Curry's paradox and ω-inconsistency

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ASTRACT: In recent years there has been a revitalised interest in non-classical solutions to the semantic paradoxes.[1] In this paper I show that a number of logics are susceptible to a strengthened version of Curry's paradox. This can be adapted to provide a proof theoretic analysis of the ω-inconsistency in Łukasiewicz's continuum valued logic, allowing us to better evaluate which logics are suitable for a naïve truth theory. On this basis I identify two natural subsystems of Łukasiewicz logic which individually, but not jointly, lack the problematic feature.

KEYWORDS: Contractionless logic, ω-inconsistency, Łukasiewicz logic, Curry's paradox, naïve truth theory.

1 Curry's paradox and Shaw-Kwei's paradox

I shall mainly be concentrating on theories whose logical vocabulary contains $\rightarrow$, $\bot$ and $\exists$. Ignoring the other connectives simplifies things in so far as the central paradoxes crucially involve only the conditional. Doing so also lessens the difference between paraconsistent and paracomplete approaches, which are primarily distinguished by the logic of conjunction, disjunction and negation.

When a theory contains a sufficient amount of arithmetic I shall assume a fixed Gödel numbering, $\ulcorner \cdot \urcorner$. In this setting I also assume familiarity with the dot notation, $\dot{\rightarrow}$, $\dot{\bot}$, etc, for generating Gödel numbers for complex formulae from Gödel numbers for their parts. By a "naïve truth theory" I shall mean any set of first order sentences in the language of arithmetic with a truth predicate which, in addition to being closed under modus ponens, has the following properties:

1. **Standard syntax**: it contains all the arithmetical consequences of classical Peano arithmetic.
2. **Intersubstitutivity**: it contains ϕ if and only if it contains $\phi[Tr(\ulcorner\psi\urcorner)/\psi]$ for any sentence ψ.
3. **Compositionality**: it contains $Tr(x) \rightarrow Tr(y)$ if and only if it contains $Tr(x \dot{\rightarrow} y)$.[2]

[1]See, for example, [2], [5], [1].

[2]Similar principles can be added if one wished to include further connectives. Note that it already follows from principle 2. that $Tr(\dot{\bot})$ is in the set iff $\bot$ is.

I shall henceforth use the notation $\vdash \phi$ to mean that the formula ϕ is in the set in question, and $\phi \vdash \psi$ to mean that ψ is in the set if ϕ is. I shall say that a set is 'consistent' if it does not contain every sentence.

I take it that these three constraints form the basis of a logic neutral naïve truth theory. Any adequate theory of truth must first contain an account of the *objects* of truth: sentences. The first condition ensures that one can formulate an adequate account of the syntax.[3] The second condition ensures that the truth predicate behaves disquotationally in all contexts. If one made the further logical assumption that every instance of $\phi \to \phi$ was in the theory then one would be able to derive from the second condition the more traditional disquotational principle, the T-schema. Finally, even with the T-schema, one cannot ensure the fully general compositionality rules hold. With the aforementioned logical assumption it is very natural to want more than just the compositionality rules. One might wish to have a compositionality *axiom*: $\forall xy((Tr(x) \to Tr(y)) \leftrightarrow (Tr(x \dot{\to} y))$. I take it that conditions (1)-(3) at least encode the basic core of the naïve conception of truth. I mention these extensions in passing only to note that they go beyond the naïve conception of truth to make assumptions about the conditional logic. This base theory is supposed to encode the most basic facts about *truth* without making assumptions about the underlying logic, allowing us to investigate that separately.

A central obstacle to the project of finding a suitable logic, particularly a suitable conditional logic, has been a number of variants of Curry's paradox. For example, one cannot add every instance of the contraction principle, W, to a naïve truth theory.

W $(\phi \to (\phi \to \psi)) \to (\phi \to \psi)$

It is worth rehearsing why this is. Suppose we have a sentence γ, the 'Curry sentence', satisfying (C): $\gamma \leftrightarrow (Tr(\ulcorner\gamma\urcorner) \to \phi)$.[4] Intuitively γ says of itself that it implies ϕ. We can then infer $\gamma \to (\gamma \to \phi)$ by substituting γ for $Tr(\ulcorner\gamma\urcorner)$ in the left-to-right direction of (C). By W and modus ponens we get (*) $\gamma \to \phi$, and thus $Tr(\ulcorner\gamma\urcorner) \to \phi$ and finally γ by the right-to-left direction of (C). From (*) and γ we have ϕ by modus ponens.

However, it is not just W that we must avoid. Shaw-Kwei [14] shows that a variant of Curry's paradox can trivialise a chain of weaker naïve truth theories. Let us use the notation $(\phi \to_{(0)} \psi)$ to mean ψ and $(\phi \to_{(n+1)} \psi)$ to mean $(\phi \to (\phi \to_{(n)} \psi))$. Then the following principles also lead to triviality

W^n $(\phi \to (\phi \to_{(n)} \psi)) \to (\phi \to_{(n)} \psi)$

Instead of choosing the ordinary Curry sentence we choose a sentence which says of itself that it implies$_{(n)}$ ϕ; to be more precise we choose a sentence γ_n, via the diagonal lemma, that satisfies $\gamma_n \leftrightarrow (Tr(\ulcorner\gamma_n\urcorner) \to_{(n)} \phi)$. By full

[3] It also seems like a natural constraint that the theory of truth should be consistent with well accepted mathematics, encoded by classical Peano arithmetic, if not all of true arithmetic.

[4] One actually needs slightly more than principles (1)-(3) to ensure there is a such a sentence, but it is evident that we should want to accommodate such a sentence.

intersubstitutivity we have (*) $\gamma_n \leftrightarrow (\gamma_n \rightarrow_{(n)} \phi)$, which by W^n reduces to $(\gamma_n \rightarrow_{(n)} \phi)$, and by (*) to γ_n. But from γ_n and $(\gamma_n \rightarrow_{(n)} \phi)$, we can deduce ϕ by n applications of modus ponens.

So, for example, a natural implicational logic without contraction is Łukasiewicz's 3-valued logic: Ł$_3$. However, although Ł$_3$ does not contain W, it does contain W^2. One might think that going to a higher finite valued logic might help, but in general the $n+1$-valued version of Łukasiewicz logic, Ł$_{n+1}$, validates W^n and is thus unsuitable for the same reason.[5]

On the other hand, in the *infinite* valued Łukasiewicz logic, Ł$_\infty$, every instance of W^n is invalid, and in fact Ł$_\infty$ can consistently support a naïve truth predicate [7]. However, Ł$_\infty$ is plagued with an apparently distinct problem - it is ω-inconsistent. This fact was first shown model theoretically by Restall in [9].

In this paper I shall demonstrate, proof theoretically, that Ł$_\infty$ is ω-inconsistent by a natural variant of Shaw-Kwei's paradox. I shall argue, however, that there is really only one principle in Ł$_\infty$ that is essential to the argument, so the argument is much more general. In §3 I discuss the prospects of two subsystems of Ł$_\infty$ that do not have this principle.

2 Number troubles

A classical extension of Peano Arithmetic is said to be ω-inconsistent iff

$\vdash \phi[\mathbf{n}/x]$ for each n, but $\vdash \exists x \neg\phi$

While an ω-inconsistent theory is not formally inconsistent, ω-inconsistency is generally considered to be an undesirable property.[6]

Once we have weakened the logic, previously equivalent ways of stating ω-inconsistency become distinct. To simplify matters I shall consider only two variants, which I shall call weak ω-inconsistency and strong ω-inconsistency respectively:

Weak ω-inconsistency: $\phi[\mathbf{n}/x] \vdash$ for each n, but $\vdash \exists x\phi$

Strong ω-inconsistency: $\vdash \phi[\mathbf{n}/x]$ for each n, but $\vdash \exists x(\phi \rightarrow \bot)$

Without the rule of reductio one cannot derive strong ω-inconsistency from weak ω-inconsistency.[7]

To run our argument we will only need to appeal to two principles about the logic. The principles in question are:

1. If $\phi \vdash \psi$ then $\exists x\phi \vdash \exists x\psi$

[5] See Restall [10] for some stronger results on the limitations of finite valued logics.

[6] It is also generally considered undesirable if the theory becomes inconsistent in ω-logic – in other words, if it cannot be consistently maintained in the presence of the infinitary ω-rule: $\{\phi[\mathbf{n}/x] \mid n \in \omega\} \vdash \forall x\phi$. Clearly ω-inconsistency entails inconsistency with the ω-rule, although the converse does not hold in general.

[7] Since I am mostly concerned with the conditional and quantifiers, and not negation, I have formulated strong ω-inconsistency in terms of the conditional and the falsum constant.

2. $(\phi \to \exists x\psi) \vdash \exists x(\phi \to \psi)$

Let me make a few remarks about these principles. Normally rule 1 would be a derived rule of a logic. It is useful in this context to take it as a primitive rule since I have made so few assumptions about the logic in question. In particular, quantificational axioms are usually formulated in terms of $\forall$, with the logic of $\exists$ following from the logic of negation. Taking this slightly less natural rule as primitive allows us to proceed without making these logical assumptions. Notice also that it is a *purely* quantificational rule – it does not say anything about the interaction between the quantifiers and connectives (such as $\to$.) I imagine that it would be very hard to formulate anything like a reasonable theory of existential quantification that didn't contain that rule. On the other hand, as we shall argue in the next section, rule 2 is actually quite distinctive to Łukasiewicz logic.

We can begin by showing that any naïve truth theory containing 1 and 2 must prove an infinitary version of the Shaw-Kwei sentence, which intuitively says of itself that, for some n, it implies$_{(n)}$ a contradiction. We can make that rigorous using the recursion theorem, which allows us to define arithmetically a function f such that $f(0,x) = x\dot{\to}\dot{\bot}$ and $f(n+1,x) = x\dot{\to}f(n,x)$.

Then using the diagonal lemma we can construct a sentence γ satisfying $\gamma \leftrightarrow \exists n Tr(f(n, \ulcorner\gamma\urcorner))$.

Theorem 2.1. *Any naïve truth theory closed under 1 and 2 can prove γ.*

Proof.

1. $Tr(\ulcorner\gamma\urcorner) \to \exists n Tr(f(n, \ulcorner\gamma\urcorner))$ by the diagonal formula and full intersubstitutivity.
2. $\exists n(Tr(\ulcorner\gamma\urcorner) \to Tr(f(n, \ulcorner\gamma\urcorner)))$ by rule 2.
3. $Tr(\ulcorner\gamma\urcorner) \to Tr(f(n, \ulcorner\gamma\urcorner)) \vdash Tr(\ulcorner\gamma\urcorner\dot{\to}f(n, \ulcorner\gamma\urcorner))$ by the naïve truth theory.
4. $\exists n(Tr(\ulcorner\gamma\urcorner\dot{\to}f(n, \ulcorner\gamma\urcorner))$ by rule 1.
5. $Tr(\ulcorner\gamma\urcorner\dot{\to}f(n, \ulcorner\gamma\urcorner)) \vdash Tr(f(n+1, \ulcorner\gamma\urcorner))$ by arithmetic.
6. $\exists n Tr(f(n+1, \ulcorner\gamma\urcorner))$ from 4 and 5 by rule 1.
7. $\exists n Tr(f(n, \ulcorner\gamma\urcorner)$ by arithmetic.
8. γ by the diagonal formula.

□

Once a theory contains γ it is already on the verge of trouble; for example it will already contain a weak ω-inconsistency.

Corollary 2.2. *Any naïve truth theory closed under 1 and 2 is weakly ω-inconsistent.*

Proof. On the one hand we have $\vdash \exists n Tr(f(n, \ulcorner\gamma\urcorner))$ by theorem 2.1.

On the other hand, observe that $Tr(f(\mathbf{n}, \ulcorner\gamma\urcorner)) \vdash \gamma \rightarrow_{(n)} \perp$ by arithmetic and full intersubstitutivity. Since we have $\vdash \gamma$ by theorem 2.1, $\gamma \rightarrow_{(n)} \perp \vdash \perp$ by n applications of modus ponens. So we have in general $Tr(f(\mathbf{n}, \ulcorner\gamma\urcorner)) \vdash$ for any n, and $\vdash \exists n Tr(f(n, \ulcorner\gamma\urcorner))$ □

From here we can obtain strong ω-inconsistency in a number of ways. Here is one. If the theory contains a 'fusion connective' (see, e.g., [11]) $\phi \circ \psi$ such that (i) $\phi, \psi \vdash \phi \circ \psi$ and (ii) $(\phi \circ \psi \rightarrow \chi) \dashv\vdash (\phi \rightarrow (\psi \rightarrow \chi))$ then we can generate a strong ω-inconsistency

Theorem 2.3. *Any naïve truth theory with rules 1 and 2 is strongly ω-inconsistent relative to the definability of a fusion connective.*

Proof. Let ϕ^n denote $(\ldots((\phi \circ \phi) \circ \phi) \ldots \circ \phi)$ (this choice of bracketing is important since we have not assumed associativity of $\circ$.) It is easy to see that $(\phi \rightarrow_{(n)} \psi)$ is equivalent to $(\phi^n \rightarrow \psi)$. It is possible to then re-run the proof of theorem 1.1 using the sentence $\gamma' \leftrightarrow \exists n Tr(g(n, \ulcorner\gamma'\urcorner) \dot{\rightarrow} \dot{\perp})$, where $g(0, x) = x$ and $g(n+1, x) = x \dot{\circ} g(n, x)$.

We have $\vdash \exists n Tr(g(n, \ulcorner\gamma'\urcorner) \dot{\rightarrow} \dot{\perp})$, and thus $\vdash \exists n (Tr(g(n, \ulcorner\gamma'\urcorner)) \rightarrow \perp)$ by naïve truth theory. Since we have $\vdash \gamma'$, we have $\vdash \gamma'^n$ for any given n, and thus $\vdash Tr(g(\mathbf{n}, \ulcorner\gamma'\urcorner))$ for each n by the arithmetical properties of g and full intersubstitutivity. □

From this we can easily give a proof theoretic version of Restall's theorem:

Corollary 2.4. (Restall.) *Infinitely valued Łukasiewicz logic, $Ł_\infty$, is strongly ω-inconistent.*

Proof. One can define a fusion connective which satisfies both (i) and (ii) as follows: $\phi \circ \psi := (\phi \rightarrow (\psi \rightarrow \perp)) \rightarrow \perp$ (see [9] §2.) □

3 The prospects for a naïve truth theory

The most significant assumption made in the proof of γ was the rule $(\phi \rightarrow \exists x \psi) \vdash \exists x(\phi \rightarrow \psi)$. It is natural then to look for logics without that rule. There are well known logics which do not contain the rule, such as intuitionistic logic and relevant logic. However these are not suitable for a naïve truth theory since they both contain the contraction axiom, W. Other logics not containing the rule are known to support a naïve truth predicate, such as Field's logic in [5] and Brady's CTQ [3][8], as well as the proposals based on relevant logic in [2], [8] and [1]. However these are all based on conditionals which are are substantially weaker than the conditional in $Ł_\infty$. I shall therefore restrict my attention to

[8]It should be noted that although [3] is concerned mainly with naïve set theory, the results extend to naïve truth theory as well.

two strong subsystems of $Ł_\infty$, which I call BCKN and BCKD.[9] Although the combination of the two logics has the problematic rule, I shall show that the rule does not belong to either system individually.

When people are talking about the Łukasiewicz predicate logic, they do not usually mean some particular axiomatic system but the logic validated by the Łukasiewicz connectives over $[0, 1]$. There is no recursive axiom system that characterises this consequence relation, so there is no question of simply "dropping" the rule from Łukasiewicz logic understood this way. However, there is a natural axiomatic system in which these questions can be sensibly addressed. It is given by adding to the propositional Łukasiewicz logic the following three principles:

- $\forall x\phi \to \phi[t/x]$ with t substitutable for x.
- $\forall x(\phi \to \psi) \to (\phi \to \forall x\psi)$ when x is not free in ϕ.
- If $\vdash \phi$, then $\vdash \forall x\phi$.

In the rest of this section I shall consider what happens when you add these principles to various other propositional logics. The first thing to note is that adding these principles to propositional Łukasiewicz logic renders provable not only the problematic rule, but the axiom $(\phi \to \exists x\psi) \to \exists x(\phi \to \psi)$ as well. For the argument I refer the reader to Petr Hájek's book, [6], lemma 5.4.15 and remark 5.4.2.

The argument for this makes essential use of Dummett's axiom

D $(\phi \to \psi) \vee (\psi \to \phi)$

Thinking about things algebraically, Dummett's axiom corresponds in a natural way to the linearity of the underlying algebra. One strategy to pursue might be to drop Dummett's axiom. A sublogic of $Ł_\infty$ that does not prove Dummett's axiom is BCK logic with 'double negation elimination' for the negation-like connective $p \to \bot$:

B $(\phi \to \psi) \to ((\chi \to \phi) \to (\chi \to \psi))$

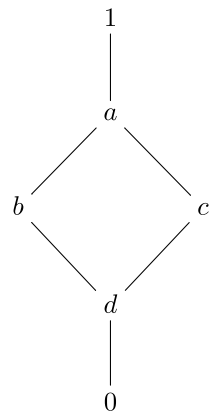
C $(\phi \to (\psi \to \chi)) \to (\psi \to (\phi \to \chi))$

K $\phi \to (\psi \to \phi)$

N $((\phi \to \bot) \to \bot) \to \phi$

[9] Both systems are strong in the sense that they properly contain the $\{\to, \bot, \forall\}$ fragments of the logics in [5] Ch. 17.4, and [3]. Most of this discussion also extends to relevant approaches to the paradoxes ([2], [8], [1]) since the ω-consistency of BCKN and BCKD would imply the ω-consistency of their strong relevant cousins BCIN and BCID, where I is the principle: $\phi \to \phi$. A full comparison, however, would require taking disjunction, conjunction and negation into account.

We can exploit the absence of linearity to generate failures of the problematic quantifier rule, $(\phi \to \exists x\psi) \vdash \exists x(\phi \to \psi)$. Consider the following non-linear model.[10]



$\to$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	d	d
b	1	1	1	c	c	c
c	1	1	b	1	b	b
d	1	1	1	1	1	a
0	1	1	1	1	1	1

It is straightforward, but tedious, to check that this validates BCKN and the quantifier axioms (interpreting $\forall$, $\exists$ and $\bot$ using the lattice ordering in the obvious way.) However, due to the non-linearity of the truth value space, it does not validate Dummett's axiom or the rule. To demonstrate the latter fact, consider a model over this truth value space with domain $\{0, 1\}$, $|p| = a$, and such that whenever $v(x) = 0$ and $u(x) = 1$, $|Fx|_v = b$ and $|Fx|_u = c$. It follows that $|(p \to \exists xFx)| = 1$ and $|\exists x(p \to Fx)| = a$. Thus the rule $(\phi \to \exists x\psi) \vdash \exists x(\phi \to \psi)$ cannot be proven from BCKN alone.

Another prospective logic can be obtained by substituting N in the above with Dummett's axiom, D. Call this BCKD. Neither the axiom $(\phi \to \exists x\psi) \to \exists x(\phi \to \psi)$ nor the corresponding rule is a theorem of BCKD. To refute $(\phi \to \exists x\psi) \vdash \exists x(\phi \to \psi)$ we can no longer exploit non-linearity, but we can instead generate failures using an infinite space of truth values. As in Ł$_\infty$, let the set of truth values be $[0, 1]$, fix the domain to $\mathbb{N}$, and let $|\phi \to \psi|_v = 1$ if $|\phi|_v \leq |\psi|_v$ and $|\psi|_v$ otherwise, let $|\exists x\phi|_v = sup\{|\phi|_u \mid u[x]v\}$ and let $|\phi \vee \psi|_v = max(|\phi|_v, |\psi|_v)$. You can check this satisfies BCKD, and the three quantifier rules above, however if for each n, with $v(x) = n$, $|Fx|_v = \frac{1}{2} - \frac{1}{n+2}$, and $|p| = \frac{1}{2}$, it follows that $|p \to \exists xFx| = 1$ since $|p| = \frac{1}{2} = |\exists xFx|$, yet for n with $v(x) = n$, $|Fx|_v < |p|$ so $|p \to Fx|_v = \frac{1}{2} - \frac{1}{n+2}$, and so $|\exists x(p \to Fx)| = \frac{1}{2}$. So as before we may conclude

[10] FOR TYPE SETTER: it would be great if these could be put side by side.

that the rule $(\phi \to \exists x\psi) \vdash \exists x(\phi \to \psi)$ cannot be proven from BCKD alone.[11]

I have, of course, only demonstrated that these logics do not prove the problematic rule, not that they can support a naïve truth predicate with a standard model of arithmetic. What the absence of this rule from BCKN and BCKD demonstrates is that what we know we can have from a conditional – as seen in the theories of [2], [5], [1] for example – is substantially weaker than what we don't know we can't have. There is thus still some interesting work to be done in determining just how strong a conditional one can combine with a naïve theory of truth.[12]

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[11] It is worth noting that BCKD is closely related to the implicational fragment of basic logic, BL. Given the rule $\phi \to \chi, \psi \to \chi \vdash (\phi \vee \psi) \to \chi$, which the model above also validates, you can prove in BCKD, F: $((\phi \to \psi) \to \chi) \to (((\psi \to \phi) \to \chi) \to \chi)$, which constitutes the basis of the implicational fragment of BL - see [4] for further details.

[12] I would like to thank an anonymous referee for some useful comments on this paper.

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