A hub-and-spoke model of geometric concepts

Mario Bacelar Valente

ABSTRACT: The cognitive basis of geometry is still poorly understood, even the 'simpler' issue of what kind of representation of geometric objects we have. In this work, we set forward a tentative model of the neural representation of geometric objects for the case of the pure geometry of Euclid. To arrive at a coherent model, we found it necessary to consider earlier forms of geometry. We start by developing models of the neural representation of the geometric figures of ancient Greek practical geometry. Then, we propose a related model for the earliest form of pure geometry – that of Hippocrates of Chios. Finally, we develop the model of the neural representation of the geometric objects of Euclidean geometry. The models are based on the hub-and-spoke theory. In our view, the existence of specific models opens the possibility of addressing the relationship between geometric figures and geometric objects, in a novel way, in terms of their neural representation.

1. Introduction

The cognitive sciences research on geometric concepts is still limited.[1] But there are more general results regarding human conceptualization and concept processing that can guide us in addressing geometric concepts (see, e.g., Baddeley et al, 2020, pp. 210-231). The purpose of the present work is to set forward a model of the neural representation of geometric concepts that underline our cognitive activity related to the practice of pure geometry.[2]

From the perspective of a philosophy of mathematical practices, we should consider actual practices in their historical context (see, e.g., Ferreirós, 2016). This is particularly the case with the present endeavor. Our purpose is not to address modern forms of pure geometry but pure geometry at its beginning, or close to it. Our object of interest is the pure geometry of Euclid's *Elements*. To arrive at a coherent model, we found it necessary to consider earlier forms of geometry, ancient Greek practical geometry and the geometry of Hippocrates of Chios.

Currently, research points to 'hybrid theories' of the neural representation of concepts combining what we may call modal and amodal neural representations (Kuhnke et al, 2023). Within these, the hub-and-spoke theory is the theory with more acceptance (Rogers & Ralph, 2022; Ralph et al, 2017). According to this theory, the neural representation of concepts is made in terms of spokes, which are modality-specific brain regions that codify modal features of concepts.[3] For example, there are the spokes that encode visual, verbal (speech), and praxis representations.[4] There are also integrative regions – the hub – which blends, in an amodal format, the different aspects codified in the spokes and gives rise to coherent concepts. The hub enables a modality-free codification of further aspects of concepts; accordingly, "[It] allows the formation of modality-invariant multi-dimensional representations that […] code the higher-order statistical structure that is present in our transmodal experience of each entity" (Ralph, 2014, p. 7). In this way, the 'hub' is "a modality-independent unified representation efficiently integrating our conceptual knowledge" (Eysenck & Keane, 2020, p. 319). This neural organization of concepts enables us to address a particular concept directly in terms of 'spokes' and a 'hub' not has brain regions but as 'parts' of the concept.

---

[1] Some examples of recent research are: Lupyan (2017), Dillon et al (2019), and Sablé-Meyer et al (2021).

[2] For other works on somewhat related issues, see, e.g., Hohol & Miłkowski (2019), Hohol (2020), Ferreirós & García-Pérez (2020), and Pantsar (2022).

[3] For this work, we will only consider the original hub-and-spoke model and not the graded hub-and-spoke model. On the differences see Ralph et al (2017).

[4] The spoke related to vision encodes representations in the visual modality related to visual features of concepts. The spoke related to the praxis can encode, e.g., representations related to object use. The spoke related to verbal (speech) descriptors encodes the 'labels' we use to name concepts.

It must be noticed that the intended model of geometric concepts is highly simplified and idealized. It will be based on these basic features of the hub-and-spoke theory (modal spokes and an amodal hub). The models will be arrived at by considering what we know about ancient practices through the history of mathematics. We do not intend to move beyond what history enables us to (or what we already know from research in neuroscience). Possible limitations of the model will be referred to (in more technical footnotes). The purpose of this work is just to set forward a tentative model of the neural representation of abstract geometric objects.[5]

One of the motivations for this work is that this model might be useful in relation to the development of a future theory of the geometric cognition underpinning pure geometry. Another motivation is that having models of the neural representation of geometric concepts gives us a new venue to address the issue of the relationship between geometric figures and geometric objects. That has already been addressed from a historical perspective (see, e.g., Valente, 2020), and also cognitive considerations have been taken into account in somewhat related issues (see, e.g., Giaquinto, 2007; Dal Magro & García-Pérez, 2019; and Ferreirós & García-Pérez, 2020). But here, we present specific models developed in the context of a theoretical framework (the hub-and-spoke theory), which provides a more 'tangible' way to address this issue (since we have models of the neural representation of geometric figures and objects).[6]

The organization of this work is as follows. In section 2, we will address features of the practical geometry of ancient Greece shared by the practical geometries of other ancient cultures. This helps in giving a more general characterization of practical geometry and arrive at models of geometric figure for it. In section 3, we consider the earliest extant form of pure geometry as revealed in Hippocrates of Chios' work. We develop a model for the case of this pure geometry, which has important differences to that of Euclid's *Elements*. In section 4, we present the corresponding model of the neural representation of geometric objects in Euclidean geometry.

2. The ancient Greek practical geometry

In this work, we will consider basic aspects of the Greek practical geometric practice that are common to other ancient practical geometries. We will include examples of these practices when helpful for us to grasp what ancient practical geometry was. A more general presentation of ancient practical geometry enables us to use examples that are more comprehensible, since there are not many direct sources related to ancient Greek practical geometry (see, e.g., Asper, 2003, pp. 109-114).[7]

A key aspect of land measurements is to have a common unit of length so that we have a common standard. In this way, different surveyors will arrive at the same measure when using different measuring instruments (e.g., rods or ropes). These measuring instruments are 'calibrated' to the adopted standard. For example, in ancient Egypt, the unit of length measure was the *cubit*. This unit corresponds to the common measure of the forearm (Imhausen, 2016, p. 47; Rossi, 2007, p. 59). For land-surveying, it seems that Egyptians used ropes having a standard length of 100 *cubits*, which were

[5] In physics and elsewhere these models are called toy models (see, e.g., Reutlinger et al, 2017). One should not be misled by their extreme simplicity. In many cases, these models enable us to treat issues that are otherwise of unsurmountable difficulty. Some examples are the spin-1/2 Ising model (in statistical mechanics; see, e.g., Yeomans, 1992), a model of the interaction of neutral pions with a point-particle of infinite mass (in quantum field theory; see, e.g., Mandl, 1959), the Horseshoe map (in dynamical systems; see, e.g., Kautz, 2011), and the Lorentz model of convection rolls in the atmosphere (also in dynamical systems; see, e.g., Kautz, 2011). For a philosophical defense of toy models see, e.g., Nguyen (2020).

[6] Here, our purpose is simply to present a new possibility to address the issue of the relationship between geometric figures and geometric objects, calling attention to basic features enabled by having models of the conceptual representation of geometric figures and objects. It is beyond the scope of the present work to develop a philosophical inquiry based on it.

[7] Much of the practice of practical geometry relied on oral transmission. According to Asper, "the mathematical passages in Aristotle or Plato (minus the general differences between written prose and oral discourse) might be a model of how mathematicians *talked* about their objects" (Asper, 2003, p. 119). In fact, in Plato's *Meno* there is an example of ancient Greek practical geometry (Plato, 1997, pp. 881-5; Valente, 2020), and the corresponding text can be seen as a written rendering of the oral teaching and discussion of geometry in ancient Greece (Saito, 2018).

divided using knots placed at 1-*cubit* intervals (Imhausen, 2016, p. 18; Rossi, 2007, p. 154). Regarding the Greek length units, a widely used unit during Hellenistic times was the *foot* which was about 30-32 cm (Lewis, 2004, p. xix).

Besides the measurement of the boundaries of fields, it was essential to calculate the areas of these fields. Agricultural plots in the middle Euphrates valley usually had rectangular shapes (Høyrup, 2002, p. 34; Mori, 2007).[8] This led to the development of the so-called surveyors' formula, which gives good results when a shape approaches that of a rectangle. Let l_1, l_2, l_3, and l_4 be the sides of a quadrilateral field plot that are measured by a surveyor (e.g., l_1 and l_3 are the 'long sides', and l_2 and l_4 are the 'short sides'). The surveyors' formula gives for the area of the field plot the value $(l_1 + l_3)/2 \times (l_2 + l_4)/2$.

Here, we find an important and common feature of practical geometries. The lengths are measured (or taken to be measured), and as such are given in terms of a unit of measure. The area is calculated from these length measures and given in terms of a unit of area. For example, in the Old-Babylonian period, the main unit of length was the *rod* (approximately 6 m), and the unit of area was the *sar*, corresponding to one square *rod* (36 m^2) (Robson, 2008, p. 294).

For different reasons, in practical geometries, some figures are widely used. These figures are clearly distinguished from all the other possible figures by naming them, even if definitions do not exist (contrary to the case of the pure geometry in Euclid's *Elements*). A good example is the circle. In ancient Mesopotamia, a circle, like other geometrical figures, was conceptualized in terms of its boundary. The circle was the shape enclosed in a circumference. In this case, both had the same name. Perhaps the best translation of the name is "thing that curves" (Robson, 2004, p. 20). The area of the circle was determined from the measure of the length of the circumference. It was given by the square of the length of the circumference divided by 12 (Robson, 2004, p. 18). The circle was drawn using a specific instrument – the compass (Høyrup, 2002, p. 105; Friberg, 2007, p. 207). The compass made it possible to have a precisely drawn figure.

In terms of a model based on the hub-and-spoke theory, we can conceive of the concept of circle as relying heavily on a 'visual spoke' that represents aspects related to the visual shape of a circle, a 'verbal spoke' that codifies the name of the circle, and a 'praxis spoke' related to the drawing and measurements on the circle. Here, going beyond the spokes considered by Ralph and co-workers (see, e.g., Ralph et al, 2017), we propose another spoke related to measure-numbers, i.e., numbers that result directly from measurements in the case of length, or indirectly in the case of areas, and are addressed in terms of abstract symbols in the context of metrological systems (see figure 1).[9, 10]

---

[8] In the case of ancient Greece, there is also evidence of the division of land in rectangular plots (Lewis, 2004, p. 3; Cuomo, 2001, pp. 7-8).

[9] There is evidence for a 'number spoke'. According to recent research, adult humans possess two distinct systems to support magnitudes: 1) a symbolic system used specifically to represent symbolic numerical magnitudes, and 2) a general magnitude system used to represent both discrete and continuous magnitudes (Sokolowski et al, 2021, p. 10). In this work we depart from the idea that the spokes consist in modality-specific representations. In our case the number spoke is amodal. Also, it might be best to address the verbal spoke in terms of an amodal representation if we are to include more than simple words in it (on the possible kind of amodal representation underlying language see, e.g., Frankland & Greene, 2020). This is not such a radical departure from the original model as it might seem. A clear-cut modal/amodal dichotomy has been challenged; it might be best to think of modal and amodal representations as limiting cases in a "continuum" that goes from "low-level sensorimotor [representations] to highly abstracted and convolved ones" (Michel, 2021, p. 671). A spoke might consist of an "hierarchical representational structure" (Michel, 2021, p. 669). In some cases, we might simplify, addressing it in terms of a modal representation; in other, more complex, situations it might be best to address it in terms of an amodal representation. See also footnote 19.

[10] We address the neural representation of the concept of geometric figure without trying to include in our account Spelke's views (see, e.g., Spelke, 2011; Spelke et al, 2010; Spelke & Lee, 2012). However, one might try to include one of the two core representational systems of geometry identified by Spelke and co-workers in the hub-and-spoke model. We hypothesize that the spatial layout system might not be too relevant for the neural representation of the concept of geometric figure, while the visual form system might be 'connected' to the visual and praxis spokes, since it is taken to "represent the shapes of 2D visual forms and movable objects" (Spelke, 2011, p. 303). Since this is too speculative, it will not be taken into account in the present work.

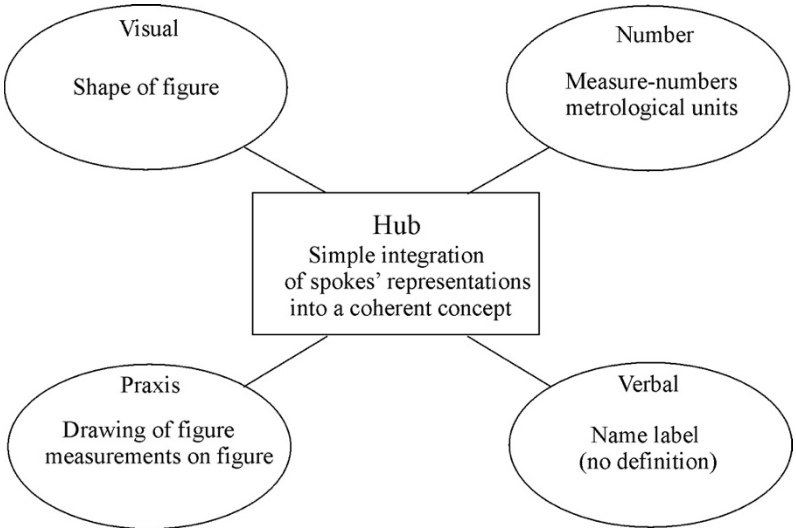

Figure 1. Hub-and-spoke model of the neural representation of geometric figure in practical geometry[11]

Another aspect shared by different practical geometries is the existence of written geometrical problems. These are couched in the terminology of practical geometry. But they are a somewhat different way of doing practical geometry. Being written problems that are disconnected from an immediate surveying activity, there are no actual measures taken into account in the problems. The measures are putative measures that could have been made according to the practice of practical geometry. This led, for example, to the adoption of conventional lengths in the problems. In Old Babylonian mathematical texts, all circumferences are taken to have the same standard length. That is so, independently of the actual length of the circumference drawn with a compass (Friberg, 2007, p. 207). So, while there are references to length measures in problems, these have not been actually measured, and neither corresponds to the actual measures of the drawn figure.

Whatever cognitive processes are at play during actual measurements these are not active during problem solving with conventionalized measures. One still uses whatever conceptual representation is at play when also making measurements, but only part of it. For example, the concept of line segment must include aspects related to the act of measuring. These are the procedures by which one attributes to the line segment a number (its length). In problems, we address these segments taking into account that there is a number associated with them—it is part of the concept—, but we disregard the use of a measuring instrument and the procedure by which the number is obtained. In our view, some aspects of the concept are only loosely taken into account.

There is a cognitive basis for this loosening of the connection of lengths to metrological units in problem solving in the context of practical geometry. We do not make use of concepts in a rigid way in which the 'full' concept is always taken into account. Research shows that conceptual processing is flexible, in the sense that "one aspect of a 'concept' may be used in one context or task, but another aspect of the concept may be used in another" (Mahon & Hickok, 2016, p. 949). A very simple way to take into account conceptual flexibility within a hub-and-spoke model of a concept is by taking a particular spoke not to be fully active during particular tasks. This can be accounted for by a reduced neural connectivity between the hub and the spoke depending on the context (Chiou & Ralph, 2019).

[11] In his study of the conceptual representation of triangles (the concept TRIANGLE), in relation to the linguistic label "triangle", Lupyan concluded that "(even formal) concepts have a graded and flexible structure, which takes on a more prototypical and stable form when activated by category labels" (Lupyan, 2017, p. 1). More specifically, he arrived at the following results: "(a) the representations of even a formally defined category like triangle reflect formally irrelevant, but perceptually relevant, properties; and (b), category names help to form a kind of idealized perceptual state—a prototype of sorts" (Lupyan, 2017, p. 9). These results fit well with the model presented here: (a) The praxis spoke might provide for the typicality effects related to, e.g., prototypical drawings being favored; and (b) The verbal spoke—the label 'triangle' —is part of the neural representation of triangle.

In this way, during problem solving, the praxis spoke of the concept of geometric figure would not be fully taken into account. The only thing that is included is the association of measure-numbers to drawn lines that in a full practical geometric practice are measured.

One example of a geometrical problem is a Hellenistic geometrical problem from a papyrus written in demotic Egyptian in the third-century BCE. The statement of the problem is as follows: “A plot of land that <amounts to> 60 square cubits, [that is rec]tangular, the diagonal (being) 13 cubits. Now how many cubits does it make [to a side]?” (Cuomo, 2001, p. 71). We have a rectangle and are asked to determine the length of its sides. These are calculated to have 12 and 5 *cubits* (Cuomo, 2001, p. 71; Friberg, 2005, p. 125). This is a problem of practical geometry; however, we do not have an actual shape whose relevant lengths were measured. In fact, in a strictly practical practice, this problem is unfeasible. We can only calculate the area after measuring the lengths of the long side and the short side.[12] In any case, as a didactic problem of practical geometry, the rectangle is conceived in terms of practical measures (in this case, the length of the diagonal) or values that are determined from practical measures (in this case, the area that is calculated from the lengths of the sides). In the problem, we consider a rectangle that is not measured nor even drawn, which has an area of 60 square *cubits*, a diagonal of 13 *cubits*, and sides that have (as calculated) 12 and 5 *cubits*.

In the context of a mathematical practice centered on the teaching and learning of geometry using written geometric problems, the praxis spoke is not fully taken into account. There are no actual measurements of lengths. The number-measures are given without being related to measurement procedures. One way to include this feature in our hub-and-spoke model is to consider that, due to conceptual flexibility, in the praxis spoke there are ‘active’ only traces of the measurement procedures (see figure 2).[13]

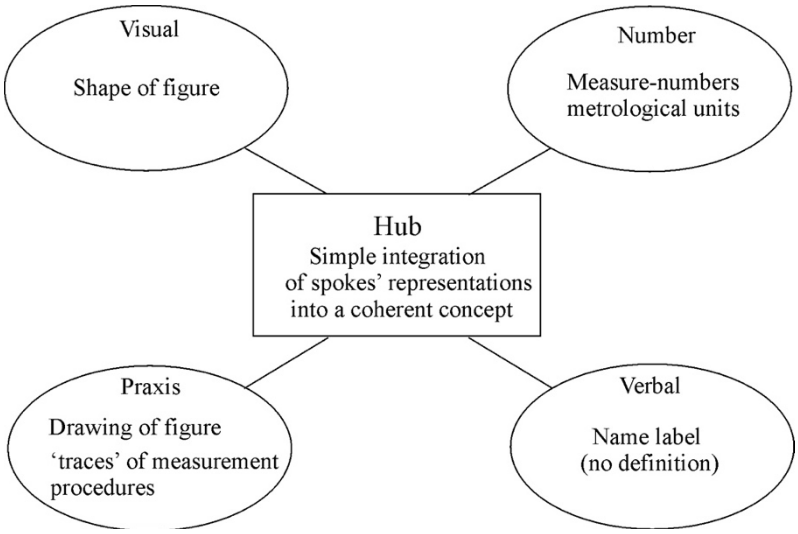

Figure 2. Hub-and-spoke model of the neural representation of geometric figure in practical geometry in the context of problem solving

[12] We might think that areas were directly measured, for example, using some sort of material square taken to be a metrological area unit; according to Damerow, this was not the case. Accordingly: “the idea to determine the size of an area by covering it with squares was an unfamiliar idea to the whole Babylonian mathematical tradition” (Damerow, 2016, p. 101).

[13] It is still the case that our conception of geometric figure is such that we might do actual measurements on the figure that contradict the adopted conventional values of lengths. In our model, the lack of actual measurements can be accommodated by taking into account the reduced neural connectivity between the hub and the spokes depending on the context (in this case, problem-solving and not the practice of surveying). A simple way to incorporate this particular case of conceptual flexibility in our model is, instead of thinking in terms of reduced neural connectivity, to consider that in the praxis spoke the aspects related to measurement are less ‘active’. That enables a more direct ‘visualization’ (see figure 2) of the difference between this step toward abstraction within practical geometry. A more ‘rigorous’ interpretation of the neural underpinning of geometric figure in the context of problem-solving might stress the role of the connectivity between the praxis and number spokes to the hub.

This implies that in the practice of practical geometry there are two related but somewhat different conceptualizations at play: the concept of geometric figure in the surveying practice and the concept of geometric figure in problem solving practices.

In the next section, we will consider early Greek pure geometry, having as a background the basic characteristics of practical geometry and the two hub-and-spoke models of the neural representation of geometric figure that we presented in the present section.

3. The early pure geometry of Hippocrates of Chios

Hippocrates' quadrature of lunules is taken to be the earliest evidence of Greek pure geometry (see, e.g., Netz, 2004; Høyrup, 2019). We know of Hippocrates' work by a text of Simplicius from the sixth-century CE. This text is based on two previous accounts, one by Alexander of Aphrodisias and the other by Eudemos. Hippocrates' work is believed to be from the early second half of the fifth century BCE. Written prose was rare; because of this Netz considers that "Hippocrates' treatise on the lunules could well be among the first treatises written in Greek mathematics" (Netz, 2004, p. 247). Regarding Eudemos account, Netz realizes an exercise of reconstruction, trying to determine what in the text is closer to Hippocrates' original. Netz assumes that the text "should have two layers, one closer to Hippocrates' original, and another closer to late fourth century mathematics" (Netz, 2004, p. 259). The main difference between these layers concerns the adoption or not of lettered diagrams, and the use of letters in the text to refers to parts of these diagrams.

Here, we want to suggest that the text might refer to an early written rendering of oral teaching by Hippocrates. This is not that a bold suggestion. We know that Hippocrates taught about astronomy and geometry (Høyrup, 2019, p. 160). According to Høyrup, the text by Alexander, reported in Simplicius' text, "draws on Hippocrates's teaching, being based either on lecture notes of his or on students' notes" (Høyrup, 2019, p. 158).

Hippocrates might have presented his arguments orally to his students accompanying them with the corresponding drawing. Even if there was a written rendering of his lectures on geometry, we take these to be the main vehicle of his approach to geometry. We suggest that like in the case of practical geometry, in pure geometry there is also an oral practice which might well be the earliest.

In what follows we will consider the passage about the first quadrature as a rewriting of an initially written rendering of oral teaching. It is as follows:

> (2) So he made his starting point by assuming, as the first among the things useful to the quadratures, that both the similar segments of the circles, and their bases in square, have the same ratio to each other. […] [(4)] he first proved by what method a quadrature was possible, of a lunule having a semicircle as its outer circumference. (5) He did this after he circumscribed a semicircle about a right-angled isosceles triangle and, about the base, <he drew> a segment of a circle, similar to those taken away by the joined <lines>. (6) And, the segment about the base being equal to both <segments> about the other <sides>, and adding as common the part of the triangle which is above the segment about the base, the lunule shall be equal to the triangle. (7) So the lunule, having been proved equal to the triangle, could be squared. (Netz, 2004, pp. 248-9)

Our purpose here is to determine what has changed in relation to practical geometry that leads us to say that here we are in the context of a pure geometric practice.

We can see that the text begins by calling to attention that the starting point of Hippocrates' argumentation is the assumption that the similar arcs of circumference of the circular figures and their bases in square have the same ratio to each other. Here, we do not have postulates like those of the *Elements*. In fact, this presupposition enters the argumentation in the same way that in problems of practical geometry. It is taken to be known by the interlocutor and without any need of justification. According to Høyrup, this presupposition was "known by Near Eastern practical geometers at least since the beginning of the second millennium BCE" (Høyrup, 2019, p. 165). Possibly, the only difference with the use of background knowledge in practical geometry problems is that the assumption that will be used during the argumentation is stated explicitly at the beginning. Notice that Hippocrates is adopting a background knowledge arising from practical geometry. As it is, by now we could still consider that Hippocrates is working within practical geometry.

The first quadrature is that of a lunule whose outer circumference can be seen as a semicircle. Thinking in terms of an oral presentation, Hippocrates after mentioning the assumption to his audience might have drawn an isosceles triangle and using a compass drawn a semicircle circumscribing it (see figure 3).

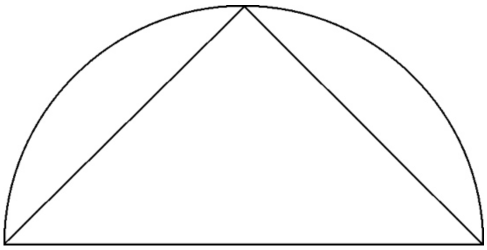

Figure 3. Initial drawing with a semicircle circumscribing an isosceles triangle

Afterward, he might have completed a square based on the triangle and using a corner of the square as the center of a circle drawn an arc segment that is similar to the two formed previously. According to Netz's rendering of the text in English, "<he drew> a segment of the circle, similar to those taken away by the joined <lines>" (Netz, 2004, p. 249) (see figure 4).

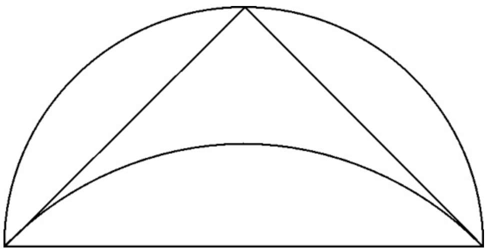

Figure 4. Completion of the drawing of the lunule

As Høyrup mentions, Hippocrates' arguments have a 'single-level', directly based on the assumptions taken into account (Høyrup, 2019, p. 179). Based on the presupposition mentioned at the beginning, Hippocrates simply mentions that the area of the circular figure about the base is equal to that of both circular figures about the other sides of the triangle. He then proceeds to add the area not included in either of these to each one of them. We have what we might call a visual operation in which we alternatively imagine each of the area addition operations (see figure 5, left and right).

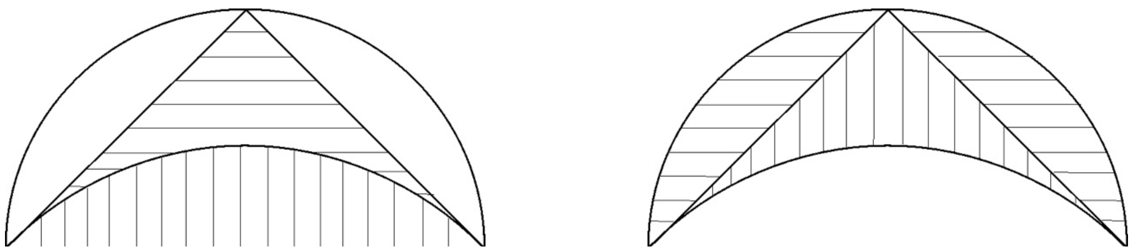

Figure 5. Two ways of doing the 'visual operation' of adding areas of figures

Evidently, the area of the two figures is the same and so "the lunule, having been proved equal to the triangle, could be squared" (Netz, 2004, p. 249). This addition of areas is something that we find in practical geometry. In fact, it is one of the basic elements of ancient Near Eastern geometry. It was clear to practitioners that "the size of a figure which consists of partial areas equal the sum of these partial areas" (Damerow, 2016, p. 115). We have what we might call the principle of conservation of area.

What is it then that makes this example a case of early Greek pure geometry and not of Greek practical geometry? The key evidence that we are not engaged in a practical geometrical practice is the lack of reference to metrological units. They are completely absent. This corresponds to a crucial conceptual change that is at the crux of the reformulation of practical geometry as pure geometry. We assist to a perfectioning of the geometrical figures that leads to what we might call the exactification of lengths.

We have seen that within practical geometry we assist to a loosening of the conception of geometric figure by not taking into account directly the measurement practices. As we have seen, we can have geometric figures that we do not measure but conceive as having lengths that cannot be even approximately like those of the figure. We mentioned the case of circles in Old Babylonian problems that adopt a conventional length for the circumference. In Hippocrates' pure geometry we have what we might call perfect figures. These are figures that look as having no irregularities and that if we were to measure them, we would find measure-numbers that are the same. That is, whatever small differences there are, they are invisible to the eye even when using the available measuring tools. This perfection of the geometric figure does not correspond to doing a more precise practical geometry. It is the opposite; not measuring the figures with better measuring techniques we take them to be perfect. In this way, e.g., the sides of a square are taken to be exactly equal. This is what we mean by the exactification of lengths. In this context, the lengths are exact and 'belong' to the figures. A length as a measure-number is the result of a measurement procedure in which, e.g., we put a measuring rod side by side with the side of a square and check that they are congruent. The length as a measure-number arises from this measurement procedure. In Hippocrates' case, we do not have this anymore. The sides of the square have lengths 'of their own', independently of whatever measurement we might make, and it is senseless to mention a metrological unit in this context.

As it is, early Greek pure geometry is the geometry of perfect figures (not yet of geometric objects).[14] Like in the case of practical geometry, these are not explicitly defined. As mentioned by Høyrup, "there is not the slightest reference to a definition in the Eudemos text" (Høyrup, 2019, p. 179). This has important consequences that we will address in the next section when comparing Hippocrates' pure geometry to that of the *Elements*. The main difference with the previous practical geometry is a further loosening of the concepts in relation to its more practical aspects related to measurements.

In terms of the very simple hub-and-spoke model of geometrical concept that we are using in this work, the main difference in relation to the concept of geometric figure of surveyors' practical geometry is in the praxis and number spokes. In the praxis spoke the main features represented are related to the drawing of figures; however, there is still a representation not so much of particular measurement procedures as of the possibility of making measurements on the figure – there are 'traces' of the praxis of measuring (this is a feature that we already have in practical geometry in the context of problem solving). Regarding the number-magnitude spoke there is no encoding of measure-numbers or metrological units. Instead, we have a representation of length, which, taking into account the praxis spoke, is dissociated from any particular measurement procedure.[15] The visual spoke is

[14] As we will see in the next section, the key differences between the perfect figures of Hippocrates' geometry and the geometric objects of Euclid's geometry lie in the further abstraction that we have with geometric objects due to the adoption of definitions that go hand-in-hand with a particular way of 'seeing' the figures (which are not any more geometric figures a such but become representations of the geometric objects).

[15] Here, we hypothesize that the number spoke is also the spoke that represents continuous magnitudes, like in the case of the non-symbolic representation of both discrete and continuous magnitudes, which is made by a general magnitude system (see footnote 9). Accordingly, we change the naming of the spoke from 'number spoke' to 'number-magnitude spoke'. Notice that this does not challenge the view that for small (non-symbolic) numerical magnitudes there is a different system responsible for subitizing (the capacity to determine 'automatically'—without any counting procedure—the quantity of a small set of items up to four; for a review see, e.g., Hyde, 2011). In the context of practical geometry, numbers are the result of measurements that are made in a sort of counting-like way. Ancient surveyors used, e.g., rods and ropes to measure lengths; they did not 'subitize' the results of measurements in a fast and 'automatic' way.

basically the same; it encodes the visual shape of a geometric figure. The verbal spoke is also the same. It encodes the ‘label’ for the figure. A ‘higher order’ change can be taken to occur in the hub that would enable to encode a conceptualization of figure as a perfect figure (see figure 6).

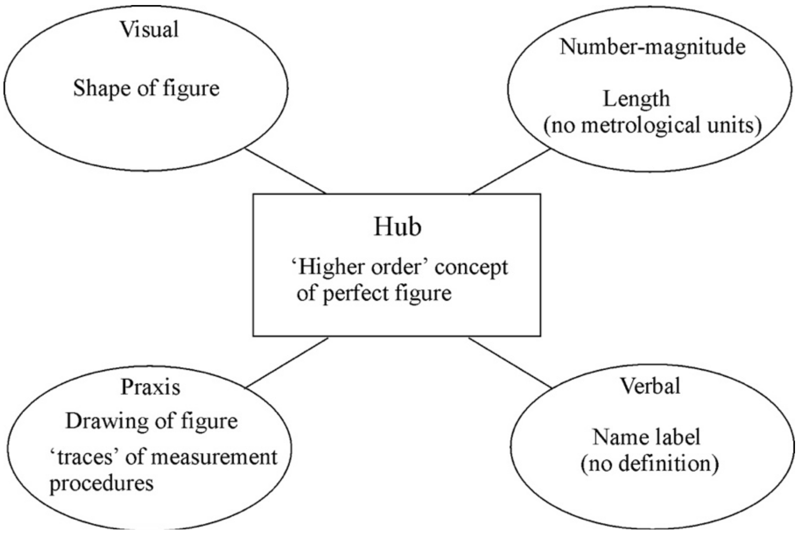

Figure 6. Hub-and-spoke model of the neural representation of geometric figure in Hippocrates’ pure geometry

So far, we have addressed Hippocrates’ pure geometry mainly in relation to the previous practical geometry. We developed hub-and-spoke models of the neural representation of geometric figures in both geometrical practices that are compatible with the main changes we have found to have occurred when going from one practice to the other.

We are now at a position where we can address the neural underpinning of the Euclidean geometrical practice. We will consider the changes that occur when going from Hippocrates’ pure geometry to Euclid’s. By taking into account the model of geometric concept in the earlier form of pure geometry and how it changed from the previous models from practical geometry, we will set forward a hub-and-spoke model of the neural representation of an abstract geometric object that makes more understandable how we go from perfect figures to abstract objects and how this can be encoded neurologically.

4. The pure geometry in Euclid’s Elements

We think that we can conceive the early stages of the geometry of the *Elements* in terms of lectures in which the teacher helps the student to memorize the enunciation of the proposition (the protasis). According to Saito the role of the protasis was to enable to memorize and refer to the propositions (Saito, 2018, pp. 928-9). It suffices to memorize the protasis and the corresponding diagram. Accordingly, “with the [unlettered] diagram in your memory, you can surely understand the protasis” (Saito, 2018, p. 929). With these two elements, a practitioner can recover the whole of the content related to a proposition as manifested in the *Elements*. In Saito’s view, “mathematical teaching was very probably performed directly and orally by drawing a diagram in front of pupils, and explaining it” (Saito, 2018, p. 924). We can conceive of a lecture as starting with the teacher reciting an enunciation (protasis) of a proposition. He would then proceed to the ‘doing’ or the ‘showing’ (Rodin, 2014, pp. 15-35); i.e., the teacher would construct a particular figure by starting to draw a diagram and pointing to the correctness of the successive steps until the final figure is constructed. Alternatively, he would show a particular result with the help of the diagram.

The diagram was probably drawn “on sand or a wax tablet” (Saito, 2018, p. 928), and, importantly, the teacher would indicate the “points and other geometrical objects by finger”. (Saito, 2018, p. 928).

In the context of an oral presentation, it is very unlikely that the teacher assigned letters to the diagram. As Netz shows, there is a close relationship between lettered diagrams and the type of written text adopted in the demonstrations of pure geometry (Netz, 1999). According to him, "the introduction of letters as tools is a reflective use of literacy". (Netz, 1999, p. 62). In this way, with the written form, "the lettered diagram is the tool which […] was made more central" (Netz, 1999, p. 66). By contrast, the oral teaching as rendered in Plato's *Meno* (Plato, 1997, pp. 881-5) reveals that "the diagram is not lettered, and the geometrical objects in it are referred to by the word 'this'" (Saito, 2018, p. 932). According to Saito, "the written text of the *Elements* with lettered diagrams shows certainly a more developed stage" (Saito, 2018, p. 932).

We might speculate that in a more organized lecture, the teacher would first present the necessary definitions, postulates, and common notions, or he would refer to them as needed.[16] This would be the counterpart of Hippocrates' practice, where we would start by mentioning the presuppositions that are used in the demonstration. We can see the Euclidean approach as resulting from the Hippocratic one when the presuppositions are "assumed as principles for which no justification is given" (Cellucci, 2013, p. 68).

Where do we find then a cognitively relevant difference between Hippocrates' pure geometry (PG_1) and Euclidean pure geometry (PG_2)?

That there is a crucial change from the Hippocratic practice can be seen in the fact that in the Euclidean practice we have explicit linguistic definitions at play. For example, a point is defined as "that which has no part" (Euclid, 1956, p. 153). Regarding lines, according to definitions 2 to 4, "A line is breadthless length. The extremities of a line are points. A straight line is a line which lies evenly with the points on itself" (Euclid, 1956, p. 153).

According to Harari, the definition of point makes reference to the idea of measurement by 'contraposition': "a point is characterized as a non-measurable entity, as it has no parts that can measure it" (Harari, 2003, p. 18). In the models of geometrical concept that we tentatively suggest in this work, when going from surveyors' practical geometry to PG_1 there is a loosening in the praxis spoke of aspects related to measurements. In terms of the notion of conceptual flexibility, the change from PG_1 to PG_2 might be accounted for by a reduced neural connectivity between the praxis spoke and the hub when compared to the neural connectivity between the verbal spoke and the hub. We would face an 'overwriting', so to speak, by the verbal spoke of any encoding related to a measurement praxis still existing in PG_2 (and 'inherited' from PG_1). This is achieved by extending the content of the verbal spoke that consist now of a definition and not just a label. The changes in the spokes go hand and hand with changes in the hub that gives rise to a coherent concept. That this can be so can be seen, e.g., in the definitions related to lines. In PG_1 we already have a perfect figure but one that has a breadth; in PG_2 we move beyond this and conceive of something that is breadthless. This linguistic term points to something that is not even visualizable—it is an abstract object. Another clear example is that of point as a geometric object. A point is defined as "that which has no part" (Euclid, 1956, p. 153). Again, something that is not even visualizable. In this way, this definition goes beyond the "non-measurability" mentioned by Harari. The point is also an entity that goes beyond a figure that, as such, we can visualize—as perfect as it might be. The verbal spoke helps to recreate the concept of perfect figure of PG_1 as the concept of abstract object of PG_2.

What does this imply regarding the diagrams that are drawn during the lectures? These are not conceived anymore as perfect figures. They are representations of geometrical objects as defined and instantiated following the postulates in the *Elements* (see, e.g., Valente, 2021). This has consequences concerning how we address the diagrams during the oral lectures (or in the written treatises).

[16] This is not such a bold suggestion as one might think. Several books of the Latin version of the *Elements* that was most influential in the Latin West during the 12th and 13th centuries have these characteristics. In book 1, in the beginning, we find the definitions, postulates, and common notions; then, we have the enunciation of proposition 1, an unlettered diagram, and a commentary giving indications about how to carry out the proof (Busard & Folkerts,1992, pp. 113-115). If we consider the books to be used by the student with the aid of a teacher this is all that is needed. Other examples are, e.g., a second-century CE fragment that contains proposition 9 of book 1 of the *Elements*. It consists only of the enunciation with an unlettered diagram (Brashear, 1994), and a fragment from the third century CE with identical characteristics (Cairncross & Henry, 2015, p. 24).

According to Ferreirós:

> The first definitions indeed suggest a *way of reading diagrams*, a perspective for seeing or conceiving what is implied by a diagram, and what is not. And this way of reading is not at all evident, especially if one previously knows only practical geometry. For the definitions and the reading that comes with them lead the practitioner to certain crucial idealizations. More importantly, the definitions suggest certain forms of response (and of indifference) to some aspects of the diagram: thus, the crossing of two drawn lines will be a (very small) planar region, but we are taught to disregard this and consider in the argumentation that one and only one point has thus been determined. (Ferreirós, 2016, p. 144)

We suggest that the conceptual change leading from PG_1 to PG_2 might have arisen in a practice based on oral lectures with PG_1 where the above-mentioned way of looking into and reading diagrams arose. That is, PG_2 does not lead to this particular way of attending to the diagrams; it would be the other way around. This way of attending to the diagrams would give rise to an early oral version of PG_2. This change would be made more explicit and stable by developing explicit linguistic definitions that help to stabilize the concepts, and further 'sedimented' by written treatises and a teaching practice that would rely more and more on these.[17, 18]

In terms of the simple model of geometrical concept that we are working with in the present paper, the verbal definition might be encoded in the verbal spoke that would be much more developed than in the cases of practical geometry and Hippocrates' pure geometry. In these cases, the content of the verbal spoke consisted only of the label used to name a geometric figure. Now we have a definition. The main difference would occur in how the hub re-represents the encoding in the spokes (see figure 7).

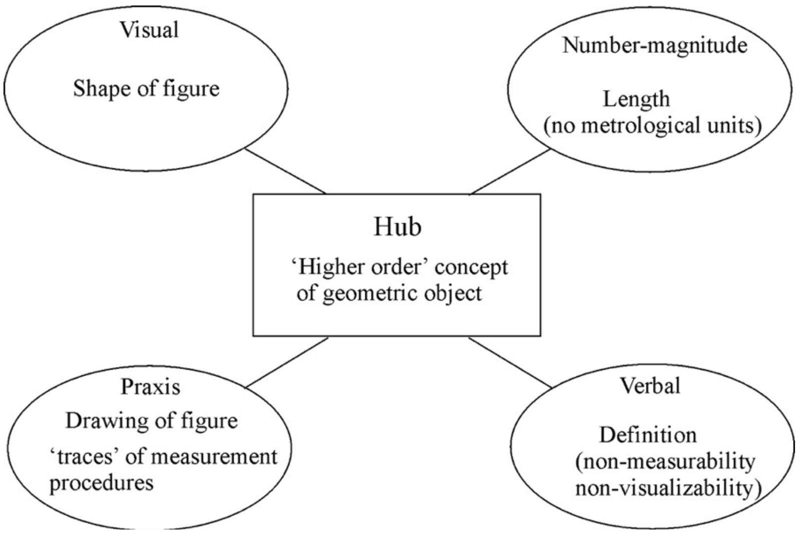

Figure 7. Hub-and-spoke model of the neural representation of geometric object in Euclidean pure geometry

The representations in the visual, praxis and number-magnitude spokes are represented in a highly abstract way by taking into account the encoding of the verbal spoke. When we look at a figure there is a particular indifference and responsiveness to its features related to the verbal definition such that we re-conceive the figure in terms of an abstract geometric object and not as a perfect figure anymore (as mentioned, the figure becomes for us a representation of the geometric object).

[17] An anonymous reviewer proposes a more involved situation: "couldn't it be the case that there was a kind of feedback-loop in which diagrams were seen in a different way which made more explicit linguistic definitions possible, which then influenced the way diagrams were seen, and so on". According to the reviewer "cognitive development rarely is as neat and unidirectional as suggested by [this paper's] explanation".

[18] Hippocrates himself is said to have written a treatise and there are references to treatises written previous to the *Elements* during the fourth-century BCE (Knorr, 1986, p. 102).

In this way, the verbal spoke becomes decisive in how the representations in the visual, praxis and number-magnitude spokes are interpreted and recombined in the hub giving rise to a 'higher order' representation of geometric abstract objects.[19]

In our view, the hub-and-spoke models of the neural representation of geometric figure/object make more intelligible what a geometric object might be since we can relate its neural representation to that of a geometric figure in Hippocrates' pure geometry and in practical geometry.[20] This makes it possible to address the issue of the relationship between geometric figures and objects considering these models. In our view, a key notion is that of conceptual flexibility that underlies the passing from one model to another (as the neurological counterpart of the gradual changes between the practices). The geometric object results from this gradual evolution from the surveyors' practical geometry passing by the practical geometry in the context of problem solving and the geometry of Hippocrates. Using our very simple models we can see the gradual change in the spokes (and the hub) leading from one conceptualization to the next. Some elements of the relevant changes occur already at the level of practical geometry, in the context of a practice of problem-solving, in the praxis spoke (there is no more actual measuring activity associated with the concept). Then, when moving to Hippocrates' geometry, we go a step further in the loosening of the grip of surveyors' practical geometry. Not only is there no codification/representation of actual measurements in the praxis spoke but also there is no representation of units of measure in the number-magnitude spoke. However, in these three stages of evolution, the visual and verbal spoke are unchanged. In Euclidean geometry, the key change can be found in the verbal spoke that now codifies a complex verbal definition that implies an important change in the way the hub integrates the different aspects codified in the spokes such that we have a coherent concept of geometric object. In this way, in the concept of geometric object, we find 'layers' of previous forms of geometry, reformulated, in particular, by the definitions (codified in the verbal spoke) and a new way of 'seeing' the corresponding figures (codified in the hub).

5. Conclusions

The main purpose of this work is to set forward a tentative model of the neural representation of abstract geometric objects. This model might be useful in relation to the development of a future

---

[19] It must be noticed that this is a tentative suggestion. In fact, the hub-and-spoke theory has not been developed by considering anything like the definitions in the *Elements*, only individual words (for concrete objects like 'apple' and 'rope', or abstract concepts like 'rule' and 'hope'. See, e.g., Hoffman et al, 2015; Kemmerer, 2015, pp. 343-4). A model in terms of cross-modal convergence zones might be more appropriate. In this kind of model instead of one main brain hub one considers a hierarchy of cross-modal convergence zones (Kuhnke et al, 2020). That is so because in a definition we engage with complex features of linguistic semantics that seem to be addressed in different brain regions than the anterior temporal lobes identified in the hub-and-spoke theory as the main hub. For example, there is evidence that what we call thematic role (e.g., agent and patient in an event verbally described as "the dog chased the cat") might be represented in two subregions of the left-mid superior temporal cortex (Frankland & Greene, 2020, pp. 289-90). The situation with Euclidean definitions is possibly much more complex at a neural level. Even if this is so, the lack of knowledge on the neural representation of geometric figures/objects and the lack of development of alternative models is such that the present highly simplified and idealized hub-and-spoke model of geometrical concepts seems to us as a useful way of framing the discussion at this point.

Independently of the issue of 'verbal spoke' being an 'acceptable' working model or not, a future development of the present models must address the issue of the role of language in geometrical concepts. Presently, there is evidence in relation to geometric problem solving. It has been found that the semantic system in the brain supports geometric problem solving, for example in problems where the practitioner must take into account rules that can be expressed linguistically, like "each interior angle for an equilateral triangle is 60º" (Zhou et al, 2018, p. 366). We take this to be indirect evidence for something like the 'amodal verbal spoke' we adopt here.

[20] Notice that the ontogeny of the geometric cognition does not have to correspond to the stages and kind of models presented here. We would, however, expect that there are some points in common in what regards an earlier neural representation of geometric figure and a later neural representation of geometric object. The reason is simple; when knowing basic school-level geometry we can understand and engage with ancient practical geometry, and the ancient geometries of Hippocrates and Euclid. Some commonalities must be at work, at the level of the neural representation of geometric concepts, for this to be possible.

theory of the geometric cognition underpinning pure geometry.

To develop the model consistent with the previous geometrical practices, we have considered a historically informed account of practical geometry. The objective was to provide a basic characterization of practical geometry. Considering these basic 'characteristics' we built two models of the neural representation of geometric figure in practical geometry using, in a very simple way, the hub-and-spoke theory of neural representation of concepts (one model corresponding to the surveyors' practice and another to the practice of problem solving).

We then address pure geometry. Before the geometrical practice related to Euclid's *Elements*, there was a development of a pure geometry that was an intermediary stage between practical geometry and the pure geometry of the *Elements*.

In this work, we present a basic characterization of this geometrical practice as revealed in Hippocrates' work on the quadrature of lunules. In our view, this pure geometry deals not with abstract objects, but still with geometric figures—perfect figures. We provide a model of the neural concept representation of perfect figures, again relying on the hub-and-spoke theory.

We then address what kind of neural representation of geometric concept we have in Euclidean pure geometry. For that, we reconstruct some aspects of the Euclidean practice taking into account how these are different from the corresponding aspects in the Hippocratic practice. Taking these differences into account together with the models of neural representation of concepts in practical geometry and Hippocrates' pure geometry we proposed a simple model for abstract geometric objects. Comparing this model with the ones related to geometric figures makes more intelligible what it is to have a concept of abstract object.

REFERENCES

Asper, M. (2003). The two cultures of mathematics in ancient Greece. In E. Robson, & J. Stedall (Eds.). *The Oxford handbook of the history of mathematics* (pp. 107-132). Oxford: Oxford University Press.

Baddeley, A., Eysenck, M. W., & Anderson, M. C. (2020). *Memory*. New York: Routledge

Brashear, W. (1994). Vier neue texte zum antiken bildungswesen. Archiv für Papyrusforschung, 40, 29-35.

Busard. H. L. L., & Folkerts, M. (1992). *Robert of Chester's(?) redaction of Euclid's Elements, the so-called Adelard II version*. Basel: Springer Basel.

Cairncross, A., & Henry, W. B. (2016). Euclid, *Elements* I.4 (diagram), 8-11, 14-25 (without proofs). *The Oxyrhyunchus Papyri, LXXXII (Graeco-Roman memoires, No. 103)*, 23-38.

Cellucci, C. (2013). *Rethinking logic: logic in relation to mathematics, evolution, and method*. Dordrecht: Springer.

Chiou, R., & Ralph, M. A. L. (2019). Unveiling the dynamic interplay between the hub- and spoke-components of the brain's semantic system and its impact on human behaviour. *NeuroImage*, 199, 114-126.

Cuomo, S. (2001). *Ancient mathematics*. London: Routledge.

Euclid (1956). *The thirteen Books of the Elements* (second edition, Vols. I-III). Translated with introduction and commentary by Sir Thomas L. Heath, from the critical edition of Heiberg. New York: Dover Publications.

Eysenck, M. W., & Keane, M. T. (2020). *Cognitive psychology: a student's handbook*. London: Psychology Press.

Dal Magro, T., & García-Pérez, M. J. (2019). On Euclidean diagrams and geometrical knowledge. *Theoria*, 34, 255-276.

Damerow, P. (2016). The impact of notation systems: from the practical knowledge of surveyors to Babylonian geometry. In M. Schemmel (Ed.). *Spatial thinking and external representation: towards a historical epistemology of space* (pp. 93-119). Berlin: Edition Open Access.

Dillon, M. R., Duyck, M., Dehaene, S., Amalric, M., & Izard, V. (2019). Geometric Categories in Cognition. *Journal of Experimental Psychology: Human Perception and Performance*, 45, 1236-1247.

Ferreirós, J. (2016). *Mathematical knowledge and the interplay of practices*. Princeton: Princeton University Press.

Ferreirós, J., & García-Pérez, M. J. (2020). Beyond natural geometry: on the nature of proto-geometry. *Philosophical Psychology*, 33, 181-205.

Frankland, S. M., & Greene, J. D. (2020). Concepts and compositionality: in search of the brain's language of thought. *Annual Review of Psychology*, 71, 273-303.

Friberg, J. (2005). *Unexpected links between Egyptian and Babylonian mathematics*. Singapore: World Scientific.

Friberg, J. (2007). *A remarkable collection of Babylonian mathematical texts: manuscripts in the Schøyen collection cuneiform texts I*. New York: Springer.

Giaquinto, M. (2007). *Visual thinking in mathematics*. New York: Oxford University Press.

Harari, O. (2003). The concept of Existence and the role of constructions in Euclid's *Elements*. *Archive for History of Exact Sciences*, 57, 1-23.

Hoffman, P., Binney, R. J., & Ralph, M. A. L. (2015). Differing contributions of inferior prefrontal and anterior temporal

cortex to concrete and abstract conceptual knowledge. *Cortex*, 63, 250-266.
Hohol, M. (2020). *Foundations of geometric cognition*. New York: Routledge.
Hohol, M., & Miłkowski, M. (2019). Cognitive artifacts for geometric reasoning. *Foundations of Science*, 24, 657-680.
Høyrup, J. (2002). *Lengths, widths, surfaces: a portrait of Old Babylonian algebra and its kin*. New York: Springer.
Høyrup, J. (2019). Hippocrates of Chios – his *Elements* and his lunes: A critique of circular reasoning. *AIMS mathematics*, 5, 158-184.
Hyde, D. C. (2011). Two systems of non-symbolic numerical cognition. *Frontiers in Human Neuroscience*, 5, 150.
Imhausen, A. (2016). *Mathematics in ancient Egypt: a contextual history*. Princeton: Princeton University Press.
Kautz, R. (2011). *Chaos: the science of predictable random motion*. Oxford: Oxford University Press.
Kemmerer, D. (2015). *Cognitive neuroscience of language*. New York: Psychology Press.
Knorr, W. R. (1986). *The ancient tradition of geometric problems*. Boston: Birkhäuser.
Kuhnke, P., Kiefer, M., & Hartwigsen, G. (2020). Task-dependent recruitment of modality-specific and multimodal regions during conceptual processing. *Cerebral Cortex*, 30, 3938-3959.
Kuhnke, P., Beaupain, M. C., Arola, J., Kiefer, M., & Hartwigsen, G. (2023). Meta-analytic evidence for a novel hierarchical model of conceptual processing. *Neuroscience & Biobehavioral Reviews*, 104994.
Lewis, M. J. T. (2004). *Surveying instruments of Greece and Rome*. Cambridge: Cambridge University Press.
Lupyan, G. (2017). The paradox of the universal triangle: Concepts, language, and prototypes. *The Quarterly Journal of Experimental Psychology*, 70, 389-412.
Mahon, B. Z., & Hickok, G. (2016). Arguments about the nature of concepts: symbols, embodiment, and beyond. *Psychonomic Bulletin & Review*, 23, 941-958.
Mandl, F. (1959). *Introduction to quantum field theory*. New York: Interscience Publishers.
Michel, C. (2021). Overcoming the modal/amodal dichotomy of concepts. *Phenomenology and the Cognitive Sciences* 20, 655-677.
Mori, L. (2007). Land and land use: the middle Euphrates valley. In G. Leick (Ed.). *The Babylonian world* (pp. 39-53). New York: Routledge.
Netz, R. (1999). *The shaping of deduction in Greek mathematics: a study in cognitive history*. Cambridge: Cambridge University Press.
Netz, R. (2004). Eudemus of Rhodes, Hippocrates of Chios and the earliest form of a Greek mathematical text. *Centaurus* 2004, 243-286.
Nguyen, J. (2020). It's not a game: Accurate representation with toy models. *The British Journal for the Philosophy of Science*, 71, 1013-1041.
Pantsar, M. (2022). On the development of geometric cognition: beyond nature vs. nurture. *Philosophical Psychology*, 35, 595-616.
Plato (1997). *Complete works*. Edited with introduction and notes by J. M. Cooper. Indianapolis: Hackett Publishing Company.
Ralph, M. A. L. (2014). Neurocognitive insights on conceptual knowledge and its breakdown. *Philosophical Transactions of the Royal Society B: Biological Sciences* 369, 20120392
Ralph, M. A. L., Jefferies, E., Patterson, K., & Rogers, T. T. (2017). The neural and computational bases of semantic cognition. *Nature Reviews Neuroscience*, 18, 42-55.
Reutlinger, A., Hangleiter, D., & Hartmann, S. (2017). Understanding (with) toy models. *The British Journal for the Philosophy of Science*, 69, 1069-1099.
Robson, E. (2004). Words and pictures: new light on Plimpton 322. In M. Anderson, V. Katz, & R. Wilson (Eds.). *Sherlock Holmes in Babylon and other tales of mathematical history* (pp. 14-26). Washington: Mathematical Association of America.
Robson, E. (2008). *Mathematics in ancient Iraq*. Princeton: Princeton University Press.
Rodin, A. (2014). *Axiomatic method and category theory*. Cham. Springer.
Rogers, T. T., & Ralph, M. A. L. (2022). Semantic tiles or hub-and-spokes? *Trends in Cognitive Sciences*, 26, 189-190.
Rossi, C. (2007). *Architecture and mathematics in ancient Egypt*. Cambridge: Cambridge University Press.
Sablé-Meyer, M., Fagot, J., Caparos, S., van Kerkoerle, T., Amalric, M., & Dehaene, S. (2021). Sensitivity to geometric shape regularity in humans and baboons: A putative signature of human singularity. *Proceedings of the National Academy of Sciences,* 118, e2023123118.
Saito, K. (2018). Diagrams and traces of oral teaching in Euclid's *Elements*: labels and references. Zentralblatt für Didaktik der Mathematik (ZDM), 50, 921-936.
Sokolowski, M., Hawes, Z., Peters, L., & Ansari, D. (2021). Symbols are special: an fMRI adaptation study of symbolic, nonsymbolic and non-numerical magnitude processing in the human brain. *Cerebral Cortex Communications*, 2, tgab048.
Spelke, E. (2011). Natural number and natural geometry. In S. Dehaene & E. Brannon (Eds.). *Space, time and number in the brain: searching for the foundations of mathematical thought* (pp. 287-317). Amsterdam: Elsevier.
Spelke, E., Lee, S. A., & Izard, V. (2010). Beyond core geometry: natural geometry. *Cognitive Science* 34, 863-884.
Spelke, E., & Lee, S. A. (2012). Core systems of geometry in animal minds. *Philosophical Transactions of the Royal Society B: Biological Sciences*, 367, 2784-2793.
Valente, M. B. (2020). From practical to pure geometry and back. *Revista Brasileira de História da Matemática*, 20, 13-33.

Valente, M. B. (2021). On the relationship between geometric objects and figures in Euclidean geometry. In A. Basu, G. Stapleton, S. Linker, C. Legg, E. Manalo, & P. Viana (Eds.). *Diagrammatic Representation and Inference* (pp. 70-78). Cham: Springer.

Yeomans, J. M. (1992). *Statistical mechanics of phase transitions*. Oxford: Oxford University Press.

Zhou, X., Li, M., Li, L., Zhang, Y., Cui, J., Liu, J., & Chen, C. (2018). The semantic system is involved in mathematical problem solving. *NeuroImage*, 166, 360-370.