On the Depth of Szemerédi's Theorem[†]

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ABSTRACT

Many mathematicians have cited depth as an important value in their research. However, there is no single, widely accepted account of mathematical depth. This article is an attempt to bridge this gap. The strategy is to begin with a discussion of Szemerédi's theorem, which says that each subset of the natural numbers that is sufficiently dense contains an arithmetical progression of arbitrary length. This theorem has been judged deep by many mathematicians, and so makes for a good case on which to focus in analyzing mathematical depth. After introducing the theorem, four accounts of mathematical depth will be considered.

Mathematicians frequently cite depth as an important value for their research. A perusal of the archives of just the *Annals of Mathematics* since the 1920s reveals more than a hundred articles employing the modifier 'deep', referring to deep results, theorems, conjectures, questions, consequences, methods, insights, connections, and analyses. However, there is no single, widely-shared understanding of what mathematical depth consists in. This article is a modest attempt to bring some coherence to mathematicians' understandings of depth, as a preliminary to determining more precisely what its value is. I make no attempt at completeness here; there are many more understandings of depth in the mathematical literature than I have categorized here, indeed so many as to cast doubt on the notion that depth is a single phenomenon at all. Yet I hope to advance our philosophical understanding of this rich cluster of values a little bit, perhaps making way for more unified accounts to follow.

My strategy in this article is to introduce Szemerédi's theorem, that every sufficiently 'dense' subset of $\mathbb{N}$ contains an arbitrarily long arithmetic progression, as a case study for focusing an investigation of mathematical depth. After discussing the content

of this result and gesturing at its proofs, I will discuss four distinct views of depth, showing how each arises in the mathematical literature and how each suffers from important problems, indicating what tasks remain for filling out each account.

1. SZEMERÉDI'S THEOREM

I shall begin by presenting Szemerédi's theorem in its historical context, in order to bring out its content with sufficient clarity for our purposes here. Its story starts with a result of Isaai Schur from 1916 that is known now as *Schur's theorem*.[1] Let h be a positive integer. Then Schur's theorem asserts that there is a positive integer $N = N(h)$ such that any coloring (or less vividly, partitioning) of $1, 2, \ldots, N$ by h many colors contains integers a, b, c of the same color such that $a + b = c$ (*cf.* [Schur, 1916]). Thus a, b, c are a 'monochromatic' solution to the equation $x + y = z$.

Nowadays we view Schur's theorem as among the first results of additive combinatorics, in which one 'aims to understand very *simple* systems: the operations of addition and multiplication and how they interact' (*cf.* [Green, 2009, p. 489] and [Tao and Vu, 2006]), but a 1927 result of van der Waerden is what really got this subject going.[2] To understand van der Waerden's theorem, recall the notion of an *arithmetic progression*, a sequence of numbers progressing in steps of the same size. For example, $2, 7, 12, 17, 22, 27, 32$ is an arithmetic progression with step size 5. Van der Waerden showed that if h and k are positive integers, then there is a positive integer $N = N(h, k)$ such that for any coloring of $1, 2, \ldots, N$ by h many colors, there is at least one monochromatic arithmetic progression of length k. His proof used the pigeonhole principle and a double induction on h and k, and is thus purely combinatorial.

Khinchin has called van der Waerden's theorem 'deep', noting that van der Waerden's work on it was the talk of Göttingen in 1927 [Khinchin, 1998, p. 9]. But for us it is just a step toward our goal of Szemerédi's theorem. A next step is Erdős and Turán's conjecture in 1936 that every sufficiently 'dense' subset of $\mathbb{N}$ contains an arithmetic progression of length three [Erdős and Turán, 1936]. More precisely, they conjectured that for any $0 < \delta \leq 1$ there is a positive integer $N = N(\delta)$ such that every subset of $\{1, 2, \ldots, N\}$ of size at least δN contains an arithmetic progression of length three. In 1952, Klaus Roth proved this conjecture, using Fourier analysis (*cf.* [Roth, 1952, 1953]); his 1958 Fields Medal citation lists this resolution among his most significant achievements [Davenport, 1960, pp. lix–lx].

At the same time Erdős and Turán also conjectured that every sufficiently 'dense' subset of $\mathbb{N}$ contains an arbitrarily long arithmetic progression, not just an arithmetic progression of length three. If true, it would entail that the coloring in van der Waerden's theorem was unnecessary, that sets of integers with essentially no structure contain arbitrarily long arithmetic progressions, as will be shown shortly.

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[1] It is not relevant to the account here, but Schur was reproving a 1908 theorem of L.E. Dickson, who sought to prove Fermat's Last Theorem. *Cf.* [Soifer, 2009, p. 301].

[2] Indeed Schur had a role in this result also; while van der Waerden called his theorem a resolution of a conjecture of Baudet, there is reason to think it was independently conjectured by Schur; *cf.* [Soifer, 2009, Chap. 34].

In 1975 Endre Szemerédi resolved this conjecture [1975], yielding the eponymous theorem that is our focus in this article. The most helpful formulation of it for this article is the following 'finitary form'.

Szemerédi's theorem. Let $k \geq 3$ be an integer and let $0 < \delta \leq 1$. Then there is a positive integer $N = N(k, \delta)$ such that any subset $A \subseteq \{1, 2, \ldots, N\}$ with $|A| \geq \delta N$ contains an arithmetic progression of length k.

This result is often formulated in terms of a density condition, using the notion of 'upper Banach density' defined as

$$d(A) = \limsup_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}.$$

Then Szemerédi's theorem says that every subset of $\mathbb{N}$ with positive upper Banach density contains arbitrarily long arithmetic progressions.

Earlier I noted that Szemerédi's theorem's entails that the coloring in van der Waerden's theorem was unnecessary. Indeed Szemerédi's theorem *implies* van der Waerden's theorem. Given h and k, pick $\delta = \frac{1}{h}$ and apply Szemerédi's theorem. Then there is an N such that any $A \subseteq \{1, 2, \ldots, N\}$ with $|A| \geq \frac{N}{h}$ contains an arithmetic progression of length k. For any h-coloring of $\{1, 2, \ldots, N\}$, at least one block A of integers of the same color will have $\frac{|A|}{N} \geq \frac{1}{h}$. Then $|A| \geq \frac{N}{h}$; so by Szemerédi's theorem A contains an arithmetic progression of length k.

I now want to consider more closely the *content* of Szemerédi's theorem. In 2012 Szemerédi won the Abel Prize, an award recently created by the Norwegian government to promote mathematical research in lieu of a Nobel Prize for mathematics. In a lecture given as a webcast upon the prize's announcement, the Fields Medalist Timothy Gowers explained the content of Szemerédi's theorem in such a clear way that I will now quote extensively from his text.

> One way of understanding Szemerédi's theorem is to imagine the following one-player game. You are told a small number, such as 5, and a large number, such as 10,000. Your job is to choose as many integers between 1 and 10,000 as you can, and the one rule that you must obey is that from the integers you choose it should not be possible to create a five-term arithmetic progression.

> For example, if you were accidentally to choose the numbers 101, 1103, 2105, 3107 and 4109 (amongst others), then you would have lost, because these five numbers form a five-term progression with step size 1002.

> Obviously you are destined to lose this game eventually, since... if you keep going for long enough you will eventually have chosen all the numbers between 1 and 10,000, which will include many five-term arithmetic progressions.

> But Szemerédi's theorem tells us something far more interesting: even if you play with the best possible progression-avoiding strategy, you will lose long before you get anywhere near choosing all the numbers. [Gowers, 2012, p. 3]

As Gowers explains, if you are trying to avoid an arithmetic progression of length k while picking numbers from $1, 2, \ldots, N$ for a large N, the largest number of numbers

you can pick is a very small percentage of N, as small as you like, as long as N is large
enough.

Gowers continues:

> If, for instance, we are trying to avoid progressions of length 23, Szemerédi's
> theorem tells us that there is some N (which may be huge, but the point is that
> it exists) such that if we play the game with N numbers, then we cannot choose
> more than $\frac{N}{1000}$ of those numbers — that is, a mere 0.1% of them — before we
> lose. And the same is true for any other progression length and any other positive
> percentage. [*ibid.*, p. 3]

Thus the power of Szemerédi's result, put succinctly, consists in revealing the existence
of remarkable additive structure within sets of integers chosen under only minimal
constraint.

To close this section I briefly want to mention *proofs* of Szemerédi's theorem.
Four main *types* of proofs have been given: Szemerédi's original 1975 proof, which
is combinatorial and in particular graph-theoretic; Furstenberg's ergodic-theoretic
proof [1977]; and Gowers's Fourier-analytic proof [1998]. A fourth approach employ-
ing hypergraph theory has also been given by Nagle, Rödl, Skokan, and Schacht
(*cf.* [Nagle *et al.*, 2006; Rödl and Skokan, 2004]), and, independently, by Gowers
[2007]. One reason for so many reproofs, especially since [Bourgain, 1986], has been
to find better bounds on N. Another is a growing sense that each proof reveals a new
facet of Szemerédi's theorem, that 'the many proofs of Szemerédi's theorem act as a
kind of "Rosetta Stone"' [Green, 2009, p. 490] — a view put forward by Tao to which
I will return in the next section.

2. THE DEPTH OF SZEMERÉDI'S THEOREM

Having explained the content of Szemerédi's theorem within its historical context, I
want next to turn to its *depth*. That the result has been judged a major accomplishment
is clear: Erdős, who had conjectured Szemerédi's theorem with Turán, recalled that

> I offered \$1,000 for [my conjecture] and late in 1972 Szemerédi found a brilliant
> but very difficult proof of [my conjecture]. I feel that never was a 1,000 dollars
> more deserved. In fact several colleagues remarked that my offer violated the
> minimum wage act. [Erdős, 1985, p. 76]

But it has also been judged deep. A typical instance of such a judgment has been given
by Tao and Vu:

> Of course, a 'typical' additive set will most likely behave like a random additive
> set, which one expects to have very little additive structure. Nevertheless, it is a
> deep and surprising fact that as long as an additive set is dense enough in its ambi-
> ent group, it will always have *some* level of additive structure. The most famous
> example of this principle is *Szemerédi's theorem* ... [a] beautiful and important
> theorem. [2006, pp. xiii–xiv]

Other judgments of the depth of Szemerédi's theorem include the announcement [Committee, 2008] of Szemerédi's 2008 Rolf Schock Prize in mathematics; Vitaly Bergelson's in [2000, p. 45]; Anthony Gardiner's in [2008, p. 964]; and others that will be cited over the course of this section.

Thus I will henceforth take for granted the widespread judgment by mathematicians that Szemerédi's theorem is deep, and turn to the philosophical question of what its depth consists in. I will consider four types of accounts of its depth, what I will call *genetic* views, *evidentialist* views, *consequentialist* views, and *cosmological* views.

In the course of assaying these four types of accounts of depth, one issue, of interest to many philosophers of mathematics, will recur: whether judgments of mathematical depth can be *objective*, and if so, in what sense. In a preliminary investigation of depth like this one, it is not my goal to decide definitively on the larger questions of whether depth is objective, what objectivity of the relevant sort would consist in, and whether it is a good/bad thing that depth is/is not objective. But given the interest of the question of the objectivity of judgments of depth, I will ask of each of the four accounts of depth I consider whether they can deliver a notion of depth that is not vague, is temporally stable, and is not essentially dependent on our contingent interests or our merely human limitations.

2.1. Genetic Views of Depth

Prior to having criteria for deep theorems, mathematicians seek indicators of depth. One common indicator is the *talent* of a theorem's provers. Raff and Zeilberger provide a good example of such a view in the following, discussing Szemerédi's theorem.

> The depth and mainstreamness of this deep theorem can be gleaned by the fact that at least four Fields medalists (Klaus Roth, Jean Bourgain, Tim Gowers, and Terry Tao) and at least one Wolf prize winner (Hillel Furstenberg) made significant contributions. [2010, p. 313]

Indeed Roth won the Fields Medal in 1958, Bourgain in 1994, Gowers in 1998, and Tao in 2006; Furstenberg won the Wolf Prize in 2006/7; and two years after Raff and Zeilberger's article, Szemerédi won the 2012 Abel Prize. These are among mathematics' most prestigious prizes, and thus, it is fair to say, these mathematicians' accomplishments have been judged to be magnificent by the mathematical community. Raff and Zeilberger's suggestion is that a result like Szemerédi's that attracts such feted contributions indicates the result's depth.

We can convert this into a criterion for depth by identifying a deep theorem as one proved by sufficiently talented mathematics. Call such an identification a *genetic* view of depth.

One problem with this as a view of depth is its subjectivity. No doubt there is widespread agreement that these mathematicians have accomplished something splendid; but such agreement is merely a sociological fact. A judgment of a mathematician's talent may be rooted in nothing more than reputation, or pedigree, or glamour. Of course such judgments *could* be rooted in a clearer and more objective criterion — but

then *that* criterion would be a better criterion for depth than merely the judgment of the theorem provers' talent.

Another problem with genetic views of depth is that such views count even the trivial theorems of talented mathematicians as deep. But not every result proved by the masters is deep. Even the best mathematicians (save perhaps Gauss) prove minor results; but a genetic view of depth cannot distinguish these from the ones rightly judged deep.

2.2. Evidentialist Views of Depth

Thus we turn to a second type of view on depth. What I want to call *evidentialist* views of depth are those that link the depth of a theorem with some quality of its *proof*. The announcement of Szemerédi's 2012 Abel Prize, for instance, focuses more on Szemerédi's proof than on the theorem's content:

> Many of his discoveries carry his name. One of the most important is Szemerédi's Theorem, which shows that in any set of integers with positive density, there are arbitrarily long arithmetic progressions. Szemerédi's proof was a masterpiece of combinatorial reasoning, and was immediately recognized to be of exceptional depth and importance. [Committee, 2012]

An evidentialist view of depth that is uncharacteristically precise by the standards of mathematical writing has been offered by the number theorist Daniel Shanks. In discussing the question of whether the law of quadratic reciprocity is deep, Shanks writes:

> We confess that although this term 'deep theorem' is much used in books on number theory, we have never seen an exact definition. In a qualitative way we think of a deep theorem as one whose proof requires a great deal of work — it may be long, or complicated, or difficult, or it may appear to involve branches of mathematics the relevance of which is not at all apparent. [1978, p. 64]

Shanks thus locates the depth of a theorem in its *laboriousness*.

There is no doubt that Szemerédi's proof of his eponymous theorem is laborious. The original article in which the proof is given includes the 'flow chart' given in Figure 1 in order to guide the reader through the proof. That Szemerédi thought such a chart necessary testifies to the proof's laboriousness. Thus according to Shanks's criterion, Szemerédi's theorem is deep.

Nonetheless Shanks's criterion has problems. Firstly, the criterion measures laboriousness in terms of several properties of proofs: length, complexity, difficulty, and impurity (wherein a proof draws on means that are not 'close' or 'intrinsic' to what it is proving). While the first three of those are related to one another (though not identical), impurity is a quite different type of property. In [Arana, forthcoming] the case for this is made in some detail, but in brief, there are many examples of short but evidently impure proofs; for one, Furstenberg's topological proof of the infinitude of primes [1955]; the proof is discussed in [Detlefsen and Arana, 2011, §4]. Moreover,
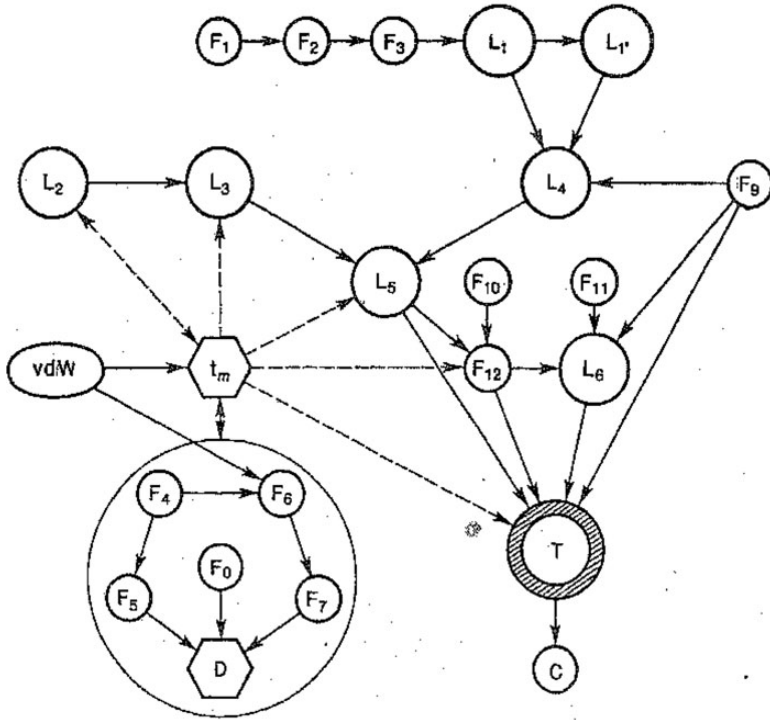
Fig. 1. Original caption: The diagram represents an approximate flow chart for the accompanying proof of Szemerédi's theorem. The various symbols have the following meanings: F_k = Fact k, L_k = Lemma k, T = Theorem, C = Corollary, D = Definitions of B, S, P, α, β, etc., t_m = Definition of t_m, vdW = van der Waerden's theorem, F_0 = "If $f : R^+ \to R^+$ is subadditive then $\lim_{n\to\infty} \frac{f(n)}{n}$ exists".

while Szemerédi's proof is undoubtedly long, complicated, and difficult, it has widely been judged pure (if not in those words), as a combinatorial proof of a combinatorial theorem.[3] Thus Shanks's category of 'laboriousness' groups together rather different properties of proof in determining whether a theorem is deep.

While one can plausibly construct at least the beginning of a story for why a theorem with a long or complex proof should be judged deep without too much trouble—the resources involved in comprehending or discovering complex proofs distinguish such theorems from less resource-intensive theorems—the same is not true for theorems with impure proofs. The Furstenberg proof of the infinitude of primes just mentioned was published in the *American Mathematical Monthly*, a journal aimed at a wide audience of mathematicians, including undergraduates. One needn't be a specialist to follow the proof, and indeed Furstenberg discovered it while still an undergraduate.

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[3] For instance, Tao writes that 'Szemerédi's theorem appears to be solely concerned with arithmetic combinatorics' [2006, p. 2], and the 2012 Abel Prize committee, in a passage quoted earlier, called Szemerédi's proof 'a masterpiece of *combinatorial* reasoning [emphasis added]'.

Yet the proof is impure. Thus the reasons for judging a theorem with a complex proof to be deep must in general differ from the reasons for judging a theorem with an impure proof to be deep. Hence Shanks's criterion of depth is less informative than would be optimal, in the following sense. If I am told that a theorem is Shanks-deep, then I must still press further to determine the reasons for its depth; whereas a criterion in terms of just impurity or just complexity would not require this further interrogation.

A potentially easy fix for Shanks's criterion would be to remove either the cluster of length/complexity/difficulty, or impurity, from the criterion. But it is not at all clear which ought to be removed, because it is unclear which is, or ought to be, more fundamental for depth. Thus this easy fix will not work.

The applicability of Shanks's criteria in the case of Szemerédi's theorem is further clouded by the latter's having multiple, apparently distinct, proofs. For while Szemerédi's proof is combinatorial and thus pure, Gowers's proof uses Fourier analysis and is evidently impure. Which proof should determine the depth of Szemerédi's theorem on Shanks's criterion?

A fix would be to count a theorem as deep if *some* of its proofs are impure. However, it would be easy to insert an irrelevant impurity into an otherwise pure proof (for instance, add a computation involving complex numbers to Szemerédi's proof but make no use of that computation in the rest of the proof). One might reply that the irrelevancy of such insertions do not make impure what would otherwise be a pure proof, because only impurities 'essential' to the identity of a given proof matter to its purity and hence depth — but how is it determined which impurities are essential to a proof being the proof that it is? To require answering that in order to solidify Shanks's criterion would simply be to replace one difficulty with another.

Another fix would be to count a theorem as deep if *all* of its proofs are laborious. Indeed all currently known proofs of Szemerédi's theorem are laborious. But that is quite contingent: tomorrow an easy proof of Szemerédi's theorem could be found. It would be preferable for judgments of depth to be more temporally stable than that.

Indeed the problem of multiple proofs is a problem with evidentialist criteria of depth generally, since, if depth is to be determined by a theorem's proof, which proof does the determining? This suggests a different type of evidentialist strategy: among the multiplicity of proofs of a given theorem, determine the theorem's depth by considering features of *just one* of its proofs. Such evidentialist strategies measure a theorem is deep if its

$$\left.\begin{array}{l}\text{shortest proof}\\\text{simplest proof}\\\text{shallowest proof}\\\text{canonical proof}\\\text{every proof of it}\\\text{most every proof of it}\end{array}\right\} \text{is deep.}[4]$$

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[4]On the notion of 'canonical' proof, attributed to Frege, see [Detlefsen, 1996, pp. 59–60].

These strategies are parasitic on a prior criterion of depth of proof, though, a criterion that seems as difficult and in need of analysis as depth of theorem.

Let us consider another evidentialist strategy: a theorem is deep if it has many different proofs. But this strategy requires an account of the individuation of proofs to work. For if two apparently different proofs are in fact the same, then what might have otherwise been judged a deep theorem perhaps should not be. The problem of individuation of proof is however a deep one (pun intended). When are two apparently different proofs genuinely different, and when are they actually the same (despite perhaps looking superficially different)? Mathematicians talk of 'rephrasing' proofs, or 'recasting' them (*cf.* [Cass and Wildenberg, 2003] for an example): but such talk indicates that the proof is still the same, only being expressed differently. One might hold that the mode of expression of a proof is part of the individuation of a proof — but it is clear that here we are in murky, largely unexplored waters, and that an account of depth that relies on resolving such issues is not yet adequately clear.[5]

Bracketing the issue of individuation of proofs, we can at least gesture toward one further evidentialist strategy. As noted in the last section, Tao has put forth the view that 'the many proofs of Szemerédi's theorem act as a kind of "Rosetta Stone" ' connecting the fields on which these proofs draw [Green, 2009, p. 490; Tao, 2006, p. 584]. On this view, Szemerédi's theorem is deep because its different proofs share features that make manifest connections between different areas of mathematics, in such a way that one is able to, in the words of André Weil, 'pass from one to the other, and to profit in the study of the first from knowledge acquired about the second' [2005, p. 340]. Theorems like Szemerédi's are thus deep, on this view, in virtue of possessing several different proofs with enough intertranslatability that a mathematician who knows one proof can translate some of her knowledge about the domain that this proof concerns into knowledge of a different domain that another proof concerns. This promises to afford her further efficiency in discovering and proving further theorems.[6] This is merely a sketch of how such an evidentialist view of depth could be developed, but I hope it is clear enough how to begin pursuing this type of view.

2.3. Consequentialist Views of Depth

Let us next turn to a third type of view on depth. A *consequentialist* view of depth measures the depth of a theorem by some quality of its consequences, or of the consequences of its proofs. Typically, the quality in question is fruitfulness, the degree to which a theorem (or a proof of a theorem) leads to yet further theorems and proofs, particularly to important theorems and proofs (for more on fruitfulness in mathematics, *cf.* [Tappenden, 2012]).

That Szemerédi's theorem is fruitful is well-documented. Of particular note is Green's and Tao's use of Szemerédi's theorem to prove that there are arbitrarily long arithmetic progressions consisting only of prime numbers [Green and Tao, 2008] — a

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[5] On this problem, see also [Dawson, 2006, §2].

[6] I have benefited greatly from discussions of intertranslatability with Sean Walsh.

result that is the first item cited in Tao's Fields Medal announcement (*cf.* [Committee, 2006, p. 1]). Tao has stressed the fruitfulness of Szemerédi's theorem:

> Remarkably, while Szemerédi's theorem appears to be solely concerned with arithmetic combinatorics, it has spurred much further research in other areas such as graph theory, ergodic theory, Fourier analysis, and number theory; for instance it was a key ingredient in the recent result that the primes contain arbitrarily long arithmetic progressions. [Tao, 2006, p. 2]

The announcement of Szemerédi's 2012 Abel Prize notes the applicability of Szemerédi's proof:

> A key step in [Szemerédi's] proof, now known as the Szemerédi Regularity Lemma, is a structural classification of large graphs . . . Over time, this lemma has become a central tool in both graph theory and theoretical computer science. [Committee, 2012]

Gowers too stresses the applicability of Szemerédi's theorem as a key reason for why it is 'fascinating' to mathematicians in his lecture accompanying the announcement of Szemerédi's Abel Prize [Gowers, 2012, p. 5].

A chief problem with consequentialist views of depth is that the value of a given consequence — such as its fruitfulness — is interest-dependent. Every theorem is fruitful with trivial logical consequences, but what a consequentialist view of depth seeks are *important* consequences. But 'importance' is not a very rigorous quality in mathematics; plenty of results formerly judged important have receded in importance as mathematicians have developed new machinery (think for instance of the former importance of results on conic sections filling up countless nineteenth-century textbooks that have now been subsumed by results in algebraic geometry). Thus if depth is interest-dependent, a theorem is deep only so long as those consequences are of interest to us, and this threatens to render depth subjective.

This leads to the following observation. Consequentialist accounts of depth can be objective only if some other values that mathematicians hold, such as 'interestingness', are objective. Those who believe in the objectivity of such values will, for that reason, be comfortable with the objectivity of consequentialist accounts of depth. But for the many who are not so confident that the other values that mathematicians adhere to are objective, this account of depth will seem unsatisfying.

Finally, a consequentialist might mark a theorem as deep if it has *deep* consequences. But then we would be in a circle, unless there is a fundamental level of depth at which all such recursions end. No such fundamental level is at hand now though.

2.4. Cosmological Views of Depth

The last type of view of depth I want to discuss is what I will call a *cosmological* view. I have chosen this word because of the Greek word *kosmos*, meaning 'order'. The functional analyst Gilles Godefroy invokes a cosmological view of depth applied to

Szemerédi's theorem in the following passage:

> Szemerédi's theorem ... is a deep combinatorial result that establishes (like Ramsey's theorems) the inescapable presence of certain structures, lumps of order in a formless dough [*grumeaux d'ordre dans une pâte informe*], even though we have a great deal of freedom of construction since the only constraint imposed on *A* is positive density. [Godefroy, 2011, p. 221]

Szemerédi has identified such a feature of his theorem as well, in an interview he gave after being awarded the 2012 Abel prize:

> In finite objects we look for patterns, different shapes, and try to understand what conditions make different patterns emerge, this is one of the most fundamental questions. A slightly pompous and philosophical way to put it is that we want to prove that there is order in any chaos. In other words, if you give me a hostile structure, I will still be able to find orderly parts in it. [Szemerédi, 2012, pp. 1–2]

Recall that Szemerédi's theorem tells us that for any arithmetic progression of length k and any desired positive density, there is an N such that we are guaranteed to find such an arithmetic progression choosing only from a subset of $\{1, 2, \ldots, N\}$ of our desired density (so its size can be very small relative to N). What Godefroy calls a 'formless dough' and what Szemerédi calls 'chaos' is the chosen dense subset of $\{1, 2, \ldots, N\}$. We choose these subsets without any other constraints besides positive upper Banach density. Szemerédi's theorem shows that these 'randomly' chosen sets must contain arithmetic progressions of the desired length. This 'unexpected' structure is what makes Szemerédi's theorem deep on what I am calling a cosmological view of depth. The idea is that there is 'serious' structure latent in the premises, despite their evident structural weakness.

To call the revealed structure 'unexpected' makes it sound like on this view, the depth of a theorem depends on our ability to see/expect structure. But if the depth of a theorem amounted to our inability to recognize order, depth would be rather subjective. Thus a fuller working out of cosmological depth would have to identify a sharper and more objective notion of orderliness.

To clarify the schematic of what I take to be going on with cosmological depth, let us consider a precisification whose distortions will hopefully not detract much from its helpfulness. Let us suppose there is a function S from statements to $\mathbb{R}$ measuring the order expressed by a statement, and a relation $A << B$ that holds when A is much less than B. Then we can say that a theorem is $S, <<$-cosmologically deep if $S(\text{premises}) << S(\text{conclusion})$. We could construe S as measuring the orderliness of the entities specified by a statement, or of the state of affairs specified by a statement. I do not know how to build S or $<<$ in a way true to practice; so I cannot say if Szemerédi's theorem is $S, <<$-cosmologically deep on this view for reasonable S and $<<$. Nor do I know how plausible it is that every agent of our rational type would judge the extensions of S and $<<$ in the same way. Nevertheless this seems to me a potentially promising approach to specifying cosmological depth better.

The cosmological view of depth, like the others I have surveyed here, does not fully capture depth as used in practice. For instance, Wiles's result that all elliptic curves arise from modular forms is surely deep, but the premise (that E is an elliptic curve) is at least as orderly as the conclusion (that E arises from a modular form). The result would thus not qualify as cosmologically deep as construed here so far. Note also that the theorem that every modular form has an elliptic curve attached to it — the Eichler-Shimura theorem — is surely also deep. We then have two deep theorems with the logical forms, respectively, of 'every A is a B' and 'every B is an A'. But then if the $\ll$ relation is antisymmetric, as seems reasonable, it would follow that both theorems cannot be cosmologically deep.[7]

3. CONCLUSIONS

I have put forward Szemerédi's theorem as a case for studying mathematical depth. Arithmetic combinatorics is a rich source of cases for thinking about depth, because its theorems tend to be simply stated and readily understandable, yet strong. Furthermore this area has been a hotbed of activity in the last couple of decades, with many articulate mathematicians at its center.

My modest goal here has been to articulate four different accounts of depth, and to indicate ways in which each is apt and inapt for characterizing depth as it occurs in mathematical practice. These accounts each shed a little light on depth as a notion with many faces; a fuller account will have to come to grips with the plurality of analyses raised here, to see if perhaps some further (and dare I say it) *deeper* unities can be found.

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[7]I am grateful to Jordan Ellenberg for the points in this paragraph.

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