Ancient Greek mathematical proofs and metareasoning

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Abstract We present an approach in which ancient Greek mathematical proofs by Hippocrates of Chios and Euclid are addressed as a form of (guided) intentional reasoning. Schematically, in a proof, we start with a sentence that works as a premise; this sentence is followed by another, the conclusion of what we might take to be an inferential step. That goes on until the last conclusion is reached. Guided by the text, we go through small inferential steps; in each one, we go through an autonomous reasoning process linking the premise to the conclusion. The reasoning process is accompanied by a metareasoning process. Metareasoning gives rise to a feeling-knowing of correctness. In each step/cycle of the proof, we have a feeling-knowing of correctness. Overall, we reach a feeling of correctness for the whole proof. We suggest that this approach allows us to address the issues of how a proof functions, for us, as an enabler to ascertain the correctness of its argument and how we ascertain this correctness.

1 Introduction

There is no unique definition of mathematical proof. Besides establishing the truth of a mathematical statement, a mathematical proof has many functions or roles. For example, convincing (that that is the case), explaining (how that is the case), and others (Dutilh Novaes 2020, 222-8). Depending on the roles we might want to stress we might adopt somewhat different definitions (see, e.g., Tall et al 2021, 15; Krantz 2011, vii-viii; Beck, Geoghegan 2010, viii). However, it is still the case that if a mathematical argumentation is not correct, strictly speaking, it is not a proof. So, the definitory role of a mathematical proof is that it establishes, or enables us to establish, the truth of a certain mathematical claim.

Depending on the context in which we will address a mathematical proof, different definitions might be valuable. For example, we might consider the following "definition":

A proof is a piece of discourse that puts forward a chain of arguments for public scrutiny, whose core function is to establish the truth of a mathematical claim.

Here, we will adopt a somewhat different working definition. The point is that a proof does not really establish the truth of a mathematical statement; the proof is an enabler for us to ascertain the truth. The reasoning is in us, not in the proof. As Rav called attention to:

> Mathematical texts abound in terms such as "it follows from … that" […] a mathematical proof in general only says that it follows, not why [it follows] […] why the consequent follows from the antecedents has to be figured out by the reader of a proof. (Rav 2007, 316-7)

So, the proof consists of a sort of scaffolding of antecedents and consequents. And it is us that must make the reasoning connecting them and come to the realization that a consequent (a conclusion) does follow from an antecedent (a premise).

A definition that, in our view, stresses this aspect of a mathematical proof is as follows:

A proof is a piece of discourse that puts forward a chain of arguments for public scrutiny, whose core function is to enable the audience to ascertain the correctness of the chain of arguments.

Here, we want to address the question of how a proof functions, for us, as an enabler to ascertain the correctness of its argument. As we will see, this question relates to the issue of how we ascertain this correctness. In the present work, we will answer both of these related questions.

We will consider the most ancient proofs there are records of. The reason is simple. If the approach developed here is to be of general application to informal mathematical proofs, it must, for starters, work for the case of extant early proofs. If this is the case, then, afterward, we might address how this approach works in the case of different mathematical practices through time (which we will not do here).

The present work is structured as follows. In section 2, we will look in some detail into a mathematical proof by Hippocrates of Chios. In section 3, we will present the framework adopted in this work. We will consider a schematic model of intentional reasoning, in which metareasoning gives rise to a feeling of correctness associated with each reasoning process. We will address Hippocrates' proof as a form of (guided) intentional reasoning. At this point, we will see how a proof functions, for us, as an enabler to ascertain the correctness of its argument, and how we ascertain its correctness. In section 4, we will address, using this framework, two proofs from Euclid's *Elements* and see how they can also be conceived as a form of (guided) intentional reasoning.

2 Hippocrates of Chios' proof of the quadrature of a lune

The earliest extant mathematical proofs are those of Hippocrates of Chios about the quadrature of lunes, which were made by Hippocrates around 450-430 BCE. We do not have Hippocrates' text, but an account of it in a text by Simplicius from the 6th century CE (Høyrup 2019a). Simplicius' text reports on what Alexander of Aphrodisias wrote around 200 CE, and on what Eudemus wrote in the late 4th century BCE (or possibly a later version of this text) (Høyrup 2019a). Here, we consider the part of Simplicius' text related to the older Eudemian text. That is made in terms of the reconstruction by Becker of the Eudemian text and Netz's translation of it into English (Netz 2004).

It is said that Hippocrates taught geometry and wrote the first collection of elements of geometry. This collection, not yet in the axiomatic format of Euclid, "is likely to have been connected to Hippocrates's teaching" (Høyrup 2019b, 36). Most likely, it consisted of a loose collection of known results and techniques – which were taken for granted – and newer developments obtained using these (Høyrup 2019b). Among the newer developments, we might expect to be Hippocrates' quadrature of lunes.

Here, we will consider the proof of the first quadrature as given in the Eudemian account of Hippocrates' text (the numbering is not part of the ancient text; it is included by Netz to ease reference). It is as follows:

(2) So he made his starting point by assuming, as the first among the things useful to the quadratures, that both the similar segments of the circles, and their bases in square, have the same ratio to each other [...] [(4)] he first proved by what method a quadrature was possible, of a lunule having a semicircle as its outer circumference. (5) He did this after he circumscribed a semicircle about a right-angled isosceles triangle and, about the base, <he drew> a segment of a circle, similar to those taken away by the joined <lines>. (6) And, the segment about the base being equal to both <segments> about the other <sides>, and adding as common the part of the triangle which is above the segment about the base, the lunule shall be equal to the triangle. (Netz 2004, 248-9)

In part 2 of the text, it is mentioned the geometric knowledge useful to arrive at the intended result: both the similar segments of the circles, and their bases in square, have the same ratio to each other. This "similar segments principle" is the only background knowledge made explicit in the text. Not mentioned are the related Pythagorean rule or the simple arithmetic of areas (additivity and subtractivity of areas). According to Høyrup, this principle, like the Pythagorean rule, and the simple arithmetic of areas, "were known since well above a millennium [BCE] in Near Eastern practical and scribal geometry" (Høyrup 2019a, 178).

Part 4 of the text specifies what quadrature is being considered; that of a lune having a semicircle as its outer circumference. Afterward, the text, in part 5, gives indications of how to draw that figure – a lune having a semicircle as its outer circumference. We circumscribe a semicircle about a previously drawn right-angled isosceles triangle. This gives rise to two segments of circle, each having a side of the triangle as its base (see fig. 1 left). Then we draw about the base of the triangle a segment of a circle similar to the segments already drawn, e.g., by completing a square and using its lower corner as the center of another circle (see fig. 1 right).

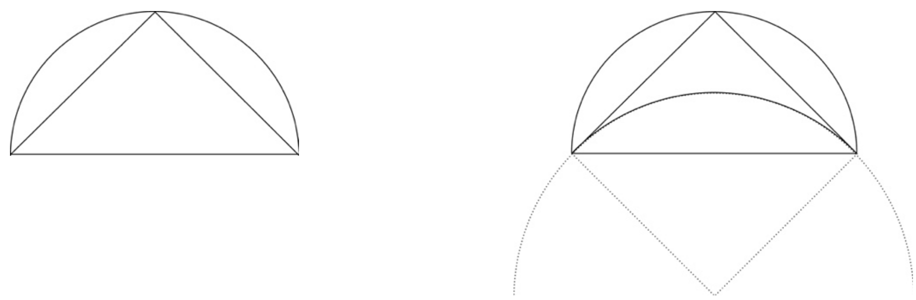


Fig. 1 Semicircle circumscribing a right-angled isosceles triangle (left). Drawing of the lune (right)

What we might call the argumentation proper is given in part 6 of the text. It can be divided into two parts. In the first, one uses the above-mentioned principle and concludes that the area of the segment of circle about the base is equal to the sum of the areas of the segments of circles about the sides. This is presented in the text already as a premise (an antecedent) for a subsequent inference: the segment about the base being equal to both segments about the other sides.

A way to visualize the "similar segments principle" and the related Pythagorean rule for our case is depicted in fig. 2 (the area of the larger square is equal to the sum of the areas of the smaller squares, and the area of the larger segment of circle is equal to the sum of the areas of the smaller segments of circles).

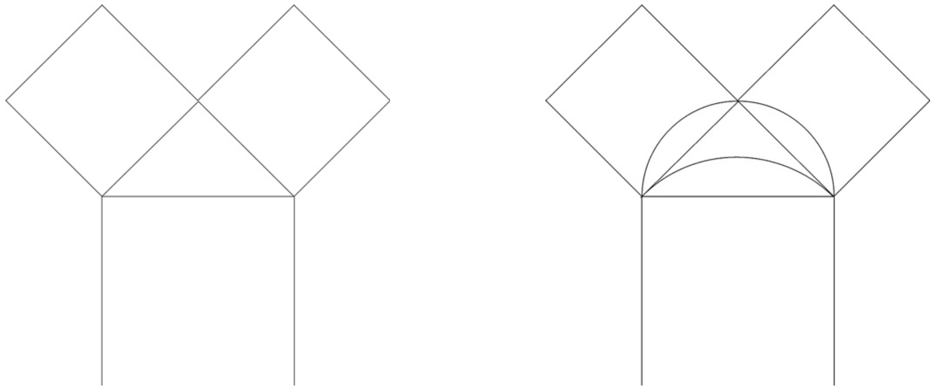


Fig. 2 The Pythagorean rule for the case of a right-angled isosceles triangle (left); Pythagorean rule plus the "similar segments principle" (right)

In the second part of the proof, the text gives us instructions on how to do simple arithmetic of areas: adding as common the part of the triangle which is above the segment about the base. Finally, the text presents the conclusion (the consequent): the lune shall be equal to the triangle.

With the aid of the figure, we "add" the part of the triangle which is above the larger segment of circle to the two small segments of circle (fig. 3 left). We also "add" this area to that of the larger segment of circle (fig. 3 right). We are adding the same area (that of part of the triangle) to areas that are equal according to the "similar segment principle" (the area made up of the two smaller segments of circle and the area of the larger segment of circle). We then conclude, as stated, that the area of the lune is equal to the area of the triangle.

Fig. 3 Simple arithmetic of areas

The argumentation in the first part of the proof is made on a single level, it is directly based on geometric knowledge useful to arrive at the intended result (Høyrup 2019a, 179); what we call the "similar segment principle". The conclusion of the first argument lays the ground for the next one which is also based on a direct application of background knowledge; in this case the arithmetic of areas.

3 Mathematical proofs as a form of (guided) intentional reasoning

Here, we develop a schematic model of intentional reasoning inspired by Frankish's model of intentional reasoning as a cyclical process (Frankish 2018). By intentional reasoning, Frankish means "deliberate, intentionally controlled reasoning" (Frankish 2018, 10), in which we work out an issue in a sequence of steps or "actions"; e.g., a mathematical problem (Frankish 2018, 10).

Frankish conceives intentional reasoning as a cyclical process (Frankish 2018, 12-4).

Let us say that we begin with a written (or spoken) sentence. In Frankish's model, "we start with [the] sentence, interpret it as a step in an argument, form a belief about the next step, add that sentence, and so on" (Frankish 2018, 13). Fig. 4 illustrates the cycles. Forming a belief and producing a new sentence would be the result of an autonomous reasoning process. In that way, while the reasoning cycle is intentional, "the processes that guide and support this reasoning will be autonomous" (Frankish 2018, 10). That means that "intentional reasoning is not wholly intentional, but guided and mediated by autonomous reasoning" (Frankish 2018, 10).

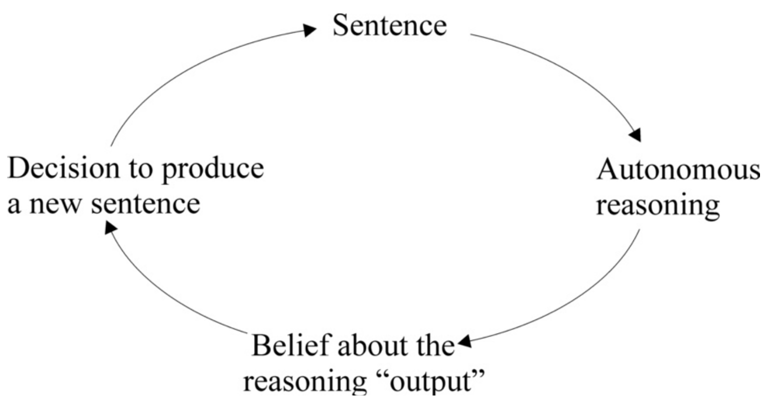


Fig. 4 Frankish's model of intentional reasoning

An example of an intentional reasoning process is making a long division. According to Frankish:

> We begin by writing out the figures in a certain format […] and autonomous processes interpret them as posing a simpler division problem [...] autonomous processes then generate a belief about the solution to this subproblem and a decision to write down further symbols expressing it. (Frankish 2018, 15)

Notice that the result of an autonomous reasoning process is not so much a "solution" but a belief about the solution. Frankish mentions, elsewhere, that "judging that p [is the case] results in my forming the belief that p [is the case]" (Frankish 2018, 7). We might say that an autonomous reasoning process gives rise not so much to a solution but to a belief that a solution is the case. In the model, arriving at a belief that p is the case is followed by the decision "to communicate [the] belief" (Frankish 2018, 7). That is, the new sentence that is produced at the end of the cycle is not so much the "solution" p but a sentence expressing the belief that p is the case.

The idea that what is expressed corresponds not to p but to the belief that p is the case is a bit awkward. If we consider the long division example, one supposedly writes new mathematical symbols for each step of the division. For each step, do the symbols express p or the belief that p is the case? The mathematical symbols seem to be expressing the output of an autonomous reasoning process, in this case, an arithmetic result, not the belief that an arithmetic result is the case.

In the model developed here, we do not adopt any notion of belief. Instead, we take into account the view that different cognitive processes are accompanied by metacognitive processes. Accordingly, "metacognition is ubiquitous because virtually all cognitive operations are monitored and controlled, before, during, and after their execution" (Fiedler, Ackerman, Scarampi 2019, 90). A crucial aspect of metacognition is that it leads to a "subjective feeling" (Fiedler, Ackerman, Scarampi 2019, 96).

Focusing on reasoning, our autonomous reasoning processes are accompanied by metareasoning processes, which provide monitoring of the reasoning processes. Metareasoning gives rise to what we might call a feeling of rightness or even a feeling of error (Ackerman, Thompson 2017). Like other affects, these "subjective feelings" have different degrees or intensities. For example, we can go from a low feeling of rightness to a strong feeling of rightness.

In Efklides' view, "'affect' is a generic term for emotions and other mental states that have the quality of pleasant-unpleasant, such as feelings, mood, motives, or aspects of the self, e.g., self-esteem" (Efklides 2006, 3). Also, according to her, metacognitive feelings or affects "have a dual character, that is, a cognitive and an affective one" (Efklides 2006, 3). In simple terms, in the case that interests us, a feeling of rightness can be accompanied by a verbal label "I feel it is right". We might say that metareasoning gives rise to a feeling-knowing of rightness (or even "wrongness", if that is the "output" of the metareasoning process).

In this way, the output of an autonomous reasoning process is not only what will be expressed as a linguistic expression ("after" language production); it is that and a metacognitive affect – a weaker or stronger feeling of rightness or correctness (in this work, we adopt the term "feeling of correctness" instead of "feeling of rightness", since we are addressing mathematical proofs).

Here, we propose a model of intentional reasoning that incorporates the notion of metareasoning. In our view, in his model, Frankish conflates two aspects of reasoning into one notion. The output of an autonomous reasoning process is not the belief that p is the case (something like "I believe that p is the case"); it is "p" plus a feeling of correctness (the feeling that the reasoning is correct; or, rephrasing, the feeling that p is the case). In this way, our schematic model is as depicted in fig. 5. Notice that in the figure, the relative position of the different components is not intended to imply that as neural processes their temporal order corresponds exactly to what is depicted (see, e.g., Vaccaro, Fleming 2018, Rouault et al. 2018, Bartley et al. 2018). Also, we have a possibly non-linguistic conclusion of the reasoning and the "intention to express the conclusion". This "intention" is conceived in terms of Levelt's model of language production (Levelt 1989), in which there is an "intention" to express verbally some conceptual structure (in our case, the non-linguistic "solution" of an autonomous reasoning process); this is not something we need to address here. The key aspects of the model are that the autonomous reasoning process is accompanied by a metareasoning process that gives rise to a metacognitive feeling of correctness of the reasoning and that metacognitive feelings have a dual aspect – a cognitive and an affective one.

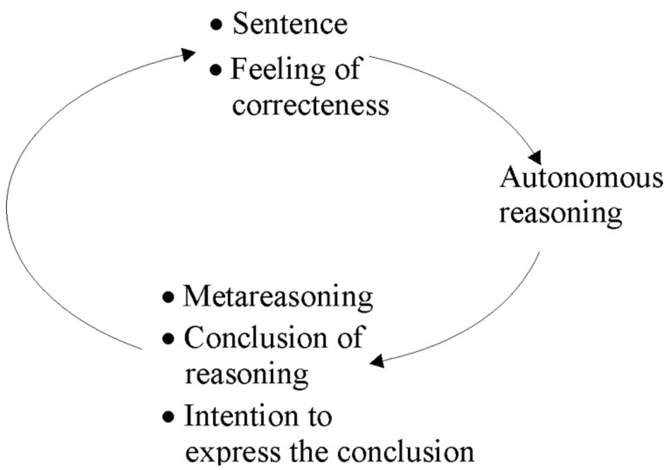


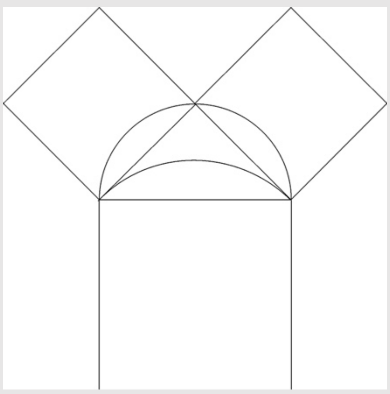
Fig. 5 Schematic model of intentional reasoning

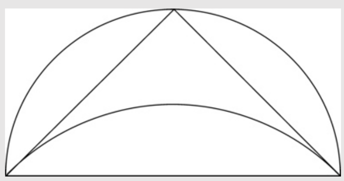
Let us now address Hippocrates' proof in terms of this model of intentional reasoning as a cyclical process. If we were considering the proof as made by Hippocrates for the first time, we would directly apply our schematic model as it is. But we want to address the proof as given in a text (in our case, Netz's translation). In our view, we can address the proof as a form of guided intentional reasoning. The written text of the proof guides us through cycles, in each of which we face the premise and the conclusion of a reasoning process. For each step/cycle of the proof, we are "induced" into producing an autonomous reasoning process that "links" the premise with the conclusion.

Hippocrates' proof (part 6 of Netz's translation) consists, as mentioned, of two parts. That means that there are two cycles of guided reasoning. Schematically it is as follows:

Cycle 1.

Initial "sentence": "similar segments principle" + figure



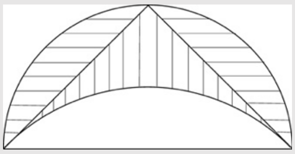


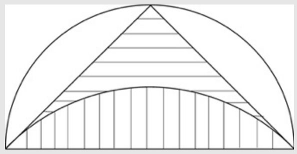
Conclusion: "the segment about the base [is] equal to both <segments> about the other <sides>" (Netz 2004, 249).

Cycle 2.

Initial "sentence": Previous conclusion + figure

We are told to: “[add] as common the part of the triangle which is above the segment about the base” (Netz 2004, 249)





Conclusion: “the lunule shall be equal to the triangle” (Netz 2004, 249).

In each of these cycles, there is an autonomous reasoning process that connects the premise to the conclusion (in the second cycle we are even given instructions on how to reason). The text guides us into producing these reasoning processes. They go hand in hand with metareasoning processes that give rise to a strong feeling of correctness for each cycle. Having a strong feeling of correctness for each cycle, we have a strong feeling of correctness for the whole proof.

Hippocrates’ proof works as a form of guided intentional reasoning. This “format” of the proof enhances the metareasoning processes that accompany the reasoning processes. We have a sequence of “small” reasoning steps, each accompanied by a metareasoning process, which gives rise to a feeling-knowing of correctness. The metacognitive “feeling” in its dual nature, of a cognitive and an affective one, is what makes us fell-know that the inferential step is correct (even if we might be wrong).

That is how a proof functions, for us, as an enabler to ascertain the correctness of its argument, and how we ascertain this correctness.

Notice that having a strong feeling of the correctness of each step, and because of this of the whole proof, does not imply that we are right. Our metareasoning depends on heuristic cues (Ackerman 2019) and might be defective. We can have a strong feeling of correctness associated with a reasoning process and be wrong. In fact, there are biases regarding the feelings of rightness generated by our metareasoning processes (see, e.g., Fiedler, Ackerman, Scarampi 2019). We must distinguish what each one of us feels-knows about a proof’s correctness from how we collectively ascertain “objectively” that a proof is correct; here, we only suggest that the “objectivity” of the correctness of a proof could be the result of a robust intersubjectivity: a strong feeling of correctness, shared by many, e.g., as the result of group discussions. There are, however, issues regarding the effectiveness of group discussions (see, e.g., Silver, Mellers, Tetlock 2021; Bang, Frith 2017); in this way, there is no simple answer.

Also, notice that, in this work, we do not address the geometric proofs *qua* geometric proofs, but as early examples of mathematical proofs. In this way, we do not try to address the “black boxes” of (geometric) autonomous reasoning and metareasoning. Currently, geometric cognition and reasoning are still poorly understood (for some views related to the issue, see, e.g., Hohol and Miłkowski (2019), Ferreirós and García-Pérez (2020) and Hohol (2020)). However, there are already some relevant results, e.g., regarding the role of diagrams and diagrammatic reasoning in proofs (see, e.g., Freksa, Barkowsky, Falomir, and van de Ven (2019); Dal Magro and García Pérez (2019); Magnani (2013); Giardino (2013); Manders (2008); Giaquinto (2007)).

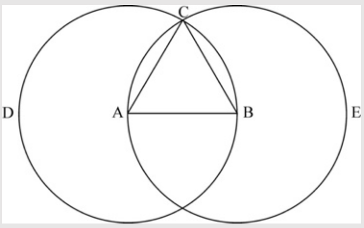
4 Two Euclidian proofs

How mathematical statements were proved changed from the time of Hippocrates to that of Euclid, roughly one and a half centuries later (see, e.g., Mueller 2006). As we have seen, in the case of Hippocrates general geometric principles were used without being proved. That is not the case with Euclid. His assumptions are much simpler than those admitted by Hippocrates. In the *Elements*, there are definitions like those of point, straight line, or circle (Heath 1956, 153); also, there are postulates, like the ones that license the construction of a straight line, or a circle (Heath 1956, 154); we also have common notions that make more precise general principles that were applied intuitively by Hippocrates. For example, we can take Hippocrates to use, in his arithmetic of areas, the following common notion: "if equals be added to equals, the wholes are equal" (Heath 1956, 155).

Since there are only a few very basic assumptions, even the simplest geometric results must be proved. That implies that, generally, proofs have more than a single level, contrary to the case of Hippocrates. In this way, a proof may rely on previously proved propositions, besides relying on the admitted assumptions. That can give rise to a very complex structure.

To address Euclidean proofs in terms of (guided) intentional reasoning, we think it is simpler if we start by addressing a proposition that does not depend on previously proved propositions. The simplest case is that of proposition 1 of book 1 of the *Elements* (I.1). The Euclidean text is as follows:

On a given finite straight line to construct an equilateral triangle.



Let *AB* be the given finite straight line.
Thus it is required to construct an equilateral triangle on the straight line AB.

With center *A* and distance *AB* let the circle *BCD* be described; [Post. 3]
again, with center *B* and distance *BA* let the circle *ACE* be described; [Post. 3]
and from the point C, in which the circles cut one another, to the points *A, B*
let the straight lines *CA, CB* be joined. [Post. I]
Now, since the point *A* is the center of the circle *CDB, AC* is equal to *AB.* [Def. I5]
Again, since the point *B* is the center of the circle *CAE, BC* is equal to *BA.* [Def. I5]
But *CA* was also proved equal to *AB;* therefore each of the straight lines *CA, CB* is equal to *AB.*
And things which are equal to the same thing are also equal to one another; [C. N. 1]
therefore *CA* is also equal to *CB.*
Therefore the three straight lines *CA, AB, BC* are equal to one another.
Therefore the triangle *ABC* is equilateral; and it has been constructed on the given finite straight line *AB.*
(Being) what it was required to do. (Heath 1956, pp. 241-242)

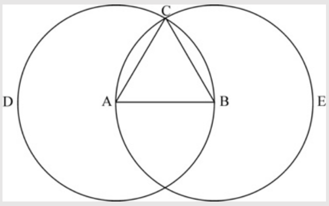
[we adopt an English translation of the standard edition of the *Elements*. On this see, e.g., Vitrac (2012)]

Analyzing the structure of this proof in terms of the schematic model of intentional reasoning as a cyclical process, while following the text of the proof, having, for each step/cycle, the verbal premise(s) and the verbal conclusion, the reader goes through autonomous reasoning and metareasoning processes. Each metareasoning process gives rise to a strong feeling of correctness associated with the conclusion of the corresponding step of the proof. Having a strong feeling of correctness for each step, we have a strong feeling of correctness of the whole proof.

Schematically, in terms of cycles of (guided) intentional reasoning, the proof is as follows (we do not include connectors like "since", "therefore", "but", and "and" (see Netz 1999, 115-6)):

Cycle 1.

Sentence$_0$: "the point *A* is the center of the circle *CDB*" → Conclusion (sentence$_1$): "*AC* is equal to *AB*"



Cycle 2.

Sentence$_2$: "the point *B* is the center of the circle *CAE*" → Conclusion (sentence$_3$): "*BC* is equal to *BA*"

Cycle 3.

Sentence$_3$ + Sentence$_4$ (a reframing of sentence$_1$): "*CA* was also proved equal to *AB*" → Conclusion (sentence$_5$): "each of the straight lines *CA, CB* is equal to *AB*"

Cycle 4.

Sentence$_5$ + sentence$_6$ (common notion 1): "things which are equal to the same thing are also equal to one another" → Conclusion (sentence$_7$): "*CA* is also equal to *CB*"

Cycle 5.

Sentence$_7$ + sentence$_5$ (implicit) → Conclusion (sentence$_8$): "the three straight lines *CA, AB, BC* are equal to one another"

Cycle 6.

Sentence$_8$ → Conclusion (sentence$_9$): "the triangle *ABC* is equilateral"

The proof of I.1 consists of 6 cycles of intentional reasoning. In each of these cycles, our metareasoning generates a strong feeling of correctness of the autonomous reasoning process being made. Overall, we reach a feeling of correctness for the whole proof.

The proof of I.1 is a bit more general than our schematic model. As we have just seen, in general, the verbal conclusion of a cycle is not, just by itself, taken to be the premise of the following cycle. We have one premise only in cycles 1, 2, and 6. In the other cycles, there is more than one premise at play. However, even if a bit more complex, we still have a form of (guided) intentional reasoning.

We will now consider another type of proof that was pervasive in ancient Greek mathematics; it is called in Latin "reductio ad absurdum" (see, e.g., Netz 1999, Heath 1981). We will call it simply a reductio proof. The form of a reductio proof is, schematically, as follows: to prove P, assume not P, derive a "contradiction"; i.e., derive an obviously false result, like a geometric statement and its opposite (e.g., $AB = CD$ and $AB \neq CD$). Conclude that P is true (Cunningham 2012, 93).

Ancient Greek proofs that adopt a reductio strategy have a very fixed argumentative structure, as described by Netz (1999, 140). Simplifying, when we intend to prove a geometrical statement P, we use "for if" to introduce the negation of P (i.e., not P) and a consequence of not P. A sequence of "arguments" follows (i.e., the verbal expressions corresponding to premises and conclusions), leading to some property, "which is impossible/absurd" (Netz 1999, 140). Afterward, we find the crucial arguments in the reductio proof. First, we have "therefore not (not P)"; second, we have "therefore P".

Here, a pause is necessary. As mentioned by Dutilh Novaes, the step of going from reaching an absurd result to concluding that P is the case (which in the Euclidean proof is made in two steps, as we have just seen), "strongly relies on a number of assumptions, and if these are not in place then the argument does not go through" (Dutilh Novaes 2016, 2625). These are related to what can be named the culprit problem and the act of faith problem. The first problem is described by Dutilh Novaes as follows:

> We start with the initial assumption, which we intend to prove to be false, but along the way we avail ourselves of auxiliary hypotheses/premises. Now, it is the conjunction of all these premises and hypotheses that leads to absurdity, and it is not immediately clear whether we can single out one of them as the culprit to be rejected. (Dutilh Novaes 2016, 2614)

To avoid the culprit problem, we must assume that "we can isolate the culprit" (Dutilh Novaes 2016, 2615). For that: "what is required is that all auxiliary assumptions/ premises used in the argument have a higher degree of certainty to us than the initial assumption that is singled out to be rejected" (Dutilh Novaes 2016, 2625).

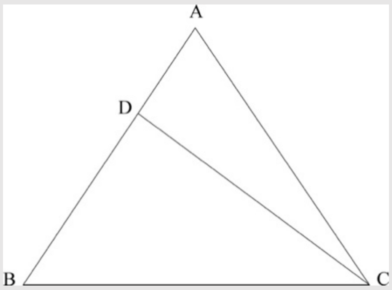
The second problem can the stated as follows: "How does one go from it being a bad idea to maintain the hypothesis to it being a good idea to maintain its contradictory?" (Dutilh Novaes 2016, 2615). According to Dutilh Novaes, "A reductio ad absurdum also starts with the tacit assumption of an exhaustive enumeration of cases: for a given proposition A, either A is the case or its contradictory is the case, and not both" (Dutilh Novaes 2016, 2615). In this way:

> If we can be sure that the enumeration of cases is truly exhaustive (i.e., excluded middle holds in the relevant situation), and that we will not end up in a situation of aporia where all options lead to absurdity, then we can safely conclude not-p after showing that p leads to absurdity. (Dutilh Novaes 2016, 2625)

Here, we want to suggest that the culprit problem and the act of faith problem are avoided in Euclidean geometry and are dealt with, each one, one step at a time, in the final stages of the argumentative structure of Euclidean reductio proofs. As mentioned, in the argumentative structure there is one point in which we reach some property that is impossible/absurd. Afterward, we deal with the culprit problem: therefore not (not P). There is an autonomous reasoning process that identifies not P as the culprit (and not some other assumption or set of assumptions). Then we face the act of faith problem: therefore P; i.e., the rejection of not P leads to an autonomous reasoning process that enables us to conclude that P is the case. We will consider an example of a reductio proof to see how this is done and how a reductio proof can be addressed from the perspective of proofs as a form of (guided) intentional reasoning.

We will consider the first reductio proof in the *Elements*; that of proposition 6 of book 1 (I.6). The text of proposition I.6 is as follows:

If in a triangle two angles be equal to one another, the sides which subtend the equal angles will also be equal to one another.



Let *ABC* be a triangle having the angle *ABC* equal to the angle *ACB;*
I say that the side *AB* is also equal to the side *AC.*
For, if *AB* is unequal to *AC,* one of them is greater. [C.N.]
Let *AB* be greater; and from *AB* the greater let *DB* be cut off equal to *A*C the less; [I.3]
let *DC* be joined. [Post. 1]
Then, since *DB* is equal to *A*C, and *BC* is common, the two sides *DB, BC* are equal to the two sides *A*C, *CB* respectively; and the angle *DBC* is equal to the angle *ACB;*
therefore the base *DC* is equal to the base *AB,* and the triangle *DBC* will be equal to the triangle *ACB,* [I.4]
the less to the greater:
which is absurd [C.N. 5].
Therefore *AB* is not unequal to *AC*;
it is therefore equal to it. (Heath 1956, 255-6)

Thus, if a triangle has two angles equal to one another then the sides subtending the equal angles will also be equal to one another. (Which is) the very thing it was required to show. [These last sentences are from the translation by Fitzpatrick (2008, 13) since in Heath's translation it is left incomplete]

The demonstration starts with a triangle *ABC* with two equal angles, *ABC* and *ACB*. What we want to show is that the corresponding sides are equal; i.e., $AB = AC$. We start our proof by assuming the "opposite" property $AB \neq AC$ (that they are "opposite" follows from the "solution" to the act of faith problem). Schematically, in terms of cycles of (guided) intentional reasoning, the proof is as follows:

Cycle 1.

Sentence$_0$: "*AB* is unequal to *AC*" → Conclusion (sentence$_1$): "one of them is greater"

[Comment: To reach the conclusion, use is made of the previously unmentioned common notion: if two quantities are unequal, then one is greater or lesser than the other (see, e.g., Joyce 1998)]

[Comment: After the first cycle of intentional reasoning that includes an autonomous reasoning process, we have what we might call the "insertion" of another construction. That is made in sentence$_2$ (by considering sentence$_1$): "Let *AB* be greater; and from *AB* the greater let *DB* be cut off equal to *A*C the less; let *DC* be joined". The new figure shows that the triangle *DBC* is smaller than the triangle *ACB*. That is our (implicit) sentence$_3$. This "property" will be important later since it will lead to a contradiction with another property that will be deduced later; on diagrams in reductio proofs, see, e.g., Dal Magro and Valente (2021)]

Cycle 2.

Sentence$_4$: "*DB* is equal to *A*C, and *BC* is common" → Conclusion (sentence$_5$): "the two sides *DB, BC* are equal to the two sides *A*C, *CB* respectively; and the angle *DBC* is equal to the angle *ACB*"

[Comment: The (guided) autonomous reasoning process, leading from sentence$_4$ to the conclusion (sentence$_5$), uses properties of the triangles as depicted in the figure and given in the text]

Cycle 3.

Sentence$_5$ → Conclusion (sentence$_6$): "the base *DC* is equal to the base *AB,* and the triangle *DBC* will be equal to the triangle *ACB*

[Comment: The (guided) autonomous reasoning process is to be made by resort to proposition I.4 (as indicated by the inclusion in the text of [I.4]]

Cycle 4.

Sentence$_6$ + (implicit) Sentence$_3$ → Conclusion (sentence$_7$): "the less [is equal] to the greater"

Cycle 5.

Sentence$_7$ → Conclusion (sentence$_8$): "[the previous conclusion] is absurd"

Cycle 6.

Sentence$_8$ → Conclusion (sentence$_9$): "*AB* is not unequal to *A*C"

[Comment: In this cycle of (guided) intentional reasoning, we reason to the rejection of the initial assumption $AB \neq AC$. It is here that the culprit problem is faced. Since there are no auxiliary assumptions besides the initial assumption (all the other "assumptions" are taken to be "true" (definitions, postulates, common notions) or proved to be true (propositions I.3 and I.4)), we use this knowledge in the reasoning and conclude that the absurd result that triangle DBC = triangle ACB and triangle DBC < triangle ACB follows from the initial assumption. Since there is one reasoning cycle to address the culprit problem, the metareasoning during this cycle generates a strong feeling of the correctness of the conclusion. In this way, dispelling any doubts we might have in this inferential step concerning the culprit problem]

Cycle 7.

Sentence$_9$ → Conclusion (sentence$_{10}$): "it is therefore equal to it"

[Comment: The act of faith problem is faced during this reasoning process. As mentioned by Fitzpatrick, here we use a previously unmentioned common notion: if two quantities are not unequal then they must be equal (Fitzpatrick 2008, 13). This common notion implies that, for quantities, there are no other cases/options that might lead to an act of faith problem. If two lengths are not unequal then these lengths must be equal; there is no other option. The reasoning process taking into account this evident property of quantities leads to the conclusion: "it is therefore equal to it"; i.e., $AB = AC$. By addressing just the act of faith problem in this last cycle, we generate a strong feeling of correctness that dispels any doubts we might have concerning the act of faith problem. Notice that we do not face any risk of having a situation of aporia where all options lead to absurdity. We have determined that if the triangle had the sides AB and AC unequal, there would be an absurd consequence; that of two triangles being at the same time equal and different. The only alternative that remains is that the sides AB and AC, of the triangle, are equal. If this also leads to absurd consequences, then pure geometry would not be consistent]

As we see, reductio proofs fit nicely within a perspective on proofs as a form of guided intentional reasoning. In our view, the present approach to mathematical proofs enlightens why reductio proofs are a hallmark of mathematics not to be found in other human practices, e.g., in legal reasoning. As we have seen, to be able to adopt a reductio proof we must face the culprit problem and the act of faith problem. If these are not avoided, "the argument does not go through" (Dutilh Novaes 2016, 2625). We see a specific feature of geometry (i.e., that these problems can be avoided) at work "directly" at the level of the structure of the proof. By being a form of guided intentional reasoning, in a proof, these problems are faced in individual and consecutive cycles of reasoning. In each cycle, metareasoning generates a strong feeling of the correctness of the conclusion; in this way, dispelling these problems.

Evidently, there are other approaches that give enlightening perspectives on mathematical proofs, in general, and reductio proofs, in particular. We are thinking, in particular, of the dialogical conception of mathematical proof (Dutilh Novaes 2018, 2020) and the view of mathematical proof as audience-reflective argumentation (Ashton 2021). However, it would go beyond the scope of the present work to address the fitting of the approach developed here with other approaches.

5 Conclusion

As mentioned in the introduction, the purpose of the present work was to determine, for ancient Greek proofs, how a proof functions, for us, as an enabler to ascertain the correctness of its argument and how we ascertain this correctness. Our answer to the first question is that Hippocrates' and Euclid's proofs work as a form of guided intentional reasoning. Each cycle corresponds to what we usually refer to as an inferential step that goes from a premise to a conclusion (the "it follows from … that …"). In each cycle, the reader "actively" connects the premise to the conclusion through an autonomous reasoning process. This "format" of the proof – i.e., that of a chain of arguments – enhances the metareasoning processes that go hand in hand with the reasoning processes. We have a sequence of "small" reasoning steps, each accompanied by a metareasoning process. It is metareasoning that produces a feeling-knowing of correctness that is associated with the verbal conclusion of an autonomous reasoning process (and that answers our second question). Feeling-knowing that the conclusion of each cycle is correct, enables us to assess the correctness of the whole mathematical proof.

6. Coda: Is an approach in terms of GIRPs really about mathematical proofs?[1]

Here, we will consider the position of a hypothetical reader, Skeptic. She/he might argue that the approach presented in the present work is not really about mathematics since it pays no attention to distinctive features of mathematical proof such as the use of notation that partially encodes rigor in its syntax, the use of diagrams, and so on. As it is, Skeptic might say that the approach to mathematical proofs as a GIRP could have been illustrated just as well with, say, legal reasoning or the verbal logic problems from general intelligence tests.

The first part of an answer would be that this is quite so. Here, we focus on commonalities, not specificities; that is, on aspects of mathematical proofs that can be shared with human reasoning with/about different human practices, activities, etc. The "distinctive features of mathematical proof" can then be addressed from within this approach in terms of GIRPs.

What this approach brings to the fore is the role of metacognition also in mathematical reasoning and, more specifically, in mathematical proofs. It is a metareasoning process related to the mathematical reasoning process that gives rise to a feeling of correctness. And yes, this is a common feature of human reasoning: "metacognition is ubiquitous because virtually all cognitive operations are monitored and controlled, before, during, and after their execution" (Fiedler, Ackerman, and Scarampi 2019, 90). Regarding our experiencing of metareasoning as a "subjective feeling" we must notice that this is not an "emotional" response. We know that the reasoning is correct because we feel that the reasoning is correct. What we mean is that this "feel" as an affect can be more of a cognitive nature and less of an affective one. As mentioned, a particular characteristic of metacognitive feelings or affects is that they "have a dual character, that is, a cognitive and an affective one" (Efklides 2006, 3). So, we do not have to think of metacognition as giving rise to bursts of emotional responses while we follow a proof. If it makes us more comfortable, we might think about the "output" of metareasoning as a "purer" strong "knowing" that the inferential step is correct: we fell-know its correctness.

Now, to the second part of the answer. Mathematical specificities are to be found in the reasoning process and accompanying metareasoning process that occurs in each cycle of a GIRP. Nowadays, geometric cognition and reasoning are still poorly understood. This is not to say that we must wait for science to open the "black boxes" of (geometric) autonomous reasoning and metareasoning to see geometric specificity at work in a geometric proof as a GIRP. We have seen an example of mathematical specificity in the present paper. Skeptic would have to agree that reductio proofs are a hallmark of mathematics not to be found in other human practices (e.g., in legal reasoning). And there

[1] Not included in published version.

is a good reason for this that is made clear by addressing mathematical proofs as GIRPs. As we have seen, to be able to adopt a reductio proof we must face the culprit problem and the act of faith problem. If these are not avoided, "the argument does not go through" (Dutilh Novaes 2016, 2625). It is a specificity of mathematics (geometry, in this case) that we do not have these problems. As we have seen, the culprit problem is faced in one of the cycles of the geometric proof as a GIRP. Since there are no auxiliary assumptions besides the initial assumption, the problem simply does not arise (as mentioned, since there is one reasoning cycle to address the culprit problem, the metareasoning during this cycle generates a strong feeling of the correctness of the conclusion). In the same way, there is another cycle in which the act of faith problem is "dissipated". It simply does not arise due to the specificity of geometry. That is an example where Skeptic can find a distinctive feature of mathematical proof at work "directly" on the mathematical proof as a GIRP.

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