}$) such that to each initial condition Q_0 at time $t = 0$ assigns the state of the system at time t,

$$Q_t = f_t(Q_0).$$

Assume now that the velocity field v is conservative, that is the dynamics of the system is reversible. Observing the duality between "observables", i.e. functions $f(q)$ on the state space, and probability distributions $\rho(q)$,

$$\int \rho_t(q) f_0(q) dq = \int \rho_0(q) f_t(q) dq,$$

the following theorem follows almost immediately

Theorem: Let $f_t : \mathbb{R}^n \to \mathbb{R}^n$ be a map solution of the equation (4.25). If the probability distribution of the initial state is ρ_0, then the distribution of the state at time t is given by

$$\rho_t(q) = \rho_0(f_t^{-1}(q)) J_t(q)^{-1}, \tag{4.26}$$

where J_t is the Jacobian of f_t.

Note that ρ_t given by equaton (4.26) is the general solution of the transport (continuity) equation

$$\frac{\partial \rho_t}{\partial t} + \nabla_q \cdot v\rho_t = 0. \tag{4.27}$$

Consider now a Hamiltonian system for which $q = (x, p)$ and v is a Hamiltonian field

$$v(x, p) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial x} \right), \tag{4.28}$$

where $H = H(x, p)$ is the Hamiltonian. In this case we have the important [1]

Theorem (of Liouville) : The Hamiltonian evolution preserves the volume in the phase space. In other words, the Jacobian J_t of the map f_t, solution of a Hamiltonian system, is equal to 1.

In this case we have

$$f_t(x,p) \equiv (X_t(x,p), P_t(x,p)), \tag{4.29}$$

and, from (4.26) and the theorem of Liouville, it follows that the transport law of probability becomes

$$\rho_t(x,p) = \rho_0 (X_{-t}(x,p), P_{-t}(x,p)). \tag{4.30}$$

Set $x_0 \equiv X_{-t}(x,p)$ and $p_0 \equiv P_{-t}(x,p)$. If x and p represents the position and the momentum at time t, then x_0 and p_0 are the initial conditions that after a time t evolve to the final conditions x and p. So we can write

$$\rho_t(x,p) = \rho_0 (x_0, p_0) \tag{4.31}$$

In the application to the classical limit of quantum mechanics, we are interested to the distribution of position $\rho_t(q)$. Given that $\rho_t(x,p)$ represents the joint probability of position and momentum at time t, the distribution $\rho_t(x)$ is given by $\rho_t(x,p)$ integrating away the momentum

$$\rho_t(x) = \int \rho_t(x,p)dp, \tag{4.32}$$

from which we get

$$\rho_t(x) = \int \rho_0(x_0,p_0)dp. \tag{4.33}$$

Now p_0 is a function of x and p and, for a fixed x, is an invertible function of p. By change of variable in the previous integral, we get

$$\rho_t(x) = \int \rho_0(x_0,p_0) \left|\frac{dp}{dx_0}\right| dx_0. \tag{4.34}$$

The function $p = p(x,x_0)$ represents the momentum of the system of a configuration starting at time $t = 0$ in x_0 and arriving at time t in x. If $S = S(x,x_0;t)$ is the action

$$S(x,x_0;t) = \int_0^t L(x(s),\dot{x}(s))ds, \quad \text{where} \quad x(0) = x_0, x(t) = x, \tag{4.35}$$

we have

$$p = \frac{\partial S}{\partial x} \tag{4.36}$$

and then

$$\left| \frac{dp}{dx_0} \right| = \left| \frac{\partial^2 S}{\partial x \partial x_0} \right| \equiv C. \tag{4.37}$$

Suppose that the initial position is fixed and that the initial momentum is random with distribution $\rho_0(p)$, that is the initial distribution is $\rho_0(x,p) = \delta(x - \bar{x}_0)\rho_0(p)$. Then, from (4.34), we get

$$\rho_t(x) = C(x, \bar{x}_0, t)\rho_0(p_0) = C(x, \bar{x}_0, t)\rho_0\left(P_t\left(x, p(x, \bar{x}_0)\right)\right), \tag{4.38}$$

which is indeed the probability distribution associated with the wave function (4.15) in the limit $\epsilon \to 0$.

Chapter 5

Local Plane Wave Structure

From the examples given in the previous chapter, quasi classical wave functions and slowly varying potentials, we can conclude that there is a particular structure of the wave function that emerges when we are in the classical regime. This structure is what we call a *local plane wave*, a wave function that locally can be regarded as a plane wave having a local wave length.

5.1 The Notion of Local Plane Wave

A precise notion of local plane wave can be given starting from the usual notion of wave length λ, that is the spatial period. (We consider here, for sake of simplicity, such a characterization for the one dimensional case.) This means that $\lambda(x,t)$ should be slowly varying over a distance of order λ

$$|\nabla\lambda(x,t)| \ll 1 \tag{5.1}$$

and, for $\psi(x,t) = R(x,t)\mathrm{e}^{i\frac{S(x,t)}{\hbar}}$, the following relations should hold

$$R(x,t) \simeq R(x+\lambda,t), \tag{5.2}$$

$$S(x,t) \simeq S(x+\lambda,t) + 2\pi\hbar. \tag{5.3}$$

By expanding in Taylor series in λ the right hand side of equation (5.2) one gets

$$R(x+\lambda,t) = R(x,t)\left[1 + \left|\frac{\nabla R(x,t)}{R(x,t)}\right|\lambda(x,t)\frac{1}{2}\left|\frac{\nabla^2 R(x,t)}{R(x,t)}\right|\lambda^2(x,t) + ...\right] \tag{5.4}$$

so that equation (5.2) implies

$$\left|\frac{\nabla R(x,t)}{R(x,t)}\right|\lambda(x,t) \ll 1, \tag{5.5}$$

$$\frac{1}{2}\left|\frac{\nabla^2 R(x,t)}{R(x,t)}\right|\lambda^2(x,t) \ll 1, \quad ... \tag{5.6}$$

Similarly, for $S(x,t)$ we obtain

$$S(x+\lambda,t) = S(x,t) + |\nabla S(x,t)|\,\lambda(x,t) + \frac{1}{2}\left|\nabla^2 S(x,t)\right|\lambda^2(x,t) + ... \tag{5.7}$$

The comparison of the expansion (5.7) with equation (5.3) gives, up to the first order terms, the definition of the local wave length $\lambda(x,t)$

$$\lambda(x,t) = \frac{\hbar}{|\nabla S(x,t)|}. \tag{5.8}$$

The smallness of the second order term, together with equation (5.8), expresses the compatibility with equation (5.1).

To sum up, a local plane wave $\psi(x,t) = R(x,t)\mathrm{e}^{i\frac{S(x,t)}{\hbar}}$ is characterized by a local wave length, defined by equation (5.8) and slowly varying in the sense of equation (5.1), and with a slowly varying amplitude $R(x,t)$ satisfying equations (5.5) and (5.6).

Note that, in the slowly varying potential case, these conditions are verified, as shown by equation (4.15), while in the short wave limit we start from a wave function for which they are satisfied from the very beginning and this shape is preserved by the dynamics.

From the very notion of local plane wave, it follows that, at any given time t, a local plane wave $\psi(x,t)$ can be thought as composed of a sum of "virtual" wave packets.

Consider in fact a partition of physical space into a union of disjoint sets Δ_k chosen in such a way that the local wave length $\lambda(x,t)$ doesn't vary appreciably inside of each of them. Denote by λ_k the almost constant value $\lambda(x,t)$ for $x \in \Delta_k$. Let χ_{Δ_k} be the characteristic function of the set Δ_k ($X_{\Delta_k}(x) = 1$ if $x \in \Delta_k$ and 0 otherwise). Since $\sum_k \chi_{\Delta_k} = 1$, we have

$$\psi(x,t) = \sum_k \chi_{\Delta_k}(x)\psi(x,t) = \sum_k \psi_k(x,t). \tag{5.9}$$

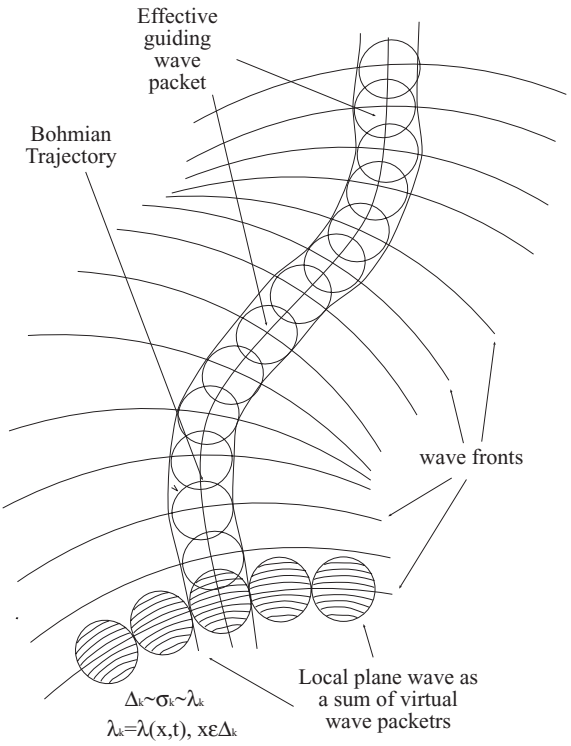

Figure 5.1: A local plane wave as a sum of "virtual" wave packets

The last equality defines the "virtual" wave packets ψ_k with wave length λ_k and with disjoint position and momentum supports. We call them "virtual" because they depend upon a partition which is arbitrary: provided that $\lambda(x,t)$ is almost constant in Δ_k, the magnitude of these sets can be of the order of several wave lengths down to a minimal size $\sigma_k \simeq |\Delta_k|^{1/3}$ of the same order of λ_k. This means in particular that it is not necessary that the wave packets decomposition at time $t' > t$ is the time evolution of the wave packets decomposition at time t. (Note that the use of characteristic functions may introduce an undesirable lack of smoothness of the wave packets, but this can be easily taken care of by substituting the χ_{Δ_k} with functions θ_k forming a smooth partition of unity [9].)

Note that in the particular case of a wave function like $\psi(x,t) = R(x,t)\mathrm{e}^{\frac{S(x,t)}{\hbar}}$, where $S \gg \hbar$, discussed in section 4.1, it can be shown that each ψ_k is actually a wave packet. Indeed, rewrite the wave function as

$$\psi^\epsilon(x,t) = R(x,t)\mathrm{e}^{\frac{i}{\epsilon}\Phi(x,t)}, \tag{5.10}$$

where $\epsilon = \frac{\lambda}{L}$ and $\Phi(x,t) \gg \epsilon$. Then the Fourier transform of the asymptotic of

$\psi^{\epsilon}(x,t)$ in the limit $\epsilon \to 0$ is again a local plane wave [32]

$$\hat{\psi^{\epsilon}}(k,t) = \frac{1}{\left|\frac{\partial^2 \Phi(x(k))}{\partial x^2}\right|^{1/2}} R(x(k)) \mathrm{e}^{\frac{i}{\epsilon}\tilde{\Phi}(x(k))}, \tag{5.11}$$

where $\tilde{\Phi}(x(k))$ is the Legendre transform of Φ, if Φ is convex. In this way there is a one to one correspondence between position and momentum. In particular, disjoint position supports correspond to disjoint momentum supports.

5.2 Local Plane Waves and the Quantum Potential

It should be stressed that condition (5.6) directly implies that

$$\frac{\hbar^2}{2m}\left|\frac{\nabla^2 R(x,t)}{R(x,t)}\right| \ll \frac{1}{2m}\left|\nabla S(x,t)\right|^2, \tag{5.12}$$

that is, the quantum potential is smaller than the kinetic energy for a given time t which, in its turn, implies the validity of the classical Hamilton-Jacobi equation. We may then conclude that the association between the emergence of classical behavior and the formation of local plane waves is indeed the hallmark of the classical limit. This conclusion receives further support from observing the expansive character of the Laplacian in Schrödinger's equation which tends to produce spreading of the wave function whenever the potential energy is dominated by the kinetic energy (that is, for bounded motion in a potential, far from the turning points).

Moreover, observe that, to have a classical limit, equation (5.12) should be valid not only at a fixed time t but also for a sufficiently large time interval. In other words, classicality requires that the local plane wave structure should be preserved by the dynamics. In this chapter we shall try to support this argument on the basis of the Ehrenfest theorem.

5.3 The Ehrenfest–Goldstein Argument

In this section we shall provide an argument [1] for our conjecture of classicality (3.12), which will rely on

1. the fact that, as soon as $\lambda_{\psi_0} \ll L$, the local plane wave is quickly produced,
2. the fact that the local plane wave is a sum of non interacting "virtual" wave packets, to each of which can be applied the Ehrenfest theorem;
3. the fact that, if $\lambda_{\psi_0} \ll L$, at time t a local plane wave gets formed with a local wave length $\lambda(x,t) \ll L$ on the macroscopic time scale.

We will deal with the first issue in section 5.7, while in this section we'll concentrate on the last two issues, basically assuming that a local plane wave is formed whenever $\lambda_{\psi_0} \ll L$. Our argument will shed light on the appropriate notion of the length scale L as well as providing support for the the stability of the local plane wave structure and therefore for the validity of equation (5.12) at any time.

The Ehrenfest theorem states that, for a sufficiently narrow wave packet, we have

$$m\frac{d^2}{dt^2}\langle X\rangle = -\langle \nabla V(X)\rangle,.$$

where $\langle\rangle$ denotes the average with respect to a the system wave function ψ. By expanding $F(x) = -\nabla V(x)$ in Taylor series around $\langle X\rangle$ one obtains

$$m\frac{d^2}{dt^2}\langle X\rangle = F(\langle X\rangle) + \frac{1}{2}\sum_{i,j}\Delta_{j,k}\frac{\partial^2 F}{\partial x_j \partial x_k}(\langle X\rangle) + ..., \tag{5.13}$$

where

$$\Delta_{j,k} = \langle X_j X_k\rangle - \langle X_j\rangle\langle X_k\rangle$$

is of order σ^2, where σ is the diameter of the packet. Therefore, the mean particle position will satisfy classical Newton equation whenever

$$\sigma^2\left|\frac{\partial^3 V}{\partial x_i \partial x_i \partial x_j}\right| \ll \left|\frac{\partial V}{\partial x_i}\right|, \tag{5.14}$$

i.e.,

$$\sigma \ll \sqrt{\left|\frac{V'}{V'''}\right|} \tag{5.15}$$

[1] S. Goldstein, private communication

where V' and V''' denotes respectively suitable estimates of the first and third derivatives (e.g., by taking a sup over the partial derivatives). Note that, for at most quadratic potentials, $V''' \equiv 0$ and we can see that the Ehrenfest theorem ensures classicality without any restriction.

Consider now a wave function ψ which is a local plane wave at some time t, i.e. of the form (5.9). Note that, wherever the particle is initially located, at time t the position X_t of the particle will be in the support of one of the wave packets ψ_k. If the condition (5.15) for σ_k holds, then at time t Newton's equation holds for the mean value $\langle X \rangle$ of the position of the particle. Therefore, at time t, we have classical behavior of the particle. However, since the minimal size of the wave packet can be taken to be of order $\lambda(x,t)$, the condition ensuring classicality at time t becomes

$$\lambda(x,t) \ll \sqrt{\left|\frac{V'(x)}{V'''(x)}\right|}. \tag{5.16}$$

Note that if

$$L = L(x) \equiv \sqrt{\left|\frac{V'(x)}{V'''(x)}\right|} \tag{5.17}$$

is interpreted as the scale of variation of the potential, equation (5.16) becomes

$$\lambda(x,t) \ll L(x). \tag{5.18}$$

The above conclusion provides support for our conjecture provided that we can ensure that $\lambda_{\psi_0} \ll L$ implies not only that a local plane wave is formed (as we already assumed) but also that the local wave length of such a local plane wave is such that $\lambda(x,t) \ll L$. In this regard, it should be noted that the local $\lambda(x,t) \ll L$ is actually implied by the global condition $\lambda_{\psi_0} \ll L$, at least for a sufficient amount of time and at most with the exceptions of periods of times which are small on the macroscopic scales, basically by appealing to conservation of energy. In fact, the only problem may arise from the phase of deceleration of the particle due to motion in a potential with turning points. If we have a potential that speeds up the particle, $\lambda(x,t)$ decreases such that $\lambda \ll L$ is not destroyed by the potential but holds better than before. But if we have turning points, where the particle inverts its direction, the situation is different since at the turning point λ may increase dramatically. However, this is a real problem

only in one dimension, because this is the only situation in which the particle really stops. The times for which the condition $\lambda(x,t) \ll L$ ceases to be valid will be typically so short that cannot influence its general validity on the macroscopic character of the motion.

5.4 Some Remarks on the Scale of Variation of the Potential

The Ehrenfest–Goldstein argument has given us a strong criterion to find a sensible definition of the scale of variation of the potential L. Independently of the Ehrenfest–Goldstein argument, one may argue that there are other possible definitions of L, more familiar than (5.17). One could consder, for example,

$$L_1 = 1/\left[\frac{1}{E}\frac{dV}{dx}\right], \quad L_2 = 1/\left[\frac{1}{V}\frac{dV}{dx}\right], \tag{5.19}$$

where E is the mean kinetic energy of the particle.

Concerning this, it should be emphasized that, if the potential is periodic, then L_1 and L_2 don't give the period. On the other hand, the L given by equation (5.17) is exactly the period for the special cases in which V is a sinusoidal function and gives a rough estimate of the period for more general periodic functions.

The scale of variation of the potential should be only a function of V, not of the particle, whose characteristics are already in λ: for this reason L_1 should not be a good choice.

If in the definition of L_2, V means the potential function $V(x)$, then L_2 is not uniquely defined because it inherits the dependence on an arbitrary constant from the potential. If V means some difference between the maximum and the minimum of the potential, the resulting notion of length scale is too restrictive because it can be defined only in the case of bounded potentials.

Be that as it may, all sensible choices of L must be of the same order and indeed they are of the same order whenever the kinetic energy is of the order of the variation of the potential on a distance L, which is typically the case. Thus, to probe the appropriateness of one or another definition, we need to consider atypical situations.

Consider

$$\epsilon = \frac{\lambda}{L_1}$$

and suppose $E \to +\infty$ so that $L_1 \to +\infty$. Then $\lambda \ll L_1$ ($\epsilon \ll 1$) is compatible with $\lambda \gg L_V$, where L_V is the period of the potential, but in this case the particle won't notice the details of V. So, in the scales defined by $\left(\frac{x}{L_1}, \frac{t}{T_1}\right)$, where $T_1 = \frac{L_1}{v}$, the motion will not be the classical motion corresponding to the Bohmian motion in the same potential. Therefore, we cannot conclude that we have a classical limit if $\epsilon \to 0$. On the other hand, suppose $E \to 0$, such that $L_1 \to 0$ ($\epsilon \to +\infty$). This is compatible with local plane wave structure with $\lambda \ll L_V$, which would suffice for classical motion even if $\lambda > L_1$, that is we have classical motion even if ϵ doesn't go to zero. We can then conclude that L_1 is not the right choice to see the classical motion.

However, in any case, the Ehrenfest–Goldstein argument provides by itself the strongest support to the correctness of L given by equation (5.17) as giving the notion of scale of variation of the potential.

Note that, for quadratic potentials, from the definition (5.17) one sees that $L = +\infty$ and thus $\epsilon \equiv 0$[2]. This means, as we have already observed in section 3.2, that for these cases the motion is always classical on suitable macroscopic scales. This is in complete agreement with the standard understanding of the classical limit in terms of the Wigner function, Feynman path integrals or Weyl quantization [35]. In fact, for Hamiltonians which are polynomials of degree ≤ 2 on phase space, Egorov's theorem (see section 8.3) holds without error, i.e. Weyl quantization commutes with the time evolution.

5.5 Some Simple Examples of L

It can be useful to compute directly in some simple special cases what is the small dimensionless parameter ϵ relevant for the classical behavior.

Consider, first of all, the free case. We have just seen in the previous section that $L = +\infty$, so that in this case we always have classical motion: the experi-

[2]By quadratic potential we mean $V(x) = ax^2 + bx + c$ (for simplicity in one dimension). Linear and constant potentials are included as limiting cases respectively for a only and a and b going to zero.

menter can choose any length scale L_o. We then have classical behavior as soon as

$$\epsilon = \frac{\lambda}{L_o} \ll 1. \tag{5.20}$$

The same can be established in the cases of the gravitational and of the harmonic potential. Given that they are a linear and a quadratic potential, $L = +\infty$, i.e. $\epsilon \equiv 0$, so there is always classical motion on the scale L_o, provided that $\lambda \ll L_o$.

A more interesting example is the case of a particle in a Coulomb potential

$$V(r) = \frac{qq'}{r}. \tag{5.21}$$

We find that L is

$$L \simeq r \tag{5.22}$$

If we consider the bound states of the hydrogen atom (with small spread in energy as is the case, e.g., for coherent states), L becomes of the order of the Bohr radius a_0. Thus, given that $\lambda/a_0 = \frac{1}{n}$ tells how many times the wave length λ is varying in the scale a_0 of the variation of the potential, we have

$$\epsilon \simeq \frac{\lambda}{a_0} \simeq \frac{1}{n}, \tag{5.23}$$

where n is the principal quantum number. Therefore, the classical limit as a limit of high quantum numbers is a particular case of our general analysis. For scattering states, $L \simeq r$, where r is simply the distance from the scattering center. This means that the scale on which the motion is classical is varying, not fixed.

Consider now the case of the Yukawa potential, that is

$$V(r) = \frac{\mathrm{e}^{-\mu r}}{r}. \tag{5.24}$$

The scale of variation of this potential, according to definition (5.17), is

$$L \simeq \sqrt{\frac{(\mu r + 1)r^2}{6 + \mu^3 r^3 + 3\mu^2 r^2 + 6\mu r}} \tag{5.25}$$

For large distances, i.e. $r \to +\infty$ (scattering states), we can say $L \simeq \frac{1}{\mu}$, the range of the potential; for small distances, i.e. $x \to 0$, we have $L \simeq r$, as in the Coulomb case.

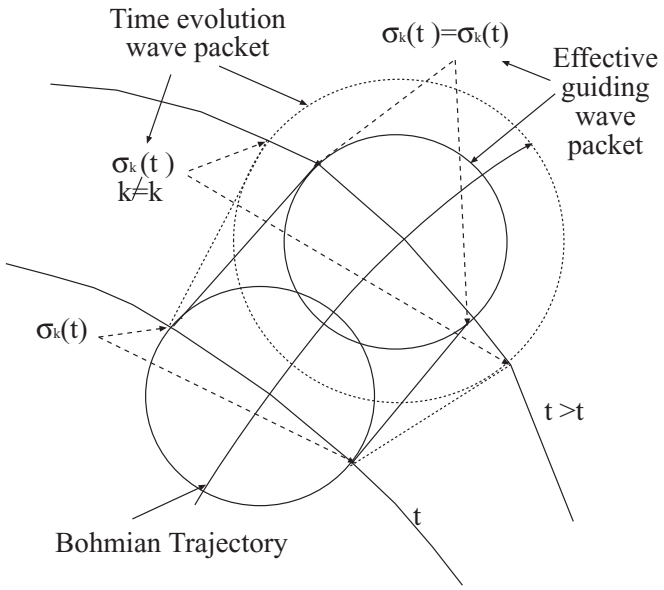

Figure 5.2: Effective guiding wave packet

5.6 Refinement of the Conjecture

It is important to observe that the Ehrenfest theorem not only suggests a natural way to define the scale on which the potential varies but also provides an explanation of the classical limit for spread out wave functions. In fact, in Bohmian mechanics each particle has its own trajectory and, for this reason, the local plane wave, at a fixed time t, undergoes a sort of collapse such that not all the wave function is relevant for the dynamics but there is an *effective* guiding wave packet for the particle, which is the part of the wave function in a local neighborhood of the trajectory at the time t. At a later time $t' = t + \Delta t$, the local plane wave may be decomposed into a different sum of wave packets. If Δt is much smaller than the time needed for σ to become of order L (i.e. to ensure that the Ehrenfest theorem is valid), then the guiding wave packet at this time t' is the one in whose support the trajectory of the particle has evolved *classically* from time t.

In this regard we would like to underline two important points. First, observe that, while in standard quantum mechanics the emergence of some classicality is always connected to the permanence of a narrow wave function during the motion, what arises from the above discussion is that the crucial feature of the

classical limit is the formation of a very spread out wave function: the local plane wave. Only in the framework of Bohmian mechanics, given that we also have configurations and not just the wave function, can we apply the Ehrenfest–Goldstein argument to spread out wave functions, thus explaining the emergence of the classical behavior in a coherent way.

Secondly, the fact that the relevant wave packet moves classically suggests very strongly that, when the condition (5.15) is satisfied, the local plane wave structure should be stable, i.e. it should not be destroyed by further motion in the potential. We have seen at the beginning of section 5.2 that it follows directly from the definition of local plane wave that the classical Hamilton-Jacobi equation is valid at a fixed time t. If the local plane wave structure is preserved, then the classical Hamilton-Jacobi equation is valid also for any time and the motion is really classical.

An important consequence of the Ehrenfest–Goldstein argument and of the above discussion is that the relevant wave length should be regarded as determined by the initial condition x_0. Suppose that the initial wave function is such that a local plane wave is produced at a certain time t (which is guaranteed by the condition $\lambda_{\psi_0} \ll L$ as we shall argue in the next section). Then the Bohmian motion starting from x_0 tends to get attached to the piece of local plane wave characterized by

$$\tilde{\lambda} = \tilde{\lambda}(\psi_0, x_0) = \lambda(X_t(x_0), t), \tag{5.26}$$

where $\lambda(X_t(x_0), t)$ is the local wave length given by (5.8) computed along the Bohmian trajectory starting from x_0. In other words, the parameter ϵ entering in the conjecture (3.12) should be regarded in general as a random variable depending on the initial condition x_0 and therefore the conjecture should be refined as follows :

For any $\delta > 0$, $I\!P^{\Psi}(D > \delta) \to 0$ when the dimensionless parameter

$$\epsilon \equiv \epsilon(\psi_0, V) = \frac{\tilde{\lambda}(\psi_0, x_0)}{L_V} \tag{5.27}$$

goes to zero in probability, uniformly in ψ_0 and V.

5.7 Local Plane Wave Formation

If $\lambda_{\psi_0} \ll L$, the particle doesn't feel in a substantial way the effect of the potential up to the macroscopic time T. During this time, the kinetic energy dominates the potential energy and the free evolution is a good approximation. This produces, asyptotically in time, a local plane wave of the form

$$\psi(t) \simeq [\text{constant}] e^{i\frac{m}{\hbar}\frac{x^2}{2t}} \hat{\psi}_0(k_0(x)) \tag{5.28}$$

(which foollows from equation (4.15)).

Moreover, it turns out that, if $\lambda_{\psi_0} \ll L$, the local plane wave is produced in a very short time. This time, however, is very difficult to determine because it depends on the initial conditions, both the initial position and the initial wave function. We can give a rough estimate of this time by considering the local plane wave as a sum of non interacting "virtual" wave packets with disjoint position and momentum supports. Consider the simple example of an initial wave function composed of two overlapping wave packets with the same position spread σ_x and with opposite momenta p and $-p$. The time this wave function needs to become a local plane wave is just the time needed to cover a space equal to σ_x, i.e.

$$\tau \simeq \frac{\hbar}{\langle E \rangle}, \tag{5.29}$$

where $\langle E \rangle \simeq \frac{p^2}{2m}$ is the mean kinetic energy, because $\sigma_x \simeq \frac{\hbar}{\sigma_p}$ and $\sigma_p \simeq 2p$. We can assume that this rough estimate of τ is somehow of general validity.

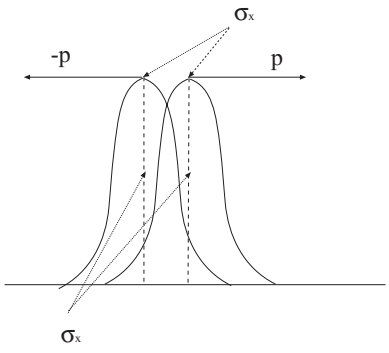

Figure 5.3: Time of formation of a local plane wave

Now consider the classical limit, that is $\lambda \ll L$. Given that $T = \frac{L}{v}$ and $v = \frac{\hbar}{m\lambda}$, this implies

$$\tau \ll T. \tag{5.30}$$

That is, taking the length scale L to be much greater than the mean de Broglie wave length λ is equivalent to considering times T much greater than the time τ of formation of a local plane wave. In other words, on the macroscopic scales $\left(\frac{x}{L}, \frac{t}{T}\right)$ on which we see classical behavior, the local plane wave has formed.

Chapter 6

Classical Limit and Decoherence

In this chapter we shall address the problem of extending our analysis to times greater than the first caustic time (see section 3.4).

6.1 The Problem of Caustics

To show how caustics can be a problem for the emergence of the classical in our model, consider, for example, the slowly varying potential case that we have discussed previously in section 4.2. In this case we have to write the propagator as the sum of propagators one for each path starting form the same initial position x_0 at time $t = 0$ and arriving at the same final position x with the same time t, so we have [26]

$$K^\epsilon(x,t;x_0,0) = \sum_j \frac{1}{(2\pi i\hbar\epsilon)^{d/2}}\sqrt{C_j(x,x_0;t)}\mathrm{e}^{\frac{i}{\hbar\epsilon}S_j^0(x,t;x_0,0)} + O(\epsilon), \tag{6.1}$$

where S_j^0 is the classical action relative to the j-th path connecting x with x_0 in a fixed time t. If the propagator is a sum of different paths, $K^\epsilon = \sum_j K_j^\epsilon$, then the wave function is a sum of different terms of the same kind as those appearing in equation (4.15). If we then apply the stationary phase method to each of them we obtain a velocity field that contains interference terms. In the limit $\epsilon \to 0$, this field has no limit but on the average we can define an average velocity field given by the weighted sum of the classical velocities, i.e.

$$v = \frac{1}{m}\frac{\sum_j \sqrt{C_j(x,0;t)}|\hat{\psi}_0(k_{0j}(x,t))|^2\nabla S_j^0(x,t;0,0)}{\sum_j \sqrt{C_j(x,0;t)}|\hat{\psi}_0(k_{0j}(x,t))|^2}. \tag{6.2}$$

This is not the classical velocity so we can conclude that, when we have interference, we haven't got the classical limit. Thus, starting from a problem describing one particle in an external potential, we are able to explain classical behavior only up to times shorter than the time t_c necessary to reach the first caustic. To sum up, caustics are a problem because different portions of the same wave function interfere with each other and we lose the classical behavior, even if $\lambda \ll L$.

6.2 Decoherence and the Problem of Caustics

We shall show that caustics do not do not cause any problems as they arise only in the highly idealized model we have considered so far: up to now we have neglected the interaction between the center of mass x of the body and the relative coordinates y (see section 3.1), as well as any perturbation due to the unavoidable interaction of the body with the external environment. These interactions produce *entanglement* between the center of mass x of the system and the other degrees of freedom y (where now y includes both the relative coordinates and the degrees of freedom of the environment). The effect of taking into account these interactions is what nowadays people call decoherence ([28],[23],[42],[43], [34]), which however is nothing but an *effective* description of all the effects that cannot be described by the external potential acting on the center of mass x. More precisely, this means that the right setting for discussing the dynamics of the body is to go beyond the approximation of motion in an external potential considered so far and to describe the system in terms of the Hamiltonian

$$H = H_{CM} + H^{(0)}_{rel,I} + H^{(0)}_{rel,E} + H^{I}_{int} + H^{E}_{int}, \tag{6.3}$$

where $H_{CM} = \left(H^{(0)} + V\right)_{CM}$ is the Hamiltonian of the center of mass coordinate x including the external potential, $H^{(0)}_{rel,I}$ and $H^{(0)}_{rel,E}$ are the "free" Hamiltonians of the relative coordinates and of the environment respectively, and the last two terms are interaction Hamiltonians between the center of mass and the internal as well as the external degrees of freedom. Those last two terms are the source of what could be called *internal* and *external decoherence* respectively.

Assume that

$$H_{int} = H^{I}_{int} + H^{E}_{int} \tag{6.4}$$

such a small perturbation of the unperturbed Hamiltonian

$$H_0 \equiv H_{CM} + H^{(0)}_{rel,I} + H^{(0)}_{rel,E}, \tag{6.5}$$

that the initial product structure

$$\Psi_0(x,y) = \psi_0(x) \otimes \phi_0(y) \tag{6.6}$$

is approximatively preserved by the dynamics, at least for sufficiently localized wave packets $\psi_0(x)$. Then, it is reasonable to expect that the effect of the perturbation is only that of selecting one of the "virtual" wave packets composing the local plane wave which is formed in absence of perturbation, forbidding in this way any kind of interference due to the caustics [1]. This is not unreasonable to expect, since one of the main achievements of the research on decoherence of the last years is to show how the elimination of these interference terms may occur. This has been proven for different models of environment, showing in this way the universality of the phenomenon of decoherence. To get a handle on how this may come about, one may consider an idealized model in which the center of mass is modelled by a material point and all the effect of the external and internal environment is described by n light particles hitting the center of mass of the body (see section 6.3).

In conclusion we may say that, for a system of N point–like particles composing a single macroscopic body with a small pair interaction U between the particles, in an external potential V, also slowly varying on the size of the body, and with an interaction H_{int} between the x coordinate and the y coordinate which is sufficiently small, we have classical limit on the macroscopic scales (3.10), if $\lambda \ll L$, because the local plane wave has been formed in the coordinate x of the center of mass of the body and decoherence avoids interference by selecting only small pieces of it. The same happens for a system of N point–like particle composing M macroscopic bodies with small pair interaction terms between the center of mass of the bodies and in an external potential which is almost constant for each particle of each body.

[1] Presumably, to solve the problem of caustics one does not need to appeal to any kind of external environment but for an extended macroscopic body H^I_{int} alone should be sufficient to eliminate the superpositions due to caustics.

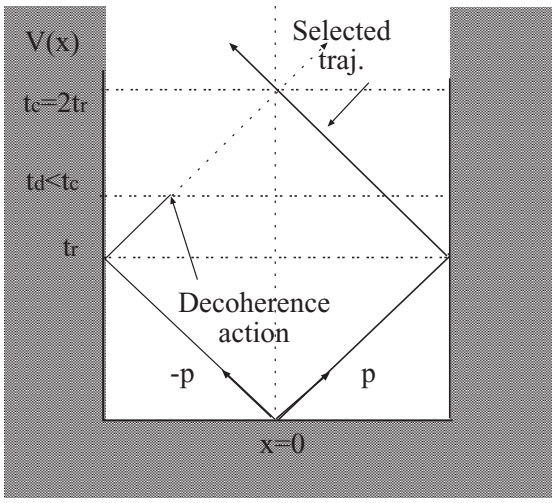

Figure 6.1: Decoherence avoids interference

Consider the example of an infinite potential well in two dimensions (one space and one time) centered around the origin and take two trajectories starting at $t = 0$ from the middle of the well ($x = 0$) with initial momenta $p_1 = p$ and $p_2 = -p$ of the same norm but with opposite directions, the first going to the right and the second to the left. After at a certain time t_r in $x = 0$ the two trajectories are reflected from the walls of the well potential. At a certain time $t_c = 2t_r$, the two classical trajectories meet again at $x = 0$. This is an example in which two paths starting from the same point $x = 0$ at $t = 0$ and arriving in the same point $x = 0$ at the same time t_c, join each other. This means that there are two velocity vectors tangent to the trajectory, p_1 on the left side and p_2 on the right, that is there is a caustic. Include now an external environment in the description of the system. To avoid interference at time t_c, it is sufficient that decoherence acts within a time $t_d < t_c$ in such a way that only one of the possible paths joining in $x = 0$ at time t_c is selected.

6.3 Simple Model of Decoherence

Consider the conditional wave function of the macroscopic body $\psi(x, t)$ as composed by two wave packets with opposite velocities. At a certain time the two wave packets interfere with each other. Let's analyse the effect of the interaction

term $H_{int} = H^I_{int} + H^E_{int}$. As a first crude model, one could take the variables y as describing the n light particles of mass μ much smaller than the mass M of the macroscopic body. In the approximation of absence of recoil of the macroscopic body after the scattering of a microscopic particle of the environment, one could argue [28] that the only effect of the environment is to select one of the two wave packets of $\psi(x,t)$. In fact, if $\psi_0(x) = \psi_{1,0}(x) + \psi_{2,0}(x)$ where $\psi_{1,0}$ is a wave packet centered around $\bar{x}_1$ and $\psi_{2,0}$ is centered around $\bar{x}_2$, after a single scattering of a light particle belonging to the environment, it can be shown [28] that the total wave function $\Psi_0(q)$ evolves as

$$\Psi_0(q) = \psi_0(x) \otimes \phi_0(y) \to \psi_1(x,t) \otimes \phi_{\bar{x}_1}(y,t) + \psi_2(x,t) \otimes \phi_{\bar{x}_2}(y,t). \tag{6.7}$$

In the above expression $\phi_{\bar{x}_i} = S\phi_i$, where S is the scattering matrix, which influences only the ϕ-functions because we are in the no recoil approximation. Note that this is not a measurement-like process because $\langle \phi_{\bar{x}_1}(y), \phi_{\bar{x}_2}(y') \rangle \simeq 1 - \delta$, where δ is a small quantity. But, as a result of a very big number n of isotropic scattering, it can be shown that [28] (see also [39]), in the limit of n big,

$$\langle \phi_{\bar{x}_1}(y), \phi_{\bar{x}_2}(y') \rangle \simeq \sum_{i=1}^{n} (1-\delta)^i \simeq \mathrm{e}^{-\delta} \to 0, \tag{6.8}$$

because δ is a small quantity. In this way we do have the same structure of a measurement process and we can conclude that the environment measures the body. It selects one of the two wave packets composing the conditional wave function of the macroscopic body.

The previous argument by Joos and Zeh is similar to the one given by Bohm and Hiley [8]. In fact, consider for example a planet exposed to the light from a star. Assume that either the photon hits the planet and is absorbed or there is no interaction between the photon and the planet. The wave function of each photon can be considered a plane wave but, after the interaction with the planet, it will be modified in such a way that beyond the planet, it forms a shadow (the wave function will be zero in the shadow and a plane wave elsewere). This shadow beyond the object is a cone limited by the diffraction angle α. Interference is possible if the wave functions of the light particles overlap and the number F of photons which don't give rise to overlap is given by the ratio of the volume of the cylinder whose base is the planet and whose height is $h \simeq \frac{R}{\alpha}$, where R is the

radius of the planet, and the total volume $V = L^3$ (consider everything enclosed in an imaginary box of size L). It can be shown that, if one considers only one incident particle, then the probability of having no interference $P_1 = F$ is small (it is proportional to $1/V$), while for n incoming particles with n very large, we have that the probability to destroy interference is

$$P_n = (1 - F)^n \simeq \mathrm{e}^{-nF} \to 0. \tag{6.9}$$

It can be easily shown that $nF = \frac{n}{V} V_{cone}$, where V is the total volume of the imaginary box and V_{cone} is the volume of the shadow cone, that is the probability of having no interference is proportional to the number of particles in the shadow. Note the similarity of the two approaches comparing equation (6.9) with equation (6.8).

Chapter 7

General Structure of the Classical Limit

So far we have assumed that the environment and the internal degrees of freedom are coupled to the center of mass so weakly that in a first approximation we can consider their contribution to the motion of the center of mass as totally negligible (reduction of the motion to the motion of the center of mass to a one body problem in an external potential), and, in a second approximation, as providing a very small perturbation to the one body problem, so that their effects are simply that of cancelling undesired interference terms. Both approximations rely on the assumption that the interaction Hamiltonian H_{int} is so small that the complete dynamical evolution does not generate any entanglement with the environment (that we call the y-system from now on) for a narrow wave packet in the center of mass coordinate (the y-system from now on).

7.1 Instability with Respect to Perturbations

We would like to now give a quantitative estimate of how smallthe effect of the environment should be in order that a narrow wave packet evolves autonomously according to Schrödinger's equation. Suppose the total initial wave function is a product and that the initial wave function ψ_0 of the x-system is a wave packet as in equation (6.6). Let $\psi_0(x)$ be a narrow wave packet, say with support of order λ, and let $U_t^{(x,y)}$ be the evolution operator generated by the total Hamiltonian

$H = H_{CM} + H_{rel} + H_{int}$, where $H_{rel} = H^{(0)}_{rel,I} + H^{(0)}_{rel,E}$ as in equation (6.3). We ask under which conditions the approximation

$$\Psi_t(x,y) = U_t^{(x,y)} \left(\psi_0(x) \otimes \phi_0(y)\right) \simeq \left(U_t^{(x)}\psi_0\right)(x) \otimes \left(U_t^{(y)}\phi_0\right)(y) \tag{7.1}$$

is justified. First of all, one should observe that this question is meaningful only if we specify the time T up to which we demand equation (7.1) to be valid. It is in fact rather clear that if we choose the time t very large, no matter how small H_{int} is, then the effect of H_{int} will in general be far from being negligible and (7.1) will break down. In our case, T is indeed fixed by our conjecture on the emergence of classicality: T must be of the order of the macroscopic time

$$T = \frac{L}{v}, \quad v = \frac{\hbar}{m\lambda_{\psi_0}}, \tag{7.2}$$

which fixes the time scale on which we expect classical behavior, provided that $\lambda \ll L$. So we demand

$$U_T^{(x,y)} \left(\psi_0(x) \otimes \phi_0(y)\right) \simeq \left(U_T^{(x)}\psi_0\right)(x) \otimes \left(U_T^{(y)}\phi_0\right)(y), \tag{7.3}$$

with T given by (7.2). For sake of concreteness, consider the interaction between the environment and the x-system as modelled by elastic collisions of light particles with the x-system [28] and assume that the rate of collision is $1/T$. The momentum transfer to the center of mass is of order p, where p is the incoming momentum of the light particle (for sake of simplicity we'll restrict to one dimensional motion). Therefore, assuming for simplicity free motion, in the time T the center of mass has covered the distance

$$\bar{L} = L + \frac{p}{m}T$$

Thus, assuming transport of the wave packet along the classical trajectories (since $\lambda \ll L$), we have that at time T the wave packet is centered in $\bar{L}$ (if initially centered in zero).

Now, if

$$\frac{p}{m}T \geq \lambda\,, \quad \text{i.e.}\,, \quad p \geq \frac{\lambda m}{T} = \frac{\hbar}{L},$$

the support of $\psi_T(x)$ will be disjoint from that of $U_T^{(x)}\psi_0(x)$ which has support around L and thus ψ_T will be orthogonal to $U_T^{(x)}\psi_0(x)$. In other words, for

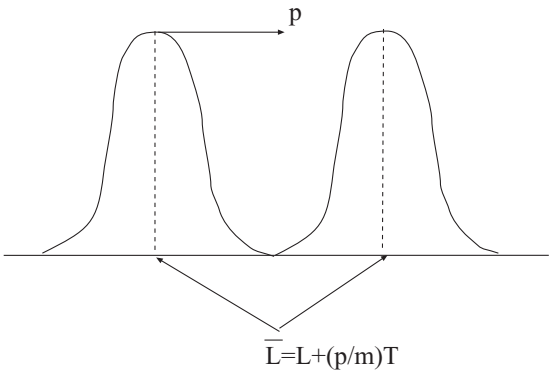

Figure 7.1: Minimal momentum transfer to destroy Schrödinger's equation

momentum transfer

$$p \gtrsim \frac{\hbar}{L} \equiv p_{crit} \tag{7.4}$$

in the time $[0, T]$, the Schrödinger's equation for the center of mass alone ceases to be a good approximation in the time scale T. Whenever (7.4) holds, we cannot describe the motion of the center of mass within the approximation of the one body problem in an external potential but we have to consider the effect of the complete dynamics given by the Hamiltonian $H = H_{CM} + H_{rel} + H_{int}$.

An estimate of the critical momentum to destroy Schrödinger's equation can also be deduced from a very elementary argument based on a pure stationary analysis. Roughly, one can proceed in the following way: consider a particle in a one dimensional box of size L. Then the spacing among energy levels is

$$\Delta E \simeq \frac{\hbar^2 n^2}{mL^2}$$

and the minimal momentum transfer necessary to produce a transition is $p = \frac{\hbar}{L}$.

7.2 Classical Limit in the General Case

In realistic situations, H_{int} is small but not so much to preserve Schrödinger's evolution. We shall then consider the Heisenberg equations for the center of mass x, which are more stable than Schrödinger's equation under perturbations[1],

[1] While the distance of the two wave functions with and without the perturbation is big in the time scale T, the distance between the operators with and without the perturbation in the same temporal scale T is of the order of the perturbation.

such that we can consider H_{int} as a real perturbation of H_0. The Heisenberg equations for the center of mass of the body are

$$\dot{x}_t = \frac{p_t}{m}, \quad \dot{p}_t = F(x_t) + F_{int}(x_t, y_t), \tag{7.5}$$

where $F_{int} = -\nabla V_{int}$. These two coupled first order equations can be written in terms of a second order equation

$$m\ddot{x}_t = F(x_t) + F_{int}(x_t, y_t) \tag{7.6}$$

where F_{int} is small. When we take the mean value on the total initial wave function $\Psi_0(q)$ we obtain

$$m\langle\ddot{x}_t\rangle = \langle F(x_t)\rangle. \tag{7.7}$$

Now the question is: under which conditions can we conclude that the actual motion of the center of mass is approximatively classical, that is

$$m\ddot{X}_t \simeq F(X_t)? \tag{7.8}$$

If $\Psi(q,t)$ is a local plane wave in x, slowly varying in y [2], that is if

$$\lambda(x,y,t) = \frac{\hbar}{|\nabla_x S(x,y,t)|} \quad , \qquad \nabla_x \lambda(x,y) \ll 1, \tag{7.9}$$

$$\left|\frac{\nabla_x^2 R(x,y)}{R(x,y)}\right| \lambda(x,y) \ll 1 \quad , \qquad \left|\frac{\nabla_x^2 S(x,y)}{S(x,y)}\right| \lambda(x,y) \ll 1, \tag{7.10}$$

where $R(x,y)$, $S(x,y)$ and $\lambda(x,y)$ are slowly varying function in y, then we can divide the local plane wave in a sum of wave packets in x and apply, provided that $\lambda \ll L$, to each of them the Ehrenfest–Goldstein argument of section 5.4 at a fixed time t to conclude that Newton's equation holds for a given time.

We have now to guarantee the validity of Newton's equation for later times: differently from what we had previously in section 5.3, we need that at different times the different guiding wave packets are selected according to classical laws. In fact now the local plane wave depends also on y and we must forbid the Bohmian trajectory to jump from one wave packet to the other in some weird way. But we know that, because of the formation of local plane waves in x, p evolves

[2] This is due to the fact that, to be in the conditions in which we have classical motion, we need that the environment doesn't perturb too much the local plane wave structure.

according to $\nabla_x S$, where the classical action $S = S^0$ is also a slowly varying function of y. It is this requirement on $p(x,y) = \nabla_x S^0(x,y)$ that guarantees that the Bohmian motion is really classical.

7.3 Local Plane Wave as a General Structure

There is an open question, in general very difficult to answer to, which is why should a local plane wave in x, slowly varying in y, be produced as soon as $\lambda \ll L$. We can give an elementary argument partially supporting this claim. If the perturbation is small, then we can assume that the wave function is given by

$$\Psi_t(x,y) \simeq \int\int K_x(x,x_0;t,0)K_y(y,y_0;t,0)\Psi_0(x_0,y_0)dx_0dy_0 = \quad (7.11)$$

$$= \int K_y(y,y_0;t,0)\tilde{\Psi}_0(x,y_0)dy_0, \quad (7.12)$$

where K_i, $i = x,y$ are the free propagators (starting from x_0 and y_0 at time $t = 0$ and arriving in x and y at time t) and

$$\tilde{\Psi}_t(x,\bar{y}) = \int K_x(x,x_0;t,0)\Psi_t(x_0,\bar{y})dx_0. \quad (7.13)$$

If Ψ_0 is a slowly varying function in y, then Ψ_t is a local plane wave in x, slowly varying in y. In fact, consider the long time asymptotic form of $\tilde{\Psi}_t(x,\bar{y})$ (suppose that the time needed to become a local plane wave in x is much shorter than the time needed for the perturbation to modify the structure)

$$\tilde{\Psi}_t(x,\bar{y}) \simeq [\text{cost}]\mathrm{e}^{i\frac{m}{\hbar}\frac{x^2}{2t}}\hat{\psi}_0(k_0(x,\bar{y})), \quad (7.14)$$

where $k_0 \simeq \frac{m}{\hbar}\frac{x}{t} + O(\bar{y})$: it is a local plane wave in x for any initial condition x_0. Then, if $\tilde{\Psi}$ is slowly varying in y, we have approximatively [3] that

$$\Psi_t(x,y) \simeq \tilde{\Psi}_t(x,y). \quad (7.15)$$

7.4 As Matters Stand

To sum up, we have seen what happens (chapters 3 and 4) to an initial product wave function in the external potential approximation. Our conjecture is that we

[3] Using some regularizer in the propagator.

have classical motion on the macroscopic scales $\left(\frac{x}{L}, \frac{t}{T}\right)$ defined by (3.10) for any sequence of initial wave functions and external potentials, provided that $\lambda \ll L$. Moreover, we have seen in chapter 5 that the emergence of local plane wave structure is crucial to establish the classical limit, almost in that approximation. After that we showed that the structure of the emergence of the classical motion remains almost the same even if we consider an initial product wave function in a more realistic model in which the coupling between the center of mass and the other degrees of freedom is taken into account. It has been shown in this chapter that Schrödinger's evolution for the conditional wave function of the x-system is quickly destroyed. Nevertheless, the conjecture is still valid, that is there is production of local plane waves in the center of mass x of the body, with weak dependence on the other degrees of freedom y. This leads to the classical motion for the x-system on the macroscopic scales defined above in the limit $\epsilon \to 0$. Then the general structure of the classical limit is captured by our conjecture independently of the presence or not of the environment. Given that, due to Schrödinger's evolution, the initial wave function spreads, and the role of the environment is just to cut it into wave packets which evolve classically because of the Ehrenfest theorem.

The key ingredient of our analysis for the explanation of the emergence of the classical world is that with each initial configuration x_0, as soon as the local plane wave has formed, is associated a guiding wave packet with a definite wave length $\lambda(x,t)$ which locally determines the particle dynamics according to the local de Broglie relation $p(x,t) = \frac{\hbar}{\lambda(x,t)}$, which, for $\lambda \ll L$, evolves according to the classical laws. This means that the classical limit can be symbolically expressed as

$$(\Psi, X) \to (P, X), \tag{7.16}$$

where (Ψ, X) is the complete quantum state description and (P, X) is the complete classical state description in terms of momentum P and position X. All the relevant macroscopic information contained in the couple (Ψ, X), in the limit, is embodied in the couple (P, X), which is the only robust quantity. In other words, as far as the macroscopic dynamics of X is concerned, only the information carried by P is relevant.

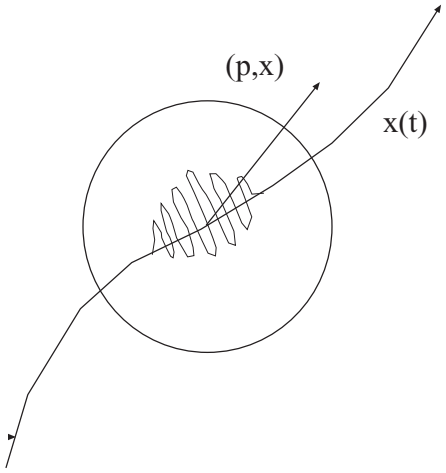

Figure 7.2: The quantum state (Ψ, X) becomes the classical stare (P, X) in the limit $\epsilon \to 0$

7.5 On the Typicality of the Classical World

We have seen that the local plane wave structure is the shape of the wave function corresponding to the emergence of the classical world. Now let's ask if the classicality is a general feature or not. Does a typical (generic) initial condition of the universe lead to a classical world or must we have special initial conditions to ensure that? This is equivalent to asking ourselves if a wave function that is a local plane wave is developed by the dynamics starting from the majority of the initial conditions of the universe.

If we consider an infinite universe, the expanding character of Schrödinger's equation makes the set of local plane waves an "attractor" for the dynamics. Thus, in this case, the local plane wave plays the role of the "equilibrium" wave function. On the other hand, for a compact universe (as is suggested by all present cosmological models) the equilibrium wave function is a wave function composed of a sum of local plane waves. This is due to interference between the waves reflected by the "edges" of the universe. Given that the states which give rise to equilibrium are the great majority and given that local plane wave is not an equilibrium wave function, we can conclude that, within a compact universe, the classical world is somehow special, atypical because it is produced by a very special kind of wave function, the local plane wave.

There are other questions that we could ask ourselves: is the atypicality of the classical world connected with the second law of thermodynamics? Is it connected with the emergence of life (see, e.g., [21])? We believe that these are relevant and interesting questions but they are beyond the scope of this work.

7.6 Quantum Chaos and the Classical World

Classical dynamics is often chaotic, that is it shows sensitive dependence on initial conditions, while quantum mechanics doesn't. But classical mechanics is, in some sense, a limit of quantum mechanics, therefore one should explain how chaos arises in the macroscopic world, given that the "true" microscopic theory is quantum mechanics.

It has been often claimed (see, e.g., [4]) that it is the singularity of the classical limit, interpreted simply as the limit $\hbar \to 0$, that produces chaos in the classical world as emerging from quantum mechanics. But what exactly do they mean by "singular limit"? Consider a dynamics depending on a given parameter α. We say that the limit $\alpha \to 0$ is singular when the features of the dynamics changes dramatically before and after the limit. With this we mean that there is no anticipation whatsoever of the behavior of the system for $\alpha > 0$ in the behavior of the system for $\alpha = 0$. So we are dealing with some discontinuous function in $\alpha = 0$.

First of all, one should realize that, if h is a constant of nature, it cannot be exactly zero. Thus it is impossible that some features of the classical world are strictly dependent on the singularity of the limit $\hbar \to 0$. In other words, the singular behavior reveals itself *only* for $\hbar$ exactly equal to zero and, given that $\hbar = 1.054 \cdot 10^{-27}$ erg·s, nothing of physical relevance can be deduced from the singularity of the limit.

As for the emergence of classical chaotic behavior, it has been shown in [14] that, while ordinary quantum mechanics is unable to explain the emergence of classical chaos, in Bohmian mechanics classical chaotic behavior naturally arises from the fact that the configurations are randomly distributed according to the Born statistical rule. So - without any misguided singularity argument - one can account also for chaos in the classical world within the framework of Bohmian

mechanics.

Chapter 8

Related Works

As we mentioned several times, one of the standard ways to look at the classical limit is that the distribution of quantum observables at a given time converges to the distribution of the corresponding classical observables, the latter being functions on phase space (see Egorov's theorem in section 8.3 for a detailed statement). In the same category of results fall those which show that the Wigner function converges to a classical distribution.

We would like to remark that such results on a quantum system imply that the one time distributions of the configuration of the Bohmian system converges to the classical distribution. If one starts to prove theorems about the classical limit along the lines we proposed, such convergences of single time distributions would be the first thing one should try. One must be aware however, that these results are much too weak to assert classical behaviour in the limit.

8.1 Convergence of Probability Distributions

From our Bohmian perspective, convergence of probability distributions follow straightforwardly from the convergence of trajectories. Convergence of probability distributions is not sufficient to explain the emergence of classical laws for the general reasons we have already pointed out in the introduction. Moreover, convergence of probability distributions is not even necessary in our model of a particle in external potential. In this regard we would like to make some remarks: consider the family of slowly varying potentials (see section 4.2). If we

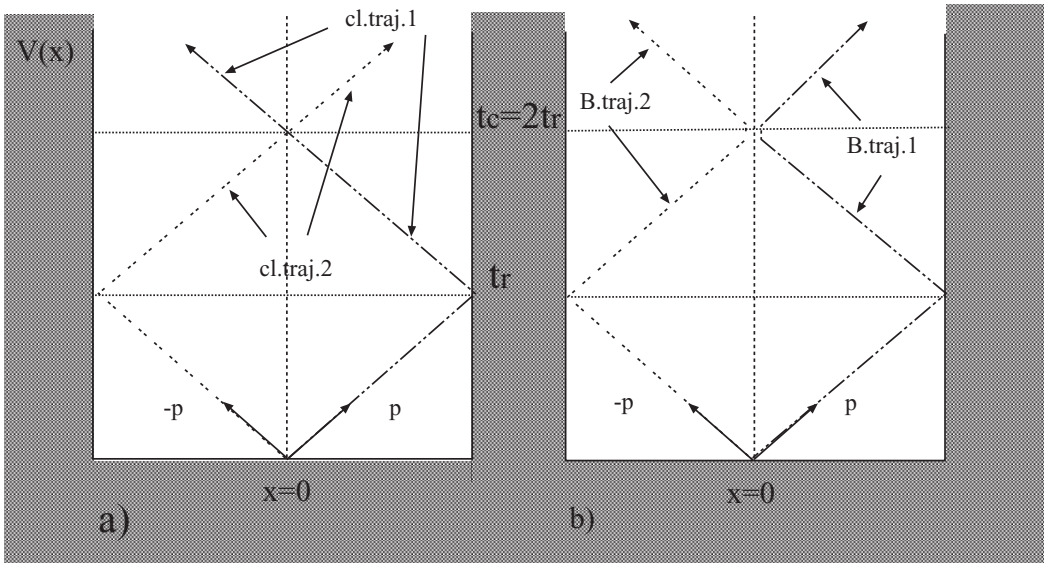

Figure 8.1: Classical and Bohmian trajectories in a one dimensional infinite well potential

look at the probability distribution of position in the presence of caustics, we see that, differently from what happens to the velocity field, the interference terms cancel each other such that the probability distributions converge to the classical distributions. In fact

$$|\psi^{\epsilon}(x,t)|^2 = \sum_j C_j(x,0;t) \left(\frac{i}{\hbar}\right)^d |\hat{\psi}_0(k_{0j}(x,t))|^2 + O(\epsilon). \tag{8.1}$$

The fact that the probability distribution of position converges to the classical one while the quantum velocity field doesn't converge to the classical velocity field (as shown at the beginning of this section) seems confusing but it can be explained as follows.

Consider the example we gave in section 6.2 of a one dimensional infinite potential well.

The classical trajectory followed by the particle with momentum $p_1 = p$ should cross the classical trajectory followed by the particle with momentum $p_2 = -p$. The first, arriving from the right part of the well, should go on to the left, while the second, after having crossed the first, should go on to the right. From equation (6.2), we have $v = 0$ at the time t_c, so that Bohmian trajectories, differently from what happens classically, don't cross each other at time t_c but are reflected

as from an invisible wall positioned in the center of the well. In this way, the Bohmian trajectory coming from the right continues to propagate in the right side while the one on the left stays on the left side. Bohmian trajectories, therefore, follow the classical ones except for those times $t > t_c$ after which the Bohmian trajectory of the particle with momentum p_1 follows the classical trajectory of the particle with momentum p_2 and vice versa. The probability distribution doesn't distinguish between different trajectories followed by different initial momenta, because actually it is the sum of both of them. This fact allows the quantum probability distribution to converge to the classical one, even if, for times longer tha caustic time formation t_c, we don't have classical behavior (Bohmian trajectories don't converge to classical trajectories). This convergence of probability distributions should not be regarded as an explanation of the classical limit, as shown by the discussion of the previous example, where the motion is highly non classical (as decoherence is not taken into account).

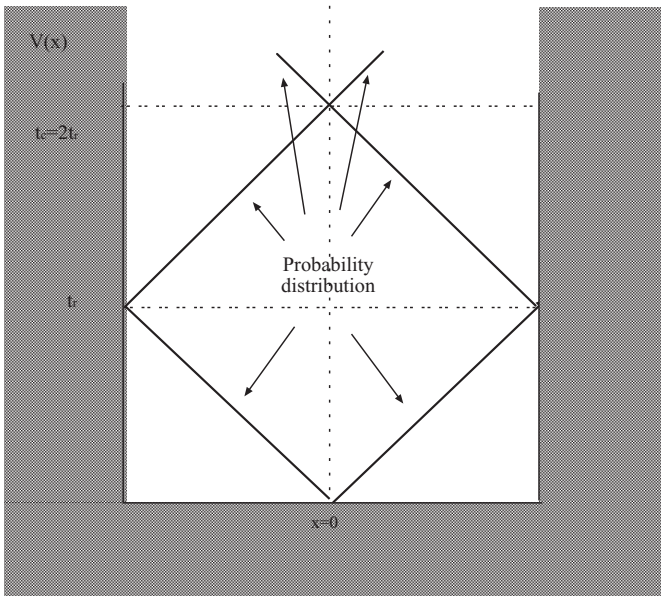

Figure 8.2: Classical and Bohmian probability distributions in a one dimensional infinite well potential

8.2 Connection with Stationary WKB

First of all, it must be stressed that the scope of WKB is to show how probability distributions become the classical ones. By itself, it tells nothing (given that it has been formulated within the framework of ordinary quantum mechanics) about the convergence between quantum and classical trajectories, as clarified in section 8.1. Nevertheless, there must be a connection between our analysis and the one given by WKB. We have just stressed in section 4.1 that the Maslov results [32] and the time dependent WKB are basically the same, but what about time independent WKB? In this case, the wave function is stationary and it is of the local plane wave form $\psi(x,t) = R(x,t)\,\mathrm{e}^{\frac{S(x,t)}{\hbar}}$. In the limit $\hbar \to 0$, it is such that S is the classical action and $|R(x,t)|^2$ is the classical probability distribution. The basic condition of validity of the WKB approximation is [29]

$$\frac{\hbar\,|\nabla V|}{[2m(E-V)]^{3/2}} \ll 1, \tag{8.2}$$

together with other conditions involving higher derivatives of V (see, e.g. [37]). Condition (8.2) is equivalent to having a slowly varying local wave length such that $\nabla\lambda \ll 1$. Recall that the local plane wave can be written as a sum of "virtual" wave packets. In the case of WKB, they may evolve in such a way that ψ remains stationary. Each effective wave packet doesn't necessarily move classically because WKB conditions by themselves do not imply the Ehrenfest condition $\lambda \ll \sqrt{\frac{V'}{V'''}}$. The role of the WKB conditions is, thus, not just that of preserving the local plane wave structure but, more strongly, that of requiring stationarity.

8.3 Convergence of Observables

Time dependent and time independent WKB are only very special ways of handling the classical limit in the framework of standard quantum mechanics. A more general way is in terms of convergence of the Wigner function or, equivalently, of expectations of semiclassical observables. Since this approach is mathematically well understood and easy to describe we give, for completeness, a short overview.

The Wigner transform $W^\hbar(\psi)(q,p)$ [5],[40] of a wave function $\psi \in L^2(\mathbb{R}^d)$ is the distribution on the classical phase space $\mathbb{R}^{2d}$ defined through

$$W^\hbar(\psi)(q,p) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} dy\, \psi^*(q+\hbar y/2)\,\psi(q-\hbar y/2)\,\mathrm{e}^{ixy\cdot p}. \tag{8.3}$$

One often would like to think of $W^\hbar(\psi)(q,p)$ as a probability distribution on phase space, but since $W^\hbar(\psi)(q,p)$ may be negative this interpretation makes no sense. It should be noted that, though $\psi^\hbar$ does not have any limit for $\hbar \to 0$, one can show that, for a large class of families $\psi^\hbar$, the Wigner transform of $\psi^\hbar$, $W^\hbar(\psi^\epsilon)(dq\,dp)$, converges (weakly on suitable test functions) to some probability measure $\mu^{\psi^\hbar}$ on the phase space in the limit $\hbar \to 0$. In this sense one might associate a classical distribution $\mu^{\psi^\hbar}$ with a family of wave functions $\psi^\hbar$ in the limit $\hbar \to 0$.

Furthermore, if $\psi_t^\hbar(q)$ is the solution of the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi_t^\hbar = -\frac{\hbar^2}{2m}\nabla^2\psi_t^\hbar + V\psi_t^\hbar \tag{8.4}$$

with initial condition $\psi_0^\hbar(q)$, then the weak limit $\text{w-lim}_{\hbar\to 0}\, W^\hbar(\psi_0^\hbar) = \mu_0$ is a probability measure on phase space. Therefore, the Wigner transform at later times converges to $\mu_t = \mu_0 \circ \Phi_{\mathrm{cl}}^{-t}$, i.e.

$$\mathrm{w}-\lim_{\hbar\to 0} W^\hbar(\psi_t^\hbar) = \mu_t = \mu_0 \circ \Phi_{\mathrm{cl}}^{-t}\,, \tag{8.5}$$

where $\Phi^t : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ denotes the classical flow generated by the classical Hamiltonian $H(q,p) = \frac{p^2}{2m} + V(q)$.

In this sense such a family $\psi^\hbar$ does not only define a distribution on phase space, but time evolutions and the limit $\hbar \to 0$ commute.

Actually, the convergence of the Wigner transform follows easily from Egorov's theorem [18], which we shall now explain. Let $a \in C_{\mathrm{b}}^\infty(\mathbb{R}^{2d}, \mathbb{R})$ be a "classical observable", i.e. a function on phase space. Then its Weyl quantization [35] $\hat{a}^\hbar$ is a bounded operator on $L^2(\mathbb{R}^d)$ whose action on wave functions $\psi \in S(\mathbb{R}^d)$ (the Schwartz space of fastly decreasing functions on $\mathbb{R}^d$), is given by

$$(\hat{a}^\epsilon\psi)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} dy\, dp\, a\left((x+y)/2, \hbar p\right)\mathrm{e}^{-i(x-y)\cdot p}\,\psi(y)\,. \tag{8.6}$$

The concept of Weyl quantization is dual to the Wigner transform in the sense that

$$\langle\psi,\, \hat{a}^\hbar\psi\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} dq\, dp\, W^\hbar(\psi)(q,p)\, a(q,p), \tag{8.7}$$

i.e. the quantum mechanical expectation value of the Weyl quantization of a function on phase space equals, by construction, the expectation of the function with respect to the Wigner distribution.

The statement of Egorov's theorem is, to leading order in $\hbar$, that time evolution and "quantization" commute. More precisely one has

Egorov's Theorem: Let $a \in C^\infty_{\rm b}(I\!R^{2d}, I\!R)$ and $H^\hbar = -\frac{\hbar^2}{2m}\nabla^2 + V(q)$ with $V \in C^\infty_{\rm b}(I\!R^d, I\!R)$, then for each $T < \infty$ there is a constant C_T such that for $t \in [-T, T]$

$$\left\| {\rm e}^{iH^\hbar t/\hbar}\, \hat{a}^\hbar\, {\rm e}^{-iH^\hbar t/\hbar} - \widehat{a \circ \Phi^t} \right\| \leq C_T \hbar, \tag{8.8}$$

where $\|...\|$ denotes the uniform norm on the bounded operators in $L^2(I\!R^d)$.

As remarked before, from Egorov's theorem the statement about the convergence of Wigner distributions follows immediately. However, Egorov's theorem is a much stronger statement because of its uniformity, but it has a surprisingly simple proof, see, e.g., [35]. Furthermore, one can also compute higher order in $\hbar$ "quantum"-corrections to the classical time evolution and all in all it appears to be an extremely strong result.

However, as emphasized before, the convergence of distributions does in no way imply anything about the convergence of the underlying trajectories in Bohmian mechanics. And indeed, we saw in section 8.1 that at caustics in configuration space the Bohmian trajectories are not even approximately classical, while the "semiclassical" distributions do not even see the caustics. This is, roughly speaking, due to the fact that the semiclassical distributions and the semiclassical observables are sensible only to "macroscopic" features of the wave function.

Appendix A

Simple Examples

It might be useful to see in a simple example what happens to a particular wave function in those special classical limits we have previously analyzed in section 4.

Consider a free Gaussian wave packet centered at zero with initial position spread σ_0 and an initial velocity $u = \frac{\hbar \bar{k}}{m}$

$$\psi_0(x) = \frac{1}{[2\pi\sigma_0^2]^{d/4}} \exp\left[-\frac{x^2}{4\sigma_0^2} + i\frac{m}{\hbar}u \cdot x\right], \tag{A.1}$$

where d is the dimension of the space.

For this wave function we can see, by direct calculation, that we obtain the same results that we have claimed are valid for a more general wave function.

The free evolution leads to

$$\psi(x,t) = \frac{\exp\left[-\frac{(x-ut)^2}{4\sigma_0^2\left(1+\frac{i\hbar t}{2m\sigma_0^2}\right)} + \frac{im}{\hbar}\left(u \cdot x - \frac{u^2 t}{2}\right)\right]}{\left[\sqrt{2\pi}\sigma_0(1+\frac{i\hbar t}{2m\sigma_0^2})\right]^{d/2}} = R(x,t)\mathrm{e}^{\frac{i}{\hbar}S(x,t)}, \tag{A.2}$$

where

$$R(x,t) = \left[\frac{1 - i\frac{\hbar t}{2m\sigma_0^2}}{\sqrt{2\pi}\sigma_0\left(1+\frac{i\hbar t}{2m\sigma_0^2}\right)}\right]^{d/2} \exp\left[-\frac{(x-ut)^2}{4\sigma_0^2\left(1+\frac{i\hbar t}{2m\sigma_0^2}\right)}\right] \tag{A.3}$$

and

$$S(x,t) = m\left(u \cdot x - \frac{u^2 t}{2}\right) + \frac{\left(\frac{\hbar t}{2m\sigma_0^2}\right)(x-ut)^2}{4\sigma_0^2\left[1+\left(\frac{\hbar t}{2m\sigma_0^2}\right)^2\right]}. \tag{A.4}$$

For later convenience, we will now write down some relevant characteristics of this wave function. The spread at time t evolves as

$$\sigma_t^2 = \sigma_0^2 \left[1 + \left(\frac{\hbar t}{2m\sigma_0^2} \right)^2 \right], \tag{A.5}$$

the velocity field $v(x,t) = \frac{1}{m}\nabla S(x,t)$ is

$$v(x,t) = u + \frac{(x-ut)}{t} \frac{\left(\frac{\hbar t}{2m\sigma_0^2}\right)^2}{2\sigma_0^2 \left[1 + \left(\frac{\hbar t}{2m\sigma_0^2}\right)^2 \right]}. \tag{A.6}$$

By integration of the velocity field one obtains the Bohmian trajectory

$$X(t) = ut + x_0 \sqrt{1 + \left(\frac{\hbar t}{2m\sigma_0^2} \right)^2}, \tag{A.7}$$

where x_0 is the position at time $t = 0$.

We would like to study the classical limit of this free Gaussian wave packet. The conjecture (3.12) we have expressed in section 3.2 states that the degrees of freedom (3.11) on the macroscopic scales (3.10) defined by λ and L tends to zero as $\epsilon = \frac{\lambda}{L} \to 0$. In this case there is no potential so $\lambda \equiv \lambda_{\psi_0}$ is the de Broglie wave length defined according to equation (7) and $L \equiv L_o$, as we have seen in section 3.2, is any scale chosen by the observer, provided that $\lambda \ll L_o$.

We shall now show how this example can be examined as an example of both of the two special families we have discussed in the chapter 4.

A.1 Quasi Classical Wave Functions

Note that the Gaussian wave packet is a short wave length function, in fact it is of the form $\psi_0(x,t) = R_0(x)\mathrm{e}^{\frac{i}{\hbar}S_0(x)}$, as one can see from equation (A.1), where

$$R_0(x) = \frac{1}{[2\pi\sigma_0^2]^{d/4}} \exp\left[-\frac{x^2}{4\sigma_0^2} \right] \tag{A.8}$$

and

$$S_0(x) = mu \cdot x, \tag{A.9}$$

don't depend on $\hbar$. Consider now the limit as $\hbar \to 0$ of this quasi classical wave function as we have seen in section 4. The wave function on the macroscopic scales $\left(\frac{x}{L_o}, \frac{t}{T}\right)$ is

$$\psi^{\hbar}(x,t) = \frac{\exp\left[-\frac{(xL_o - utT)^2}{4\sigma_0^2\left(1+\frac{i\hbar tT}{2m\sigma_0^2}\right)} + \frac{im}{\hbar}\left(u \cdot xL_o - \frac{u^2tT}{2}\right)\right]}{\left[\sqrt{2\pi}\sigma_0(1+\frac{i\hbar tT}{2m\sigma_0^2})\right]^{d/2}}. \tag{A.10}$$

Given that both $T = \frac{L_o}{v}$ and L_o are kept constant in the limit $\hbar \to 0$, there is no real need to go on these macroscopic scales to see classical behavior: let's remain on the microscopic scales.

Note that also the wave function at time t is a short wave length function, and we can write it as

$$\psi^{\hbar}(x,t) = R^{\hbar}(x,t)\mathrm{e}^{\frac{i}{\hbar}S^{\hbar}(x,t)}, \tag{A.11}$$

where $R^{\hbar}$ and $S^{\hbar}$ are given from equations (A.3) and (A.4).

In the limit $\hbar \to 0$, we obtain

$$\psi^{(0)}(x,t) = R^{(0)}(x,t)\mathrm{e}^{\frac{i}{\hbar}S^{(0)}(x,t)} + O(\hbar). \tag{A.12}$$

$R^{(0)}(x,t) = R_0(x-ut) = |\frac{dx}{dx_0}|^{-1/2}R_0(x_0)$, where $x_0 = x - ut$, is the initial amplitude translated along the classical trajectory

$$R^{(0)}(x-ut) = \frac{1}{[2\pi\sigma_0^2]^{d/4}}\exp\left[-\frac{(x-ut)^2}{4\sigma_0^2}\right], \tag{A.13}$$

and $S^{(0)}(x,t)$ is the free classical action

$$S^{(0)}(x,t) = -\frac{mu^2t}{2} + mu \cdot x, \tag{A.14}$$

as we expected from Maslov's theorem.

Note that the wave packet doesn't spread, in fact

$$\sigma_t^{(0)} = \sigma_0 + O(\hbar). \tag{A.15}$$

Observe that the asymptotic of $\psi^{\hbar}(x,t)$ in the limit $\hbar \to 0$ is still a wave packet and, for this reason, a local plane wave.

The velocity field is $v^{(0)}(x,t) = \frac{\nabla S^{(0)}}{m}$, that is

$$v^{(0)}(x,t) = u + O(\hbar^2). \tag{A.16}$$

The limiting Bohmian motion as $\hbar \to 0$ is given by

$$X^{(0)}(t) = ut + x_0 + O(\hbar). \tag{A.17}$$

Note that the initial conditions are

$$X_0^{(0)} = x_0, \quad V_0^{(0)} = u, \tag{A.18}$$

in agreement with what we have established in section 4.

A.2 Slowly Varying Potentials

Consider again a free Gaussian wave packet, but now in the limit we studied in section 4.2.

The free case is an extreme case of a slowly varying potential where the only free parameter is $L = L_o \to +\infty$, while λ remains constant. Given that the macroscopic scales are defined as $\left(\frac{x}{L_o}, \frac{t}{T}\right)$, where $T = \frac{L_o}{v} = \frac{mL_o\lambda}{\hbar}$, in this limit both T and L_o are going to $+\infty$. The macroscopic scales can also be written in terms of the dimensionless parameter $\epsilon = \frac{\lambda}{L_o} \to 0$ as $\left(x\frac{\lambda}{L_o}, t\frac{\lambda}{L_o}\right)$. In this way we rescale space and time as $x' = x\epsilon$, $t' = t\epsilon$ and we take the limit $\epsilon \to 0$.

The wave function rescales as

$$\psi^\epsilon(x,t) = \frac{\exp\left[-\frac{(x-ut)^2}{4\epsilon^2\sigma_0^2\left(1+\frac{i\hbar t}{2m\epsilon\sigma_0^2}\right)} + \frac{im}{\hbar\epsilon}\left(u\cdot x - \frac{u^2 t}{2}\right)\right]}{\left[\sqrt{2\pi}\sigma_0\epsilon(1+\frac{i\hbar t}{2m\epsilon\sigma_0^2})\right]^{d/2}}. \tag{A.19}$$

We can rewrite it as

$$\psi^\epsilon(x,t) = R^\epsilon(x,t)\mathrm{e}^{\frac{i}{\hbar}S^\epsilon(x,t)}, \tag{A.20}$$

where R^ϵ and S^ϵ are given by

$$R^\epsilon(x,t) = \left[\frac{1 - i\frac{\hbar t}{2m\epsilon\sigma_0^2}}{\sqrt{2\pi}\sigma_0\left(1+\frac{i\hbar t}{2m\epsilon\sigma_0^2}\right)}\right]^{d/2} \exp\left[-\frac{(x-ut)^2}{4\epsilon^2\sigma_0^2\left[1+\frac{i\hbar t}{2m\epsilon\sigma_0^2}\right]}\right] \tag{A.21}$$

and

$$S^{\epsilon}(x,t) = m\left[u\cdot x - \frac{u^2 t}{2}\right] + \frac{\left(\frac{\hbar t}{2m\epsilon\sigma_0^2}\right)(x-ut)^2}{4\epsilon^2\sigma_0^2\left[1+\left(\frac{\hbar t}{2m\epsilon\sigma_0^2}\right)^2\right]}. \tag{A.22}$$

Taking the limit $\epsilon \to 0$, the wave packet becomes

$$\psi^{(0)}(x,t) = \left[i\sqrt{\frac{2}{\pi}}\frac{m\sigma_0}{t\hbar}\right]^{d/2} \exp\left[-(x-ut)^2\left(\frac{m\sigma_0}{\hbar t}\right)^2 + \frac{i}{\hbar\epsilon}\frac{mx^2}{2t}\right] + O(\epsilon). \tag{A.23}$$

We can rewrite as

$$\psi^{(0)}(x,t) = R^{(0)}(x,t)\mathrm{e}^{\frac{i}{\hbar\epsilon}S^{(0)}(x,t)} + O(\epsilon). \tag{A.24}$$

We have that

$$\begin{aligned} R^{(0)}(x,t) &= \left[i\sqrt{\frac{2}{\pi}}\frac{m\sigma_0}{t\hbar}\right]^{d/2} \exp\left[-(x-ut)^2\left(\frac{m\sigma_0}{\hbar t}\right)^2\right] = \\ &= \left(\frac{i}{\hbar}\right)^{d/2}\sqrt{C(x,0;t)}\left|\hat{\psi}_0^{(0)}(k_0(x,t))\right|^2, \end{aligned} \tag{A.25}$$

where

$$\begin{aligned} \hat{\psi}_0^{(0)}(k_0(x,t)) &= \left(\sqrt{\frac{2}{\pi}}\sigma_0\right)^{d/2} \exp\left[-(k_0-\bar{k})^2\sigma_0^2\right] = \\ &= \left(\sqrt{\frac{2}{\pi}}\sigma_0\right)^{d/2} \exp\left[-(x-ut)^2\left(\frac{m\sigma_0}{\hbar t}\right)^2\right] \end{aligned} \tag{A.26}$$

is the Fourier transform of the initial wave function. In fact $\bar{k} = \frac{m}{\hbar}u$ is the mean initial wave number, $k_0 = \frac{m}{\hbar}\frac{x}{t}$ is the initial momentum which makes the phase stationary and $C(x,0;t) = \left[\frac{m}{t}\right]^{d/2}$ is the factor in front of the propagator. $S^{(0)}(x,t)$ is the classical free action with initial position x_o

$$S^{(0)}(x,t) = \frac{mx^2}{2t}. \tag{A.27}$$

From equation (A.24) we see that the dynamics again produces a local plane wave. The wave function is totally spread out, in fact

$$\sigma^{\epsilon 2} = \sigma_0^2\epsilon^2\left[1+\left(\frac{\hbar t}{2m\epsilon\sigma_0^2}\right)^2\right], \tag{A.28}$$

so that the limiting spread for $\epsilon \to 0$ is

$$\sigma_t^{(0)} = \frac{\hbar}{2m\sigma_0} t + O(\epsilon). \tag{A.29}$$

The velocity field is

$$v^{(0)}(x,t) = \frac{x}{t} + O(\epsilon). \tag{A.30}$$

The Bohmian trajectory in the limit $\epsilon \to 0$ is

$$X^{(0)}(t) = \left[u + \frac{\hbar}{2m\sigma_0^2} x_0 \right] t + O(\epsilon). \tag{A.31}$$

The initial conditions in this case are

$$X_0^{(0)} = 0, \quad V_0^{(0)} = u + \frac{\hbar}{2m\sigma_0^2} x_0. \tag{A.32}$$

The probability distributions of initial conditions are given by

$$\rho(x_0 = 0, 0) = |\psi_0^{(0)}(0)|^2 = \delta(D) \tag{A.33}$$

and

$$\rho(v_0, 0) = \left(\frac{m}{\hbar} \right)^d |\hat{\psi}_0(k_0(v_0))|^2, \tag{A.34}$$

as predicted by equation (4.24).

Appendix B

Uniformity of the Convergence in Simple Examples

In appendix A we have shown through the examples how the one-parameter families lead to classical behavior. What we still have not proved is the uniformity of the convergence of the deviation from classicality D defined by equation (3.11) in the macroscopic scales (3.10) to zero for any sequence of initial wave function and external potential (ψ_0^λ, V^L) in the limit $\epsilon = \frac{\lambda}{L} \to 0$.

In this section we'll analyze again the free Gaussian wave packet and we'll show that the probability distribution of the deviation from classicality in the macroscopic scales tends uniformly to a delta-function centered at zero as $\epsilon \to 0$ for these cases.

Now we want to see in what sense the conjecture that we have stated in section 3.2 is true.

There are different conditions that might lead to classical motion, e.g.:

-A *SUPER STRONG CONDITION* Given the mean de Broglie wave length $\lambda = \frac{\hbar}{mv}$, where $v = \hbar\sqrt{\langle\psi_0, -\nabla^2\psi_0\rangle)}$, then the deviation from classicality on the macroscopic scales defined by λ and L $D(xL, tT) \to 0$ as $\epsilon \to 0$.

-B *STRONG CONDITION* Given the mean de Broglie wave length $\lambda = \frac{\hbar}{mv}$, where $v = \hbar\sqrt{\langle\psi_0, -\nabla^2\psi_0\rangle)}$, then the deviation from classicality on the macroscopic scales defined by λ and L on the Bohmian trajectories X_t $D(X_t) \to 0$ as $\epsilon \to 0$.

-C *SEMI STRONG CONDITION* Given a more refined wave length func-

tion of the initial condition x_0 $\lambda(x_0) = \frac{\hbar}{mv(x_0)}$, where $v(x_0)$ is the derivative of the Bohmian trajectory in the limit $\epsilon \to 0$, then the deviation from classicality on the macroscopic scales defined by λ and L $D(xL, tT) \to 0$ as $\epsilon \to 0$.

-D *SEMI WEAK CONDITION* Given the wave length $\lambda(x_0) = \frac{\hbar}{mv(x_0)}$, where $v(x_0)$ is the derivative of the Bohmian trajectory in the limit $\epsilon \to 0$, then the deviation from classicality on the macroscopic scales defined by λ and L on the Bohmian trajectories X_t $D(X_t) \to 0$ as $\epsilon \to 0$.

-E *WEAK CONDITION* Given the mean de Broglie wave length $\lambda = \frac{\hbar}{mv}$, where $v = \hbar\sqrt{\langle\psi_0, -\nabla^2\psi_0\rangle}$, then the probability distribution of the deviation from classicality on the macroscopic scales defined by λ and L $\rho(D) \to \delta(D)$ as $\epsilon \to 0$.

Let's calculate the quantum force $F_Q = -\nabla\left[\frac{\hbar^2}{m}\frac{\nabla^2 R}{R}\right]$ for a free gaussian wave function. We have, from equation (A.10), that

$$F_Q(x,t) = \frac{\hbar^2}{4m\sigma_0^4}\frac{x - ut}{\left[1 + \left(\frac{\hbar t}{2m\sigma_0^2}\right)^2\right]^2}. \tag{B.1}$$

The macroscopic scales in which the motion must be classical are $\left(\frac{x}{L_o}, \frac{t}{T}\right)$, where $T = \frac{L_o}{v} = \frac{mL_o\lambda}{\hbar}$.

From equation (3.11), in the macroscopic scales $x = x'L_o$, $t = t'T$ defined by λ and L_o (suppressing the primes) we have

$$D(x,t,\lambda,L_o,\sigma_0,m,u) = \left(\frac{\lambda L_o}{2\sigma_0^2}\right)^2 \frac{x - \frac{u}{v}t}{\left[1 + \left(\frac{\lambda L_o}{2\sigma_0^2}\right)^2 t^2\right]^2}, \tag{B.2}$$

where

$$v = \frac{\hbar}{m\lambda} \quad \text{and} \quad u = \frac{\hbar}{m\bar{\lambda}}, \tag{B.3}$$

In general D is a function of x, t, all the parameters which defines the initial wave function (namely $\hbar$, σ_0, m, u) and the relevant length scales λ and L_o.

In terms of the dimensionless parameters $\alpha = \frac{\lambda L_o}{2\sigma_0^2}$, D becomes

$$D(x,t,\alpha) = (x - \frac{u}{v}t)\frac{\alpha^2}{\left[1 + \alpha^2 t^2\right]^2}. \tag{B.4}$$

The Bohmian trajectories on the macroscopic scales are

$$X_t = \frac{u}{v}t + \frac{x_0}{L_o}\sqrt{1 + \alpha^2 t^2} \tag{B.5}$$

such that D on the trajectories is given by

$$D_{X_t}(\alpha, x_0, L_o) = \frac{x_0}{L_o} \frac{\alpha^2}{\left[1+\alpha^2 t^2\right]^{3/2}}. \tag{B.6}$$

The probability distribution of D is given by

$$\rho(y(x,t)) = |\psi(y(x,t))|^2 \left|\frac{dx}{dy}\right| \tag{B.7}$$

where $|\psi(y(x,t))|^2$ is computed on the macroscopic scales, that is

$$|\psi(y(x,t))|^2 = \frac{\exp\left[-\frac{y^2}{2\left[\alpha^2\left(\frac{\sqrt{2}\sigma_0^2}{L_o}\right)(1+\alpha^2t^2)^2\right]^2}\right]}{\sqrt{2\pi}\left[\frac{\sqrt{2}\sigma_0}{L_o}\sqrt{1+\alpha^2t^2}\right]}. \tag{B.8}$$

This leads to

$$\rho(y(x,t)) = \frac{1}{\sqrt{2\pi}\bar{\sigma}_0}\exp\left[-\frac{y^2}{2\bar{\sigma}_0^2}\right], \tag{B.9}$$

where

$$\bar{\sigma}_0 = \frac{1}{\mu}\frac{\alpha^2}{\left[1+\alpha^2t^2\right]^{3/2}}, \tag{B.10}$$

where $\mu = \frac{L_o}{\sqrt{2}\sigma_0}$. Note that $\alpha = \epsilon\mu^2$, given that $\epsilon = \frac{\lambda}{L_o}$.

The mean de Broglie wave length λ obtained from the mean kinetic energy according to equation (7) is given by

$$\lambda = \frac{\hbar}{mv}, \tag{B.11}$$

where

$$v = \sqrt{u^2 + \left[\frac{\hbar}{\sqrt{2}m\sigma_0}\right]^2}. \tag{B.12}$$

The more refined wave length depending on the initial condition x_0 is given by

$$\lambda(x_0) = \frac{\hbar}{mv(x_0)} \tag{B.13}$$

where

$$v(x_0) = \dot{X}_t = u + \frac{\hbar}{2m\sigma_0}\frac{x_0}{\sigma_0}. \tag{B.14}$$

For those particular sequences for which $\epsilon \to 0$ in such a way that $\alpha = \frac{\lambda L_o}{2\sigma_0^2}$ is constant, then $D \simeq$ constant. Moreover, $D(X_t) \simeq \frac{x_0}{L_o}$ is constant. This expression

is small except if $x_0 \simeq L_o$, that is except for those initial conditions belonging to the tails of the gaussian. These are counterexamples of the validity of the conjecture in the *super strong* and the *strong* sense.

To weaken the conjecture and to take into account the fact that these initial conditions have small probability we have two possibilities.

One possibility is to assign to each initial condition a different wave length $\lambda \equiv \lambda(x_0)$ given by equation (B.13). Then the conjecture may be true in the following senses: $D \to 0$ as $\epsilon \to 0$ for any $\lambda = \lambda(x_0)$ (*semi strong* sense), or on the Bohmian trajectories $D(X_t) \to 0$ as $\epsilon \to 0$ (*semi weak* sense).

A second possibility is to assign to each initial condition x_0 the same wave length, that is the de Broglie wave length obtained from the kinetic energy according to equation (7). In this case the conjecture should hold in the *weak* sense, that is the probability distribution of D $\rho(D) \to \delta(D)$ as $\epsilon \to 0$.

It can be seen from equation (B.4) that if

$$\alpha \simeq \text{const} \quad \text{and} \quad \frac{u}{v(x_0)} = \frac{\lambda}{\overline{\lambda}} = \frac{1}{1 + \frac{x_0}{2\sigma_0^2}\overline{\lambda}} \simeq \text{const}, \tag{B.15}$$

that is, $x_0 \simeq \frac{\sigma_0^2}{\overline{\lambda}}$, then

$$D \simeq \left(x - \frac{\lambda}{\overline{\lambda} t}\right) \frac{\alpha^2}{[1 + \alpha^2 t^2]^2} \not\to 0 \tag{B.16}$$

for $\epsilon \to 0$.

From equation (B.6), we an see that if

$$\alpha \simeq \text{const} \quad \text{and} \quad \frac{u}{v(x_0)} \simeq \frac{\lambda}{\overline{\lambda}} \simeq \text{const}, \tag{B.17}$$

then we have

$$D(X_t) \simeq \frac{\lambda}{\overline{\lambda}} \frac{\alpha}{[1 + \alpha^2 t^2]^{3/2}} \simeq \text{const} \tag{B.18}$$

in the limit $\epsilon \to 0$. So, the conjecture in the *semi weak* and *semi strong* sense doesn't hold.

Now let's analyse what happens to $\rho(y(x, y))$ if the wave length is the mean de Broglie λ . We can see from equation (B.9) that $\rho(D) \to \delta(D)$ any time that $\bar{\sigma}_0 \to 0$. From equation (B.10) we see that $\bar{\sigma}_0$ depends on α and μ such that we have to prove that it goes to zero in all the possible limits in which $\alpha = \epsilon\mu^2$ and μ can combine each other, provided that $\epsilon \to 0$.

a) $\epsilon \to 0, \mu \simeq$ constant: this limit corresponds to $\lambda \ll L_o$ and $L_o \simeq \sigma_0$ and $\bar{\sigma}_0 \simeq \epsilon^2 \to 0$.

b) $\epsilon \to 0, \mu \to 0$: this limit corresponds to $\lambda \ll L_o \ll \sigma_0$ and $\bar{\sigma}_0 \simeq \epsilon^2\mu^3 \to 0$.

c) $\epsilon \to 0, \mu \to +\infty, \epsilon\mu \to$ constant: this limit corresponds to $\lambda \ll L_o$ and $\lambda \simeq \sigma_0$ and $\bar{\sigma}_0 \simeq \frac{1}{\mu} \to 0$.

d) $\epsilon \to 0, \mu \to +\infty, \epsilon\mu \to +\infty$: this limit corresponds to $\sigma_0 \ll \lambda \ll L_o$ and $\bar{\sigma}_0 \simeq \frac{1}{[\epsilon\mu]\mu^2} \to 0$.

e) $\epsilon \to 0, \mu \to +\infty, \epsilon\mu \to 0$: this limit corresponds to $\lambda \ll \sigma_0 \ll L_o$ and

$$\bar{\sigma_0} \simeq \frac{1}{\mu}\frac{[\epsilon\mu^2]^2}{[1+(\epsilon\mu^2)^2]^{3/2}}, \tag{B.19}$$

and depends on what $\epsilon\mu^2$ does. Let's analyze what happens in all the different situations.

e-1) $\epsilon\mu^2 \simeq$constant: this means $\lambda L_o \simeq \sigma_0^2$ and $\bar{\sigma}_0 \simeq \frac{1}{\mu^2} \to 0$.

e-2) $\epsilon\mu^2 \simeq 0$: this means $\lambda L_o \ll \sigma_0^2$ and $\bar{\sigma}_0 \simeq \frac{\epsilon\mu^2}{\mu} \to 0$.

e-3) $\epsilon\mu^2 \simeq +\infty$: this means $\lambda L_o \gg \sigma_0^2$ and $\bar{\sigma}_0 \simeq \frac{1}{\mu[\epsilon\mu^2]^2} \to 0$.

So we can conclude that there are some initial conditions for which we haven't got the classical limit according to the *semi strong* conjecture but their probability to happen is so small that D goes to zero as $\epsilon \to 0$ with high probability. Therefore, the examples we have analyzed show that the conjecture we have given in section 3.2 is satisfied.

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