Conceptions of infinity and set in Lorenzen's operationist system

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Abstract

In the late 1940s and early 1950s Lorenzen developed his operative logic and mathematics, a form of constructive mathematics. Nowadays this is mostly seen as the precursor to the more well-known dialogical logic[1] and one could assumed that the same philosophical motivations were present in both works. However we want to show that this is not always the case. In particular, we claim, that Lorenzen's well-known rejection of the actual infinite as stated in Lorenzen (1957) was not a major motivation for operative logic and mathematics. In this article, we claim that this is in fact not the case. Rather, we argue for a shift that happened in Lorenzen's treatment of the infinite from the early to the late 1950s. His early motivation for the development of operativism is concerned with a critique of the Cantorian notion of set and related questions about the notion of countability and uncountability; only later, his motivation switches to focusing on the concept of infinity and the debate about actual and potential infinity.

1 Introduction

In his work on the philosophy of mathematics, Paul Lorenzen was motivated by the challenges mathematics was confronted with at the beginning of the 20th century and the so-called foundational crisis of mathematics[2]. To tackle the problems raised during this crisis, he developed a new foundational system for mathematics in the 1950s, a system he called "operative logic" and "operative mathematics"[3]. Nowadays, the work on operative *logic* is mostly referred to as a precursor of his later, much better known, work on dialogical

[1]Notable exceptions are the works of Schroeder-Heister (2008), Coquand and Neuwirth (2017) and Kahle (in this volume).

[2]At least this pertains to his broader philosophical motivations. At the beginning, mathematical reasons were more prevalent, as Lorenzen recognized the impact of his work in lattice theory on consistency proofs (see Coquand and Neuwirth (2017), Neuwirth (20xx. To appear.)). I would like to thank one of the referees for pointing this out to me.

[3]A complete presentation can be found in Lorenzen (1955). For all quotes from this book and all the other German texts, the translations to English are my own.

logic. Indeed, this approach was developed as an answer to the shortcoming of operative (proto-)logic.[4] In this article we want to focus on his operative *mathematics*, which is an elaborate attempt at building a constructive version of mathematics that still preserves (most of) modern analysis. Although he later abandoned the specific way in which operative mathematics was set up, he continued to pursue the general ideas and motivations operative mathematics rested on.

At the beginning of the 1950's however Lorenzen still perceived his operative logic and mathematics to be nothing less than a "new way to overcome the foundational crisis" (Lorenzen, 1956b).[5] This new way is situated between the two main factions that developed answers to the crisis, the "Hilbertians" and the "intuitionists". In Lorenzen's view the question of a "good" foundation for analysis remained unanswered by both accounts and so he sought out a new way towards a foundation by using methods and ideas of both approaches while overcoming the problems they were faced with. Central to this endeavor is a thorough treatment of the real numbers. On Lorenzen's view, it was a (mistaken) treatment of the reals that led to the development of Cantorian set theory and the connected notion of mathematical infinity, which in turn gave rise to the problems that were at the heart of the foundational crisis.

Nowadays, Lorenzen is mostly known for his rejection of actual infinity. Indeed, Lorenzen (1957) claims that the next big challenge for mathematics is to show that "the infinitely large (more precisely, the actual infinite) is to be demonstrated to be disposable" [6]. Therefore, it could seem that the rejection of actual infinity was also a central motivating factor for his development of operative mathematics.

In this article we want to argue that this is in fact not the case. Instead, we claim that the rejection of actual infinity was *not* prevalent in his development of operative mathematics; in particular, the notion of infinity was *not* his main focus, but rather the notion of set was. Our hypothesis is that a *shift occurred in Lorenzen's treatment of infinity*: In the beginning he focused on the notion of set and connected to this, the notions of countable

[4]For a detailed account of how and why Lorenzen abandoned operative logic and developed dialogical logic, see Lorenz (2001).

[5]"Die operative Logik und Mathematik stellt einen neuen Weg der Überwindung der Grundlagenkrise dieser Wissenschaften dar." Lorenzen (1956b), a short but very informative outline of the main ideas of his operationist system, also sketches a larger project he seemed to have in mind, by planning on exploring the ramifications operative mathematics has for wider applications in the sciences.

[6](Lorenzen, 1957, 11), translation taken from (Lorenzen, 1987, 202)

and uncountable sets[7]; only in the late 1950s his focus shifted towards the question of potential and actual infinity[8]

To make this hypothesis plausible, we will first give an overview of how Lorenzen's operative approach is embedded in the general discussion on the foundational crisis in mathematics and, in particular, how it relates to the major approaches of formalism and intuitionism (section 2). To explain the first part of the proposed shift in Lorenzen's work, we will take a closer look at how Lorenzen eliminates the classical notion of set and how this impacts the notion of countability and uncountability in operative mathematics (section 3). In section 4 we will compare his treatment of the notion of infinity from the early and late 1950's and argue for the shift in Lorenzen's treatment of infinity. In the last section we will suggest possibilities for future work on the explanation of why this shift occurred and the systematic question whether operative mathematics is a valid framework for potential infinity.

2 Operationalism and the foundational crisis

Before going into the details of how Lorenzen attempted to tackle the question of the notions of set and infinity in an "operative" way, we want to show how Lorenzen situated his approach in the discussion about the foundational crisis in mathematics. This chapter therefore also serves as a short overview of Lorenzen's thought on the foundations of mathematics in the 1950's, as most of his work from this time is quite unknown to an international audience.[9]

In the late 1940s and early 1950s, Lorenzen was working on an constructive account of mathematics that had at its center the notion of operations via certain schematic rules. In Fraenkel et al. it is described as follows:

> For Lorenzen, the main (though not the only) subject of mathematics is the treatment of calculi—this should by no means be misunderstood as a claim that mathematics is a calculus, which

[7] Sets are countable if there is an injective function from the set to the set of the natural numbers; it is uncountable if there is not such function

[8] To explain what these terms mean, Lorenzen (and others) often point to Aristotele's definition in his *Metaphysics*, book 9, chapter 6. Mathematical definitions of these concepts vary; they usually contain a description of some "construction process", but differ in to what point the construction should proceed and whether this is spelled out via induction, computation or other approaches. In section 4.2. we discuss a possibilities of interpreting Lorenzen's notion of construct as a definition of potential infinity.

[9] One reason is that most of his papers were published in German and no English translations are available.

> Lorenzen would very definitely reject—where a calculus is understood to be a system of rules for schematic operations with figures, which may but need not be marks on paper; they might as well be pebbles (calculi) or any other physical objects. In addition to this precisification of the subject-matter of mathematics - and only slightly connected with it - Lorenzen stipulates that the methodical frame be as wide as compatible with the conditio sine qua non that all mathematical statements be definite.[10]

(Fraenkel et al., 179)

Lorenzen acknowledges several sources for the ideas he uses in this operative approach, the most important among them being Weyl (1918). His approach is quite comprehensive, giving not only a new way of building up different areas of mathematics (with analysis as the most important one), but complementing this with a new logical system, the operative logic, which in turn is founded on the so-called protologic.

He sees this as a kind of reverse account to Hilbertian mathematics and meta-mathematics:[11]

> After it became clear that an axiomatisation of the naive theory is not enough, but is in need of a metamathematics, the task arises to justify the metamathematical modes of inference. The object of metamathematics are certain formal systems [Kalküle], i.e. axiomatized theories. Reversing this line of research, in "operationalism" it is the formal systems (i.e. operating with symbol strings [Figuren] as such) that are put at the beginning, metamathematics is supplanted by protologic. (Lorenzen, 1956b)[12]

[10]For an example of such a system of rules, see the system Z below, for a definition of definiteness see section 4.2.

[11]Note that Lorenzen's view on other ways out of the foundational crisis, as presented for example by Hilbert or Brouwer, are only described here insofar as they serve as a demarcation line for his operationalism. In his later work, for example (Lorenzen, 1960, p.119), he indeed argued that the foundational crisis has been overcome precisely because we have a fruitful interplay between Hilbert's and Brouwer's approach.

[12]"Nachdem deutlich geworden ist, daß eine Axiomatisierung der naiven Theorien nicht genügt, sondern noch eine Metamathematik erfordert, stellt sich das Problem, die metamathematischen Schußweisen zu begründen. Gegenstand der Metamathematik sind gewisse Kalküle, nämlich die axiomatisierten Theorien. In genauer Umkehrung dieser Untersuchungsrichtung werden im 'Operativismus" beliebige Kalküle (also das Operieren mit Figuren als solches) an den Anfang gestellt, an die Stelle der Metamathematik tritt so eine Protologik."

In Lorenzen's view, Hilbert's Program and the distinction between mathematics and meta-mathematics is not able to adequately address the problem of a foundation for mathematics for very basic, structural reasons:

> According to Hilbert's Program for the foundation of mathematics, the task to provide consistency proofs is assigned to meta-mathematics. [...] As every proof is only as good as the methods it uses, it can be objected to such proofs that the contentual inference [inhaltliche Schließen] is neither formalized nor indeed justified.
>
> While attempting such a formalization it became apparent that the distinction between mathematics and metamathematics is not suited for the problem of foundation. The proof methods in metamathematics are non others than those used in mathematics. A foundation of metamathematics is therefore nothing but the foundation of a part of mathematics [...].[13] (Lorenzen, 1951a, 162)

So, instead of justifying the use of an axiomatic system in meta-mathematics (for example by providing a consistency proof) in what could be perceived as a justification after the fact, Lorenzen puts the justification at the beginning of his investigation via the protological principles:

> One can obtain certain protological principles without assuming logical or mathematical knowledge, which are sufficient to establish customary logic and mathematics […]. (Lorenzen, 1956b)

These basic principles are, for example, operations on symbol strings that have their origin in basic practices like pre-mathematical forms of counting. They give rise to formalized operations or rules through which the usual objects in mathematics can be defined. As one example, if one thinks about numbers in the way of "strokes on paper" of the form I, II, III, ... this can be written down as rules on symbol strings such as the following rule system Z:

1. Begin with I.
2. If you have reached x, add xI.

[13] Of course this is only Lorenzen's account of Hilbert's program and it would be interesting to see, if it is a faithful one. For a non-Lorenzen viewpoint see for example Sieg (1999).

So, general rules on how to operate with symbol strings are given via formal systems which then constitute the objects of operative mathematics. In this way Lorenzen can utilize the usefulness of an axiomatic methodology and at the same time have a sound foundation through the protological principles.[14]

Such an approach shows the importance of action and applicability in mathematics that are characteristic for Lorenzen's thoughts:

> Such rules are not examined as to whether they are "true" or not—they are only examined as to whether they are "useful" or not, i.e. if acting on these rules, meaning the construction of symbols, is suitable to some purpose. This, however, is no longer a mathematical question, but belongs to applications. (Lorenzen, 1951a, 163)

This applicability is also the biggest difference between Lorenzen's operative mathematics and intuitionistic approaches like Brouwer's and Weyl's. For Lorenzen a good foundation of mathematics should preserve the full power of contemporary analysis. This means, in particular, that one should be able to use *tertium non datur*, which is made possible in operative mathematics.

> In contrast to intuitionistic attempts we are now—after an unobjectionable foundation of logic—allowed to always use the tertium non datur. The uncomfortable restriction to "decidable properties", "enumerable real numbers", etc. is no longer neccessary. An approach towards this intended direction was made already by Weyl in 1918—however it had to fail, because a justification of the tertium non datur was still lacking. (Lorenzen, 1951a, 166)

Lorenzen addresses this issue in operative logic by distinguishing between the effective predicate calculus (*effektive Quantorenkalkül*) (which is in essence the intuitionistic approach) and the fictional predicate calculus (*fiktive Quantorenkalkül*) which includes the tertium non datur and is therefore the classical predicate calculus. As the word "fictional" suggests, the justification of such a calculus means to "justify a fiction"(Lorenzen, 1955, 79). This justification can be given "in most of the cases" (Lorenzen, 1955,

[14] Lorenzen was not in general opposed to using axiomatic methods, as long as they are not used as a foundation: "Axiomatizations have no other purpose than to simplify operating on constructible sets, functions, real numbers etc." (Lorenzen, 1951a, 165). Lorenzen (1960, 118) calls this the "systematic priority of axiomatic mathematics over constructive mathematics."

84) by a careful analysis of the concepts of effective and fictional derivability or un-derivability. Using these means Lorenzen achieves a compatibility of operative analysis with classical analysis—although he notes that "[t]he operative conception also implies that the intuitionistic opinion is right in that 'in fact' only effective derivability is of interest" (Lorenzen, 1955, 84).

Operative logic and mathematics is deemed to be a third way out of the crisis in that it overcomes both the foundational weakness of formalistic approaches and the lack of full applicability of the intuitionistic approaches, while at the same time borrowing some of their strengths. Incidentally, this also holds for Russell's solution to the crisis. As we will see in the next section, Lorenzen uses a version of Russell's ramified type theory to avoid impredicativity:

> With the laudable exception of the intuitionists, who went their own way, this approach (Russell's ramified type theory) has been abandoned in favour of unramified type theory only because ramified type theory [Stufenlogik] was allegedly too complicated, because the stock of modern mathematics could not be "saved" in a satisfactory manner. However, the impredicativity of this logic has never been justified, at least until now: the error of infinite regress cannot be avoided this way. Yet ramified type theory [verzweigte Stufen] can be much simplified by introducing something like untyped branches [ungestufter Zweige], instead of appealing to unramified types [...]. And, with this, one remains "predicative"[...]. (Lorenzen, 1956a, 275)

Contrary to the relative unrenownedness of Lorenzen's operativism in today's literature[15], the book was well-received by his contemporaries. Nevertheless, it was also met with criticisms that mainly pertained to the foundational aspect of operationalism: As protologic carries much of the foundational load, it is not surprising that in the reception of Lorenzen's work a great part of the criticism was directed against the protological principles and the claim that it gives a sound foundation. William Craig (1957, 319) singled this out as the main problem of Lorenzen's approach:

> The main weakness of the book is its failure to indicate as clearly as a work on foundations should the strength of the methods employed and thus of the underlying assumptions. Claims that only the principles of the chapter on *Protologic* are employed and that no understanding of other logical notions is required are unconvincing and seem unnecessary.

[15] A notable exception is Fraenkel et al..

Gerhard Frey (1957, 633) even claims that Lorenzen "circumvents [...] the actual philosophical questions on purpose" when asserting that mathematical and logical knowledge is not needed for operating with symbol strings.

According to Lorenz (2001), Lorenzen himself was convinced to abandon operative (proto)-logic after a discussion with Tarski when he was visiting the Institute of Advanced Study in Princeton in the Fall 1957/58. Then, he began to develop what is now known as dialogical logic. But this rethinking of the foundations did not diminish the significance of Lorenzen's mathematical work. Thoralf Skolem points this out by writing in his review that "although one may doubt whether Lorenzen's theory is the best conception of mathematics, the reviewer believes that the book will have a sound influence on the mathematical world" (Skolem, 1957, 290). This is also the essence of Wolfgang Stegmüller's extensive review of Lorenzen's book:

> In particular the part that in the reviewers eyes represents the most important contribution of Lorenzen towards a foundation of mathematics, namely the theory of the real numbers, is described in detail in this book; in fact the thoughts developed in it are in the most part independent and can therefore be isolated from the operative framework of the theory and transferred to different system constructions. This point is not to be underestimated in an overall assessment of Lorenzen's achievement; as whatever one's opinion of Lorenzen's operative interpretation of logic and mathematics may be—his foundation of analysis is in most parts independent of this interpretation and will without doubt be inspiring and fruitful for all future attempts in this direction. (Stegmüller, 1958, 161-162)

Despite his above mentioned critique, Craig also concurs with Lorenzen's claim that the operative approach signifies a third way out of the crisis in much the same way as described above: "The reviewer believes that the book [...] presents a legitimate and probably fruitful third approach to foundations. Its evident advantage over intuitionism is the preservation of classical logic and arithmetic and of much larger portions of the rest of mathematics, and over formalism the interpretation of these." (Craig, 1957, p.318)

3 Elimination of the classical notion of set

In the second half of the 19th century, set theory was developed in search for a sound foundation of analysis (see Ferreirós, 2007). As is well-known, the first approach of so-called "naive" set theory led to paradoxes that were then addressed by an appropriate axiomatization. Nowadays the Zermelo-Fraenkel-Choice (ZFC) axiomatization is considered to be standard in set theory.

But for Lorenzen this whole development is unsatisfactory: Not only is axiomatization not a valid form of foundation for him, he believes that the whole concept of set that underlies Cantorian set theory is flawed. He makes this point in one way or another in nearly all of his papers concerned with foundational work. As he writes in the introduction to his (1955, 4): "In spite of Cantor's 'definition' of set—of which as it is known nothing can be deduced in as much as nothing can be deduced from Euclid's 'definition' of a point—a set in mathematics is never built through a 'collection into a whole' [...]."

The fact that classical analysis rests on this (in Lorenzen's view) mistaken concept of set is one of big challenges which analysis, and in turn modern mathematics, has to face. So a central requirement for a valid foundation for mathematics is the elimination of the classical concept of set[16] and its replacement with an alternative account of set, while at the same time making sure that the fundamental theorems of classical analysis are preserved.

Lorenzen develops this alternative in a string of papers that are concerned with the concept of finite sets (Lorenzen, 1952c), the concept of set and its use in topology (Lorenzen, 1952a) and especially the concept of set in analysis (Lorenzen, 1951b,c). These papers lead up to Lorenzen (1955) where he explains in detail how the classical concept of set can be eliminated.

The core idea of Lorenzen's operationist system is the following: instead of having an informal definition of set that relies on a quasi-intuitive understanding of some kind of "collection", Lorenzen mathematically defines what a set is via abstractions of formulas[17]. Formulas and relations themselves are defined by giving rules that produce them.

As an example for such a procedure, consider the way Lorenzen (1951b, 2) shows how unary relations are built for some atoms $x, y, \ldots$:

1. If σ is a relation, then $\sigma(x)$ is a formula for a variable x.

2. Let $a, B, A(x)$ be formulas, then the following are formulas:

   $A \wedge B$, $A \vee B$, $\rightarrow B$, $A \leftrightarrow B$, $\neg A$, $\forall x A(x)$, $\exists x A(x)$.

[16] As an example of an instance where Lorenzen formulates this as a challenge, see Lorenzen (1954, 67).

[17] For the purpose of this paper we will not differentiate between "Aussage" and "Aussageform", which we will both translate as "formula". Their difference lies in the use of free variables, see (Lorenzen, 1955, 178).

3. If $A_1, \ldots, A_\kappa$ are formulas and σ a relation, then the following is a relation:

$$\mathbf{l}_\sigma(A_1 \to \sigma(x_1), \ldots, A_\kappa \to \sigma(x_\kappa))$$

Clause three means that σ is the relation inductively defined by the rules

$$A_1 \to \sigma(x_1)$$
$$\ldots$$
$$A_\kappa \to \sigma(x_\kappa)$$

Over such a language, sets are now given through formulas[18]:

Let $A_1(x), A_2(x)$ be formulas, such that for all x

$$A_1(x) \leftrightarrow A_2(x)$$

then $M_x A_1(x) = M_x A_2(x)$, where $M_x A(x)$ is the set of x such that $A(x)$.

Although the final setup in Lorenzen (1955) is more elaborate than this example,[19] the basic idea remains the same: operative rules tell us how to build up certain symbol strings that then are formulas, relations and so on. On this basis objects like sets and functions can be built and therefore precise definitions for them can be given.

Before we look more closely at how this setup enables Lorenzen to give an operative account of the real numbers, we will consider the case of arithmetic. Here we can see in more detail how deeply the operative conception of set differs from the classical notion. For arithmetic Lorenzen develops an account on how to treat finite sets, that is, not as a specific instance of the general concept of set, but rather as an object that can be treated independently from general sets:

> Since with the constructive foundation [of mathematics] one can already operate with symbols of arbitrary (finite) length before

[18]Form the definition it may seem that there is no difference between Lorenzen's way of defining sets via abstractions as given here and (some form of) comprehension in standard set theory. However, Lorenzen had a more general notion in mind. When introducing abstractions in (Lorenzen, 1955, 100) to built terms from objects he comments that starting with Frege and Russell one normally conceives of abstractions as classes:

> With this, abstractions should be reduced to the introduction of "classes". However we will see below, that classes are nothing more than a special case of abstract objects.

(Lorenzen, 1955, 101)

[19]For instance, the defining schemata for relations have to satisfy certain criteria, such as foundation, to avoid circularity.

> giving the definition of a formula—and therefore also before giving the definition of the notion of set—one would not want the notion of finite set to depend on the later choice of a definition of formula [...]. It is only the colloquial name "finite set" which misleads us into believing that one would have to define "set" as a basic concept and then "finite" as a specific difference. (Lorenzen, 1952c, 331)

Consider a certain kind of symbol strings $x, y, \ldots$ that is built by writing the symbol strings x and y and so on in the given way (these symbol strings x can for example be formulas of a certain calculus). Such a $x, y, \ldots$, called a *system*, can be derived from the following calculus:

I. x (x is a system),

II. $\mathbf{r} \to \mathbf{r}, x$ (if $\mathbf{r}$ is a system, then also $\mathbf{r}, x$).

Finite sets can then be defined by abstraction: two systems give the same finite sets if they consist of the same objects, irrespective of the number and place of these objects in the system (so x, x, y, z gives rise to the same finite set as z, y, y, x, z), where the meaning of "consists of the same objects" is expressed through certain formulas.[20] So Lorenzen goes the exact opposite way to the classical approach when considering finite and general sets: He first defines finite sets as separate entities and only later adjusts the process of abstraction for general formulas to arrive at general sets.

Indeed, he is able to set up the whole of arithmetic with the use of systems and finite sets only. For instance, numbers are defined via the system Z in section 2 and cardinal numbers are defined as the length of systems. In §13, where he defines basic numbers, Lorenzen includes two interesting philosophical comments. The first concerns the way in which operative arithmetic does not face the problem of incompleteness in the same way in which classical axiomatic approaches to arithmetic do. Because operative mathematics rests on protological principles, it is able to provide (operative) proofs "by reasoning in terms of content" (*inhaltlich beweisen* (Lorenzen, 1955, 135)) for sentences that would be undecidable in an axiomatic system because of Gödel's Incompleteness Theorem.

The second comment is one of the few times he explicitly refers to the underlying concept of infinity for operative mathematics in his (1955). He points out that the system Z that defines numbers contains the idea of iteration and therefore also the, as he calls it, "purest form of potential infinity"(Lorenzen, 1955, 133). This process of iteration or potential infinity that

[20] Accounts for this way of treating finite sets can be found for example in Lorenzen (1952c, 1955).

is implicit in every rule of a calculus he assumes to be understandable to everybody, as it lies at the heart of the operative concept of applying rules again and again. Note that this does not necessarily means that he *restricts* himself to potential infinity or that he rejects actual infinity in operative mathematics. Instead the reference to potential infinity has the purpose of explaining why the process of iteration meets the underlying requirements of protologic,[21] namely to be understandable without assuming any prior logical knowledge.[22]

Of course, when considering analysis, finite sets are not enough. Instead, one has to use the general definition of sets via abstraction as outlined above. Taking this as a starting point, Lorenzen then develops the so-called "language strata" (*Sprachschichten*) in which the objects of analysis appear step by step. The idea of language strata counters the underlying problem of impredicativity: Lorenzen (1955, 165) describes it, with Weyl, as the "mathematical process" that goes from considering things like numbers as object, to also consider formulas about these objects as objects themselves; a process that in classical set theory allows the existence of the (unrestricted) power set.[23]

The language strata are built up in the following way (see Lorenzen, 1955, chap. 5): The basic language stratum S_0 consists of the basic objects that can be build by operations of symbol strings. As we showed above this also includes basic numbers. S_1 then consists of all of the elements of the basic language and the first formulas that can be build via iteration of the following operations:

1. Inductive definitions, which are usually formed through the use of an operator of induction $\mathbf{I}_\sigma$ (*Induktionsoperator*). [24]
2. combinations with logical symbols $\rightarrow, \neg, \wedge, \vee, \forall, \exists$.

Sets of the first language stratum S_1 are then sets that can be formed in the way pointed out above through formulas of S_1. Now we can iterate this process and form S_2 in the same way we formed S_1 from S_0. In particular, we get a formula that represents the enumeration for the objects of S_1 and is itself an object of S_2. So, with S_2, we now have two types of sets: the sets that could already be formed from S_1 and the "new" sets that were formed

[21]See section 2 for an example of the basic principles of protologic.

[22]We will discuss the question of potential and actual infinity in more detail in section 4.

[23]As we have seen in section 2, Lorenzen places himself in the tradition of Russell's type theoretic approach; see also Lorenzen (1956a, 275).

[24]We introduced this operator in the definition of a relation, see above.

through formulas from $S_2 - S_1$. This process can now be iterated and gives us the following sequence: $S_0 \subseteq S_1 \subseteq S_2 \dots$

Interestingly, Lorenzen extends this method of iteration to countably infinite ordinals: S_ω is defined to be the union of the S_n for all finite n (and more general for a (countable) limit ordinal θ, S_θ is the union of all S_ν for $\nu < \theta$). $S_{\omega+1}$ is then again built over S_ω through the usual iteration of inductive definitions and application of logical symbols. The question upon which countable ordinal to stop this iteration process is open,[25] though it is essential that it does go beyond S_ω.[26]

In all language strata that are higher than the basic S_0 we are provided with iterations of sets: set of sets of basic objects, sets of sets of sets of basic objects and so on; and if we want every set to appear as an element, we have to iterate up to a limit ordinal. At the same time, every higher language strata also produces new sets of basic objects as can be shown via Cantor's diagonalization method: there is no enumeration of the elements of S_ν in S_ν, but there is one in $S_{\nu+1}$[27]. As a consequence the power set becomes a relative notion, namely relative to a specific language stratum, where every higher language stratum adds more sets.

In the same way we can also see that countability and uncountability are relative notions. As the enumeration of a language stratum S_ν is not element of S_ν, the set of all basic objects in S_ν is not countable in S_ν but it is in $S_{\nu+1}$. Therefore uncountability will always only be relative to some language stratum and this entails trivial solutions to the questions of the Axiom of Choice and the Continuum Hypotheses.

However the difference between countability and uncountability is a central notion for classical mathematics and therefore Lorenzen expands on the issue in Lorenzen (1956a). Here he defines a replacement notion for uncountability and shows how it can be used to transfer the countability-uncountabulity differentiation into operative mathematics. Consider the following definition (Lorenzen, 1956a, 276): Fix a limit θ_1 and a second limit $\theta_2 > \theta_1$. All language strata should be considered as constructed up to S_{θ_2}. Then we call all strata up to index θ_1 "primary" and all higher strata up to θ_2 "secondary". Similarly, a set is called "primary" if it is definable through a formula in a primary language stratum, and "secondary" otherwise.

Lorenzen immediately notes that the primary–secondary distinction does not seem to fit the usual countable–uncountable distinction. It can be shown that every infinite primary set contains secondary subsets whereas subsets of

[25]Lorenzen (1955, 189) names $\omega + 1, 2\omega, \omega^2$ or ϵ_0; Lorenzen (1956a, 275) names ω^ω.

[26]Lorenzen (1955, 189) argues that "[f]or analysis, in the way we will develop it in chapter 6, it is only necessary that we transcend ω."

[27]For a detailed construction of this argument see (Lorenzen, 1955, 191-192)

countable sets are always countable. But, he argues, the notion of secondary set can be *used* in operative mathematics as a replacement for uncountability. For this he considers several examples from integration theory, measure theory and topology in which instances he shows that the replacement works as intended. One example is the following: Real numbers are primary as they can be introduced through primary convergent sequences of rational numbers (that are primary themselves). But intervals of real numbers are secondary because every interval of reals consists of reals of arbitrarily high primary strata (Lorenzen, 1956a, 276-277). He therefore argues that the notions of secondary set and uncountable set are "essentially" equivalent:

> Summing up, I would like to claim that the use of secondary strata (obviously not without boundaries, only up to the unproblematic ordinal numbers, like ω^ω) in connection with the linguistic means [sprachliche Mitteln] available in every stratum gives us enough mobility to mostly follow the lines of reasoning [Gedankengängen] of modern mathematics.

(Lorenzen, 1956a, 279)

At this point Lorenzen developed an operative analysis which is "in essence" (*"im wesentlichen"*) similar to classical analysis. By classical analysis Lorenzen means "in virtue of content", so not an axiomatic system of analysis but concrete, non-abstract analysis or—as he puts it—everything that can be found in "current textbooks" (Lorenzen, 1955, 196). In particular, this includes everything that is needed for the natural sciences, like theoretical physics. He therefore achieved his main goals: To present a foundation for mathematics that on the one hand is meaningful by using a mathematically definable concept of set; while on the other hand is still preserving the essence of classical analysis. For our purposes, the former is the more informative conclusion. We can now state that the elimination of the classical concept of set was a main motivation for Lorenzen's work on operative mathematics and, in his view, he achieved his goal of replacing it with a more acceptable one. In the next chapter we will examine how this endeavour relates to Lorenzen's views on the concept of infinity.

4 The question of infinity

4.1 A shift in focus

Let us now return to our initial claim about a shift in Lorenzen's treatment of infinity. As we mentioned in the introduction, Lorenzen is know for his rejection of actual infinity in his (1957). Here he uses operative mathematics as an example for a foundation of analysis that only uses potential infinity instead of requiring actual-infinite sets. He concludes with the statement that actual infinity should be eliminated from modern mathematics (Lorenzen, 1957, 11). This could suggest that operative mathematics was motivated by the search for a framework that allows for such an elimination. Indeed this claim is made in the literature: In Stegmüller's review of Lorenzen's book on operative mathematics, he remarks that "from the beginning the author is concerned in allowing nothing more that the 'potential infinite' in the foundation of mathematics, [...] while abandoning the idea of 'completed infinities' "(Stegmüller, 1958, 177). Heyting (1957) reviews both papers (1957; 1954) at once. He judges the first one as the philosophical and the second one as the more mathematical paper that together explain Lorenzen's operationist system. He points out that "the author agrees with intuitionists in so far as he does not accept the notion of actual infinity in mathematics." (Heyting, 1957, 368) In both cases however the respective reviews appeared *after* Lorenzen (1957); in the cases of Heyting that is, of course, necessarily the case. Prima facie that does not seem to be problematic, as all of these works were tightly connected by the common topic of the operationist approach and its foundational role, as well as by publication date. But, surprisingly, when regarding these works in a chronological order and not in retrospect, as a single body of work, an explicit rejection of actual infinity is not endorsed by Lorenzen until his (1957). Rather, the question of infinity is not at all prevalent in developing his operationist system. As we have shown in the last chapter, what is heavily discussed is the question of the notion of set and the deficiencies the Cantorian notion of set has, which makes it unsuitable for a "good" foundation for mathematics.

This leads us to propose that a shift occurred in Lorenzen's focus on foundational questions which manifests itself by first concentrating on the notion of set and later on the notion of infinity. Let us give the following classification:

First phase: In the early 1950s Lorenzen focuses on the *notion of set*. He sees it as his central task to eliminate the classical notion of set and to replace it with a rigorous, operative definition. This provides a solution of the problem of uncountable sets, as *uncountability* only remains as a relative notion, whereas all sets are countable in an absolute sense.

Second phase: In the late 1950s Lorenzen's focus is on *the notion of infinity*. This is now primarily framed as the *distinction between potential and actual infinity* and it culminates in the goal of eliminating the actual infinite from mathematics.

To make the timeframe of the above classification a little more precise, we will count the articles that lead up to Lorenzen (1955) and papers that expand on this, like Lorenzen (1956a), towards the first phase, whereas Lorenzen (1957) marks the shift towards considering the notion of infinity as the more fundamental question and, therefore, is already a representative of the second phase.

Let us point out more clearly what this shift does and does not imply: It does not imply that Lorenzen's topic of research changed between the two phases outlined above. Instead, it is a change of focus regarding the issue which notion (set or infinity) is the more fundamental and, therefore, whether eliminating one or the other gives us good reasons to argue that the mathematical outcome is free of the inherent flaws of other foundational approaches. The technical way of explicating such a good foundation can—and indeed is—the same, namely the framework of operative mathematics, but the *reason* for why one should consider this to be a good foundation differs. Naturally, the notion of set and notion of infinity are interconnected: In the first phase, the argument that the newly developed notion of set only relies on potential infinity is used as an added argument for its soundness; in the second phase, the intended potentialist notion of infinity is explicated by using the operative notion of set.[28] Nevertheless, it still matters heavily which of the notions is primarily regarded for foundational purposes, not least because changing one does not have to entail changing the other.[29]

So our main goal here is to show that such a shift in focus exists in Lorenzen's work. However with proposing the occurrence of such a shift, we make no claim on if operative mathematics constitutes a potentialist framework regarding infinity or not. Indeed we will treat this as a separate question that due to place and time constraints will have to be considered in a different article. The same holds for the question of *why* such a shift occurs. An analysis of this requires a much wider consideration of Lorenzen's philosophical development than we are able to present in this context. We will therefore limit ourselves to pointing out some considerations towards both questions at the end of this article.

In the rest of this article we will argue for Lorenzen's proposed change in focus in the following way: Having already described that the elimination of the Cantorian notion of set was a main motivation for the development of operative mathematics in the first phase (see section 3), we will show how this

[28] Examples where Lorenzen argues along these lines will be given below.

[29] The same holds for a connection that seems even more fundamental, namely, the connection between intuitionistic logic and potential infinity. Although it is generally assumed that they are dependent on each other, Linnebo and Shapiro (2019) show that there is an explication of potential infinity that still allows for classical logic.

changes towards regarding the question of infinity as more foundational in the second phase. As this is spelled out quite clearly in Lorenzen (1957), the main burden of proof for the proposed shift lies in arguing that the question of infinity was not prevalent as the main foundational motivation in the first phase.

Let us therefore start by addressing this point first. What we intend to show is that in the first phase Lorenzen places no special emphasis on the question of potential and actual infinity and indeed does not commit himself explicitly to a standpoint in this debate. We will present two arguments: First, we point out that Lorenzen simply does not reject actual infinity explicitly, even in cases where it would be obvious to do so; and, second, we document his argument that the problems usually associated with infinity are in fact problems that arise because of the Cantorian notion of set and, by doing so, he reduces the question of infinity to the question of the notion of set.

4.2 Constructs and infinity

Both lines of argument can be seen quite clearly in his article "On the consistency of the concept of infinity" (1952b). In the first sentence he states: "The problematic nature of the concept of infinity is independent from that of set." This may seem surprising at first, as one major motivation of operative mathematics is to address the problems in modern mathematics by eliminating the classical notion of set. But in the following it becomes clear, that with "concept of infinity" he means something very specific, namely the notion of the infinity as it appears *in arithmetic*: "If one poses the question of the consistency of the notion of infinity, one wants to know, if no contradictory statements can be proven over ℕ, i.e. if arithmetic (the theory of ℕ) is consistent" (Lorenzen, 1952b, 591). He then explains this further by arguing why the natural numbers are the right choice for such an investigation into the concept of infinity:

> Of course one could take any infinite set instead of ℕ - however, with such a formulation one would unnecessarily enter into the difficulty of the concept of set. What makes ℕ so unproblematic as a set is that each one of its elements is constructible. Starting from 1, one obtains all natural numbers when constructing an additional $a + 1$ to a. We will call every set a "construct" whose elements can be constructed in such a (eventually much more complicated) way. (Lorenzen, 1952b, 591)

He elucidates this point in the conclusion of the article:

> In cases where the dangers of contradiction still exist in connection with an infinite set, e.g. the set of all real numbers, all real functions, as they are used in classical analysis—in all these cases one will need to find "the error" not in the concept of infinity, but in the concept of set. (Lorenzen, 1952b, 594)

How is this presentation of a concept of infinity to be understood? Being aware of Lorenzen's decisive rejection of actual infinity in later articles, one could in retrospect read this as the difference between potential and actual infinity. Indeed, several points seem to support this thesis, at least when looking at the first term of the distinction, namely, potential infinity. The way in which "constructs" are introduced accords with the usual way in which potential infinity is explained, i.e. there is some kind of procedure that can be used again and again with the possibility to go on without an end. So, it could be that for Lorenzen constructs are an acceptable way of describing infinity because they are compatible with the framework of potential infinity by being the outcome of a well-defined process.

If we claim that the underlying motivation in restricting the concept of infinity to that of constructs is founded on the potential-actual infinity distinction, this immediately raises the question: Why doesn't Lorenzen say so? Instead, Lorenzen never uses the term "potential infinity" in the whole article. A similar point could be made for his treatment of potential-actual infinity in his book on operative mathematics (Lorenzen, 1955). Here, he does mention both concepts already in the introduction when he describes Brouwer's intuitionistic program (Lorenzen, 1955, 2). But even if he uses the introduction many times to point out which philosophical and mathematics thoughts influenced him in his work, he never commits to a standpoint in the actual-potential infinity debate. And, as we have seen in section 3, even when he mentions potential infinity later on in the book, he never goes so far as to explicitly reject actual infinity. One example for such a remark is: "We already presupposed the capacity to conceive of every rule of a calculus as something potentially infinite on the part of the reader when dealing with protologic" (Lorenzen, 1955, 133). The maximal commitment we find here is that he does not have to stay in a finitist framework because of his assumption that agents carrying out the schematic operations have the capacity to work with the potential infinite.

Still, this kind of argument does not seem sufficient. There can be a number of reasons why he never explicitly stated the connection to the actual-potential infinity debate. He could have considered it to be unimportant to make this philosophical point in a more mathematical paper; he could have regarded it as so obvious that there is no need to bring it up; he could have been unaware of his underlying motivations (though this is quite unlikely).

To settle the question more satisfactorily, let us look more closely at the notion of "constructs" he introduces in the paper. This term seems to capture the concept of infinity he wants to consider (and therefore the acceptable concept of infinity), whereas problems arise with sets that are not constructs. So our question from above reduces to the problem whether the introduction of constructs is motivated by questions about infinity or by questions about the concept of set. As we have seen above, the argument can be made that the definition of constructs is motivated by the concept of potential infinity. What, then, about actual infinity? Could it be that Lorenzen simply understands everything which is not a construct, for example the set of the real numbers, as an actual infinite set and therefore rejects it as an instance of acceptable infinity? He seems to hint at something like this when talking about what he (later in the article) calls the "schism in mathematics":

> Whether it is justified to demand a ban of all those sets, which cannot be reduced to constructs, this is still a debated question. If one lets this question stand open, the interesting situation arises which is on display in current mathematics: there are actually two mathematics, a constructive one (in particular in the intuitionistic form) and a classical one (the axiomatic variety also belongs here).(Lorenzen, 1952b, 594)

In the same year "On the consistency of the concept of infinity" is published, Lorenzen gives a talk at the second "Colloque International Logique Mathematique" in Paris. This talk is later published in Lorenzen (1954), where he provides a more detailed account of the notion of construct. It is especially interesting to see how he frames the introduction of constructs: It is to provide an answer to the challenge of eliminating the naive concept of set from analysis (Lorenzen, 1954, 67). The notion of construct is explained, similarly to (Lorenzen, 1952b, 591), as a foundation for the basic notion of "set" (Lorenzen, 1954, 70). He then discusses how and why it could be legitimate to restrict the naive notion of set in such a way and presents his construction of language strata, where each language stratum S_n and indeed also S_ω and $S_{\omega+1}$ are constructs (Lorenzen, 1954, p71).

At this point it becomes quite clear that Lorenzen regards the concept of set as much more foundational as the concept of infinity. When restricting the concept of set to the acceptable version of constructs, then the concept of infinity indeed becomes consistent with the arguments presented in Lorenzen (1952b). But this is a consequence of settling the question about what a "good" concept of set is. So the question of the right concept of infinity is secondary to the question of the correct concept of set, indeed the former is resolved by resolving the latter.

It only remains to be shown whether he still holds this position in the final version of operative mathematics in Lorenzen (1955). Here he does not use the notion of construct; instead, he replaces it with a more elaborate setup in which definiteness is the key notion.

The notion of definiteness (*Definitheit*) is very basic for operative logic and mathematics. In short, a symbol string is definite if it is derivable in a calculus ("definite via proof") or if it is not derivable in a calculus ("definite via refutation"). He gives an inductive definition of being "definite" already in the introduction of his book (1955, 5):

1. Every formula that is decidable via schematic operations is called definite.
2. If a notion of being definite via proof or being definite via refutation is determined for a formula, then the formula itself is called definite.

He then builds up the operative notion of set in the way we described in section 3, always making sure that hi logical and mathematical setup remains definite. This becomes particularly important when deciding where to terminate the construction of language strata. As we have seen above, Lorenzen is not set on the exact point where the construction of language strata has to end, it is only important that one makes the step to S_ω resp. $S_{\omega+1}$. At the same time it is equally important that the construction *does* end at some point![30] The reasons for the restriction of this iterative process is given in (1955, 189):

> However, there would be no *definite* meaning in saying that the iteration should be continued "arbitrarily" long—in the same way clauses from modern set theory, like the following, are not definite: "the index of the language strata should run through Cantor's II. number class (II. Zahlenklasse)".

So the notion of definiteness takes the place of the notion of construct in providing a boundary line for which sets should be permitted or excluded, as all operative sets have to appear in some language stratum. In a second step, this restriction of the concept of set gives then rise to the operative concept of infinity where uncountable infinities only appear relative to language strata.

[30] "... obviously not without boundaries, only up to the unproblematic ordinal numbers..." (Lorenzen, 1956a, 279)

We can therefore conclude that in the first of the proposed phases, Lorenzen indeed considered the concept of set the most fundamental *philosophical* notion in developing operative mathematics[31]. Resolving the problems introduced by the classical notion of set was his main motivation. Again, this does not mean that he did not care about the question of actual-potential infinity. But it was not prevalent in his foundational motivations; rather the question of how to deal with the infinite gets resolved by using the "right" conception of set. We will see how this changed in the next subsection.

4.3 Rejection of actual infinity

To show Lorenzen's shift in focus on the concept of infinity in the second of the proposed phases, we will concentrate on the article "The Actual-Infinite in Mathematics" (Lorenzen, 1957). The reasons for considering this article only are two-fold: First, the article gives a very strong formulation of the motivation behind his work in what we called the second phase. Especially the programmatic part at the end of the article constitutes what we claim to be the content of the second phase. Second, soon after the publication of this article, Lorenzen began to abandon operative logic and mathematics. According to Lorenz (2001, 35), the reason was a flaw in the concept of definiteness that ultimately made it inappropriate for Lorenzen's purposes.[32] In his (1957) however, Lorenzen still considers operative mathematics, but this time motivated by eliminating the notion of actual infinity from mathematics.

This explains why the mathematical content of this paper is not new in comparison to his prior work. The philosophically motivated interpretation of this mathematical content, however, is novel, as Lorenzen looks at operative mathematics through the lens of the question of potential and actual infinity. Here we can find all of the direct references to these conceptions of infinity we were unsuccessfully looking for in the earlier works.

When introducing the construction of the natural numbers via the usual rules, Lorenzen states:

[31] Again, this does not mean that he didn't have major mathematical motivations for the development of the operationist system, see also footnote 2.

[32] "It was Alfred Tarski who [...] convinced him of the impossibility to characterize arbitrary (logically compound) propositions by some decidable generalization of having a decidable proof-predicate or a decidable refutation-predicate. [...] Hence, Lorenzen's attempt of an inductive definition of 'definite' in order to find a characterization of propositions which relinquishes the synonymy of 'definite' and 'decidably definite' had to be accepted as inappropriate" (Lorenz, 2001, 35).

> [To] assert that infinitely many such numbers are really exist, that they can really be constructed by following this rule would of course be false.[...]
>
> In philosophical terminology we say that the infinity of the number sequence is only potential; that is, it exists only as a possibility but does not actually (i.e., not in reality) exist. (Lorenzen, 1987, 196), translated from (Lorenzen, 1957, 4-5).

He concludes that "in arithmetic there is [...] no motivation to introduce the actual infinite" [33]. He then discusses how actual infinity comes up in geometry and the construction of the real numbers and finally discusses modern set theory:

> It is this actual infinity of real numbers (latent in modern mathematics since the seventeenth century) that Cantor first brought explicitly to light, and it is the basis for the present acceptance of the Cantorian conception of infinity.
>
> With the admission of the set of all real numbers as a legitimate object for mathematics there is simultaneously admitted the "power set," that is, the set of all subsets. (Lorenzen, 1987, 200), translated from (Lorenzen, 1957, 8).

All of this is neither historically nor mathematically new. But Lorenzen frames this development completely under the motivational point of view of the actual-potential infinity debate. What consequences does he then draw for the future development of mathematics? He states:

> The key to the indicated solution lies in replacing Cantor's power set of the set C of all cardinal numbers with an appropriate potentially infinite set. [...] Thus, instead of the power set, we have to construct an appropriate potentially infinite set of propositional forms. (Lorenzen, 1987, 201), translated from (Lorenzen, 1957, 10).

This is really the locus where the proposed shift comes into play. He still concludes that the concept of (a certain kind of) set has to be replaced by a conceptually different notion, but the motivation, explicitly stated, is to eliminate actual infinity. The replacement of the concept of set now only becomes the way in which such a reduction to potential infinity is carried out. The new concept of set has to fulfill the requirement of being only potentially

[33] (Lorenzen, 1987, 197), translated from (Lorenzen, 1957, 5)

infinite; so the concept of infinity here is the primary concern, the concept of set is secondary. Lorenzen states this quite clearly in his programmatic appeal with which he concludes the article:

> If the conception developed here is correct, it represents for modern mathematics a reform [...].
>
> Just as at that time the infinitely small was to be eliminated from mathematics, now also the infinitely large (more precisely, the actually in- finite) is to be demonstrated to be dispensable.
>
> (Lorenzen, 1987, 201/202), translated from (Lorenzen, 1957, 10-11).

5 Conclusion and outlook

We have shown above that Lorenzen shifts his focus from considering the concept of set as the more fundamental notion in his foundational work in operative mathematics, until the mid-50s, to focusing on the concept of the infinite and, in particular, the actual-potential infinity distinction in the later 1950s. This seems to indicate that Lorenzen's operative work was not simply a precursor for his later work, but a body of work with its own philosophical and mathematical motivations.

As mentioned before, two questions still remain open: First, why does this shift occur? And secondly, regardless of Lorenzen's motivations, does operative mathematics represent a framework for potential infinity? The answer to both questions requires more work than can be done within the context of this article, so let us just point out some consideration towards possible answers.

Regarding the question of why such a shift occurred, a simple answer presents itself. The 1950's marked a significant change for Lorenzen from an institutional point of view. Having been appointed a professor of mathematics at the University of Bonn since 1952, he changed to a professorship in philosophy at the University of Kiel in 1956. From a thematic point of view, this may not seem surprising since he had been working in the foundations of mathematics for some years. Nevertheless, Lorenzen perceived it as a change himself, as he remarked upon it several times in correspondence with the philosopher Oskar Becker (who was Lorenzen's colleague in Bonn), as can be seen in this slightly sarcastic passage:

> Besides, as I am a "philosopher" now, this is all right with me, because it is common with "philosophical" book that they are being completely misunderstood. (Letter from Lorenzen to Becker,

August 22nd, 1957, OB 5-1-1, Philosophical Archive, University of Konstanz.)[34]

So one explanation for this shift is that the context of debate changed. One could argue that within the first phase Lorenzen primarily wanted to inform the mathematical debate on foundational issues as it developed in the aftermath of the so-called foundational crisis. In the second phase, however, Lorenzen re-framed his operationist approach to inform the philosophical debate about mathematical infinity. To fully argue for such a thesis, more work has to be done. But, reading Lorenzen's papers, one can always find hints that he was very aware of which discussions would be considered interesting for the mathematical community and its discourse on foundational issues; and which would be better suited for the philosophical community.[35] So, this is a viable candidate for an explanation of this shift.

The systematic question whether the operationist approach is indeed a framework for potential infinity (and does not assume any kind of actual infinity) has no clear answer from the outset. Obviously, Lorenzen considered this to be the case, as he names operative mathematics as one solutions to replace the usual power set with a potentially infinite set of formulas (Lorenzen, 1957, 10). But this view did not remain uncontested. Niebergall (2004) analyses the underlying assumption of infinity Lorenzen draws from in setting up basic numbers (via the usual rules presented in operative mathematics) and thoroughly examines Lorenzen's notions of "always-counting-on" and, more generally, rule-based infinite processes (see Niebergall, 2004, 171). In the end he concludes that such approaches to infinity can never stay completely potential, one always has to assume some kind of actual infinity "be it of infinite objects or be it of infinitely many objects" (Niebergall, 2004, 171).[36]

Looking at operative mathematics, this analysis seems to hold in an even stronger way. Let us consider the fundamental framework in which operative analysis is developed, namely the framework of language strata. The

[34]"Da ich jetzt ja außerdem 'Philosoph' bin, ist mir das auch insofern recht, weil es bei 'philosophischen' Büchern ja gang und gäbe ist, dass sie völlig missverstanden werden."

[35]One such occurrence is the following quote (more can easily be found in his papers): "Taking the cavalier attitude that most mathematicians display toward philosophical questions, we could try simply to ignore these "sophistical"' problems." (Lorenzen, 1987, 197/198), translated from (Lorenzen, 1957, 6).

[36]Although this conclusion can be contested: One of the referees pointed out that Niebergall critique only seems to hold when reconstructing it in standard set theory and this is not the way Lorenzen intended it to be read. It would be interesting to contrast these different viewpoints in more detail and to investigate to which amount modern analysis (like the one of Linnebo and Shapiro (2019)) can have an impact on Lorenzen's notion of potential infinity.

fact that the first language stratum contains infinitely many object can be explained by the rule-based procedures of schematic operations (although this is exactly what Niebergall criticizes). However, the most relevant step in the construction of the language strata occurs when defining S_ω and $S_{\omega+1}$. As we have seen, S_ω is the union of all the language strata from before. Note that this does not yet mean that we are actually building a new language over atoms in this step. This only happens when defining $S_{\omega+1}$. So we now consider the infinite union of all of the infinite objects in the S_n and in $S_{\omega+1}$ we begin to add new objects to this infinite union of infinite objects. Therefore, we not only allow sets whose elements can be defined by a construction process, but we have to consider infinitely many versions of different construction processes at once (to know if an x is in S_ω we have to search over infinitely many S_n and check if x appears in one of them). Lorenzen states that the reason for allowing such procedures is motivated by mathematical expediency:

> If we want to attain that every set also appears as an element, we have have to go up to a limit ordinal with the index of the language strata. For example, one can go beyond the language strata with finite index by building a set whose elements are $u, \{u\}, \{\{u\}\} \ldots$. This set is only representable in $S_{\omega+1}$. (Lorenzen, 1955, 190)

As Lorenzen (1955, p.189) himself remarks, this step is crucial for being able to mirror classical analysis in operative mathematics. Even if Niebergall's arguments from above could be sidestepped, the setup with language strata seems to make it challenging to argue that operative mathematics provides a framework for potential infinity.

Perhaps an answer to this challenge could be found in recent work on actual and potential infinity by Linnebo and Shapiro (2019). Here the authors address Niebergall's general claim that there is no clear way of expressing potential infinity without either staying finite or falling back on actualist assumptions about infinity (see Linnebo and Shapiro, 2019, 166). They propose a way out of Niebergall's dilemma by using modal logic to explicate different potentialist positions. Indeed, they are able to show that there is a way of being potentialist about infinity while still using classical logic (instead of having to use intuitionistic logic). It could be an interesting future project to see how Lorenzen's operational system, or indeed similar constructive approaches, fare when analyzed with Linnebo and Shapiro (2019)'s modal explication.

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