A *Pre-formal* Proof of *Why* No Planar Map Needs More Than Four Colours

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Abstract. Although the Four Colour Theorem is passé, we give an elementary *pre-formal* proof that *transparently* illustrates *why* four colours suffice to chromatically differentiate any set of contiguous, simply connected and bounded, planar spaces; by showing that there is *no minimal* 4-coloured planar map $\mathcal{M}$.

Keywords. contiguous area, four colour theorem, 4CT, planar map, pre-formal proof, simply connected.

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1. Introduction

Although the Four Colour Theorem is considered passé (see §1.A.), we give an elementary *pre-formal* proof that *transparently* illustrates *why* four colours suffice to chromatically differentiate any set of contiguous, simply connected and bounded, planar spaces by showing that:

(1) If, for some natural numbers m, n, every planar map of less than $m+n$ contiguous, simply connected and bounded, areas can be 4-coloured;

(2) And, there is a *minimal* 4-coloured planar map $\mathcal{M}$, of $m+n$ such areas, where creation of an additional contiguous, simply connected and bounded, area C within $\mathcal{M}$ yields a map $\mathcal{C}$ which necessitates that C require a 5^{th} colour;

(3) Then:

- (a) If A_m is a set of m contiguous, simply connected, and bounded areas of $\mathcal{C}$, *none* of which shares a non-zero boundary segment with C; and B_n is a set of n contiguous, simply connected and bounded, areas of $\mathcal{C}$, *some* of which share *at least* one, non-zero, boundary segment with C, then $m = 0$;
- (b) *No* two areas $b_{n,i},\ b_{n,j}$ of B_n can share *two*, distinctly separated, non-zero boundary segments;
- (c) *No* two areas $b_{n,i},\ b_{n,j}$ of B_n can share a non-zero boundary segment that has *no* point in common with C if each area of B_n abuts the area C *only* once;
- (d) *Some* area $b_{n,i}$ of B_n *must* share *at least* two, distinctly separated, non-zero, boundary segments with C;
- (e) *No* area $b_{n,i}$ of B_n can share *two*, distinctly separated, non-zero boundary segments with C.

We conclude that there is *no minimal* 4-coloured planar map $\mathcal{M}$.

1.A. A historical perspective

It would probably be a fair assessment that the mathematical significance of any new proof of the Four Colour Theorem 4CT *continues* to be perceived as lying not in any ensuing theoretical or practical utility of the Theorem per se, but in whether the proof can address the philosophically 'unsatisfying', and occasionally 'despairing' (see [Tym79]; [Sw80]; [Gnt08], [Cl01]) lack of, mathematical 'insight', 'simplicity' and 'elegance' in currently known proofs of the Theorem (eg. [AH77], [AHK77], [RSST], [Gnt08])—an insight and simplicity this investigation seeks in a *pre-formal*[1] proof of 4CT.

For instance we note—amongst others—some candid comments from Robertson, Sanders, Seymour, and Thomas's 1995-dated (apparently pre-publication) web-survey[2] of their proof [RSST]:

> "The Four Color Problem dates back to 1852 when Francis Guthrie, while trying to color the map of counties of England noticed that four colors sufficed. He asked his brother Frederick if it was true that any map can be colored using four colors in such a way that adjacent regions (i.e. those sharing a common boundary segment, not just a point) receive different colors. Frederick Guthrie then communicated the conjecture to DeMorgan. The first printed reference is due to Cayley in 1878 ([Cay79]).
>
> . . .
>
> The next major contribution came from Birkhoff whose work allowed Franklin in 1922 to prove that the four color conjecture is true for maps with at most 25 regions. It was also used by other mathematicians to make various forms of progress on the four color problem. We should specifically mention Heesch who developed the two main ingredients needed for the ultimate proof - reducibility and discharging. While the concept of reducibility was studied by other researchers as well, it appears that the idea of discharging, crucial for the unavoidability part of the proof, is due to Heesch, and that it was he who conjectured that a suitable development of this method would solve the Four Color Problem.
>
> This was confirmed by Appel and Haken in 1976, when they published their proof of the Four Color Theorem [1.2] (*sic*).
>
> **Why a new proof?**
>
> There are two reasons why the Appel-Haken proof is not completely satisfactory.
>
> - Part of the Appel-Haken proof uses a computer, and cannot be verified by hand, and
> - even the part that is supposedly hand-checkable is extraordinarily complicated and tedious, and as far as we know, no one has verified it in its entirety."
>
> *. . . Thomas et al: [RSSp], Pre-publication web survey.*

> "It has been known since 1913 that every minimal counterexample to the Four Color Theorem is an internally six-connected triangulation. In the second part of the proof, published in [**4**, p. 432], Robertson et al. proved that at least one of the 633 configurations appears in every internally six-connected planar triangulation. This condition is called "unavoidability," and uses the discharging method, first suggested by Heesch. Here, the proof differs from that of Appel and Haken in that it relies far less on computer calculation. Nevertheless, parts of the proof still cannot be verified by a human. The search continues for a computer-free proof of the Four Color Theorem."
>
> *. . . Brun: [Bru02], §1. Introduction (Article for undergraduates)*

> "Being the first ever proof to be achieved with substantial help of a computer, it has raised questions to what a proof really is. Many mathematicians remain sceptical about the nature of this proof due to the involvement of a computer. With the possibility of a computing error, they do not feel comfortable relying on a machine to do their work as they would be if it were a simple pen-and-paper proof.
>
> The controversy lies not so much on whether or not the proof is valid but rather whether the proof is a valid proof. To mathematicians, it is as important to understand why something is correct as it is finding the solution. They hate that there is no way of knowing how a computer reasons. Since a computer runs programs as they are fed into it, designed to tackle a problem in a particular way, it is likely they will return what the programmer wants to find leaving out any other possible outcomes outside the bracket.

[1] The need for distinguishing between *belief-based* 'informal', and *evidence-based* 'pre-formal', reasoning is addressed by Markus Pantsar in [Pan09]; see also [An21], §1.D.

[2] See [RSSp]; also [Thm98], [Cl01], and the survey [Rgrs] by Leo Rogers.

Many mathematicians continue to search for a better proof to the problem. They prefer to think that the Four Colour problem has not been solved and that one day someone will come up with a simple completely hand checkable proof to the problem."

... Nanjwenge: [Nnj18], Chapter 8, Discussion (Student Thesis).

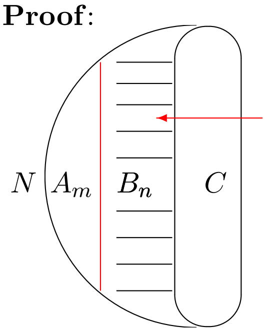
"The heavy reliance on computers in Appel and Haken's proof was immediately a topic of discussion and concern in the mathematical community. The issue was the fact that no individual could check the proof; of special concern was the reductibility [*sic*] part of the proof because the details were "hidden" inside the computer. Though it isn't so much the *validity* of the result, but the *understanding* of the proof. Appel himself commented: "...there were people who said, 'This is terrible mathematics, because mathematics should be clean and elegant,' and I would agree. It would be nicer to have clean and elegant proofs." See page 222 of Wilson."

... Gardner: [Grd21], §11.1, Colourings of Planar Maps, pp.6-7 (Lecture notes).

2. A *pre-formal* proof of the 4-Colour Theorem

Proposition: No planar map requires more than four colours.

Proof: ***Planar Map*** $\mathcal{C}$



Only the immediate portion of each area $c_{n,1}, c_{n,2}, \ldots, c_{n,r}$ of B_n abutting C is indicated.

Fig.1

Definitions

(1) Consider the surface of the hemisphere (planar map $\mathcal{C}$) in Fig.1 where:

(a) A_m denotes a region of m contiguous, simply connected and bounded, surface areas $a_{m,1}, a_{m,2}, \ldots, a_{m,m}$, *none* of which shares a non-zero boundary segment with the contiguous, simply connected, surface area C (as indicated by the red barrier which, however, is *not* to be treated as a boundary of the region A_m);

(b) B_n denotes a region of n contiguous, simply connected and bounded, surface areas $b_{n,1}, b_{n,2}, \ldots, b_{n,n}$, *some* of which share *at least one* non-zero boundary segment $c_{n,i}$ with C. In other words, for each $1 \leq i \leq r$, $c_{n,i} = b_{n,j}$ for some $1 \leq j \leq n$;

(c) C is a single contiguous, simply connected and bounded, area created by annexing one or more contiguous, simply connected, portions of each area $c_{n,i}$ in the region B_n;

(d) N is treated as an orientation pole of the hemisphere.

Hypothesis

(2) Since four colours suffice for maps with fewer than 25 regions, we assume the existence of some m, n which define a *minimal* configuration of the region $\{A_m + B_n + C\}$ where:

(a) *any* configuration of p contiguous, simply connected and bounded, areas *can* be 4-coloured if $p \leq m + n$, where $p, m, n \in \mathbb{N}$, and $m + n \geq 25$;

(b) *any* configuration of the $m + n$ contiguous, simply connected and bounded, areas of the region $\{A_m + B_n\}$ *can* be 4-coloured (*before* the creation of C by annexing some portions from each area $c_{n,i}$ of B_n);

(c) the region $\{A_m + B_n + C\}$ in the planar map $\mathcal{C}$ is a *specific* configuration of $m + n + 1$ contiguous, simply connected and bounded, areas that *cannot* be 4-coloured (by the *minimality* condition); whence the area C *necessarily* requires a 5^{th} colour.

Lemma **(Annexation): Creating additional areas in $\mathcal{C}$ does *not eliminate* the need for a 5'th colour.**

(3) Note that once the area C is created, creating additional areas does *not eliminate* the need for a 5^{th} colour, since:

(a) If a new area is created strictly within the region $\{A_m + B_n\}$, the 5^{th} colour is necessitated for the newly created area by the *minimality* assumption.

(b) If the area C is sub-divided into the two areas C_1 and C_2, then either one can be absorbed back into the original areas in B_n from which it was formed by annexation, reducing the configuration again to a *minimal* one; thus necessitating the 5^{th} colour.

Lemma **(a): If A_m is a set of m contiguous, simply connected, and bounded areas of $\mathcal{C}$, *none* of which shares a non-zero boundary segment with C; and B_n is a set of n contiguous, simply connected and bounded, areas of $\mathcal{C}$, *some* of which share *at least* one, non-zero, boundary segment with C, then $m = 0$.**

(4) Consider, now, the mirror image of Fig.1, with mirrored regions A'_m, B'_n, area C', and orientation pole S (Fig.2).

Planar Map $\mathcal{C}'$ (mirror image of $\mathcal{C}$)

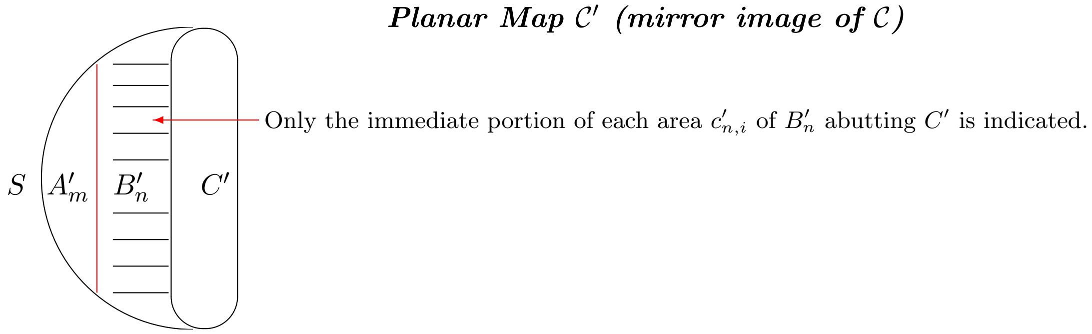


Fig.2

(5) By our hypothesis that the region $\{A_m + B_n\}$ (ergo its mirror image $\{A'_m + B'_n\}$) *can* be 4-coloured, joining the two halves into a sphere, where each area of the region B_n is aligned with its mirror image, would extinguish both C and C', yielding a 4-coloured configuration of *at least* $n+2m$ contiguous, simply connected and bounded, areas forming the region $\{A_m+B_n+B'_n+A'_m\}$ (now on the surface of the sphere formed by melding the two hemispheres).

(6) However, creation of $\{A_m + B_n + B'_n + A'_m\}$ from $\{A_m + B_n\}$, initially by annexing areas of $\mathcal{M}$ to form $\mathcal{C}$, and thereafter of $\mathcal{C}$ to form successor maps, would contradict the Annexation Lemma *if* A_m contained *at least* one area which does *not* share a non-zero boundary segment with C.

(7) Hence the region A_m is empty under *minimality*, and $m = 0$. Moreover, by repeated reasoning, the region B_n cannot contain *any* area which does *not* share a non-zero boundary segment with C; whence *each* area of the region B_n abuts the area C *at least* once.

Lemma **(b) *No* two areas $b_{n,i}$, $b_{n,j}$ of B_n can share *two*, distinctly separated, non-zero boundary segments.**

(8) If region A_m is empty, and *each* area of region B_n abuts the area C *at least* once, then:

(a) *no* two areas $b_{n,i}$ and $b_{n,j}$ of region B_n can share *two*, distinctly separated, non-zero boundary segments,

(b) since the l areas of the region, say A_l $(l > 0)$, enclosed by such boundary segments of the areas $b_{n,i}$ and $b_{n,j}$ would *not* then share *any* non-zero boundary segment with the area C; contradicting (7).

***Lemma* (c) *No* two areas $b_{n,i},\ b_{n,j}$ of B_n can share a non-zero boundary segment that has *no* point in common with C if each area of B_n abuts the area C *only* once.**

Planar Map $\mathcal{C}$

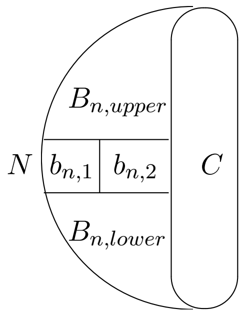


Fig.3

(9) Moreover, if the region A_m is empty, and each area of the region B_n abuts the area C *only* once, then *no* two areas of B_n—say $b_{n,1}$ and $b_{n,2}$ (see Fig.3)—can share a non-zero boundary that *does not* intersect C:

(a) since that would divide $\{B_n - b_{n,1} - b_{n,2}\}$ into two *non-empty* regions, say $B_{n,upper}$ and $B_{n,lower}$;

(b) such that *no* area of the region $B_{n,upper}$ shares a non-zero boundary with *any* area of the region $B_{n,lower}$; whence:

(i) some areas in each of the regions $B_{n,upper}$ and $B_{n,lower}$ would necessarily require the 2 colours not shared with the areas C, $b_{n,1}$ and $b_{n,2}$; since:

- if one of the regions, say $B_{n,upper}$, requires only 1 of the 2 colours,
- then annexing one of the areas of $B_{n,lower}$, say $b_{n,lower}$, which has this colour, say x, into C would reduce $\mathcal{C}$ identically to $\mathcal{M}$,
- whilst still requiring 5 colours (since $b_{n,\ lower}$ would now abut areas with all the four colours of $\mathcal{M}$),
- thereby violating *minimality*;

(ii) whilst each of the regions $\{B_{n,upper} + b_{n,1} + b_{n,2} + C\}$ and $\{B_{n,lower} + b_{n,1} + b_{n,2} + C\}$ would necessarily require C to have the 5^{th} colour—and violate *minimality*—in order to avoid violating *minimality* when combined (superimposed suitably) to form $\mathcal{C}$!

***Lemma* (d) *Some* area $b_{n,i}$ of B_n *must* share *at least* two, distinctly separated, non-zero, boundary segments with C**

Planar Map $\mathcal{C}$

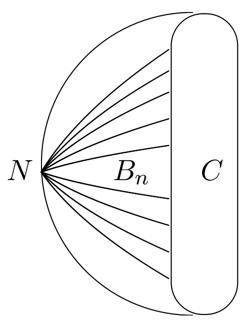


Fig.4

(10) Hence, if the region A_m is empty, and each area of the region B_n abuts the area C *only* once, then all the areas of the region B_n can be treated as bounded by longitudinals that meet at the orientation pole N of the hemisphere (see Fig.4).

(11) However, the region B_n would then require at most 2 colours if n is even, and 3 colours if n is odd; whence the area C would not require a 5^{th} colour, contradicting *minimality*.

***Lemma* (e) *No* area $b_{n,k}$ of B_n can share *two*, distinctly separated, non-zero boundary segments with the area C.**

(12) Hence some area in the region B_n, say $b_{n,k}$, must abut the area C at *at least* two, distinctly separated, non-zero boundary sections.

(13) However, the region $\{C + b_{n,k}\}$ can then be treated as an equatorial band which divides the areas in the region $\{B_n - b_{n,k}\}$ into the *two* regions B_L (see Fig.5) and B_R (not shown); such that *no* area of the region B_L abuts *any* area of the region B_R.

Planar Map $\mathcal{C}_L$

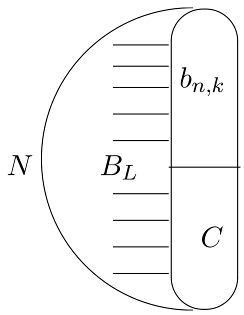


Fig.5

(14) Hence the regions $\{C + b_{n,k} + B_L\}$ and $\{C + b_{n,k} + B_R\}$ can be treated as two, distinctly separated, hemispherical maps $\mathcal{C}_L$ and $\mathcal{C}_R$:

(a) each of which has less than the number of areas required for *minimality*; but

(b) each of which necessarily requires a 5^{th} colour;

thus contradicting the assumption that $\{A_m + B_n + C\}$ is a *minimal* configuration.

We conclude that *no* planar map requires more than 4 colours. □

Concluding comment: In conclusion, we note that the above *pre-formal* proof of the Four Colour Theorem highlights the significance of differentiating between (see [An21], §*5. What is knowledge?*):

- Plato's *knowledge* as *justified true belief*, which seeks a *formal* proof in a first-order mathematical language in order to *justify* a *belief* as *true*; and
- Piccinini's *knowledge* as *factually grounded belief*, which seeks a *pre-formal* proof in order to *justify* the axioms and rules of inference of a first-order mathematical language which can, then, *formally* prove the *belief* as *justifiably true* under a *well-defined* interpretation of the language.

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