Labeled Calculi and Finite-valued Logics

Matthias Baaz[1,⋆] Christian G. Fermüller[2,⋆⋆]
Gernot Salzer[2] Richard Zach[2,3,⋆⋆⋆]

[1] Institut für Algebra und Diskrete Mathematik E118.2,
Technische Universität Wien, A–1040 Vienna, Austria
[2] Institut für Computersprachen E185.2,
Technische Universität Wien, A–1040 Vienna, Austria
[3] Group in Logic and the Methodology of Science,
University of California, Berkeley, CA 94720–3840

Abstract

A general class of labeled sequent calculi is investigated, and necessary and sufficient conditions are given for when such a calculus is sound and complete for a finite-valued logic if the labels are interpreted as sets of truth values (sets-as-signs). Furthermore, it is shown that any finite-valued logic can be given an axiomatization by such a labeled calculus using arbitrary "systems of signs," i.e., of sets of truth values, as labels. The number of labels needed is logarithmic in the number of truth values, and it is shown that this bound is tight.

Keywords: finite-valued logic, labeled calculus, signed formula, sets-as-signs
MR Subject Classification: 03B50, 03B22, 03F05

1 Introduction

Looking at the definition of Gentzen's sequent calculus LK for classical logic, one is struck by the obvious connection between the introduction rules and the truth-table semantics. This connection can be exploited: the rules can be used to extract models; this is the essential idea of Schütte's completeness proof for LK. A similar connection can also be established for LJ. A construction similar to Schütte's provides a completeness proof relative to Kripke models for intuitionistic logic [17].

These are examples for two contrary aims in logic. The first is to find calculi that characterize a given semantics, the second is to find semantics for a logic that is only given as a calculus. Intuitionistic logic and modal logics were originally investigated in pursuit of the second aim, and currently, e.g., linear logic is so investigated. For classical logic and many-valued logics one has been in the position of the first aim: one had the matrix semantics and was looking

⋆Corresponding author. Email address: `baaz@logic.tuwien.ac.at`
⋆⋆Research supported by FWF Grant J01103-MAT.
⋆⋆⋆Research supported by FWF Grant P10282-MAT.

for axiomatizations. The results for many-valued logic have often been similar to the classical case, using many-placed sequent calculi or tableaux systems with truth values as signs (cf. [2, 4, 6, 11, 13, 16]). In a way, the first aim has been reached, at least for all finite-valued logics, although there are further questions, e.g., of finding calculi with specific properties (see, e.g., [1]).

The connection between syntax and semantics *in general* is rather opaque and little understood. In the present paper we investigate this connection for the class of *labeled calculi*.[4] A labeled calculus works on multisets of labeled propositional formulas (in LK, we would have two labels, expressed by the left and right sides of the sequent).

We give necessary and sufficient conditions for a labeled sequent calculus to be considered as a finite-valued logic in the sense that the labels can be interpreted as sets-as-signs with respect to which the calculus is sound and complete. It turns out that the central question is: Which cut rules can be eliminated in the style of Gentzen?

In the reverse direction, we show that for any finite-valued logic and any suitable system of sets-as-signs (i.e., sets of truth-values) it is possible to construct a sequent calculus (which satisfies the above conditions). We also obtain bounds on the number of truth values of a logic extracted from a labeled calculus by means of Sperner's Lemma. These bounds imply that roughly $\log_2 m$ signs suffice for an m-valued logic.

The paper is structured as follows. Section 2 provides basic concepts concerning labeled sequent calculi. Section 3 reviews definitions concerning finite-valued logics. Section 4 deals with cut elimination; it describes an algorithm for checking whether cuts are eliminable in the style of Gentzen. Section 5 defines the notion of a rich calculus and shows how to construct a finite-valued logic for a given rich sequent calculus. Section 6 deals with the opposite direction: given a finite-valued logic it shows how to obtain a corresponding labeled sequent calculus. Section 7 establishes the exponential relationship between the number of truth values and the number of labels.

2 Labeled Sequent Calculi

A propositional language consists of an enumerable set of propositional variables A, B, …, and a fixed finite supply of propositional connectives $\Box_1$, $\Box_2$, …. The arity of a connective $\Box$ is denoted by $\mathrm{ar}(\Box)$. Formulas are defined as usual by induction, i.e., propositional variables are formulas, and if $\Box$ is a connective and $A_1, \ldots, A_{\mathrm{ar}(\Box)}$ are formulas then $\Box(A_1, \ldots, A_{\mathrm{ar}(\Box)})$ is a formula. Furthermore, let $\mathbf{L}$ be a finite set of objects called *labels*. A *labeled formula* is an expression of the form $l{:}\,A$, where $l \in \mathbf{L}$, and A is a propositional formula.

[4] A remark concerning the relationship between the labeled calculi studied here and Gabbay's labeled deductive systems (LDS) seems in order. Both frameworks employ a system of labels to mark formulas in a derivation. Methodologically, the two frameworks have different aims: Gabbay constructs LDS for given logics, we construct logics for given labeled calculi. LDS are mainly used to deal with substructural logics; in our labeled calculi, all structural rules are present and there is no label algebra involved.

Labels may be thought of as syntactic markers by which to keep track of the status of formulas in the course of a derivation. Classical examples of the use of labeled formulas in deductive systems are the signed variants of Beth tableaux for classical [14] and many-valued logics [5, 8], where truth values or sets of truth values function as labels. Calculi using labeled formulas are called *labeled calculi*.

A *labeled sequent* is a finite multiset of labeled formulas. We denote labeled sequents by uppercase Greek letters, and write Γ, Δ for $\Gamma \cup \Delta$ and $\Gamma, l{:}\, A$ for $\Gamma \cup \{l{:}\, A\}$. An *atomic* sequent is a sequent consisting only of formulas $l{:}\, A$, where A is a propositional variable.

Remark 2.1 To follow Gentzen's original notion of sequents more closely we could have defined sequents as finite sequences of labeled formulas. Then a Gentzen sequent $\Gamma \to \Delta$ would correspond to a sequent with two labels marking the formulas on the left hand and right hand side of the arrow, respectively. For our purpose the order of formulas within a sequent is irrelevant, therefore we use multisets instead of sequences.

A *labeled sequent calculus* consists of a finite set of rules. In this paper, we concentrate on a particular kind of labeled sequent calculi, which might be called *strictly analytic*. Since no other sequent calculi are going to be considered, the qualifier will be omitted. In the following, $A, A_1, A_2, \ldots$ denote formula variables.

WEAKENING. For every $l \in \mathbf{L}$,

$$\frac{\Gamma}{\Gamma, l{:}\, A}\ \text{weak:}\, l$$

CONTRACTION. For every $l \in \mathbf{L}$,

$$\frac{\Gamma, l{:}\, A, l{:}\, A}{\Gamma, l{:}\, A}\ \text{cntr:}\, l$$

Weakening and contraction are called *structural rules*.

PROPOSITIONAL RULES. An introduction rule for a propositional connective $\Box$ at position $l \in \mathbf{L}$ is of the form

$$\frac{\Gamma, \Delta_1 \quad \cdots \quad \Gamma, \Delta_\ell}{\Gamma, l{:}\, \Box(A_1, \ldots, A_{\mathrm{ar}(\Box)})}\ \Box{:}\, l$$

where the formulas in Δ_i, $1 \leq i \leq \ell$, are of the form $l'{:}\, A_j$ for some $1 \leq j \leq \mathrm{ar}(\Box)$ and $l' \in \mathbf{L}$.

The labeled formula $l{:}\, \Box(A_1, \ldots, A_{\mathrm{ar}(\Box)})$ is called *main formula*, the formulas in Δ_i are called *auxiliary formulas*, and the formulas in Γ are the *side formulas* of a rule.

Remark 2.2 The rules as stated above are called *multiplicative*: the side formulas of all premises and of the conclusion are identical. In additive rules each premise may have side formulas different from the other premises; the conclusion contains all these side formulas. In our case the choice is just a matter of taste since the calculi contain unrestricted weakening and contraction rules.

Calculi may be complemented by axioms and cuts.

AXIOMS. An axiom schema is of the form $l_1{:}\,A, \dots, l_\ell{:}\,A$ for some $\{l_1, \dots, l_\ell\} \subseteq \mathbf{L}$, $\ell \geq 1$. Every instance of an axiom schema is called axiom. In an *atomic* axiom, A has to be an atomic formula.

An axiom schema can be identified with the set $\{l_1, \dots, l_\ell\}$ of involved labels. The set of all axiom schemas in a given sequent calculus is denoted by $\mathcal{A}x$; it can be regarded as a subset of $2^{\mathbf{L}}$. An axiom is called *proper* if $\ell \geq 2$ and none of its proper subsets is an axiom.

Without loss of generality we assume that all supersets of an axiom are again axioms; the supersets of an axiom could be derived by the weakening rule anyway. In particular, the set $\mathbf{L}$ of all labels is an axiom, provided there is at least one axiom.

CUT RULES. A cut (rule) is of the form

$$\frac{\Gamma, l_1{:}\,A \quad \cdots \quad \Gamma, l_\ell{:}\,A}{\Gamma}\ \mathrm{cut}{:}\,l_1, \dots, l_\ell$$

for some $\{l_1, \dots, l_\ell\} \subseteq \mathbf{L}$, $\ell \geq 1$.

A cut rule can be identified with the set $\{l_1, \dots, l_\ell\}$ of involved labels. The set of all cut rules in a given sequent calculus is denoted by $\mathcal{C}uts$; it can be regarded as a subset of $2^{\mathbf{L}}$. A cut is called *proper* if $\ell \geq 2$ and none of its proper subsets is a cut.

Without loss of generality we assume that all supersets of a cut rule are again cut rules; applications of cut rules based on supersets can always be replaced by the smaller cut rule, discarding the parts of the proof concerning the superfluous labels. In particular, the set $\mathbf{L}$ of all labels is a cut, provided there is at least one cut.

Remark 2.3 It would of course be possible to allow cuts with more than one occurrence of the cut formula in a premise. However, such an extended cut rule (together with weakenings) implies the soundness of several simple cut rules of the above form, namely those obtained by deleting all but one occurrence of the cut formula in each premise. On the other hand, given the latter cut rules the original cut can be simulated.

A *derivation* of a labeled sequent Δ from labeled sequents $\Gamma_1, \dots, \Gamma_n$ is an upward rooted finite tree of sequents, where each leaf node is an instance of an axiom or one of the Γ_i, each internal sequent follows from the ones immediately above it by applying one of the rules, and the root sequent is Δ. A *proof* of Δ is a derivation from axioms alone. A *refutation* of sequents $\Gamma_1, \ldots, \Gamma_n$ is a derivation of the empty sequent from these sequents. To express that a formula F has a proof in a particular calculus $\mathbf{C}$ we write $\mathbf{C} \vdash F$.

3 Finite-valued Propositional Logics

Definition 3.1 A *matrix* $\mathbf{M}$ for a propositional language consists of a finite set, V, of at least two *truth values* and a total *truth function* $\widetilde{\Box}: V^{\mathrm{ar}(\Box)} \to V$ for every connective $\Box$. A matrix (together with the corresponding propositional language) is called a $|V|$*-valued propositional logic*.

Unless stated otherwise we assume in the following that a fixed matrix $\mathbf{M}$ is given.

Definition 3.2 An *interpretation* $\mathbf{I}$ is a function mapping propositional variables to truth values. It is extended to a valuation function $\mathrm{val}_{\mathbf{I}}$ on formulas, defined inductively by $\mathrm{val}_{\mathbf{I}}(A) = \mathbf{I}(A)$ for variables A, and $\mathrm{val}_{\mathbf{I}}(\Box(A_1, \ldots, A_{\mathrm{ar}(\Box)})) = \widetilde{\Box}(\mathrm{val}_{\mathbf{I}}(A_1), \ldots, \mathrm{val}_{\mathbf{I}}(A_{\mathrm{ar}(\Box)}))$.

Definition 3.3 A subset of V is called *sign*. A set of signs, S, over V is called *system of signs* if for every $v \in V$ there is a subset $S_v \subseteq S$ such that $\bigcap S_v = \{v\}$.

Note that $|S| \geq 2$ since $|V| \geq 2$. Furthermore, each system of signs contains a set of non-empty signs, whose intersection is empty.

Proposition 3.4 *Suppose S is a system of signs over V. Then, for all $v \in V$ there is $T_v \subseteq S$ such that $V \setminus \{v\} = \bigcup T_v$.*

Proof By definition, for each $u \in V$ there is $S_u \subseteq S$ such that $\{u\} = \bigcap S_u$. So we can write

$$V \setminus \{v\} = \bigcup_{u \in V \setminus \{v\}} \bigcap S_u = \bigcap_i \bigcup S'_i$$

for some $S'_i \subseteq S$ by distributivity. Obviously, for each i, $V \setminus \{v\} \subseteq \bigcup S'_i$ and so for at least one i, $V \setminus \{v\} = \bigcup S'_i$. $\Box$

Definition 3.5 A *labeled formula expression* is a formula built up from labeled formulas using $\mathbin{\text{⩓}}$, $\mathbin{\text{⩔}}$, and $\neg$. Semantically, a sequent $\{l_1\colon A_1, \ldots, l_n\colon A_n\}$ can be identified with the labeled formula expression $l_1\colon A_1 \mathbin{\text{⩔}} \cdots \mathbin{\text{⩔}} l_n\colon A_n$. A *label assignment* la is a function mapping labels to signs, $la\colon \mathbf{L} \to 2^V$.

A labeled formula $l\colon A$ is intended to mean that A takes (under a given interpretation $\mathbf{I}$) a truth value in $la(l)$. So we write $\mathbf{I} \models_{la} l\colon A$ iff $\mathrm{val}_{\mathbf{I}}(A) \in la(l)$ and extend this in the obvious way to signed formula expressions. More precisely:

Definition 3.6 Let $\mathbf{I}$ be an interpretation. The valuation function for labeled formula expressions, $\mathrm{lval}_{\mathbf{I}}$, is defined as

(a) $\mathrm{lval}_{\mathbf{I}}(l{:}\,A) = \top$ if $\mathrm{val}_{\mathbf{I}}(A) \in la(l)$, and $= \bot$ otherwise.

(b) $\mathrm{lval}_{\mathbf{I}}(\neg F) = \top$ if $\mathrm{lval}_{\mathbf{I}}(F) = \bot$, and $= \bot$ otherwise.

(c) $\mathrm{lval}_{\mathbf{I}}(F_1 \curlywedge F_2) = \top$ if $\mathrm{lval}_{\mathbf{I}}(F_1) = \mathrm{lval}_{\mathbf{I}}(F_2) = \top$, and $= \bot$ otherwise.

(d) $\mathrm{lval}_{\mathbf{I}}(F_1 \curlyvee F_2) = \bot$ if $\mathrm{lval}_{\mathbf{I}}(F_1) = \mathrm{lval}_{\mathbf{I}}(F_2) = \bot$, and $= \top$ otherwise.

We write $\mathbf{I} \models_{la} F$ if $\mathrm{lval}_{\mathbf{I}}(F) = \top$ and $\mathbf{I} \not\models_{la} F$ otherwise. F is called *satisfiable* (*valid*) if $\mathbf{I} \models_{la} F$ holds for some (all) $\mathbf{I}$. To express that a formula F is valid in a particular logic $\mathbf{M}$ we write $\mathbf{M} \models_{la} F$.

In other words, labeled formula expressions are nothing but formulas in classical two-valued logic based on a particular kind of atomic formulas, namely labeled formulas.

4 Cut Elimination

The central proof-theoretic property of an analytic calculus is the eliminability of all applications of *given cut rules* (Gentzen's Hauptsatz). In this paper, however, we use the calculation of eliminable cut rules for a *given cut-free calculus* to determine the truth values of the many-valued logic hidden in the calculus. The correspondence between reduction of cuts and resolution has been used in [3] to establish a cut-elimination theorem for certain first-order finite-valued calculi.

Definition 4.1 (Reducibility) A cut rule C given by the labels $\{l_1, \ldots, l_n\}$ is reducible with respect to cut rules $\mathrm{C}_1, \ldots, \mathrm{C}_m$ and introduction rules $\Box{:}\, l_1, \ldots, \Box{:}\, l_n$ for some operator $\Box$ if there is a refutation of the set $\{\Delta_j^{l_i} \mid 1 \leq j \leq \ell_i,\ 1 \leq i \leq n\}$ of sequents using only structural rules and the cut rules $\mathrm{C}_1, \ldots, \mathrm{C}_m$. The $\Delta_j^{l_i}$ are the auxiliary formulas of the introduction rule $\Box{:}\, l_i$ (see section 2); furthermore, the formula variables $A_1, \ldots, A_{\mathrm{ar}(\Box)}$ are regarded as propositional variables. To distinguish this kind of refutation from others we call it *reducing* refutation.

The refutation should be regarded as a schematic derivation. Obviously we may add arbitrary sequents Γ as side formulas and substitute arbitrary propositional formulas for the A_i and still obtain a valid derivation in the calculus.

Definition 4.2 (Local Admissibility) A set $\mathcal{C}$ of cut rules is *locally admissible* if each cut rule in $\mathcal{C}$ is reducible with respect to $\mathcal{C}$ and all introduction rules.

Note that in general there may be several introduction rules for an operator and a label. In this case local admissibility means that one has to check reducibility with respect to each of these rules.

Definition 4.3 (Global Admissibility) A set $\mathcal{C}$ of cut rules is *globally admissible* if it is locally admissible and for each derivation from atomic sequents using arbitrary cut formulas there is a derivation of the same end-sequent using only atomic cuts.

Proposition 4.4 *Reducibility is decidable.*

This follows immediately from the following lemma. The basic idea is to use propositional hyperresolution in classical (two-valued logic) for the construction of the refutations. Here we only review the basic definitions of hyperresolution; for a comprehensive treatment as well as for the proof of its completeness and correctness we refer the reader to [7, 10].

Definition 4.5 (Hyperresolution) A *literal* is a negated or unnegated propositional variable. A *clause* is a finite multiset of literals. It is called negative (positive) if it contains only negated (unnegated) literals. The clause $C \cup D$ is called *(binary) resolvent* of the two clauses $\{A\} \cup C$ and $\{\neg A\} \cup D$, where A is a propositional variable. A clause C is a called *factor* of D if $C = D$, or if C is a factor of a factor of D, or if $C = \{A\} \cup E$ and $D = \{A, A\} \cup E$ for some literal A and some clause E.

A *clash sequence* $(C; D_1, \dots, D_n)$ is a tuple of clauses where the D_i are positive clauses called *satellites* and C is a non-positive clause called *nucleus* of the clash sequence. Let $C_0 = C$, and let C_i be a binary resolvent of D_i and C_{i-1} for $0 < i \leq n$. If D_n is a positive clause, then the factors of D_n are called *hyperresolvents* of $(C; D_1, \dots, D_n)$.

A *hyperresolution deduction* of some clause D from some *input clauses* $C_1, \dots, C_n$ is an ordered tree of clauses with root D, where the leaf nodes are input clauses and each internal clause is the hyperresolvent of the clash sequences formed by its immediate predecessor clauses.

Definition 4.6 The clause corresponding to a sequent $\Gamma = \{l_1{:}\, A_1, \dots, l_n{:}\, A_n\}$ is given by

$$\mathrm{cls}(\Gamma) \quad = \quad \{P_{l_1:A_1}, \dots, P_{l_n:A_n}\}$$

where the $P_{l_i:A_i}$ are propositional variables uniquely associated with the labeled formulas $l_i{:}\, A_i$.

The clause corresponding to a cut rule $\mathrm{C} = \{l_1, \dots, l_n\}$ and a formula A is given by

$$\mathrm{cls}(\mathrm{C}, A) \quad = \quad \{\neg P_{l_1:A}, \dots, \neg P_{l_n:A}\}$$

where the $P_{l_i:A}$ are propositional variables uniquely associated with the labeled formulas $l_i{:}\, A$.

Lemma 4.7 *A cut rule* $\mathrm{C} = \{l_1, \dots, l_n\}$ *is reducible with respect to cut rules* $\mathrm{C}_1, \dots, \mathrm{C}_m$ *and introduction rules* $\Box{:}\, l_1, \dots, \Box{:}\, l_n$ *for some operator* $\Box$ *iff clause set* $\mathcal{D}$ *is unsatisfiable, where* $\mathcal{D}$ *consists of the following clauses:*

1. *For each set Δ^{l_i} of auxiliary formulas occurring in the premise of an introduction rule for $l_i\colon \Box(A_1, \ldots, A_{\mathrm{ar}(\Box)})$, $\mathcal{D}$ contains the clause* $\mathrm{cls}(\Delta^{l_i})$, *where $A_1, \ldots, A_{\mathrm{ar}(\Box)}$ are regarded as propositional variables.*

2. *For each cut rule C_i, $1 \leq i \leq m$, and each propositional variable A_j, $1 \leq j \leq n$, $\mathcal{D}$ contains the clause* $\mathrm{cls}(\mathrm{C}_i, A_j)$.

Proof *If:* We use the completeness of hyperresolution, i.e., the fact that for any unsatisfiable set of input clauses there exists a hyperresolution deduction of the empty clause. We claim that any such hyperrefutation ρ of the input clauses as defined in the proposition can be translated into a reducing refutation, i.e., into a sequent calculus refutation of the premises of $\Box\colon l_1, \ldots, \Box\colon l_n$ using only structural rules and the cut rules $C_1, \ldots, C_m$. Let $C', D_1, \ldots, D_m$ be the sequence of clauses that form the predecessor nodes of an internal clause E in ρ. By definition, E is a hyperresolvent corresponding to the clash sequence $(C'; D_1, \ldots, D_m)$. The satellite clauses D_i as well as E has to be positive, and C' has to be a non-positive input clause. The only such clauses are the ones corresponding to the cut rules. Let Δ_i and Σ be the labeled sequents obtained from the clauses D_i and E, respectively, by replacing the literal $P_{l:A}$ by the labeled formula $l\colon A$. We translate each hyperresolution step

$$\frac{C' \quad D_1 \quad \ldots \quad D_n}{E}$$

of ρ into a sequent calculus inference

$$\frac{\Delta_1 \quad \ldots \quad \Delta_n}{\Sigma}\; C'$$

Furthermore, each factoring step corresponds to one or more contractions.

Only if: Suppose the cut is reducible. Then, by definition 4.1, there is a reducing refutation of the sequents containing the auxiliary formulas of the introduction rules. Inverting the above translation we associate a clause with each sequent and a clash sequence with each application of a cut rule in the refutation. This way we obtain a hyperresolution derivation of the empty clause from the set of clauses corresponding to the used cut rules and the premises of the introduction rules. By the correctness of hyperresolution we conclude that this set of clauses is unsatisfiable. □

Lemma 4.8 *Any locally admissible set of cuts is globally admissible.*

Proof Similar to the usual proof (see e.g. [17]) we move some non-atomic cut highest in the proof tree upwards until it is immediately below the direct inferences of the cut formula. Then we use the reducibility of the cut rule with respect to the outermost connective of the cut formula and the occurring inference rules. This process is continued until all cuts are atomic. □

Theorem 4.9 *If a set of cuts is locally admissible and the set $\mathcal{A}x$ of axioms is closed under application of the cut rules, then any sequent derivable is cut-free derivable.*

Proof By lemma 4.8, any locally admissible set of cuts is globally admissible. If $\mathcal{A}x$ is closed under cuts the atomic cuts may be eliminated as well. $\square$

This variant of Gentzen's proof of the cut-elimination theorem warrants the following

Definition 4.10 A set of cuts is *Gentzen eliminable* if it is locally admissible and the axioms are closed under cuts.

Example 4.11 Note that not all sets of redundant cuts are Gentzen eliminable. For instance, consider the following simple labeled calculus on $\mathbf{L} = \{a, b, c, d\}$, with $\mathcal{A}x = \{L \subseteq \mathbf{L} \mid \{a, b\} \subseteq L\}$, $\mathcal{C}uts = \{L \subseteq \mathbf{L} \mid \{c, d\} \subseteq L\}$, and introduction rules:

$$\frac{a{:}\, A}{a{:}\, \Box A}\ \Box{:}\, a \qquad \frac{b{:}\, A}{b{:}\, \Box A}\ \Box{:}\, b$$
$$\frac{c{:}\, A}{c{:}\, \Box A}\ \Box{:}\, c \qquad \frac{c{:}\, A}{d{:}\, \Box A}\ \Box{:}\, d$$

The derivable sequents are exactly those containing $\{a{:}\, \Box^k A, b{:}\, \Box^l A\}$ as subsequents. So clearly anything derivable using cuts can be derived without such a cut, but the cut $\{c, d\}$ is not reducible.

5 From Labeled Calculi to Finite-valued logics

In the following we assume a labeled calculus with at least one proper axiom and one proper cut; furthermore, the axioms are assumed to be closed under cuts.

For our main theorems we need the notions of *pre-axioms* and *pre-cuts* of a given calculus. A pre-axiom (pre-cut) is a set of labels which itself is not yet an axiom (a cut) but which becomes one no matter which label is added. More formally, the sets of pre-axioms and pre-cuts are defined as

$$\begin{aligned} pre\mathcal{A}x &= \{L \subseteq \mathbf{L} \mid L \notin \mathcal{A}x,\ L \cup \{l\} \in \mathcal{A}x \text{ for all } l \in L^c\} \\ pre\mathcal{C}uts &= \{L \subseteq \mathbf{L} \mid L \notin \mathcal{C}uts,\ L \cup \{l\} \in \mathcal{C}uts \text{ for all } l \in L^c\} \end{aligned}$$

where L^c denotes the complement of L with respect to $\mathbf{L}$, $L^c = \mathbf{L} \setminus L$.

Example 5.1 Gentzen's sequent calculus for two-valued logic can be viewed as a labeled calculus with two labels, say t and f. The formulas on the left hand side of a sequent are implicitly marked by f, those on the right hand side by t. The only axiom as well as the only cut is $\{t, f\}$. Therefore we have $pre\mathcal{A}x = pre\mathcal{C}uts = \{\{t\}, \{f\}\}$.

The following two properties are the key for characterizing those labeled sequent calculi which represent many-valued logics.

(P1) *Eliminability of compound axioms.* Compound axioms are derivable from atomic axioms.

(P2) *Refutability of pre-axioms.* The set $\{\{l{:}\,A \mid l \in L\} \mid L \in pre\mathcal{A}x\}$ of sequents, where A is some arbitrary propositional variable, is refutable using cuts only.

We start by stating some basic facts.

Proposition 5.2

(a) The set $\mathbf{L}$ *of all labels is an axiom as well as a cut.*

(b) Each set of labels not containing a cut can be extended to a pre-cut.

(c) Each set of labels not containing an axiom can be extended to a pre-axiom.

Next we show that pre-axioms and pre-cuts are *dual* to each other.

Lemma 5.3 (Duality of pre-axioms and pre-cuts) *Suppose a calculus satisfies property P2. Then the following holds:*

(a) If L is a pre-axiom then L^c is a pre-cut.

(b) If L is a pre-cut then L^c is a pre-axiom.

Proof Remember that L^c is an abbreviation for $\mathbf{L} \setminus L$.

(a) We first show by an indirect argument that no subset of L^c is a cut. Suppose $K \subseteq L^c$ is a cut. Since L is a pre-axiom, adding any label from its complement leads to an axiom. In particular, $L \cup \{l\}$ is an axiom for all $l \in K$. From these axioms we may again derive L using cut rule K. Since axioms are closed under cut rules, L has to be an axiom, which contradicts the assumption that it is a pre-axiom. Hence no subset of L^c can be a cut, and no proper subset can be a pre-cut. If there is a pre-cut in L^c, it has to be equal to L^c.

    It remains to show that L^c is indeed a pre-cut. Suppose it is not. No subset of L^c is a cut, therefore it can be extended to a pre-cut K. Now observe that $K \cap L'$ is non-empty for all pre-axioms L': for $L' = L$ this follows from the fact that L^c, the complement of L, is a proper subset of K, whereas the other pre-axioms have to contain some label not occurring in L which therefore has to be in K. This means that from the set of sequents $\{\{l{:}\,A\} \mid l \in K\}$ we may derive by weakenings all sequents $\{l{:}\,A \mid l \in L'\}$ corresponding to pre-axioms L'. By condition P2, the set of pre-axioms is refutable, i.e., from the set $\{\{l{:}\,A\} \mid l \in K\}$ we derive the empty sequent.

In other words, K is a cut, contradicting the assumption that it is a pre-cut.

Summarizing, no proper subset of L^c is a pre-cut, therefore it can be extended to one. As we showed in the last paragraph, no proper superset of L^c is a pre-cut hence L^c has to be a pre-cut itself.

(b) Similar to above we first show that no proper subset of L^c is a pre-axiom. Suppose $K \subsetneq L^c$ is a pre-axiom. By the first part of the lemma its complement K^c has to be a pre-cut. But K^c is a proper superset of pre-cut L, and by definition all extensions of a pre-cut are cuts, including K^c. Contradiction.

It remains to show that no proper superset of L^c is a pre-axiom. This will imply the assertion of the lemma since any set which is not an axiom can be extended to a pre-axiom. Now suppose $K \supsetneq L^c$ is a pre-axiom. By the first part of the lemma its complement, K^c, is a pre-cut, and it is a proper subset of L. This contradicts the assumption that L is a pre-cut, since by definition any extension of a pre-cut is a cut. $\square$

Definition 5.4 A labeled sequent calculus is called *rich* if it has a non-empty set of axioms $\mathcal{A}x$ with at least one proper axiom, a non-empty set $\mathcal{C}uts$ of Gentzen eliminable cuts with at least one proper cut, and satisfies properties P1 and P2.

Proposition 5.5 *It is decidable if a labeled calculus can be extended to a rich calculus by adding cuts.*

Theorem 5.6 *Let* $\mathbf{C}$ *be a rich labeled sequent calculus. Then there is a finite-valued propositional logic* $\mathbf{M}$*, a system* S *of signs, and a label assignment* $la\colon \mathbf{L} \stackrel{1-1}{\to} S$ *such that*

$$\mathbf{C} \vdash \Delta \iff \mathbf{M} \models_{la} \Delta$$

for all labeled sequents Δ*. Furthermore, the rules are sound for* $\mathbf{M}$ *and* la*.*

We construct a new calculus, $\mathbf{C}^*$, whose rules can be regarded as macro-rules abbreviating derivations in $\mathbf{C}$. The new calculus has the property that it does not just transform labeled literals but sets of labeled literals of the form $\{l\colon A \mid l \in L\}$ (abbreviated as $L\colon A$), where L is a pre-axiom. From $\mathbf{C}^*$ one obtains directly the truth tables of a finite-valued logic corresponding to $\mathbf{C}$.

$\mathbf{C}^*$ contains one introduction rule per connective $\Box$ and pre-axiom $L \in pre\mathcal{A}x$. This rule is constructed in several steps by *combining* rules.

Definition 5.7 The *combination* of two rules

$$\frac{\Gamma, \Delta_1 \quad \cdots \quad \Gamma, \Delta_m}{\Gamma, \Delta}\, r \qquad \text{and} \qquad \frac{\Gamma, \Delta'_1 \quad \cdots \quad \Gamma, \Delta'_n}{\Gamma, \Delta'}\, r'$$

is defined to be the rule

$$\frac{\Gamma, \Delta_1, \Delta'_1 \quad \cdots \quad \Gamma, \Delta_1, \Delta'_n \quad \cdots \quad \Gamma, \Delta_m, \Delta'_1 \quad \cdots \quad \Gamma, \Delta_m, \Delta'_n}{\Gamma, \Delta, \Delta'} \; r \times r' \quad .$$

Obviously, the combination of rules is commutative and associative. The iterated combination $r_1 \times \cdots \times r_\ell$ is also written as $\prod_{i=1}^{\ell} r_i$.

Lemma 5.8 *Each application of $r \times r'$ can be replaced by applications of r and r'.*

Let $pre\mathcal{A}x = \{L_1, \ldots, L_n\}$ be the set of pre-axioms. We define the new rule

$$\frac{\Gamma, \Delta_1 \quad \cdots \quad \Gamma, \Delta_n}{\Gamma} \; A\colon \bot$$

where $\Delta_i = \{l\colon A \mid l \in L_i\}$; it expresses the refutability of pre-axioms (property P2).[5]

We first define an intermediary calculus $\mathbf{C}^+$ by combining all rules of $\mathbf{C}$ for the same label and connective into one rule and extending the premises to pre-axioms.

Step 1. If for some connective $\Box$ and some label l the calculus $\mathbf{C}$ contains no rule, add the rule

$$\frac{\Gamma}{\Gamma, l\colon \Box(A_1, \ldots, A_{\mathrm{ar}(\Box)})} \; \Box\colon l$$

which is an instance of the weakening rule.

Step 2. If for some connective $\Box$ and some label l there are several introduction rules $(\Box\colon l)_1, \ldots, (\Box\colon l)_\ell$, replace them by the single rule $\prod_{i=1}^{\ell} (\Box\colon l)_i$.

Step 3. For each rule $\Box\colon l$ containing the variables $A_1, \ldots, A_{\mathrm{ar}(\Box)}$, construct a rule $(\Box\colon l)' = \Box\colon l \times \prod_{i=1}^{\mathrm{ar}(\Box)} (A_i\colon \bot)$.

Observe that all premises of a rule $(\Box\colon l)'$ of $\mathbf{C}^+$ either contain an axiom or are of form $\{l\colon A_1 \ldots A_{\mathrm{ar}(\Box)} \mid l \in L\}$ where $L \in pre\mathcal{A}x$.

To obtain the macro-rules of $\mathbf{C}^*$ we combine the corresponding rules of $\mathbf{C}^+$.

Step 4. For each pre-axiom L construct the rule $(\Box\colon L)'' = \prod_{l \in L} (\Box\colon l)'$.

Step 5. The final rules $\Box\colon L$ are obtained from $(\Box\colon L)''$ by removing all premises containing an axiom and by removing duplicate formulas in sequents.

Note that in the last step we may obtain rules without any premises at all.

Proposition 5.9

[5] $A\colon \bot$ should be regarded as a schema with parameter A rather than as a single rule.

(a) Every cut-free derivation in calculus $\mathbf{C}$ *can be transformed into a cut-free derivation in* $\mathbf{C}^+$*, and vice versa.*

(b) Every derivation in $\mathbf{C}^+$ *can be transformed into a derivation in* $\mathbf{C}^*$*, and vice versa.*

Proof

(a) $\mathbf{C} \Rightarrow \mathbf{C}^+$: By adding weakenings to extend the premises to pre-axioms.

$\mathbf{C}^+ \Rightarrow \mathbf{C}$: By iterated application of lemma 5.8.

(b) $\mathbf{C}^+ \Rightarrow \mathbf{C}^*$: By induction on derivations. Suppose a sequent is derived in $\mathbf{C}^+$ using an introduction rule, the main formula is $l\colon A$. By the definition of the rules of $\mathbf{C}^*$, the premises of the introduction rules for each pre-axiom containing l are derivable from the premises of the $\mathbf{C}^+$ rule, and so $L\colon A$ is derivable in $\mathbf{C}^*$ for any pre-axiom L such that $l \in L$. Furthermore, if L is a pre-axiom such that $l \notin L$, then $L\colon A \cup \{l\colon A\}$ is an axiom and so derivable as well. Since the set of all pre-axioms is refutable, we have the required derivation in $\mathbf{C}^*$.

$\mathbf{C}^* \Rightarrow \mathbf{C}^+$: By iterated application of lemma 5.8. $\square$

The logic corresponding to $\mathbf{C}$ can now be constructed as follows. As set of truth values we choose the set of pre-cuts: $V = \mathit{preCuts}$. The label assignment is defined as

$$la(l) \quad = \quad \{L \in \mathit{preCuts} \mid l \in L\}$$

i.e., the set of truth values corresponding to label l are all pre-cuts containing l. For each connective $\square$ we define a truth relation $\tilde{\square}$ in the following way: for each rule $\square\colon L$ and each premise $L_1\colon A_1, \ldots, L_{\mathrm{ar}(\square)}\colon A_{\mathrm{ar}(\square)}$, the relation contains the $(\mathrm{ar}(\square){+}1)$-tuple

$$(\, L_1^c,\, \ldots,\, L^c_{\mathrm{ar}(\square)},\, L^c \,)\ .$$

Note that each component of the tuple is a truth value: all L_i's are pre-axioms, therefore their complements are pre-cuts.

It remains to show that the relations $\tilde{\square}$ are total functions and that calculus $\mathbf{C}$ is correct and complete with respect to this semantics.

Proposition 5.10 *Suppose the sequent* $\Pi = \{l\colon \square(A_1, \ldots, A_{\mathrm{ar}(\square)}) \mid l \in M\}$ *is cut-free derivable in* $\mathbf{C}^+$*, and* $M = M' \cup M''$ *where* $M' \cap M'' = \emptyset$*. Then there is a cut-free derivation of the form*

$$\dfrac{\dfrac{\Delta_1, \Gamma_1^1 \quad \ldots \quad \Delta_1, \Gamma_{r_1}^1}{\vdots}\quad \ldots \quad \dfrac{\Delta_n, \Gamma_1^n \quad \ldots \quad \Delta_n, \Gamma_{r_n}^n}{\vdots}}{\begin{array}{c}\Delta_1 \cup \{l\colon \square(A_1, \ldots, A_{\mathrm{ar}(\square)}) \mid l \in M''\} \quad \ldots \quad \Delta_n \cup \{l\colon \square(A_1, \ldots, A_{\mathrm{ar}(\square)}) \mid l \in M''\}\\ \vdots \\ \Pi\end{array}}$$

where Δ_i, Γ_i^j is an axiom, Δ_i is a union of pre-axioms for the A_i, and $\Delta_i \cap \Gamma_i^j = \emptyset$. The M'' component uses only formulas from the Γ's and the M' component only formulas from the Δ's, i.e., there are derivations

$$\begin{array}{ccc} \Delta_1 \quad \dots \quad \Delta_n & \qquad & \Gamma_1^i \quad \dots \quad \Gamma_{r_i}^i \\ \vdots & & \vdots \\ \{l\colon \Box(A_1, \dots, A_{\mathrm{ar}(\Box)}) \mid l \in M'\} & & \{l\colon \Box(A_1, \dots, A_{\mathrm{ar}(\Box)}) \mid l \in M''\} \end{array}$$

Proof Analogous to the proof of Gentzen's midsequent theorem; see [17]. $\Box$

Lemma 5.11 *Let $L_1\colon A_1, \dots, L_{\mathrm{ar}(\Box)}\colon A_{\mathrm{ar}(\Box)}$ be any premise of a rule $\Box\colon L$ in $\mathbf{C}^*$. Then for all $l' \in L^c$, $l'\colon \Box(A_1, \dots, A_{\mathrm{ar}(\Box)})$ is derivable from the set of unit sequents $l\colon A_i$, where $l \in (L_i^c)$ and $1 \le i \le \mathrm{ar}(\Box)$.*

Proof Let $l' \in L^c$. Then $L \cup \{l'\}\colon \Box(A_1, \dots, A_{\mathrm{ar}(\Box)})$ is an axiom, and so by P1 is cut-free derivable in $\mathbf{C}^+$ from atomic axioms. We may assume the A_i to be propositional variables. Taking $M' = L$ and $M'' = \{l'\}$ we know that $l'\colon \Box(A_1, \dots, A_{\mathrm{ar}(\Box)})$ is derivable from certain Γ_j^i by proposition 5.10. By definition of $\mathbf{C}^*$, each premise of the $\mathbf{C}^*$-rule $\Box\colon L$ occurs among the premises of the $\mathbf{C}^+$-rule $\Box\colon l$ for any $l \in L$. Thus, the premise in question must be contained in some Δ_k, and in fact, be equal to some Δ_k since these are all pre-axioms. consequently Γ_j^k must be non-empty. If $l\colon A_i \in \Gamma_j^k$ then $l \in L_i^c$ (by proposition 5.10, $\Gamma_j^k \cap \Delta_k = \emptyset$), and Γ_j^k is derivable from the unit sequent $l\colon A_i$. $\Box$

Proposition 5.12 *The truth relations $\widetilde{\Box}$ are functional and total with respect to their last component.*

Proof $\widetilde{\Box}$ *is functional.* Suppose $\widetilde{\Box}$ is not functional. Then some premise occurs twice in two rules, $\Box\colon L_1$ and $\Box\colon L_2$. By lemma 5.11, $l'\colon \Box(A_1, \dots, A_{\mathrm{ar}(\Box)})$ is derivable from the set of the units sequents $l_j\colon A_i$ for all $l' \in L_1^c$ and $l' \in L_2^c$. Since L_1 is a pre-cut and L_2 contains at least one additional label l, we may apply cut rule $L_1 \cup \{l\}$ and derive the empty sequent from the unit sequents $l_j\colon A_i$. But for every i, the set of all l_j forms only a pre-cut, which cannot derive the empty sequent. Contradiction.

$\widetilde{\Box}$ *is total.* Suppose $\widetilde{\Box}$ is not total. Then some sequent $\Delta = L_1\colon A_1, \dots, L_{\mathrm{ar}(\Box)}\colon A_{\mathrm{ar}(\Box)}$ does not occur among the premises of $\Box\colon L$ for any pre-axiom L. Consider its complement, the set of unit sequents $l_j\colon A_i$, where $l_j \in L_i^c$ and $1 \le i \le \mathrm{ar}(\Box)$. Each premise of a rule $\Box\colon L$ for all pre-axioms L has to contain a signed formula $l_j\colon A_i$ also occurring in this set of unit sequents; otherwise it would be identical to Δ. But then every premise is derivable from the unit sequents, and therefore also all conclusions, i.e., $L\colon \Box(A_1, \dots, A_{\mathrm{ar}(\Box)})$ is derivable for all pre-axioms L. By property P2, we can derive the empty sequent from these conclusions and thus from the unit sequents $l_j\colon A_i$. But for every i, the set of all l_j forms only a pre-cut, which can not derive the empty sequent. Contradiction. $\Box$

Lemma 5.13 *Let $L{:}\,A$ be a labeled sequent such that $L \in \mathit{preAx}$. Then* $\mathrm{lval}_{\mathbf{I}}(L{:}\,A) = \bot$ *iff* $\mathrm{val}_{\mathbf{I}}(A) = L^c$.

Proof The expression $L{:}\,A$ is an abbreviation for $\bigvee\!\!\!\!\bigvee_{l \in L} l{:}\,A$. Therefore $L{:}\,A$ is false in $\mathbf{I}$ iff $\mathrm{val}_{\mathbf{I}}(A) \notin \bigcup_{l \in L} la(l)$. By definition, $la(l)$ is the set of all pre-cuts containing l, i.e., $\bigcup_{l \in L} la(l)$ is the set of all pre-cuts containing any label of L. This means that $\mathrm{val}_{\mathbf{I}}(A)$ has to be a subset of the complement of pre-axiom L. Since truth values are pre-cuts by construction of the logic and L^c but none of its proper subsets is a pre-cut (cf. proof of lemma 5.3(a)), we conclude that $\mathrm{val}_{\mathbf{I}}(A) = L^c$. □

Proof (of theorem 5.6)
Soundness.

1. *The axioms are valid.* Let Δ be an axiom. We have to show that $\bigcup_{l \in \Delta} la(l) = \mathit{preCuts}$. Assume the contrary, i.e., for some pre-cut Γ, $\Gamma \notin \bigcup_{l \in \Delta} la(l)$. By the construction of the label assignment this implies $\Gamma \cap \Delta = \emptyset$. In other words, it is possible to derive the empty sequent from the axiom Δ and the pre-cut Γ. Contradiction.

2. *The structural rules are sound.* Trivial.

3. *The cut rules are sound.* Obviously true since $\bigcap_{l \in L} la(l) = \emptyset$ for $L \in \mathit{Cuts}$.

4. *The propositional introduction rules are sound.* We show the soundness of the macro-rules in $\mathbf{C}^*$. By proposition 5.9 this implies the soundness of the rules in $\mathbf{C}$.

   Consider the introduction rule for $L{:}\,\Box(A_1, \dots, A_{\mathrm{ar}(\Box)})$, where L is a pre-axiom, and let $\mathbf{I}$ be an interpretation falsifying this expression. By lemma 5.13 this is the case iff $\mathrm{val}_{\mathbf{I}}(\Box(A_1, \dots, A_{\mathrm{ar}(\Box)})) = \widetilde{\Box}(\mathrm{val}_{\mathbf{I}}(A_1), \dots, \mathrm{val}_{\mathbf{I}}(A_{\mathrm{ar}(\Box)})) = L^c$. Let L_i' be the value of A_i in $\mathbf{I}$. By construction of $\widetilde{\Box}$, the introduction rule must have a premise $L_1{:}\,A_1, \dots, L_{\mathrm{ar}(\Box)}{:}\,A_{\mathrm{ar}(\Box)}$ such that $L_i' = L_i^c$. But by lemma 5.13, $\mathbf{I}$ falsifies each sequent $L_i{:}\,A_i$, and therefore it falsifies the whole premise.

Completeness. We first show the completeness of $\mathbf{C}^*$ with respect to sequents consisting entirely of expressions $L{:}\,A$ where L is a pre-axiom. $\mathbf{C}^*$ is complete in the sense that for every expression $L{:}\,\Box(A_1, \dots, A_{\mathrm{ar}(\Box)})$ there is a rule decomposing $\Box(A_1, \dots, A_{\mathrm{ar}(\Box)})$ into simpler expressions. Therefore every maximal derivation tree is finite, with each branch ending in an atomic sequent. There are two possibilities:

1. The atomic sequent contains two expressions $L{:}\,A$ and $L'{:}\,A$ with $L \neq L'$, i.e., some $l \in L'$ does not occur in L. Since L is a pre-axiom, $L \cup \{l\}$ is an axiom, making the sequent true in every interpretation.

2. For all propositional variables A, the atomic sequent contains at most one expression $L{:}\,A$ (regarding the sequent as set). We construct an interpretation $\mathbf{I}$ by defining $\mathrm{val}_{\mathbf{I}}(A) = L^c$ for variables occurring in the sequent,

and choosing an arbitrary truth value for all other variables. **I** falsifies the atomic sequent, and by construction of the truth tables, it also falsifies every sequent on the branch down to the root.

As a consequence, every valid sequent labeled with pre-axioms has a derivation ending with axioms.

Now consider a sequent $\Delta = L_1{:}\, A_1, \ldots, L_n{:}\, A_n$, where the L_i are arbitrary sets of labels. If Δ is valid, then every extension of Δ by further labeled formulas is valid, in particular those sequents obtained by extending the L_i to pre-axioms. Each of the latter sequents has a proof in $\mathbf{C}^*$. Using the fact that the set of pre-axioms is refutable these proofs can be combined to a proof of Δ.

$S = \{la(l) \mid l \in \mathbf{L}\}$ *is a system of signs.* We have to show that every truth value $v \in V$ can be obtained as the intersection of some signs. (Note that by construction $V = \mathit{preCuts}$.) Without loss of generality we may consider the intersection of *all* signs containing v, i.e., we show that $\bigcap\{la(l) \mid l \in \mathbf{L}, v \in la(l)\}$ is exactly the set $\{v\}$. Assume the contrary, namely that the intersection contains a second truth value v'. This means that whenever $v \in la(l)$ for some label l, we also have $v' \in la(l)$. But by construction of la this is equivalent to $v \subsetneq v'$, which is a contradiction: both truth values are pre-cuts, and one pre-cut cannot be a proper subset of another one. $\Box$

Example 5.14 Let $\mathbf{C}$ be the labeled sequent calculus with $\mathbf{L} = \{a, b, c\}$, $\mathcal{A}x = \{\mathbf{L}\}$, and the introduction rules

$$\frac{b{:}\, A, c{:}\, A, a{:}\, B \quad a{:}\, A, c{:}\, A, b{:}\, B \quad a{:}\, A, b{:}\, A, c{:}\, B}{a{:}\, \Box(A, B)}\ \Box{:}\, a$$

$$\frac{b{:}\, A, c{:}\, A, b{:}\, B \quad a{:}\, A, c{:}\, A, a{:}\, B, c{:}\, B \quad a{:}\, A, b{:}\, A, b{:}\, B}{b{:}\, \Box(A, B)}\ \Box{:}\, b$$

$$\frac{b{:}\, A, c{:}\, A, c{:}\, B \quad a{:}\, A, c{:}\, A \quad a{:}\, A, b{:}\, A, a{:}\, B}{c{:}\, \Box(A, B)}\ \Box{:}\, c$$

As a matter of convenience we write $bc{:}\, A$ instead of $b{:}\, A, c{:}\, A$ etc. throughout this example. For the set of pre-axioms, pre-cuts, and cuts we obtain

$$\begin{aligned} \mathit{pre}\mathcal{A}x &= \{\{a,b\}, \{a,c\}, \{b,c\}\} \\ \mathit{preCuts} &= \{\{c\}, \{b\}, \{a\}\} \\ \mathit{Cuts} &= \{\{a,b\}, \{a,c\}, \{b,c\}, \mathbf{L}\} \end{aligned}$$

The pre-cuts are obtained as the dual of the pre-axioms, and *Cuts* is the set of all extensions of pre-cuts. Of course, the elements of *Cuts* need not really be cuts; this is only the case if $\mathbf{C}$ satisfies P1 and P2.

P1 requires that all compound axioms can be reduced to atomic ones. Since we only have one operator and one axiom, we just have to check that each premise of the combined rule $\Box{:}\, a \times \Box{:}\, b \times \Box{:}\, c$

$$\frac{bc{:}\, A, abc{:}\, B \quad abc{:}\, A, ab{:}\, B \quad \cdots \quad abc{:}\, A, bc{:}\, B \quad ab{:}\, A, abc{:}\, B}{abc{:}\, \Box(A, B)}\ \Box{:}\, abc$$

contains an axiom.

P2 requires that the pre-axioms are refutable:

$$\dfrac{\dfrac{ab\colon A \quad ac\colon A}{a\colon A}\ \text{cut}\colon bc \qquad \dfrac{ac\colon A \quad bc\colon A}{c\colon A}\ \text{cut}\colon ab}{\bot}\ \text{cut}\colon ac$$

(contractions are done implicitly).

To check that the cuts are globally admissible, we have to prove that each cut is reducible with respect to all introduction and cut rules. As an example we show that cut: bc is reducible with respect to $\mathcal{C}uts$ and the rules $\Box\colon b$ and $\Box\colon c$ by giving a reducing derivation of the premises of $\Box\colon b$ and $\Box\colon c$. We first derive the pre-axioms $bc\colon A$ and $ab\colon A$:

$$\dfrac{ab\colon A, a\colon B \quad ab\colon A, b\colon B}{ab\colon A}\ \text{cut}\colon ab \qquad \dfrac{bc\colon A, b\colon B \quad bc\colon A, c\colon B}{bc\colon A}\ \text{cut}\colon bc$$

The third pre-axiom, $ac\colon A$, is directly given as a premise of $\Box\colon c$. The refutation of the pre-axioms (see above) completes the reducing derivation.

In the construction of $\mathbf{C}^*$, the first two steps are vacuous for this particular calculus. Step 3 requires that each introduction rule is combined with $A\colon\bot$ and $B\colon\bot$. For $(\Box\colon b)' = \Box\colon b \times A\colon\bot \times B\colon\bot$ we obtain

$$\dfrac{bc\colon A, ab\colon B \quad bc\colon A, bc\colon B \quad ac\colon A, ac\colon B \quad ab\colon A, ab\colon B \quad ab\colon A, bc\colon B}{b\colon \Box(A,B)}\ (\Box\colon b)^*$$

(duplicate labeled formulas and premises containing axioms have been removed). Similarly we obtain the rules for $(\Box\colon a)'$ (six premises) and $(\Box\colon c)'$ (seven premises). Computing the combined rules for each pre-axiom according to step 4 and deleting premises containing axioms we end up with the rules

$$\dfrac{ab\colon A, bc\colon B \quad bc\colon A, ab\colon B}{ab\colon \Box(A,B)}\ \Box\colon ab$$

$$\dfrac{ab\colon A, ac\colon B \quad ac\colon A, ab\colon B \quad ac\colon A, bc\colon B \quad bc\colon A, ac\colon B}{ac\colon \Box(A,B)}\ \Box\colon ac$$

$$\dfrac{ab\colon A, ab\colon B \quad ac\colon A, ac\colon B \quad bc\colon A, bc\colon B}{bc\colon \Box(A,B)}\ \Box\colon bc$$

Finally we are in the position to describe the logic $\mathbf{M}$ corresponding to $\mathbf{C}$. The set of truth values is given by the set of pre-cuts: $V = \{\{a\}, \{b\}, \{c\}\}$. The label assignment is particularly simple: $la(l) = \{l\}$ for all $l \in \mathbf{L}$. The truth table for $\tilde{\Box}$ can be read off the macro rules by looking at the complements of the respective labels:

$\tilde{\Box}$	$\{a\}$	$\{b\}$	$\{c\}$
$\{a\}$	$\{a\}$	$\{b\}$	$\{c\}$
$\{b\}$	$\{b\}$	$\{a\}$	$\{b\}$
$\{c\}$	$\{c\}$	$\{b\}$	$\{a\}$

Example 5.15 Let $\mathbf{C}$ be the labeled sequent calculus with $\mathbf{L} = \{a, b, c, d\}$, $\mathcal{A}x = \{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \mathbf{L}\}$, and the introduction rules

$$\frac{ac{:}\, A \quad ad{:}\, A \quad bc{:}\, A}{a{:}\, \Box A}\ \Box{:}\, a \qquad \frac{ac{:}\, A \quad bd{:}\, A \quad cd{:}\, A}{b{:}\, \Box A}\ \Box{:}\, b$$
$$\frac{ab{:}\, A \quad ad{:}\, A \quad bd{:}\, A}{c{:}\, \Box A}\ \Box{:}\, c \qquad \frac{ab{:}\, A \quad bc{:}\, A \quad cd{:}\, A}{d{:}\, \Box A}\ \Box{:}\, d$$

We again write $ab{:}\, A$ instead of $a{:}\, A, b{:}\, A$ etc. For the set of pre-axioms, pre-cuts, and cuts we obtain

$$\begin{array}{rcl} pre\mathcal{A}x &=& \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\} \\ pre\mathcal{C}uts &=& pre\mathcal{A}x \\ \mathcal{C}uts &=& \mathcal{A}x \end{array}$$

$\mathbf{C}$ has property P1: the compound axioms $l_1 l_2 l_3{:}\, \Box A$ for $l_1 \neq l_2 \neq l_3$ as well as $abcd{:}\, \Box A$ can be reduced to atomic axioms. Note that no premise occurs in more than two rules, and each premise contains two labels. Therefore each premise of $\Box{:}\, l_1 \times \Box{:}\, l_2 \times \Box{:}\, l_3$ contains at least three different labels and therefore is an axiom.

$\mathbf{C}$ has property P2: for every $l \in \mathbf{L}$, it is possible to derive the unit sequent $l{:}\, A$ using a cut, e.g.,

$$\frac{ad{:}\, A \quad bd{:}\, A \quad cd{:}\, A}{d{:}\, A}\ \text{cut}{:}\, a, b, c$$

These unit sequents derive the empty sequent by one more application of a cut rule. To see that all cuts are admissible, observe that any three introduction rules together contain all pre-axioms as premises, which are refutable.

For the construction of $\mathbf{C}^*$ we only have to perform steps 4 and 5, since all premises are already pre-axioms. The rules of $\mathbf{C}^*$ are obtained from the combinations $\Box{:}\, l \times \Box{:}\, l'$ for all $l \neq l'$. Since any two rules share exactly one premise, and every premise with three labels is already an axiom, we obtain the following macro rules:

$$\frac{ac{:}\, A}{ab{:}\, \Box A} \quad \frac{ad{:}\, A}{ac{:}\, \Box A} \quad \frac{bc{:}\, A}{ad{:}\, \Box A} \quad \frac{bd{:}\, A}{bc{:}\, \Box A} \quad \frac{cd{:}\, A}{bd{:}\, \Box A} \quad \frac{ab{:}\, A}{cd{:}\, \Box A}$$

Thus, $\mathbf{M}$ is a six-valued logic since there are six pre-cuts. The label assignment is given by

$$\begin{array}{rcl} la(a) &=& \{\{a, b\}, \{a, c\}, \{a, d\}\} \\ la(b) &=& \{\{a, b\}, \{b, c\}, \{b, d\}\} \\ la(c) &=& \{\{a, c\}, \{b, c\}, \{c, d\}\} \\ la(d) &=& \{\{a, d\}, \{b, d\}, \{c, d\}\} \end{array}$$

Finally, the truth table for $\widetilde{\Box}$ is given by

$\widetilde{\Box}$			$\widetilde{\Box}$	
$\{a,b\}$	$\{a,c\}$		$\{b,c\}$	$\{b,d\}$
$\{a,c\}$	$\{a,d\}$		$\{b,d\}$	$\{c,d\}$
$\{a,d\}$	$\{b,c\}$		$\{c,d\}$	$\{a,b\}$

The next two examples show that P1 and P2 are necessary conditions independent of each other.

Example 5.16 Consider a labeled sequent calculus **C** with $\mathbf{L} = \{a,b\}$, $\mathcal{A}x = \{\{a,b\}\}$ and four introduction rules

$$\frac{a\colon A}{a\colon \Box A} \quad \frac{a\colon A}{b\colon \Box A} \quad \frac{b\colon A}{a\colon \Box A} \quad \frac{b\colon A}{b\colon \Box A}$$

C trivially satisfies P1 requiring the reducibility of compound axioms to atomic ones:

$$\cfrac{\cfrac{a\colon A, b\colon A}{a\colon A, b\colon \Box A}}{a\colon \Box A, b\colon \Box A}$$

However, **C** does not fulfill P2: **C** has no cuts, since the only possibility $\{a,b\}$ would allow to derive the empty sequent from axioms:

$$\cfrac{\cfrac{\cfrac{\cfrac{a\colon A, b\colon A}{a\colon A, a\colon \Box A}}{a\colon \Box A, a\colon \Box A}}{a\colon \Box A} \qquad \cfrac{\cfrac{\cfrac{a\colon A, b\colon A}{a\colon A, b\colon \Box A}}{b\colon \Box A, b\colon \Box A}}{b\colon \Box A}}{\bot}\ \text{cut}\colon a,b$$

Therefore the set $\{\{a\},\{b\}\}$ of pre-axioms cannot be refuted.

It not hard to see that there is no finite-valued logic corresponding to **C**. All four possible truth tables for $\Box$ make some introduction rule unsound.

Example 5.17 Consider a labeled sequent calculus **C** with $\mathbf{L} = \{a,b\}$, $\mathcal{A}x = \{\{a,b\}\}$ and a connective $\Box$ in the language, but without introduction rules for $\Box$. Clearly, **C** does not satisfy P1, since there are no rules to decompose the compound axiom $a\colon \Box A, b\colon \Box A$. On the other hand, P2 holds since the pre-axioms can be refuted using the cut $\{a,b\}$.

It not hard to see that there is no finite-valued logic corresponding to **C**. All four possible truth tables for $\Box$ makes **C** incomplete. As an example, let $\widetilde{\Box}(a) = \widetilde{\Box}(b) = a$. The labeled sequent $A\colon \Box a$ is valid for this matrix, but it is neither an axiom nor derivable.

6 From Finite-valued Logics to Labeled Calculi

Theorem 6.1 *Let* $\mathbf{M}$ *be a finite-valued propositional logic with a set* V *of truth values, and let* $S \subseteq 2^V$ *be a system of signs. Then there is a rich labeled sequent calculus* $\mathbf{C}$ *with labels* $\mathbf{L}$*, as well as a label assignment* $la\colon \mathbf{L} \overset{1-1}{\to} S$ *such that*

$$\mathbf{M} \models_{la} \Delta \quad \Longleftrightarrow \quad \mathbf{C} \vdash \Delta$$

for all labeled sequents Δ *and all rules of* $\mathbf{C}$ *are sound with respect to* $\mathbf{M}$*. The cuts are the sets* $\{la^{-1}(\alpha_1), \ldots, la^{-1}(\alpha_n)\}$ *such that* $\bigcap\{\alpha_1, \ldots, \alpha_n\} = \emptyset$*.*

Corollary 6.2 *Let* $\mathbf{C}$ *be a sequent calculus without cut rules such that there is a finite-valued propositional logic* $\mathbf{M}$*, a system* S *of signs, and a label assignment* $la\colon \mathbf{L} \overset{1-1}{\to} S$ *such that*

$$\mathbf{C} \vdash \Delta \quad \Longleftrightarrow \quad \mathbf{M} \models_{la} \Delta$$

for all labeled sequents Δ *and such that all rules of* $\mathbf{C}$ *are sound with respect to* $\mathbf{M}$*, then* $\mathbf{C}$ *can be extended to a rich sequent calculus by adding cuts.*

Since la is a 1-1 mapping we will use $la(l)$ instead of l to label formulas, this way avoiding to define $\mathbf{L}$ and la explicitly.

Systems of signs allow a unified treatment of different formalisms. In the positive calculi of Rousseau [11] and Takahashi [16], the signs are just the singleton sets. In the negative calculi of Schröter [13] and Carnielli [6], the signs are all sets containing all but one truth value.[6] Other examples for systems of signs are up- and downsets with respect to totally ordered truth values [8] or distributive lattices [9], or signs forming the supremum or infimum with respect to semi-lattices [12, 18].

Definition 6.3 A *partial normal form* for a connective $\Box$ and a sign α is a conjunction of disjunctions of labeled formulas

$$\bigwedge_i \bigvee_j \beta_{ij}\colon A_{k_{ij}} \ ,$$

$1 \leq k_{ij} \leq \mathrm{ar}(\Box)$, which is true in an interpretation $\mathbf{I}$ iff $\mathrm{val}_{\mathbf{I}}(\Box(A_1, \ldots, A_{\mathrm{ar}(\Box)})) \in \alpha$.

Proposition 6.4 *For any connective* $\Box$ *and sign* $\alpha \in S$*, there is a partial normal form for* $\Box$ *and* α*.*

Proof Let T be the set $\{\langle v_1, \ldots, v_{\mathrm{ar}(\Box)}\rangle \mid \widetilde{\Box}(v_1, \ldots, v_{\mathrm{ar}(\Box)}) \in \alpha\}$. If the sets $\{v_i\}$ were among the labels, each tuple $\langle v_1, \ldots, v_{\mathrm{ar}(\Box)}\rangle$ could be characterized by the signed expression $\bigwedge_i \{v_i\}\colon A_i$, which is only true in an interpretation satisfying $\mathrm{val}_{\mathbf{I}}(A_i) = v_i$. However, since S forms a system of signs, each truth value can be obtained as the intersection of signs, i.e., each pseudo-formula $\{v_i\}\colon A_i$

[6] The duality of positive and negative calculi was analyzed in [2] and [18].

can be replaced by the proper labeled formula expression $\bigwedge_{\beta \in S_{v_i}} \beta{:}\, A_i$ where $\bigcap S_{v_i} = \{v_i\}$. Forming the disjunction of these expressions for every tuple in T, we obtain the labeled formula expression

$$\bigvee_{t \in T} \bigwedge_{i=1}^{\operatorname{ar}(\Box)} \bigwedge_{\beta \in S_{t(i)}} \beta{:}\, A_i$$

which is true in $\mathbf{I}$ iff $\operatorname{val}_{\mathbf{I}}(\Box(A_1, \ldots, A_{\operatorname{ar}(\Box)})) \in \alpha$. Using distributivity, the formula can be rewritten as conjunction of disjunctions, as required by the proposition. $\Box$

There is a close connection between the conjuncts in a partial normal form and the premises of introduction rules in sequent calculi.

Definition 6.5 Let $\bigwedge_{i=1}^{\ell} \Delta_i$ be a partial normal form for $\Box$ and α. The *introduction rule* for $\Box$ and α is the rule

$$\frac{\Gamma, \Delta_1 \quad \cdots \quad \Gamma, \Delta_\ell}{\Gamma, \alpha{:}\, \Box(A_1, \ldots, A_{\operatorname{ar}(\Box)})} \; \Box{:}\, \alpha$$

where we identify disjunctions of labeled formulas with labeled sequents.

Axiom sequents are sequents which are always true. Cut rules remove contradictory formulas. The structural rules are given by definition. In summary we have the following:

Definition 6.6 A *sequent calculus* $\mathbf{C}$ for a logic $\mathbf{M}$ with respect to a system of signs S is given by the following rules:

1. for every sign $\alpha \in S$, a weakening rule weak: α and a contraction rule cntr: α,
2. for every connective $\Box$ and every sign $\alpha \in S$ an introduction rule $\Box{:}\, \alpha$,
3. for all signs $\alpha_1, \ldots, \alpha_\ell \in S$ such that $\bigcup_{i=1}^{\ell} \alpha_i = V$, an axiom schema $\alpha_1{:}\, A, \ldots, \alpha_\ell{:}\, A$, and
4. for all signs $\alpha_1, \ldots, \alpha_\ell \in S$ such that $\bigcap_{i=1}^{\ell} \alpha_i = \emptyset$, a cut rule cut: $\alpha_1, \ldots, \alpha_\ell$.

Proof (of theorem 6.1) *Soundness.* The proof of the soundness of $\mathbf{C}$ with respect to $\mathbf{M}$ is an easy induction on the length of the proof.

Completeness. We have to show that whenever $\mathbf{M} \models \Delta$, we also have $\mathbf{C} \vdash \Delta$. Suppose we have a valid sequent Δ. $\mathbf{C}$ is complete in the sense that for every connective $\Box$ and every label l there is an introduction rule $\Box{:}\, l$ decomposing the conclusion into simpler sequents in the premises. Constructing the proof tree backwards we obtain a finite tree with atomic sequents as leafs. It remains to show that all leafs contain axioms.

Assume the contrary, i.e., some atomic sequent Γ has the property that for every propositional variable A the set $L_A = \{\alpha \mid \alpha{:}\, A \in \Gamma\}$ forms no axiom. By definition this means that $\bigcup L_A \neq V$. Let v_A be any truth value in $V \setminus \bigcup L_A$, and let $\mathbf{I}$ be an interpretation defined by $\mathbf{I}(A) = v_A$ for all propositional variables A. Clearly, $\mathbf{I}$ falsifies Γ. By construction of the introduction rules, every interpretation falsifying any premise also falsifies the conclusion. Therefore $\mathbf{I}$ also falsifies the sequent Δ at the root, contradicting the validity of Δ.

Property P1. We have now to argue that compound axioms are derivable from atomic ones. This follows from the fact that in the above completeness argument only atomic axioms are used.

Property P2. To establish that $\mathbf{C}$ satisfies property P2 we use proposition 3.4. We get that $V \setminus \{v\}$ is represented by a pre-axiom for all $v \in V$. Consequently, the set of clauses corresponding to pre-axioms in the sense of definition 4.6 is unsatisfiable. By completeness of hyperresolution, there is a hyperresolution derivation of the empty clause from this set of clauses. Like in the proof of lemma 4.7 we can translate this derivation into a refutation within $\mathbf{C}$.

Gentzen-eliminability of cuts. For a set of labels $\{l_1, \ldots, l_n\}$ let $\mathcal{D}$ consist of the following clauses:

1. For each set Δ^{l_i} of auxiliary formulas occurring in the premise of an introduction rule for $l_i{:}\, \Box(A_1, \ldots, A_{\mathrm{ar}(\Box)})$, $\mathcal{D}$ contains the clause $\mathrm{cls}(\Delta^{l_i})$, where $A_1, \ldots, A_{\mathrm{ar}(\Box)}$ are regarded as propositional variables.
2. For each cut rule C_i, $1 \leq i \leq m$, and each propositional variable A_j, $1 \leq j \leq n$, $\mathcal{D}$ contains the clause $\mathrm{cls}(\mathrm{C}_i, A_j)$.

Suppose that $\mathcal{D}$ is satisfiable. Then the functionality of $\widetilde{\Box}$ implies that there is also an $\mathbf{M}$-interpretation satisfying the set of sequents $\{\{l_1{:}\, \Box(A_1, \ldots, A_{\mathrm{ar}(\Box)})\}, \ldots, \{l_n{:}\, \Box(A_1, \ldots, A_{\mathrm{ar}(\Box)})\}\}$. By contraposition, it follows from lemma 4.7 that the set of cut rules is locally admissible. Obviously, the set of axioms is closed under application of cuts. Thus we obtain the Gentzen-eliminability of the cut rules.

Finally, note that there is a least one proper axiom and one proper cut as required by definition 5.4 since there are at least two truth values and therefore at least two signs. $\Box$

Proof (of corollary 6.2) Add the cuts $\{la^{-1}(\alpha_1), \ldots, la^{-1}(\alpha_n)\}$ such that $\bigcap\{\alpha_1, \ldots, \alpha_n\} = \emptyset$ to $\mathbf{C}$. The resulting calculus can be shown to be rich by the same arguments as in the proof above using the propositional rules to obtain partial normal forms. $\Box$

Summarizing we may now state our main result as follows.

Theorem 6.7 *Let* $\mathbf{C}$ *be a labeled sequent calculus without cut rules. There is a finite-valued propositional logic* $\mathbf{M}$*, a system* S *of signs, and a label assignment* $la{:}\, \mathbf{L} \overset{1-1}{\to} S$ *such that*

$$\mathbf{C} \vdash \Delta \iff \mathbf{M} \models_{la} \Delta$$

for all labeled sequents Δ and such that all rules of **C** *are sound with respect to* **M** *iff* **C** *can be extended to a rich sequent calculus by adding cuts. Moreover, this property is decidable.*

7 The Size of Many-valued Logics Contained in Labeled Calculi

Using Sperner's lemma we now estimate the number of truth values of many-valued logics represented by labeled calculi.

Definition 7.1 A *Sperner set* over a set L is a set of subsets of L such that no subset contains another.

Lemma 7.2 (Sperner [15]) *Let V be a Sperner set over a set with m elements. Then $|V| \le \binom{m}{[m/2]}$.*

Note that $\binom{m}{[m/2]}$ is a tight upper bound, which can be obtained by choosing all subsets of cardinality $[m/2]$ as the elements of the Sperner set.

Theorem 7.3 *Let V be the truth values of a many-valued logic represented by a sequent calculus with label set* **L**. *Then $|V| \le \binom{|\mathbf{L}|}{[|\mathbf{L}|/2]} \sim 2^{|\mathbf{L}|}\sqrt{3/(2\pi|\mathbf{L}|)}$.*

Proof If a sequent calculus with labels **L** represents a many valued logic with truth values V, then by theorems 5.6 and 6.1, the number of truth values equals the number of pre-cuts, where the pre-cuts are obtained from the eliminable cuts. By lemma 5.3, the number of pre-cuts equals the number of pre-axioms. The set of pre-axioms forms a Sperner set over **L**. □

Corollary 7.4 *A many-valued logic with n truth values can be represented by a sequent calculus with at most m labels where $n \le \binom{m}{[m/2]}$; i.e., $m = O(\log_2 n)$.*

Proof Let **L** be a set of m elements. Construct a Sperner set V over **L** by choosing all $v \in V$ such that $|v| = [m/2]$; obviously there are exactly $\binom{m}{[m/2]}$ of them. V is the intended set of truth values. For $l \in \mathbf{L}$, let $\alpha_l = \{v \in V \mid l \in v\}$. Then $S = \{\alpha_l \mid l \in \mathbf{L}\}$ is a set of signs. The rest follows from theorem 6.1. □

Summarizing, many-valued logics may be represented by sequent calculi with a logarithmic number of labels.

Acknowledgments

Many thanks to Angel Gil and Ventura Verdú for reading an earlier draft of the paper and suggesting several improvements. Thanks also to the referees for their valuable comments.

References

[1] A. Avron. Natural 3-valued logics—characterization and proof theory. *J. Symbolic Logic*, 56(1):276–294, 1991.

[2] M. Baaz, C. G. Fermüller, and R. Zach. Dual systems of sequents and tableaux for many-valued logics. *Bull. EATCS*, 51:192–197, 1993. paper read at *2nd Workshop on Tableau-based Deduction*, Marseille, April 1993.

[3] M. Baaz, C. G. Fermüller, and R. Zach. Elimination of cuts in first-order finite-valued logics. *J. Inform. Process. Cybernet. EIK*, 29(6):333–355, 1994.

[4] P. Borowik. Multi-valued n-sequential propositional logic (Abstract). *J. Symbolic Logic*, 52:309–310, 1985.

[5] W. A. Carnielli. Systematization of finite many-valued logics through the method of tableaux. *J. Symbolic Logic*, 52(2):473–493, 1987.

[6] W. A. Carnielli. On sequents and tableaux for many-valued logics. *J. Non-Classical Logic*, 8(1):59–76, 1991.

[7] C.-L. Chang and R. C.-T. Lee. *Symbolic Logic and Mechanical Theorem Proving.* Academic Press, London, 1973.

[8] R. Hähnle. *Automated Deduction in Multiple-Valued Logics.* Oxford University Press, Oxford, 1993.

[9] R. Hähnle. Commodious axiomatization of quantifiers in multiple-valued logic. In *26th Int. Symp. on Multiple-Valued Logics, Santiago de Compostela, Spain*, pages 118–123. IEEE Press, Los Alamitos, May 1996.

[10] A. Leitsch. *The Resolution Calculus.* Springer, 1997.

[11] G. Rousseau. Sequents in many valued logic I. *Fund. Math.*, 60:23–33, 1967.

[12] G. Salzer. Optimal axiomatizations for multiple-valued operators and quantifiers based on semi-lattices. In *13th Int. Conf. on Automated Deduction (CADE'96)*, LNCS (LNAI) 1104, pages 688–702. Springer, 1996.

[13] K. Schröter. Methoden zur Axiomatisierung beliebiger Aussagen- und Prädikatenkalküle. *Z. Math. Logik Grundlag. Math.*, 1:241–251, 1955.

[14] R. Smullyan. *First-order Logic.* Springer, New York, 1968.

[15] E. Sperner. Ein Satz über Untermengen einer endlichen Menge. *Math. Z.*, 27:544–548, 1928.

[16] M. Takahashi. Many-valued logics of extended Gentzen style I. *Sci. Rep. Tokyo Kyoiku Daigaku Sect A*, 9(231):95–110, 1967.

[17] G. Takeuti. *Proof Theory.* Studies in Logic 81. North-Holland, Amsterdam, 2nd edition, 1987.

[18] R. Zach. *Proof Theory of Finite-valued Logics.* Diplomarbeit, Technische Universität Wien, Vienna, Austria, 1993.