Mathematical Modality
An investigation in higher-order logic

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Modal logic has played a large role in contemporary philosophy of mathematics. Many ideas about mathematical objects, especially sets, have been fruitfully studied using distinctive mathematical modalities according to which some mathematical truths, and perhaps also mathematical objects, are contingent.[1]

From the perspective of more traditional modal metaphysics, the postulation of these mathematical modalities carries some substantive commitments. They conflict with two commonly held theses, found in Kripke (1980): (i) that the truths of mathematics are metaphysically necessary (and that there is no metaphysical contingency about what mathematical objects there are), and (ii) that metaphysical necessity is the broadest kind of necessity there is. Given the first it follows that these mathematical modalities are neither identical to, nor restrictions of the more familiar notion of metaphysical necessity as it is normally understood: modal reality is richer than we thought.

This invites a number of further questions about the structure of modal reality, now understood not as questions about the features of metaphysical modality specifically, but about modalities in general and their relationships. The goal of this paper is to explore some of these questions in the context of a specific higher-order theory of modalities. In this framework, a notion of broad necessity emerges which has all modalities—mathematical, metaphysical or otherwise—as restrictions.

My focus will be on the implications of views that posit mathematical contingency or indeterminacy about statements that concern the 'width' of the set theoretic universe—a prime example being Cantor's continuum hypothesis. I will argue that this requires rejecting two further orthodoxies concerning the structure of modal reality.

The first departure from orthodoxy is that we must embrace a radical rejection of Brouwer's principle—we must acknowledge the possibility that there are truths that

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[1]These include, but are not limited to: Shapiro (1985), Hellman (1989), Parsons (1983), Fine (2006), Linnebo (2013), Studd (2013), Hamkins and Linnebo (2022), Scambler (2021), Builes and Wilson (2022), Brauer (2020). In the set-theoretic case, examples of contingent statements put forward in the literature can be roughly divided into those about height, such as large cardinal hypotheses, and those about width, such as the continuum hypothesis. This trend is certainly not limited to contemporary philosophy of mathematics: there are, for instance, many connections between intuitionistic mathematics and modal logic.

are possibly impossible (section 5). The rejection is radical because we must accept the existence of such truths when ‘possible’ and ‘impossible’ are understood in their *broadest* senses. (It is widely thought that Brouwer’s principle fails for the **mathematical** modalities. One consequence of the more radical rejection, by contrast, is that it rules out backwards looking modalities like those posited by Studd (2013).)

The second departure from orthodoxy (section 6) is that we must reject the so-called Leibniz biconditionals, stating that what is possible, in the broadest sense of possible, is what is true in some broadly possible world; an assumption that has almost been taken for granted in modal metaphysics, and has been very influential in epistemology, philosophical logic, natural language semantics, and many other disciplines.

Common to both of these arguments is the idea that, if there is width contingency, information encoding the membership conditions for merely possible subsets of an actual set cannot exist in actuality or else the sets would actually exist after all (by the separation axiom). Brouwer’s principle ensures that there are actual individuals (things that might have been sets) corresponding to every merely possible set, and these individuals contain information about their possible membership conditions. The Leibniz biconditionals give us the ability to single out, using maximally specific properties, merely possible sets, letting us do the same. Section 7 argues that these conclusions hold even when we weaken the quantificational logic to accommodate contingent existence. It is common in the literature on quantified modal logic to appeal to a distinction between the ordinary “inner” quantifiers, and the “outer” quantifiers which are governed by a classical quantificational logic. Thus even if the individuals and properties encoding this information about possible membership conditions don’t properly exist, but exist only in the outer sense, the same sort of reasoning can still be applied. The view thus must accept a radical kind of contingency about what there is: while the standard view is that there is only contingency about what there is according to the inner quantifiers—the outer quantifiers have a “constant domain”—the view under consideration must allow even the classical outer quantifiers to have an “expanding domain” interpretation.

In the final section of the paper I turn to the question of whether mathematical contingency is viable, despite these negative results, and if so how pervasive it is. I conjecture that against a minimal background logic of mathematical modality it is consistent that there is a wide range of width contingency, and suggest that the resulting picture still has attractions. On the other hand, some authors have suggested that mathematical indeterminacy is so pervasive it could even arise in arithmetical contexts.[2] I end by pointing out that, against the same minimal background logic for the mathematical modalities, arithmetic is determinate and non-contingent (cf. Goodsell (2022)), raising the prospect of a more general project to figure out what makes a mathematical statement capable of being indeterminate or contingent.

[2]See Warren and Waxman (2020), and Field (1998).

1 Set-Theoretic Contingency: Height and Width

Let's begin by delineating some different motivations for positing mathematical contingency. We will look at three different motivations for positing contingency about the sets in the literature, and distinguish two distinctive sorts of contingency which I'll gloss as *height* and *width* contingency.

Motivations for positing height contingency can be traced back to Cantor himself. Cantor's view was that the transfinite ordinals—mathematical objects representing the order-types of well-orders—continued indefinitely through the operations of taking successors and limits. Some have taken Cantor's remarks to suggest a kind of mathematical contingency. Not any collection of sets form a set, on pain of Russell's paradox. But they nonetheless *could* have formed a set—a stage V_κ of a possible larger set-theoretic universe. Charles Parsons (Parsons (1983)), and several subsequent authors, have been more explicit about the modal in this formulation.[3] For now we'll give this idea the following gloss:[4]

Height Extensibility Necessarily, the sets (whatever they may be) are possibly a proper initial segment of all the sets.

Here the sense of possibility in play is, presumably, not metaphysical possibility but a primitive kind of mathematical possibility in need of further explication (I will offer some tentative suggestions in section 4).

More recently there has been significant interest in a different kind of indefinite extensibility inspired by Paul Cohen's method of forcing. Joel Hamkins, in a number of papers, has suggested that, even when we restrict ourselves to a particular infinite stage V_α—'the sets of rank α'—one can always consider a larger set theoretic universe that contains more sets of that rank.[5] For instance, the method of forcing lets one describe, within any given set-theoretic universe, a larger one that contains more sets of natural numbers.[6] We have an explicitly modal articulation of related ideas in Scambler (2021), Pruss (2020), Builes and Wilson (2022).[7] Let's give this idea the following gloss:

Width Extensibility Necessarily, the sets of rank α (whatever they may be) are possibly properly contained in the sets of rank α.

[3] See Fine (2006), Linnebo (2013), Studd (2013). I primarily draw from the latter two papers.

[4] Actually Zermelo's idea seems to be importantly different from that of Parsons', formalized below. Zermelo above is concerned with the structure of ZF-relations generally, without selecting any particular one for attention, so his form of indefinite extensibility is one formulable in the language of pure higher-order logic alone (see Bacon (2023b)). By contrast set-theory is often taken to be the study of one particular ZF-relation, membership, and Parsons, Linnebo and Studd each formulate their versions of indefinite extensibility in terms of it.

[5] See, for instance, Hamkins (2012).

[6] One can even describe what these new sets of natural numbers will have to look like, although they will be in some sense ω-inconsistent from the perspective of the present universe.

[7] Hamkin's also uses modal logic in his work to spell out the multiverse view–Hamkins (2003), Hamkins et al. (2015)–but it seems clear that his uses of the modal operator $\Box A$ are really abbreviations for something quantificational: $\Box A$ means A holds in all forcing extensions of the universe, where this is a statement that can be articulated in the extensional language of first-order set theory.

Again, the notion of possibility here is a primitive mathematical one, which may be identical to or orthogonal to the one appealed to above. These latter authors are typically interested in Width Extensibility because they want to make sense of the idea that all sets are countable in a strictly modal sense:

Countabilism Every set is 'countable' in the sense that for any set x, it is possible that there is an injection from the natural numbers to x.

For some motivations for Countabilism see Meadows (2015) and Builes and Wilson (2022).[8] Countabilism also follows from the following schema, where A can be any first-order formula of set-theory and $p \Vdash A(\hat{x}_1, \ldots, \hat{x}_n)$ is a statement in the language of first-order set-theory that expresses in the object language the claim that $A(x_1, ..., x_n)$ is true in every forcing extension by a generic filter containing p:[9]

Forcing Possibilism If there is a partial order $\mathbb{P}$ and $p \in \mathbb{P}$ such that $p \Vdash A(\hat{x}_1, \ldots, \hat{x}_n)$, then it's possible that $A(x_1, \ldots, x_n)$.

Forcing Possibilism legitimizes a certain practice that seems commonplace among set-theorists. The set-theorist I have in mind sets out theorizing in the language of set theory. They may then consider various partial orders $\mathbb{P}$ belonging to the cumulative hierarchy, and its associated collection of dense subsets D, and ask seemingly modal questions of the form 'what *would* the set-theoretic universe have looked like if there had been a filter that had intersected every element of D?'. For instance, $\mathbb{P}$ might consist of finite bits of information about a potential function from ω to $\{0, 1\}$ ordered by informativeness, and the postulated filter then consists of a collection of these bits of information that approximate a total function $f : \omega \to \{0, 1\}$ which differs from every actually existing function over some finite bit of information (since, for any actual function, the set of finite partial functions not contained in that function is dense).[10] Consequently, by positing the possibility of such a filter we describe a possible set-theoretic universe containing a new function from ω to $\{0, 1\}$. One might attempt to make sense of this practice by interpreting the set-theorist's quantifiers as initially ranging over a restricted portion of the 'real' sets, and the possibility containing the new filter as simply arising from enlarging the range of those quantifiers. But this approach assumes there is a background universe of 'real' sets—consisting of *all* the sets there are—yet the procedure for describing new sets can be applied just as readily to this background universe of all sets as it can to any

[8] Builes and Wilson (2022) argue that while height extensibilism can be motivated by a certain sort of attitude to Russell's paradox—the non-self-membered sets do not in fact form a set, they could have done—Countabilism follows from taking a parallel attitude toward Cantor's theorem.

[9] Countabilism follows since, for any set, x, the partial order of finite partial functions from ω to x forces the claim that $\hat{x}$ is countable, so Forcing Possibilism implies that x is possibly countable. This principle also implies the principle **HE** from Scambler (2021). A proof that HE and Countabilism are equivalent correspond to theorems 3 and 4 of Scambler (2021) (see p.1092). One difference is that Scambler's framework uses plural quantification.

[10] More precisely, $\mathbb{P}$ consists of the finite partial functions $\omega \rightharpoonup \{0, 1\}$ ordered by inculsion. Every actually existing function $f : \omega \to \{0, 1\}$ determines a filter of finite partial functions F (its finite subsets), but will also be disjoint from one of the actual dense subsets of $\mathbb{P}$, namely $\mathbb{P} \setminus F$. Thus if there had been a filter of partial functions that intersected every element of D, it's union would have to be a totally defined function on ω that doesn't actually exist.

of its restrictions. Granted, it is possible still to reduce this seemingly modal talk to extensional quantification over possibilities, in the style of Lewis (1986), even when it is applied to the entire universe of sets.[11] Each element of $\mathbb{P}$ may be thought of as a 'possibility'—in our running example, the possibility specifies the behaviour of a new function on ω on finitely many of its arguments. In the language of set theory, one can define a relation of a sentence being 'true at' a possibility, from which we may paraphrase any claims of possibility and necessity extensionally, in terms of existential or universal quantification over possibilities. Nonetheless, I find the idea that the set-theorist is describing *genuine contingency* about the set-theoretic universe to be deeply attractive.

Apart from Countabilism and Forcing Possibilism, contingency about the width of the set-theoretic hierarchy can also be motivated by considerations of set-theoretic indeterminacy. Cantor's continuum hypothesis, which we will abbreviate CH, is the claim that every infinite set of real numbers (identified with a certain set in our iterative hierarchy) can either be put in one-to-one correspondence with the set of all real numbers or can be put in one-to-one correspondence with the natural numbers. This claim is, surprisingly, left unsettled by presently accepted mathematics: no currently accepted axiomatic theory (whether first-order or higher-order) implies CH or implies its negation.[12] Perhaps this is a symptom of a deeper kind of indeterminacy about the truth value of this statement. According to this picture, our state of ignorance about the continuum hypothesis is akin to our ignorance about whether a borderline heap is a heap or not: there is simply no fact of the matter, and so additional investigation would yield no headway. Indeterminacy seems to be a kind of contingency, and in order for the continuum hypothesis to be contingent in this sense, it must be contingent what real numbers there are.

2 Why do we need modalities?

What does the modal way of formulating these questions afford us? There is a way of thinking about mathematical contingency that isn't genuinely modal. Consider, by analogy, the way that modality is treated by David Lewis: he uses modal operators, but he ultimately paraphrases those operators away in extensional terms, using first-order quantification over concrete possible worlds.[13]

The use of modal operators in Hamkins (2012), like in Lewis, is similarly superficial, and is ultimately spelled out in terms first-order quantification over universes—indeed the modal operators in that setting can be eliminated entirely in terms of the first-order set-theoretic primitives using forcing theoretic ideas. Similarly, a common story about indeterminacy, *supervaluationism*, might make room for some sort of

[11]Lewis bases his reduction on a Kripke semantics, which trades on complete possibilities. While these possibilities are partial, there is a parallel semantic treatment of modality in terms of quantification over partial possibilities (see Humberstone (1981)), and so a parallel reduction of modal talk to extensional quantification is possible in this context as well.

[12]It is sometimes suggested that second-order set theory settles the continuum hypothesis (see Kreisel (1967) p150), but these authors have a semantically defined theory in mind when they talk about by 'second-order set theory'.

[13]Lewis (1968), and for further relevant discussion, Williamson (2013) section 7.4.

set-theoretic indeterminacy without positing any genuine set-theoretic contingency. Indeterminacy, for the supervaluationist, is more perspicuously expressed by a metalinguistic predicate than by a propositional operator. A *sentence* is indeterminate when there are several candidate, or "admissible", interpretations of the vocabulary appearing in the sentence, some of which make the sentence true and others which make the sentence false.

In this section I'll argue that these imitative uses of mathematical contingency cannot properly capture indeterminacy or contingency about the width of the set-theoretic universe. This inability follows, essentially, from the existence of categoricity theorems that can be formulated and derived in a minimal (axiomatic) higher-order logic. By contrast, I will suggest that genuinely modal formulations of contingency and indeterminacy are not subject to these results.

Higher-order logic is a very natural framework for investigating these questions. First, following (Williamson (2013) p.422, Williamson (2003b) §4), we can express the existence of genuine (as opposed to metalinguistic) contingency and indeterminacy using a higher-order generalization into sentence position: $\exists_t p(\Diamond p \wedge \Diamond\neg p)$ or $\exists_t p(\neg\Delta p \wedge \neg\Delta\neg p)$. The truth of these sorts of existential generalizations do not get preserved under attempts to paraphrase away modality, and may be thought to capture the difference between genuine and ersatz contingency.[14] Second, it seems to be the appropriate framework for articulating the extensional paraphrases of mathematical contingency claims. There are well-known difficulties with simply identifying the supervaluationists' admissible interpretations of '$\in$', or Hamkins' universes, with first-order individuals, such as set-theoretic models.[15] By contrast, if we can quantify directly into the position of a binary predicate, such as $\in$, things run much more smoothly. This is because a higher-order generalization does not need to be understood as a notation for quantifying over another sort of individual, such as sets, classes or properties—this would gain us nothing. We can rather think of them as devices for making generalizations into grammatical positions other than that of a singular term. The move from 'John talks' to '$\exists X$ John Xs' has the same status as the move from 'John talks' to '$\exists x\ x$ talks' — it is immediate and logical, and in neither case is it dependent on the existence of abstract objects, like sets, classes, or properties. For if 'John talks' does not logically imply the existence of abstract objects, nothing that 'John talks' logically implies can either. Similarly, we do not need to rely on set-theory to specify the intended interpretation of the higher-order quantifiers (a set-theoretic interpretation of $\exists R$ would not license the move from $A(\in)$ to $\exists R.A(R)$), or even to characterize its logic.[16] Our approach will be axiomatic—much like axiomatic set theory does not require a set-theoretic semantics to proceed, neither does higher-order logic. Indeed, an argument due to Harris (1982) suggests we do not need to specify the meanings of the higher-order quantifiers in independent

[14]For a discussion of one sort of problem that arises paraphrasing the propositional quantifiers in David Lewis's extensional framework, see Dorr (2005).

[15]See section 1 of Williamson (2003a).

[16]It is common in mathematical logic to identify higher-order logic with the set of sentences true in full extensional Henkin models. Apart from eluding axiomatization, this logic has the drawback of validating the Fregean principle of extensionality in which propositions, properties and relations are individuated by coextensiveness, ruling out any kind of genuine contingency in the logic.

terms: the axioms and rules governing the higher-order quantifiers pin them down uniquely, in the sense that any other generalizing device satisfying them are logically equivalent when they appear in any sentence.

Returning to the issue at hand, we can use higher-order generalizations to provide a supervaluationist account of the indeterminacy of the continuum hypothesis:

Supervaluationism The symbol '$\in$', as used by mathematicians, has multiple admissible interpretations. Under some such interpretations 'CH' is true, and under others it is false.

Here 'admissibility' should be a higher-order predicate that combines with a binary predicate to form a sentence, and the quantification in question is higher-order. Using e, t and $\sigma \to \tau$ respectively to indicate expressions with the type of a name, a sentence, and of an expression that combines with an expression of type σ to form an expression of type τ, we can represent the admissibility predicate as an expression, Adm of type $(e \to e \to t) \to t$. We can then formulate the supervaluationist claim flatfootedly as:

1. $\exists_{e \to e \to t} RS(\mathrm{Adm}\, R \wedge \mathrm{Adm}\, S \wedge \mathrm{CH}\, R \wedge \neg\, \mathrm{CH}\, S)$

where $\mathrm{CH}\, R$ is the result of replacing $\in$, in the statement of the continuum hypothesis in first-order set theory, with the second-order variable R, and $\exists_{e \to e \to t}$ is the device for making generalizations into the position of a binary predicate. Statements about universes can be given a similar higher-order rendition, without falling into the paradoxes that would ensue if they were treated as further individuals. (Note, however, that having sharply distinguished quantification from singular quantification over properties, we will follow a common practice of using talk of relations and properties in English to indicate formal sentences that involve higher-order generalizations; strictly speaking the indicated sentences do not quantify over properties or relations.)

Now what sort of relations could be admissible interpretations of '$\in$', or represent the membership relation of a universe? The issues at stake here are parallel; we will focus on the notion of admissibility to keep discussion brief. There are a great many properties of relations that can be expressed in a higher-order language. An admissible notion of membership, R, should of course be extensional: if two individuals in the field of R are such that the same things bear R to them — i.e. R counts them as sets with the same members — then they should be the same. Similarly, since sets are built up in stages, it's natural to think that an admissible notion of membership should be well-founded, which can also be given a straightforwardly higher-order formulation. Indeed, for each axiom of second-order ZF—listed in figure 1—there is a corresponding property of relations which ought to be had by any candidate notion of membership. These properties are obtained by replacing $\in$ in the principles in figure 1 with R, and restricting the quantifiers to the field of R—i.e. replacing $\forall_e x$ with $\forall_e(\exists y(Rxy \vee Ryx) \to$ and doing similar things for $\exists_e$ (the restriction is necessary even when $R = \in$ because we are interpreting $\forall_e$ as an absolutely general quantifier that ranges over tables and chairs as well as sets). The conjunction of these five properties is a single second-order sentence ZF^R in a single higher-order variable R. Using the device of λ-abstraction we get a definition belonging to the language of

Extensionality $\forall_e xy(\forall_e z(z \in x \leftrightarrow z \in y) \to x =_e y)$

Union $\forall_e x \exists_e y \forall_e z(z \in y \leftrightarrow \exists w \in x.z \in w))$

Powerset $\forall_e x \exists_e y \forall_e z(z \in y \leftrightarrow z \subseteq x)$

Foundation $\forall_e x(\exists_e y.y \in x \to \exists_e y \in x \neg \exists z \in x.z \in y)$

Replacement $\forall_{e \to e \to t} R \forall_e x(\forall_e yzz'(Ryz \wedge Ryz' \to z = z') \to \exists_e z \forall_e y(y \in z \leftrightarrow \exists w \in x.Rwy))$

Infinity $\exists_e x(\exists y \in x(\forall z.z \notin y) \wedge \forall y \in x(y \cup \{y\} \in x)$.

Figure 1: The axioms of ZF

pure higher-order logic:

$$\mathrm{ZF} := \lambda R.\, \mathrm{ZF}^R$$

Thus, our hypothesis is that admissible precisifications of membership are ZF relations:

2. $\forall_{e \to e \to t} R(\mathrm{Adm}\, R \to \mathrm{ZF}\, R)$

In order for there to be indeterminacy in the supervaluational sense, there must be variation between the ZF relations. However Zermelo (2010) famously showed that the variation you can have between ZF relations is very limited: ZF relations can differ about their 'height'—how long the iteration process continues—but cannot differ about the 'width'—what the sets are like at a particular stage. If you take the series of stages $V_0^R, V_1^R, ...$ of a ZF relation R—constructed by repeatedly applying Rs internal powerset operation to its emptyset—they will be isomorphic to the stages $V_0^S, V_1^S, ...$ of another ZF relation S, provided those stages both are reached in the respective constructions. It follows that the structure of the pure sets is pinned down uniquely up to any given stage: the only freedom one has concerns how long the sequence of stages extends. In particular we have

3. $\forall_{e \to e \to t} RS.(\mathrm{ZF}(R) \wedge \mathrm{ZF}(S) \to (\mathrm{CH}(R) \leftrightarrow \mathrm{CH}(S))$.

Combined with the claim that admissible relations are ZF relations, we refute the supervaluational account of the indeterminacy of CH: 1-3 are inconsistent. In general, these extensional/metalinguistic alternatives to genuine contingency seem to only make space for height contingency, not width contingency.

Now let's consider what happens if we posit *real* contingency about the set-theoretic universe: not mere indeterminacy about what our set-theoretic words refer to, but rather indeterminacy in *the sets themselves*, concerning how they are related to one another by the membership relation.[17] This indeterminacy would not be metalinguistic, but a kind of contingency concerning the pattern of the membership relation among the individuals. Here we must thus assume a notion of propositional indeterminacy that is not reduced or explained in terms of linguistic indeterminacy,

[17] See Bacon (2018b), Goodsell (2022). On the possibility of non-linguistic indeterminacy, Goodsell writes "On this conception, for arithmetic to be indeterminate is for the numbers themselves to have an indeterminate structure, independently of how we speak about them" p.128.

but is simply another propositional operator alongside the other more familiar modal operators.

The logical situation with respect to Zermelo's theorem is somewhat different here. Zermelo's theorem is, in some sense, an 'intra-world' constraint: no two ZF relations from the same possibility can differ in width. But there is no obvious way to get an 'inter-world' analogue of Zermelo's theorem. It's instructive to look at one strategy for proving such a theorem, and seeing where it might fail. The strategy would be to pick the actual sets and membership relation out in a modally rigid way, and then use Zermelo's theorem to compare the rigidified membership relation with the sets at possibilities where they might have changed. Let us suppose, then, that we can rigidly pick out to the things which are in fact sets and rigidly pick out the membership relation. Call this rigid relation $\in^*$. One might hope to argue, as above, that necessarily, $\in^*$ and $\in$ agree in width, i.e. that the *actual sets* are isomorphic to the sets under the membership relation, whatever that might be at the relevant possibility, by appealing to the necessity of Zermelo's theorem. In which case $\in$ and $\in^*$ would agree about CH, and of course the value of CH according to $\in^*$ is not contingent given $\in^*$ is by stipulation rigid. However, in order to apply Zermelo's theorem, we need that $\in^*$ is not only a ZF relation in actual fact, but necessarily a ZF relation. But crucially $\in^*$ could fail to satisfy the separation axiom, especially if we consider possibilities at which there are new properties for the second-order quantifier to range over. For instance, if it is possible that there be a set, $x \subseteq \mathbb{N}$, of natural numbers that doesn't in fact exist (as one would expect for the contingency of CH) then there is a new property, $\lambda y.y \in x$, which does not define a subset* of $\mathbb{N}$ according to $\in^*$.

Similar morals may be drawn for the Width Extensibilist. What we observe, firstly, is that the possibility of a ZF relation containing more sets of rank α cannot be actually witnessed, for by Zermelo's theorem any two ZF relations are isomorphic up to a given rank (provided they both extend that far). Nonetheless, Width Extensibility is on first-looks consistent with Zermelo's theorem because the actual sets of rank α, whatever they might be, could possibly fail to contain *all* the sets of rank α, assuming there could have been more properties and thus more conditions with which to define subsets of sets with rank below α. We see then that both sorts of width contingency require the possibility not merely of 'new' type e entities, but also of 'new' type $e \to t$ properties.

3 The Structure of Modal Reality

The sorts of mathematical contingency posited will have wider implications for the structure of modal reality. In this section and the next three we articulate some of these connections.

Several things require untangling before we can draw these implications. For a start, what do we mean by the structure of modal reality? Often metaphysicians mean by this various theses formulated in terms of a particular kind of modality, Kripke's notion of 'metaphysical necessity'. But as this modality is used by Kripke and subsequent philosophers, mathematics is metaphysically necessary (see Kripke (1980) p36).

It follows that the mathematical contingency and indeterminacy appealed to above cannot be explained in terms of metaphysical contingency or any restriction of it, and the possibilities on which mathematical indeterminacy and extensibility theses are predicated are not metaphysical possibilities. These theses concern the structure of modality in general, but not the structure of metaphysical necessity.

To theorize about the structure of modal reality in its entirety, we must be able to talk about all the modal notions there are, metaphysical modality and otherwise, and talk about the logical relationships between these modal notions. Crucial to this enterprise is the ability to specify what it means for an operator to be a *modality*, and to specify the logical relationships between modalities—when one modality is *as broad as* another. (For instance, we have seen above that metaphysical necessity is not as broad as mathematical necessity or determinacy.[18]) Indeed, higher-order logic provides us with the perfect framework to carry this out, for in the language of higher-order logic one can quantify directly into the positions occupied by sentences and by sentential operators allowing one to formulate definitions of these notions. Once this is done it is possible to then introduce a notion of *broad necessity*, an operator defined as possessing every necessity: what is broadly possible concerns what is possible in any sense of 'possible'. We will argue that the study of the structure of broad necessity has a good claim to being the study of the 'structure of modal reality' simpliciter.[19] The possibilities posited by this notion can be seen, by definition, to subsume the determinacy-theoretic possibilities and mathematical possibilities. It follows that any possibilities in which the continuum hypothesis has a different truth value, or in which there are more ordinals than there in fact are, will automatically be broad possibilities. So principles about the structure of broad necessity can have a direct bearing on the question of mathematical contingency and *vice versa*.

In order to start theorizing about modalities we face a choice. If we assume a certain thesis about the granularity of reality—roughly, that propositions, properties and relations are individuated relatively coarsely, by provable equivalence in a minimal system of higher-order logic called H—it is possible to give completely reductive definitions of *being a modality*, *being as broad as*, and *broad necessity*. If we wish to be neutral on the matter of propositional granularity, we appear to need another primitive. A higher-order predicate, Nec, *being a necessity*, of type $(t \to t) \to t$, is a natural primitive for this purpose.[20] We will take the former route of pursuing a logicist account of modality at the expense of neutrality of grain, but if you do not accept this theory of granularity everything I say can in a precise sense be translated into the latter framework by disregarding our definition of 'Nec', and replacing subsequent uses of it with the primitive.[21]

[18] The case that metaphysical necessity is not as broad as determinacy can be made even with respect to non-mathematical claims of vagueness, given the supervenience of the vague propositions on the precise; see Bacon (2018a), Bacon (2018b).

[19] Certainly it has a better claim to this than the study of metaphysical necessity, given the remarks above.

[20] One could instead take broad necessity as the primitive, an approach taken in Dorr et al. (2021). However, by taking *being a necessity* as primitive we can provide a *justification* for the posit of a broadest necessity, rather than imposing that assumption by fiat.

[21] The logicist account is spelled out in more detail in Bacon (2018a) and Bacon (2023a) chapter

PC A whenever A is a tautology.

UI $\forall_\sigma F \to Fa$, where $F : \sigma \to t$, $a : \sigma$

REF $a =_\sigma a$

LL $(a =_\sigma b) \to (Fa \to Fb)$

$\beta\eta$ $A \to B$ where A and B are immediately $\beta\eta$ equivalent.

MP If $\vdash A$ and $\vdash A \to B$, then $\vdash B$.

Gen If $\vdash A \to B$, and $x : \sigma$ does not occur free in A, $\vdash A \to \forall_\sigma x B$.

RE If $\vdash P\overline{x} \leftrightarrow Q\overline{x}$, then $\vdash P =_{\overline{\sigma} \to t} Q$ provided no variable in $\overline{x}$ is free in P or Q.

Figure 2: Classicism, C

The system we will work in, *Classicism*, or simply C, is axiomatized in figure 2. The language of Classicism is that of higher-order logic: it contains infinitely many variables of each type, constants $\to$ of type $t \to t \to t$ and $\forall_\sigma$ of type $(\sigma \to t) \to t$ for each type t, and complex terms are made exclusively by application and abstraction: you can make a term (MN) of type τ from terms M and N of types $\sigma \to \tau$ and σ, and you can make a term $\lambda x.M$ of type $\sigma \to \tau$ from a term M of type τ and a variable, x, of type σ. The other logical operations—disjunction, existential quantification, and identity at each type—can be introduced by definition in any of the standard ways, and as usual we will write these in infix notation where appropriate, and suppress λ when it appears after a quantifier. The first seven axioms and rules in figure 2 we shall call H, and encode a relatively neutral higher-order logic: it consists of the standard axioms of classical logic for the quantifiers and truth-functional connectives, and an axiom governing the behaviour of λ-terms. The Rule of Equivalence ensures that the theory proves the claim that two propositions, properties or relations R and S are identical whenever it can prove that R and S are coextensive.[22] It is the last rule that that distinguishes Classicism from more structured theories of granularity: it implies, for instance, that *being old and wise* and *being wise and old* are the very same property ($\lambda x.(Fx \wedge Gx) =_{e \to t} \lambda x.(Gx \wedge Fx)$) on account of their being provably coextensive from the laws of classical logic.

Using the purely logical language of higher-order logic it is possible to say that a given operator, X of type $t \to t$, has a 'normal modal logic'. Roughly, it is normal if the smallest collection of propositions containing (i) the tautologies, (ii) closed under modus ponens, (iii) containing the claim that X satisfies the normality axiom, and (iv) closed under X-necessitation are all true. Because we can quantify into sentence position we can state what it means for an operator to be closed under modus ponens with a single generalization:

$$\text{MP-Closed} := \lambda X.\forall p(X(p \to q) \to Xp \to Xq)$$

7, and the theory with a primitive necessity predicate, Nec, in Bacon and Zeng (2022). The latter shows that the theory Classicism used in Bacon (2018a) and Bacon (2023a) is interpretable in their theory, and that their theory is indeed neutral about the granularity of reality.

[22] Other presentations of this system, with different axioms and rules, can be found in Bacon (2018a) and Bacon and Dorr (forthcoming).

Given a modal operator $\Box$, we can similarly say what it means for a 'collection' of propositions, represented by an operator Y of type $t \to t$, to be closed under necessitation for $\Box$: $\forall p(Yp \to Y(\Box p))$.

$$\text{Nec-Closed} := \lambda XY.\forall p(Yp \to Y(Xp))$$

We can then state that p is in the normal modal logic for $\Box$ by saying that p belongs to any collection of propositions that contains the tautology, is closed under modus ponens and necessitation for X, and contains the claim that X is closed under modus ponens (i.e. the K axiom):

$$\begin{aligned}\text{InNormalModalLogicOf} := \lambda Xp.\forall Y(Y\top \wedge \text{MP-Closed}\,Y \wedge \\ Y(\text{MP-Closed}\,X) \wedge \text{Nec-Closed}(X,Y) \to Yp)\end{aligned}$$

Definition 3.1 (Weak Necessity)**.** *An operator, X, is a weak necessity iff every proposition in its 'normal modal logic' is true.*

$$\text{WNec} := \lambda X.\forall p((\text{InNormalModalLogicOf}\,X)p \to p)$$

The notion of a weak necessity is sufficient for applications of normal modal logic: if one considers an interpreted propositional modal language in which '$\Box$' is interpreted by a weak necessity, then every theorem of the smallest normal modal logic, K, will be true. A weak necessity is not only normal but necessarily so with respect to itself. In metaphysics, however, a stronger notion of necessity is in play: a true necessity mustn't be contingently normal with respect to any *other* kind of necessity. A logically perfect agent's knowledge may be normal, and known by them to be so, but it is not usually physically necessary, say, that they are logically perfect; so this agents knowledge is not a necessity in the operative sense.

Definition 3.2 (Strong necessity)**.** *An operator X is a strong necessity iff, for every weak necessity Y, it is Y-necessarily a weak necessity.*

$$\text{Nec} := \lambda X.\forall Y(\text{WNec}\,Y \to Y(\text{WNec}\,X))$$

We can now spell out what it means for one necessity to be as broad as another: there must be a strict implication from one necessity to the other. It would be arbitrary to single out any particular necessity to articulate this strict implication, so we require the implication to be strict in every possible sense. In fact, broadness is a special case of the more general notion of *entailment*. In the below we write $\overline{x}$ for a sequence of varibles $x_1...x_n$, $\overline{\sigma}$ for a sequence of type $\sigma_1...\sigma_n$, and $\overline{\sigma} \to \tau$ for the type $\sigma_1 \to ... \to \sigma_n \to \tau$.

Definition 3.3 (Entailment)**.** *Given two relations, R and S, of type $\overline{\sigma} \to t$, R entails S iff for every necessity Z, it's Z-necessary that any things standing in R stand in S.*

$$\leq_{\overline{\sigma}} := \lambda RS.\forall_{t\to t}Z(\text{Nec}\,Z \to Z\forall\overline{x}(R\overline{x} \to S\overline{x}))$$

We can also introduce 'multi-premise' entailment. If X of type $(\overline{\sigma} \to t) \to t$ represents a collection of propositions, properties or relations we say it entails another *proposition, property or relation R iff anything entailing everything in X entails R, and we write this $X \leq R$:*

$$\leq := \lambda X R \forall S(\forall T(XT \to S \leq T) \to S \leq R)$$

Given two necessity operators, X and Y, we say that X is as broad as Y iff, X entails Y, i.e. $X \leq_{t\to t} Y$.

We modeled our notion of a necessity on the idea of a normal modal operator. In a normal modal logic one can prove that if some finite list of propositions, $p_1, ..., p_n$, are each necessary, so is anything that they jointly entail. The analogous infinitary principle, that anything entailed by an arbitrary collection of necessary propositions is also necessary by contrast, cannot be proven from the principles of normal modal logic.[23] Arguably there are necessities, such as *having an objective chance of 1*, that do not satisfy this further principle, so we do not build it in to our definition. At any rate, we can specify this further property:[24]

Definition 3.4 (Infinitely closed necessity). *A necessity, X, is infinitely closed iff, whenever a proposition is entailed by the collection of all necessary propositions, that proposition is also necessary: $\forall q(\mathrm{X} \leq q \to Xq)$.*

$$\mathrm{Nec}_\infty := \lambda X(\mathrm{Nec}\, X \wedge \forall q(\mathrm{X} \leq q \to Xq))$$

Of course, for any necessity, X, there is another necessity X^∞ defined as being a proposition that is entailed by the X necessities: $X^\infty := \lambda p.X \leq p$.

Proposition 1. *If X is a necessity, its infinite closure X^∞ is an infinitely closed necessity operator.*

Finally, we define broad necessity as being necessary in every sense of necessity

Definition 3.5 (Broad Necessity). *p is broadly necessary iff it's X-necessary for every necessity X*

$$\Box := \lambda p.\forall X(\mathrm{Nec}\, X \to Xp)$$

In order to justify the title 'broad necessity' one must show that $\Box$ does indeed meet our criteria for being a necessity, and that it is as broad as any other necessity. These are verified by the following theorem.[25]

Theorem 2. *The following are theorems of Classicism*

1. $\mathrm{Nec}\, \Box$
2. $\mathrm{Nec}_\infty\, \Box$
3. $\Box \forall X(\mathrm{Nec}\, X \to \Box \leq X)$

[23] If we add infinite conjunctions to propositional modal logic, this strengthening is valid in the usual Kripke semantics, but not in variant semantics such as the topological semantics for S4, and so is not derivable from K augmented with the logical laws governing infinitary conjunction.

[24] See Bacon and Zeng (2022) p160.

[25] Proofs of all the theorems to follow may be found in Bacon (2023a) chapters 7 and 8.

Next we list some theorems of Classicism that concern the logic of broad necessity.

Theorem 3. *The following are theorems of Classicism or rules under which it is closed:*

K $\forall_t p \forall_t q(\Box(p \to q) \to \Box p \to \Box q)$

T $\forall_t p(\Box p \to p)$

4 $\forall_t p(\Box p \to \Box\Box p)$

CBF$^\sigma$ $\forall_{\sigma\to t} F(\Box\forall_\sigma x\, Fx \to \forall_\sigma x \Box Fx)$

NE$^\sigma$ $\forall_\sigma x \Box \exists_\sigma y.x =_\sigma y$

Necessitation *If A is a theorem of Classicism, so is $\Box A$*

Note that the first three axioms and Necessitation ensures the theorems of S4 for $\Box$ belong to Classicism. The first three axioms straightforwardly fall out of the fact that $\Box$ is the broadest necessity. K is guaranteed by the fact that $\Box$ is a necessity. T follows from the fact that the truth operator ($\lambda p.p$) is a necessity, and $\Box$ is as broad as it; 4 follows from the idea that the composition of two necessities is a necessity, so that $\Box$ must be as broad as $\lambda p.\Box\Box p$.

Some care is needed when interpreting the theorems CBF$^\sigma$ and NE$^\sigma$ of Classicism. In this paper we concieve of the symbols $\forall_\sigma$ and $\exists_\sigma$ as devices of generalization. The job description of a generalization, like $\forall_\sigma x.Fx$, is to express, in a single sentence, something that without it could only be approximated with an infinite schema consisting of the generalizations instances—formulas of the form Fa. It is this inference from $\forall_\sigma x.Fx$ to Fa in particular that is key in deriving CBF$^\sigma$ and NE$^\sigma$. But these theorems are not so attractive when we instead read the first-order $\forall_e$ and $\exists_e$ in terms of the ordinary quantificational idioms of English — words like 'everything', 'something' and 'exists'. Read that way NE^e tells us that everything necessarily exists (in any sense of 'necessarily' you might choose). Many philosophers—*contingentists*—regard this is as obviously false. For these philosophers our $\forall_e$ and $\exists_e$ do not correspond in any important sense to what *exists*. Nonetheless, contingentists usually have these generalizating devices at their disposal under the guise of the so-called "outer" or "possibilist" quantifiers. Often the outer quantifiers can be defined in terms of the modal operators and the contingentists prefered quantifiers, but even if this is not possible, they can also be introduced by stipulation via suitable introduction and elimination rules. We'll return to this issue in more detail in section 7; the main point here is that contingentists should be understanding the ensuing discussion in terms of these outer quantifiers.

4 Mathematical Necessity

This concludes our general theory of modality in higher-order logic. In order to apply it to the present topic of mathematical modality and indeterminacy we must introduce new non-logical operator constants to the logical language to stand for these operations. I will use the symbol $\blacksquare$, and will read it as the relevant sort of mathematical necessity or as determinacy depending on the application. For convenience we will use the terms 'mathematically necessary' and 'mathematically possible' in a way that is neutral between these interpretations. Call the language of pure higher-order logic $\mathcal{L}$, and the result of adding $\blacksquare$ to it $\mathcal{L}^{\blacksquare}$.

We must, of course, assume $\blacksquare$ is a necessity. However, it seems plausible that it is also closed under arbitrary logical consequences so we will make the stronger assumption:

Mathematical Necessity $\mathrm{Nec}_\infty\,\blacksquare$

Let us call the result of adding Mathematical Necessity to Classicism $\mathsf{C}^{\blacksquare}$.

Of course, "mathematical necessity" is a term of art, and it is certainly open to someone to posit a notion of necessity that is not infinitely closed and attempt to theorize about mathematical contingency in terms of that notion instead. But I think the extra generality gained by weakening infinitary closure is minimal. No progress will have been made if the continuum hypothesis, say, is *technically* mathematically contingent, but the mathematical necessities still settle the continuum hypothesis, in the sense that they either collectively entail it or entail its negation; the same goes for any other claim about the width of the set-theoretic hierarchy. Thus on this picture we shouldn't care merely about contingency with respect to the mathematical modality, but also about contingency with respect to the closure of mathematical necessity under logical consequence, $\blacksquare^\infty$ defined as $\lambda p.\blacksquare \leq p$ (recall proposition 1).[26]

A second point that is relevant here is that the infinite closure of $\blacksquare$ can in many contexts be derived. For instance, on the interpretation of $\blacksquare$ as the determinacy operator, one can derive infinite closure from the assumption that infinitary conjunction is precise, and the assumption that applying precise operations to precise arguments yields precise results (see Bacon (2020b) section III).[27] Or, on the reductive interpretations of $\blacksquare$ discussed next (e.g. as broad necessity), infinite closure is also derivable by theorem 2.2 above.[28]

[26] The two set-theoretic assumptions we make about $\blacksquare$ (outlined below) are either equally plausible when made about $\blacksquare^\infty$ or follow from the original assumptions about $\blacksquare$, so someone insisting on theorizing in terms of a notion of necessity that is not closed should take our uses of "mathematical necessity" to be referring instead to the closure of their notion. These assumptions are: (i) that sets are modally rigid, and (ii) that the axioms of set theory are necessary. It seems just as plausible that sets should be modally rigid with respect to the closure of mathematical necessity as with respect to mathematical necessity. The claim that the axioms of set theory of mathematically necessary implies that the axioms of set theory are entailed (trivially) by mathematical necessities.

[27] Hartry Field (Field (2003)) has suggested that rejecting the closure of the determinacy operator under infinitary consequence is key to making sense of the paradoxes of higher-order vagueness, which is relevant on that interpretation of $\blacksquare$. But this is not the only solution: Bacon (2020b) provides a different route to avoiding those paradoxes which avoids the expressive challenges that Field account faces (see also Priest (2010), Bacon (2015) §3.1).

[28] There is also a more local point to be made, namely that the results in this paper only rely on infinite closure in a small number of places and these appeals can be replaced by a variety of other plausible assumptions that would have the same effect. The only point this assumption is used in section 5 is in the proof that if broad necessity satisfies the Barcan formula then $\blacksquare$ does too, and in section 6 in the the proof that, given the Leibniz biconditionals, mathematical possibility is truth is some mathematically possible world. Either of these weaker assumptions alone would then suffice for the results in those sections. We mentioned already the assumption that $\blacksquare = \Box$ automatically

There is a long standing question for the modal extensibilists about the interpretation of mathematical modality (see §2.3 of Studd (2013)). Øystein Linnebo simply writes:

> *This is not metaphysical modality in the usual post-Kripkean sense.* Rather, the modality [...] is related to that involved in the ancient distinction between a potential and an actual infinity. (Linnebo (2013)p207)

But this tells us very little, and different authors have posited all sorts of modalities to fill this role. Fine (2006), for instance, posits, an 'interpretational' modality, whereas Scambler a dynamic one relating to the abilities of an ideal reasoner (Scambler (2021) p1100). Studd (2013), rejects these proposals, and likens the mathematical modalities more to tense operators, although does not find an interpretation he is fully happy with. To my mind, these replacements offer no more clarity.

The present framework, however, has an alternative to offer, namely that the relevant sort of necessity is just broad necessity. Any charge of unclarity here is easily met, for the notion of broad necessity is as clear as the logical operations from which it is defined — quantification and the truth-functional operations.

The Broad Necessity of Mathematics $\blacksquare =_{t\to t} \Box$

Under this hypothesis, the subsequent discussion would be greatly simplified. Nonetheless, there are some philosophical views we wish to remain neutral about that require us to keep them separate. Clearly any mathematical possibility is possible in the broadest sense, so it is the converse entailment that is at stake: is every broad possibility mathematically possible? One might worry that broad possibility is *too* broad. For instance, some authors have entertained the hypothesis that there is a notion of *logical* necessity in which even mathematical theories, such as $\mathrm{ZF}^{\in}$, could be contingent.[29] But the failure of our hypothesis above doesn't rule out precisely defined notions filling the roles that we care about. For instance consider:

$$\Box_{\mathrm{ZF}} := \lambda p.\, \mathrm{ZF}^{\in} \leq p$$

$$\Box_{\neq} := \lambda p.\exists_t q(q \wedge \Diamond q \leq p)$$

The former builds in the necessity of $\mathrm{ZF}^{\in}$, whereas the latter the necessity of distinctness.[30] The latter also has the virtue that it can be reductively defined in purely logical terms, and even someone who believed in logical possibility could maintain the necessity of $\mathrm{ZF}^{\in}$ with the force of $\Box_{\neq}$. Note that these reductive accounts of

ensures the infinite closure of $\blacksquare$ due to the infinite closure of $\Box$, but actually strengthening the assumption that sets are rigid with respect to mathematical necessity to the same assumption with respect to broad necessity would also suffice.

[29]These ideas can be formulated precisely in the present higher-order framework of Classicism — see, for instance, Bacon (2020a), Bacon and Dorr (forthcoming), Bacon (2023a) chapter 8. These views generally imply that distinct individuals can be broadly possibly identical; on the other hand the idea that distinct sets are *mathematically* possibly identical would cause trouble for the attractive idea that it's mathematically necessary which elements a set has. Using Set Rigidity, stated below, one can prove by transfinite induction that sets are mathematically necessarily distinct (Lemma 23 in appendix A).

[30]See Dorr et al. (2021) and Bacon and Dorr (forthcoming).

$\blacksquare$ are all infinitely closed necessities, and so in the presence of these identifications Mathematical Necessity is redundant. Of course, one might also wish to maintain that claims about the concrete realm are not mathematically contingent, in which case even these reductions are not plausible.[31] We will not take sides on any of these issues—going forward we treat $\blacksquare$ as a primitive.

Once we have singled out a suitable closed necessity $\blacksquare$, we can formulate various theses about the interaction of mathematical necessity with mathematical primitives. Let us now add to the language of mathematical necessity, $\mathcal{L}^{\blacksquare}$, a binary predicate $\in$ of type $e \to e \to t$. Call the resulting language $\mathcal{L}^{\in\blacksquare}$. Here we make only two assumptions about the interaction of $\blacksquare$ and $\in$. First, we will assume that it is mathematically necessary that $\in$ satisfies the axioms of higher-order ZF.

The Necessity of Set Theory $\blacksquare\mathrm{ZF}^{\in}$

As we noted before, this is compatible with the view that $\mathrm{ZF}^{\in}$ is logically contingent, and so contingent in the broadest sense.

Second, we assume that sets are 'rigid' in the sense that they cannot gain or lose members. We can require rigidity with respect to many different modalities. Rigidity with respect to the broadest modality implies rigidity with respect to any weaker modality. Since we wish to remain neutral about the extent of broad contingency—perhaps there could be contingency about the make up of a set relative to some notion of 'contingency'—we will only require sets to be rigid with respect the mathematical modality/determinacy operator $\blacksquare$. Here is how we say that a set, x, cannot gain members: if any property, F, possibly applies to some member of x then there is in fact a member of x to which F possibly applies (for otherwise x could have members that are not among its actual members). Here is how we say that it cannot lose members: if, for any property F, there is some member of x that is possibly F, then it's possible that some member of x is F (for otherwise there is some actual member of x that is possibly not a member of x). This means we want $\forall_{e\to t}F(\exists y \in x \wedge \blacklozenge Fx \leftrightarrow \blacklozenge\exists_e y(y \in x \wedge Fx))$. This is essentially the dualized form of the Barcan formula for the quantifiers restricted to $\in x$. In general we will define what it means for a relation $R : \sigma_1 \to ... \to \sigma_n \to t$ to be rigid as follows, writing $\overline{x}$ for a sequence of varibles $x_1, ..., x_n$, $R\overline{x}$ for $Rx_1...x_n$, $\forall\overline{x}$ for $\forall_{\sigma_1}x_1...\forall_{\sigma_n}x_n$, and $\overline{\sigma} \to t$ for $\sigma_1 \to ... \to \sigma_n \to t$.

$$\mathrm{Rigid}_{\blacksquare} = \lambda R\forall_{\overline{\sigma}\to t}S(\blacksquare\forall\overline{x}(R\overline{x} \to S\overline{x}) \leftrightarrow \forall\overline{x}(R\overline{x} \to \blacksquare S\overline{x}))$$

So we can now state our principle that sets are rigid:

Sets are Rigid $\forall_e x(\mathrm{Set}\, x \to \mathrm{Rigid}_{\blacksquare}\, \lambda y.y \in x)$

Rigidity here is stated with respect to $\blacksquare$, although there is a stronger notion of rigidity stated in terms of broad necessity.

Observe that this principle is *not* the claim that the *binary* membership relation is rigid: it is the claim that, for each set x, the unary property of belonging to x is rigid. If membership were rigid there could be no contingency in the pattern of

[31] One reason we might want a notion of mathematical necessity like this would be to articulate a modal sense in which mathematics is conservative over the concrete. See Dorr (2010).

membership claims. Note, too, that this principle is restricted to sets: it's consistent with this principle that a non-set, x, could become a set, in which case belonging to x would not be rigid.[32]

Let me head off one potential confusion. If x is the set of all sets of rank α, the claim Sets are Rigid implies that x cannot gain or lose elements. However, this does not mean that there couldn't have been more sets of rank α, it rather implies that if there had been more sets of rank α x wouldn't have contained them all. This confusion becomes particularly tempting when we start using putative singular terms like V_α or $P(\mathbb{N})$ to refer to sets. V_α is not itself a term in the language of set theory, it is really a definite description and so the property of belonging to V_α, $\lambda x.x \in V_\alpha$, can fail to be rigid consistently with the principle Sets are Rigid.[33].

One might think of our notion as a 'vertical' notion of rigidity: a rigid property cannot change its extension from one mathematical possibility to a later one. There is also a 'horizontal' notion of rigidity ruling out changes of extension between two mathematical possibilities abreast of each other, which becomes relevant when $\blacksquare$ fails to satisfy the convergence axiom: $(\blacklozenge\blacksquare p \wedge \blacklozenge\blacksquare q) \rightarrow \blacklozenge\blacksquare(p \wedge q)$.[34] There is actually broad agreement from both height and width extensibilists[35] that the modal logic of mathematical necessity is at least $\mathsf{S4.2}$, which includes the convergence principle, so we can treat this as something of a side issue. But if we are not assuming convergence, then we might want to strengthen Sets are Rigid along these lines, and one of our arguments (Theorem 7) will require this strengthening if convergence is not assumed.[36]

We will call the system we get by adding these principles to Classicism $\mathsf{C}^{\blacksquare\in}$. The additions to Classicism are summarized in figure 3. Of course, theses we have considered earlier can now be formulated precisely:

Forcing Possibilism $\exists_e xy(\mathrm{PO}\, x \wedge y \in x \wedge x \Vdash A(\hat{x}_1, \ldots, \hat{x}_n) \rightarrow \blacklozenge A(x_1, ..., x_n))$ when $A(x_1, \ldots, x_n)$ is a first-order formula of $\mathcal{L}^{\in}$.

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[32] A plausible strengthening of Sets are Rigid prefixes it with $\blacksquare$; given this strengthening *belonging to* x, where x is a non-set, would have to become rigid once x is a set.

[33] Observe too that our principle entails that individuals with no members — the empty set and urelements–necessarily have no members. It doesn't quite imply that urelements are necessarily urelements: for all we've said an urelement might become identical to the emptyset because the system we are in does not rule out the necessity of distinctness

[34] According to the ordinary notion of rigidity defined above there could be two 'divergent' mathematical possibilities where x is a set and has different extensions in both: perhaps, even, there is some y such that $\blacklozenge(\mathrm{Set}\, x \wedge y \in x)$ and $\blacklozenge(\mathrm{Set}\, x \wedge y \notin x)$. The rigidity of sets would ensure $\blacklozenge\blacksquare(\mathrm{Set}\, x \wedge y \in x)$ and $\blacklozenge\blacksquare(\mathrm{Set}\, x \wedge y \notin x)$, but without convergence this state of affairs is consistent. By contrast, if every pair of mathematical possibilities recognize as mathematically possible a common mathematical possibility, rigidity ensures the extension of x is the same at the common possibility, and thus is the same at the original pair.

[35] See, for instance, Hamkins and Linnebo (2022), Scambler (2021), Linnebo (2013) and Studd (2013).

[36] If x is a possible set (x may or may not be an actual set) then a condition of its being horizontally rigid is that if something possibly belongs to x it necessarily does so whenever x is a set. So in the absence of convergence we can strengthen Sets are Rigid by adding the axiom: $\forall_e xy(\blacklozenge y \in x \rightarrow \blacksquare(\mathrm{Set}\, x \rightarrow y \in x))$. Given convergence and $\blacksquare$Sets are Rigid, this turns out not to be a strengthening of our theory of mathematical modality. For if the principle were false, then we would have that for some x and y, $\blacklozenge(y \in x)$ and $\blacklozenge(\mathrm{Set}\, x \wedge y \notin x)$. $\blacksquare$Sets are Rigid (and lemma 23) would let us then prove that $\blacklozenge\blacksquare y \in x$ and $\blacklozenge\blacksquare y \notin x$, which is inconsistent using the convergence axiom.

Mathematical Necessity $\mathrm{Nec}_\infty \blacksquare$

The Necessity of Set Theory $\blacksquare \mathrm{ZF}^\in$

Set Rigidity $\forall_e x(\mathrm{Set}\, x \to \mathrm{Rigid}_\blacksquare\, \lambda y.y \in x)$

Figure 3: $\mathsf{C}^{\blacksquare\in}$ adds these axioms to Classicism.

Countabilism $\forall_e x(\mathrm{Set}\, x \to \blacklozenge \exists_e f : \mathbb{N} \to x(\forall_e zw(fz =_e fw \to z =_e w)))$

here $f : \mathbb{N} \to x$ means that f is a function from $\mathbb{N}$ to x, $\mathrm{PO}\, x$ states that x is a partial order, and $\Vdash$ is the forcing relation definable in the language of first-order set-theory. These further principles will not be part of our neutral theory of mathematical necessity and sets. As we pointed out earlier, Countabilism can be derived from Forcing Possibilism. Note that Forcing Possibilism allows us to derive 'metaphysical' versions of famous independence results in the object language:

Contingency of CH $\blacklozenge \mathrm{CH} \wedge \blacklozenge \neg \mathrm{CH}$

The framework of $\mathsf{C}^{\blacksquare\in}$ + Forcing Possibilism thus enables a thorough going modal approach to independence proofs, letting us convert existing independence results for metalinguistic claims about derivability into results about genuine mathematical contingency.[37]

5 Brouwer's principle and the Barcan formula

In this section and the next we'll examine two questions: given the width contingency hypothesis, what is the logic of mathematical necessity? and what are the implications for the structure of modal reality more generally? We'll begin in this section with two key modal principles that might be thought to govern broad necessity: the Barcan formula and Brouwer's principle.

Consider a typical claim made by the modal extensibilist: that there could have been more sets than there in fact are. There are two models you might have of this possibility. The first maintains that there is a "constant" (i.e. non-contingent) domain of individuals, and contingency about which sets there are is contingency about which of those individuals are sets.[38] The second model maintains that there

[37]Strictly speaking, Forcing Possibilism lets us derive contingency from forcing arguments. A strengthening of Forcing Possibilism for proper classes is needed, for instance, to accommodate proper class forcing—although we need to be careful how we formulate it, as forcing with an arbitrary proper class may not preserve the axioms of ZF. Note that not all independence results can correspond to contingency claims — we will see later that $\mathsf{C}^{\blacksquare\in}$ proves the non-contingency of arithmetical statements, so that Gödelian arguments for the independence of certain arithmetical statements from our modal mathematical theory will not correspond to any contingency. There is something special about the method of forcing according to this.

[38]This model of indefinite extensibility is explored explicitly in Uzquiano (2015). But interestingly it could also fit the theory of indefinite extensibility found in Linnebo (2013) and Studd (2013), when we are explicit that $\forall_e$ means the *outer* quantifier: while these authors make claims to the effect that new individuals can come into existence using their contingentist quantifiers, they typically also

are no actual individuals that could have been the new sets: if there had been new sets, then they would have been new individuals altogether.

The first model of set-theoretic contingency works well for the height extensibilist. Essentially we are positing, in addition to the sets, a bunch of actual proper classes that could have constituted a new layer of sets. But this model is not so good for the width contingentist, because they are positing new subsets of things *we already recognize to be sets*. Those new subsets cannot be characterized by properties we already have access to, in actuality, because otherwise we could already obtain them by the separation axiom, which lets us create subsets of sets we already have from any condition. The problem is that once we have actual individuals hanging around corresponding to those merely possible subsets, those individuals contain the information we need to define those subsets: if x is a merely possible set of natural numbers, say, we will show (theorem 7) that it is actually a set because we can still talk about what would have belonged to x if it had been a set: $\{y \in \mathbb{N} \mid \blacksquare(\mathrm{Set}\, x \to y \in x)\}$. The rigidity of sets ensures that this is the same as the set x would have been if it had been a set.[39]

It is worth emphasizing that this problem for the width contingentist can manifest itself in a number of weaker logical settings. Consider, for instance, the contingentist who prefers a weaker quantificational logic for quantificational words like 'everything' and 'something'. For the contingentist, our generalizing devices that we have notated $\forall_e$ and $\exists_e$ correspond instead to what they call "outer quantifiers", and their uses of words like 'everything' and 'something' should be interpreted instead by restricting our quantifiers $\forall_e$ and $\exists_e$ to things that *exist*. But it makes no difference to our argument whether our merely possible set of natural numbers, x, exists in the outer or inner sense: if we have some means of quantifying over it, then the information about its possible membership conditions is accessible to us and we can construct the set in question. On this picture the sort of contingency about what there is is radical: it involves contingency even about what *possible* things there are, as stated using the possibilist/outer quantifiers.

Resuming our assumption that the quantifiers are classical (interpreting them as outer quantifiers, if necessary), how do we articulate this idea of there being new individuals that don't actually exist? A property, F, characterizes some merely possible individuals if it's possible for something to have had it, but no actual thing could have had it. To say that there are no such properties is to say that there couldn't be any new individuals: $\forall_{e\to t}F(\Diamond\exists_e xFx \to \exists_e x\Diamond Fx)$. This principle (in its contrapositive form) is often called the Barcan formula, BF^e, and corresponds to the other direction of CBF^e (the converse Barcan formula). While CBF^e tells us that things can't go out of existence, BF^e says things can't come into existence. As we noted in theorem 3, CBF^e is in fact a *theorem* of Classicism, whereas BF^e is not. This represents a deep asymmetry in the logic of the classical quantifiers. (A similar asymmetry persists even in the context of weaker quantificational logics: one can help oneself, either through definition or otherwise, to wider outer quantifiers that have a classical

appeal to modalized quantifiers under which the Barcan formula, discussed below, is valid under a certain translation.

[39]It is here that we may need the stronger notion of rigidity in footnote 36, if we are not assuming convergence.

quantificational logic, but it's consistent that even these classical quantifiers fail to satisfy the Barcan formula. One might have thought that, just as we can introduce quantifiers which validate the converse Barcan formula and the necessity of existence, it should also be possible to introduce an even wider quantifier that is guaranteed to satisfy the Barcan formula. But, unless one makes further assumptions about the logic of broad necessity, it is not possible.[40])

Now let us ask what implications all of this has for the logic of *broad* necessity. Of course, if we are to admit the mathematical possibility of new individuals then we should also admit the broad possibility of such individuals so we should expect the Barcan formula to fail for both mathematical and broad necessity alike. Perhaps more surprising is the fact that we must also reject the necessity of the Brouwerian principle:

B $\forall_t p(p \to \Box\Diamond p)$

To deny this is to admit the possibility of truths that are possibly impossible *in the broadest sense of possible and impossible*. Brouwer's principle for broad necessity is a part of the 'orthodox' package of views about modal reality, exemplified in, for instance, Lewis (1986) and Stalnaker (1976), and in the explicitly higher-order context Williamson (2013), Fritz (MS), and Goodsell and Yli-Vakkuri (MS). This package usually comes along with the view that metaphysical necessity is the broadest necessity, and its logic is S5. Of course, this picture holds that B is not only true, but broadly necessary. This is equivalent to saying that truth entails being possibly necessary:

$\mathsf{B}^{\leq}$ $\lambda p.p \leq_{t\to t} \lambda p.\Box\Diamond p$

Let us write C5 for the result of adding $\Box$B, or $\mathsf{B}^{\leq}$, to Classicism. C5 contains all the theorems of the modal logic S5 for broad necessity.

In order for there to be contingency about what individuals there are, in the widest possibilist sense of *there are*, there must be contingency about what is possible, and this is essentially what Brouwer's principle rules out. Why is this so? The usual model theoretic explanation of this rests on a certain possible worlds model theory in which worlds are possible *relative* to other worlds, and the Brouwerian principle corresponds to the symmetry of this relation of relative possibility.[41] If one world, w, considers another world, v, to be possible and to contain individuals in its domain that do not belong to ws domain, then every world possible relative to v must contain those individuals (since individuals exist necessarily). This means the original world w cannot be possible according to v, so the relation of relative possibility is not symmetric. In short, Brouwer's principle lets you look backwards and this is problematic for the width contingentist, since (putting it very informally)

[40] Related points are the subject of in progress work with Cian Dorr, Peter Fritz and Ethan Russo. In the setting of Free Classicism (defined in appendix B) one can formalize and prove the claim that if there is a classical quantifier it is unique, and anything behaving logically like a free quantifier is provably a restriction of it. Nonetheless, while one can prove necessitism and the converse Barcan formula for the classical quantifier in this context, one cannot prove the Barcan formula.

[41] See Cresswell and Hughes (1996) pp17-21.

it lets you 'send information back' about the membership conditions for subsets of the natural numbers in the form of the individuals that are those subsets there.

This explanation is unsatisfactory due its reliance on a particular model theory, as well as possible worlds assumptions that we will have reason to question shortly. Our job in the rest of the section is to make all of the above reasoning precise. First, we will give a straightforward object language argument (due to Arthur Prior) that Brouwer's principle for the broadest modality implies the Barcan formula for the broadest modality. Then we will show that the Barcan formula for broad modality implies the Barcan formula for the mathematical modality. Finally, we will show that the Barcan formula for the mathematical modality lets us prove that V_α is rigid for each ordinal α, ruling out width contingency. For the reasons outlined above, we cannot prove in an analogous manner that V is rigid: there can be contingency about which things are sets, there just cannot be contingency about which things are subsets of the sets we already have—all such contingency comes from the height and not the width of the universe.

Let us begin by explaining Prior's derivation of the Barcan formula for broad necessity from Brouwer's principle for broad necessity.[42] The Barcan formula says:

$\mathsf{BF}^e_\Box$ $\forall_{e\to t}F(\forall_e x\Box Fx \to \Box\forall_e x\, Fx)$

We will argue contrapositively, and show that if there is a property only applying to new/merely possible individuals, then there could have been be a truth which is possibly impossible. F characterizes merely possible individuals if there could have been Fs but there is nothing which could have been F. There are two possible cases: either the true claim that nothing is possibly F is itself possibly impossible, in which case we have a truth that's possibly impossible. Or else it's not, so that it's necessarily possible that nothing is possibly F. This means (given the necessity of existence) that there couldn't have been something that is necessarily possibly F. But there could have been an F, which means that the claim that that thing is F is not necessarily possible — i.e. is possibly impossible. In the latter case, the truth that is possibly impossible is a merely possible truth.

This style of argument can be run at any type whatsoever. In fact, we appealed to nothing special about broad necessity in this argument. For any necessity, X, let us write BF^σ_X and B_X for the Barcan formula and Brouwerian principle concerning X (i.e. $\forall_{\sigma\to t}F(\forall_\sigma x.X(Fx) \to X(\forall_\sigma x\, Fx))$ and $\forall_t p(p \to X\neg X\neg p)$). Prior's argument establishes that, for any necessity whatsoever, the X-necessity of Brouwer's axiom for X, i.e. $X\mathsf{B}_X$, implies the Barcan formula for X, BF^σ_X. So to summarize:

Theorem 4 (Prior)**.**

1. *C proves* $\forall X(\mathrm{Nec}\, X \to X\mathsf{B}_X \to \mathsf{BF}^\sigma_X)$
2. *$\mathsf{C5}$ proves* $\mathsf{BF}^\sigma_\Box$*.*

If mathematical contingency requires failures of $\mathsf{BF}^e_\Box$, as we have been suggesting, it means we must reject the orthodox logic of $\mathsf{C5}$. To complete this line of argument,

[42] I offer an informal argument below, the formal version of this proof is found in Prior (1967) p146 and attributed to E.J. Lemmon. It is based on an earlier argument due to Prior (1956).

we next need to establish that the Barcan formula for broad necessity implies the Barcan formula for mathematical necessity. Intuitively, if there couldn't be new things in the broadest sense of 'could', then there couldn't be new things in any more restrictive sense. One might naïvely take this to mean that the broad Barcan formula implies the Barcan formula for any modality whatsoever. But this is not quite true. Some counterexamples to the Barcan formula have nothing to do with the possibility of new individuals, but to do with the failure of the necessity to be closed under infinite conjunctions. For every individual there's a chance of 1 that if it's a point on the dartboard the dart won't land on it, but it doesn't follow that there's a chance of 1 that the dart won't land on any point on the dartboard; so if *having chance 1* is a necessity, it doesn't respect the Barcan formula irrespective of the status of the broad Barcan formula. However, the Barcan formula for broad necessity implies the Barcan formula for any necessity that is infinitely closed. It follows, too, that Brouwer's axiom for broad necessity implies the Barcan formula for every necessity that is infinitely closed. We summarize this with the following theorem of the orthodox system $\mathsf{C5}$

Theorem 5.

1. $\mathsf{BF}^\sigma_\Box \to \forall X(\mathrm{Nec}_\infty X \to \mathsf{BF}^\sigma_X)$
2. *In* $\mathsf{C5}$, $\forall X(\mathrm{Nec}_\infty X \to \mathsf{BF}^\sigma_X)$

The proof is included in appendix A. As a straightforward corollary, we see that the behaviour of mathematical modalities and determinacy are tightly constrained by the behaviour of broad necessity:

Corollary 6 ($\mathsf{C}^\blacksquare$)**.** $\Box\mathsf{B}$ *implies* $\mathsf{BF}^\sigma_\blacksquare$ *and* $\mathsf{BF}^\sigma_\Box$ *implies* $\mathsf{BF}^\sigma_\blacksquare$.

Now, finally, we will show that Barcan for $\blacksquare$ (and thus Barcan for $\Box$, and Brouwer for $\Box$) entails the rigidity of V_α, and that the rigidity of V_α in turn refutes the various width contingency hypotheses we were interested in, such as the indeterminacy of the continuum hypothesis and countabilism.

The first-order of business is to define the property V_α: being a set whose rank is no greater that α. This is done by transfinite recursion:[43]

$$V_0 := \lambda x.\bot$$

$$V_\alpha := \lambda x \forall y(y \in x \to \exists \beta \in \alpha V_\beta y).$$

Where α is an ordinal (i.e. a transitive set that is totally ordered by membership: $\forall \beta\beta' \in \alpha(\beta \neq \beta' \to \beta \in \beta' \vee \beta' \in \beta)$). It is usual in set-theory texts to use V_α as a name for a set, whereas here it is a predicate. Our choice discourages the temptation to think of V_α as automatically rigid in virtue of being a set, as we earlier cautioned against.

The claim that there is no contingency about the width of the universe, then, is the claim that for every ordinal α, V_α is rigid. Note that everything we say here is

[43] This can be made into an explicit definition in the usual way $V_\gamma = \lambda x(\forall Y(\forall y \neg Y0y \wedge \forall \alpha(\mathrm{Ord}\,\alpha \wedge \forall y \forall \beta \in \alpha(y \subseteq Y\beta \to Y\alpha y)) \to Y\gamma x)$.

entirely consistent with height contingency: for all we say, there could be new ordinals γ and as a result new sets belonging to V_γ.

Theorem 7 ($\mathsf{C}^{\blacksquare\in}$)**.** *Given* $\mathsf{BF}^e_\Box$ *(for broad necessity), being of stage* α *(i.e.* V_α*) is rigid for every ordinal* α*.*

What is going on here? Given the Barcan formula, the only way there could be 'new' sets is if there are already individuals hanging around that could have been those new sets. Thus we arrive at the 'first model' mathematical contingency, where there is a modally constant domain of individuals containing a whole bunch of non-sets, and the contingency concerns which of those individuals are sets. As we pointed out earlier, this model of set-theoretic contingency is unfriendly to width contingency; the argument in the appendix A essentially takes that informal reasoning and makes it precise.

The significance of this result is that, for any ordinal α, there cannot be new sets of rank α. Among other things, this implies the non-contingency of the continuum hypothesis, and thus its determinacy on one way of reading $\blacksquare$. For in order for it to be indeterminate whether the continuum hypothesis is true one has to introduce new sets with small ranks ($\omega + n$ for finite n): new sets of natural numbers, or new bijections between sets of reals and reals. One can similarly refute countablism: if there is no injection from $\mathbb{N}$ to x then this fact is necessary, for there cannot be any new injections given $\mathsf{BF}^e_\Box$. To make these remarks precise we introduce some useful concepts and propositions, which are proven in appendix A.

I will say that a formula of first-order set-theory is absolute with respect to a transitive model M and class $\mathcal{C}$ of transitive models $N \supseteq M$ extending M iff (i) when it is satisfied by some elements of M it is also satisfied by those elements in any member of $\mathcal{C}$, (ii) if it is not satisfied by those elements in M it is not satisfied by them in any extension in $\mathcal{C}$. This has an obvious modal analogue:

Definition 5.1 (Modal Absoluteness)**.** *A formula* $A(\overline{x}, \overline{y})$ *is* modally absolute *iff the formulas*

- $\forall_e \overline{x}(\mathrm{Set}\,\overline{x} \wedge A(\overline{x}) \to \blacksquare A\overline{x})$
- $\forall_e \overline{x}(\mathrm{Set}\,\overline{x} \wedge \neg A(\overline{x}) \to \blacksquare\neg A\overline{x})$

are both true, where $\overline{x}$ *is short for a sequence of variables* $x_1...x_n$*, and* $\mathrm{Set}\,\overline{x}$ *is short for the conjunction* $\mathrm{Set}\,x_1 \wedge ... \wedge \mathrm{Set}\,x_n$*.*

A sufficient condition for a formula of first-order set-theory to be absolute is if all of the quantifiers in the formula are restricted by formulas that are not only absolute, but do not change their extensions across models. The modal analogue of this stronger property is rigidity. In the present higher-order setting, we can similarly define a class of first-order sentences that are provably modally absolute: the smallest set of sentences containing $x \in y$ and containing $\neg A$, $A \wedge B$, $\forall_e x(C \to A)$ whenever A and B are in the set, and C is a rigid property of sets ($\mathrm{Rigid}_\blacksquare(\lambda x.C)$ and $\lambda x.C \leq_{e\to t} \mathrm{Set}$ are true).

Theorem 8 ($\mathsf{C}^{\blacksquare\in}$)**.** *Suppose $A(\overline{x})$ is a first-order set-theoretic formula with free variables $\overline{x}$. If all the quantifiers in $A(\overline{x})$ are restricted to rigid properties of sets, then A is modally absolute.*

The proof of this theorem is provided in the appendix. In practice we could dispense with the metalinguistic notion of modal absoluteness: cases where we apply theorem 8 to a particular formula $A(\overline{x})$ can be replaced by proving in the object language the two formulas in definition 5.1 from the assumption that the relevant predicates are rigid (those restricting the quantifiers in A). But we need the concept to state theorem 8, and the theorem provides useful perspective on what we are actually doing when carry out an argument that a particular formula is modally absolute because it is general, whereas these particular arguments are not. There too the metalinguistic ascension is often dispensible, and harmless.

Given theorem 8 and Set Rigidity, any formula that's provably equivalent to one whose quantifiers are all restricted by set membership will be modally absolute. This lets us derive the following useful facts:

Theorem 9 ($\mathsf{C}^{\blacksquare\in}$)**.** *For any ordinal α, the following conditions are modally absolute.*

1. *being an ordinal less than α.*
2. *being a limit ordinal less than α.*
3. *being the smallest limit ordinal, the successor of the smallest limit ordinal, the successor of the succcessor of the smallest limit ordinal...*

moreover, the properties in 3. are rigid.

Note that while the property of being an ordinal is modally absolute, it needn't be rigid: every ordinal is necessarily an ordinal, but we have not ruled out the possibility of there being further ordinals, in agreement with our previous claim that these results are compatible with height contingentism.

Let us say that a first-order set-theoretic sentence is *arithmetical* if all the quantifiers in the sentence are restricted by the predicate 'belongs to the smallest limit ordinal'. Notice that theorems 9.3 and 8 immediately imply arithmetical sentences are non-contingent in our background theory $\mathsf{C}^{\blacksquare\in}$.

Corollary 10 ($\mathsf{C}^{\blacksquare\in}$)**.** *$\blacksquare A \vee \blacksquare\neg A$ is derivable in $\mathsf{C}^{\blacksquare\in}$ for any arithmetical sentence A*

Thus if there is set-theoretic contingency, it is not to be found in the arithmetical statements of set-theory. We return to this asymmetry in the appendix.

While the non-contingency of arithmetic is unavoidable given necessity of ZF, necessity is closed under entailment, and that sets are rigid, more interesting mathematical claims, like CH, can be contingent. However, another corollary of the above is that in the presence of $\mathsf{BF}^e_\Box$ or $\Box\mathsf{B}$, even this sort of contingency disappears.

Corollary 11 ($\mathsf{C}^{\blacksquare\in}$)**.**

1. *$\mathsf{BF}^e_\Box \to \blacksquare CH \vee \blacksquare\neg CH$.*

2. $\mathsf{BF}^e_\Box \to \forall_e x(\text{Uncountable}\, x \to \blacksquare\, \text{Uncountable}\, x)$

where $\text{Uncountable}\, x := \forall_e x(\forall_e f : \mathbb{N} \to x \neg\, \text{Injection}\, f)$

The consequents of these conditionals are thus outright theorems of $\mathsf{C5}$:

Corollary 12 ($\mathsf{C5}^{\blacksquare\in}$)**.**

1. $\blacksquare CH \vee \blacksquare\neg CH.$

2. $\forall_e x(\text{Uncountable}\, x \to \blacksquare\, \text{Uncountable}\, x)$

Thus The Contingency of CH and Countabilism are inconsistent in $\mathsf{C5}$. The reason this is true, roughly, is that CH is about sets of rank $V_{\omega+2}$: it's equivalent to a formula whose quantifiers are restricted to $V_{\omega+2}$. But given the modal absoluteness of V_α and of $\omega + 2$ (proven above) it follows by theorem 8 that CH is modally absolute. Note that the modal absoluteness of CH implies CH $\to \blacksquare$ CH and $\neg$ CH $\to \blacksquare\neg$ CH, so $\blacksquare CH \vee \blacksquare\neg CH$ follows from an instance of excluded middle.

More generally, by theorem 7, any set theoretic statement that is equivalent to a sentence that can be formulated using quantifiers restricted to V_α for some α will be determinately true or false, and non-contingent in other senses of contingency. Thus these arguments extend straightforwardly to other contentious axioms of set theory such as the generalized continuum hypothesis up to some cardinal κ, Martin's axiom for partial orders up to cardinality κ, and so on. They do not extend to claims about the 'height' of the universe, such as large cardinal hypotheses.

Might one take this to be an argument against width contingency? We certainly do not have anything like a straightforward logical reductio of width contingency, for Classicism on its own includes neither the Barcan formula or Brouwer's principle, and there very are natural models in which they fail.[44]

Theorem 13. *The following are **not** theorems of Classicism*

B $\forall_t p(p \to \Box\Diamond p)$

5 $\forall_t p(\Diamond p \to \Box\Diamond p)$

$\mathsf{BF}^\sigma_\Box$ $\forall_{\sigma\to t} F(\forall_\sigma x \Box Fx \to \Box\forall_\sigma x\, Fx)$

One might, however, still see an objection to width contingency here. After all, isn't $\mathsf{S5}$ in some sense the standard logic of necessity? I am not persuaded. If there ever was an implicit decision within the philosophical community about which logic of necessity is 'standard' it happened before mathematical modalities and determinacy operators were being discussed widely, and most likely was made with Kripke's notion of metaphysical necessity in mind. The failures of Brouwer's principle posited here are entirely compatible with its holding for the more restricted notion of metaphysical necessity. It follows too that the Barcan formula may be valid for metaphysical necessity, and that the continuum hypothesis is either metaphysically necessarily

[44]Several sorts of models are described in the appendices to Bacon (2018a) and Bacon and Dorr (forthcoming) and in chapter 18 of Bacon (2023a).

true or necessarily false. And this too is entirely compatible with our diagnosis of the continuum hypothesis as indeterminate and thus contingent in the broadest sense (in this case, then, it will be indeterminate whether CH is a metaphysical necessity or impossibility). When it comes to positive arguments for the S5 principle, they are thin on the ground. Some considerations are abductive, and come from the relative simplicity and power of S5—the only schemas of propositional modal logic it doesn't imply are clearly invalid, whereas S4 leaves the validity of many modal principles open.[45] But of course, nobody thinks that the theoretical virtues of simplicity and power can outweigh the countervailing virtue of truth—after all $\perp$ is simple and very powerful. The theorist already convinced of the indeterminacy of the continuum hypothesis may find much less mileage in these abductive considerations. Other arguments for the S5 principle are far less compelling, for they often appeal to the model theory of modal logic in a patently illegitimate way—e.g. appealing to the idea that to be the broadest necessity it must quantify over 'all' possible worlds in some set-theoretic model, without taking into account that in the intended model (if there is one!) what worlds in the model represent genuine possibilities could well be contingent.[46] Finally, we should also emphasize that broad necessity, as it has been introduced here, is not necessarily a notion we had pretheoretically—intuitions about how it should behave should be taken with a generous pinch of salt, and it is generally better to simply work with its formal definition, *being necessary for every necessity*, and see where our philosophical theorizing takes us.

6 The Leibniz Biconditionals

Let us now turn to another pervasive idea in modal metaphysics, the notion of a *possible world*. Possible worlds can be wielded as a purely model theoretic tool for establishing metalogical properties like consistency and invalidity in modal languages. In a model of a modal language sentences might be interpreted by arbitrary sets of possible worlds, and these might serve as the domain for quantifiers binding sentence variables if the language has them. In the present higher-order setting, this ensures various theorems of Classicism are valid—Boolean identities, like $\forall_t pq((p \wedge q) =_t (q \wedge p))$—but also ensures validities beyond Classicism. Because there are propositions modeled by the singleton of a possible world, $\{w\}$, every consistent proposition is entailed by one of these special world propositions, leading to distinctive validities. World propositions are special because they are either fully contained or disjoint from any other set of possible worlds.

However, metaphysicians often take possible world talk to be more than a mere model theoretic tool. Someone taking the possible world model of propositions meta-

[45]Several broadly abductive arguments are made in Williamson (2013). Scroggs (1951) shows that the only modal logics extending S5 contain schemas to the effect that there are only n possibilities, for some finite n. Fine (1974) shows there are continuum many modal logics extending S4.

[46]See Bacon (2018a) §5.4 for a critical discussion of these arguments. The point here is that the mathematical objects of the relevant model that in fact are representing genuine possibilities may not represent genuine possibilities had things been sufficiently different. We should also keep track of the fact that if there is mathematical contingency, the model itself might change its mathematical structure.

physically seriously should believe that these special world propositions exist.[47] Given our previous observation that singletons are consistent, and contained or disjoint from (i.e. contained in the complement of) any other proposition, we will adopt the following definition of a world proposition:

$$\text{World} = \lambda w.(\Diamond w \wedge \forall_t p(w \leq_t p \vee w \leq_t \neg p))$$

World propositions are broadly possible propositions such that any other proposition is either entailed by it or inconsistent with it. The latter condition ensures that worlds settle all questions. The possible world metaphysician ought, then, to subscribe to the *Leibniz Biconditionals*: that something is possible if and only if it is entailed by a world proposition.

LB^t $\forall_t p(\Diamond p \leftrightarrow \exists w(\text{World}\, w \wedge w \leq_t p))$

As with Brouwer's principle, we might also consider the necessitation of the Leibniz biconditionals, $\Box\mathsf{LB}^t$. The necessitation is stronger and equivalent to the claim that *being possible* is the same as *being true at a possible world*—a claim which might be thought to better capture the idea that possibility can be *analyzed* in terms of possible worlds:

$\mathsf{LB}^{t=}$ $\Diamond =_{t\to t} \lambda p \exists w(\text{World}\, w \wedge w \leq_t p)$

It is worth noting that the possible worlds metaphysics encoded in LB^t is a substantive further commitment—it is not already a theorem of Classicism. Indeed, it doesn't follow from the Barcan formula, or even the Brouwerian axiom.[48]

Theorem 14. *LB^t is not a theorem of $\mathsf{C5}$.*

I have here brushed over an important choicepoint that arises in contexts where the propositional Barcan formula, $\mathsf{BF}^t_\Box$, fails. In this setting there could be 'new' questions concerning the truth of propositions that do not in fact exist: in that case, we might want to consider a strengthening of our definition of World ensuring that worlds necessarily settle all the questions, even new ones. This strengthening can be obtained by prefixing a $\Box$ to the second conjunct in our definition: a strong world is possible and *necessarily* settles every question.[49]

$$\text{SWorld} := \lambda w.(\Diamond w \wedge \Box\forall_t p(w \leq_t p \vee w \leq_t \neg p))$$

Anything that's a strong world is a world, and the result of replacing world with strong world in LB^t yields a strengthening we will call the Strong Leibniz Biconditionals:[50]

[47] Whether world propositions simply *are* possible worlds, as Prior and Fine maintained (Prior and Fine (1979)), or simply guaranteed by the existence of possible worlds will not be important in what follows. Once you have taken enough possible world machinery seriously, including notions like *possible world* and *true at*, then for any world w, there is the proposition that *every proposition true at w is true simpliciter*. Propositions of this form can play the role of world propositions in what follows.

[48] See the first model described in appendix D of Bacon and Dorr (forthcoming).

[49] See Bacon (2023a) chapter 7.

[50] The right-to-left direction of LB^t is in fact a theorem of Classicism, since anything entailed by a possible proposition (such as a world proposition) must be possible. The left-to-right direction of LB^t follows from SLB^t, for if p is possible it is entailed by a strong world it is entailed by a world, since every strong world is a world.

SLB^t $\forall_t p(\Diamond p \leftrightarrow \exists w(\mathrm{SWorld}\, w \wedge w \leq_t p))$

I myself am of the view that stronger notion of world better fits the notion at issue in possible world metaphysics. But since the results I prove here do not need the full strength of the strong Leibniz biconditionals, I'll work with the weaker notion in this section. Theorems we prove later from the Leibniz biconditionals thus can also be proven with the strong Leibniz biconditionals so that nothing turns on our choice about how to define world.

Like other principles we have encountered, such as the Barcan formula, there are generalizations of the Leibniz biconditionals to other types. For instance, a property theoretic version states that a property is possible (i.e. possibly instantiated) iff it is entailed by a world property. In general:

$\mathsf{LB}^{\overline{\sigma}}$ $\forall_{\overline{\sigma}\to t} R(\Diamond_{\overline{\sigma}} R \leftrightarrow \exists_{\overline{\sigma}\to t} W(\mathrm{World}_{\overline{\sigma}}\, W \wedge W \leq_{\overline{\sigma}\to t} R))$

where these notions are defined as follows.

Definition 6.1. *Let $\overline{x}$ be a sequence of variables $x_1...x_n$ of types $\overline{\sigma} = \sigma_1, ..., \sigma_n$.*

$$\Diamond_{\overline{\sigma}} := \lambda R \Diamond \exists \overline{x} R \overline{x}$$

$$\neg_{\overline{\sigma}} := \lambda R \lambda \overline{x} \neg (R \overline{x})$$

$$\mathrm{World}_{\overline{\sigma}} := \lambda W(\Diamond_{\overline{\sigma}} W \wedge \forall_{\overline{\sigma}\to t} S(W \leq_{\overline{\sigma}\to t} S \vee W \leq_{\overline{\sigma}\to t} \neg_{\overline{\sigma}\to t} S))$$

For those used to thinking in the possible worlds framework, an intension of type $e \to t$ (i.e. a function from worlds to extensions) is a world property at a given world w if it has a non-empty extension at exactly one world that's possible relative to w, and at that world its extension contains exactly one individual. Thus $\mathsf{LB}^{e\to t}$ is valid in model theories where the second-order quantifiers range over arbitrary functions from worlds to extensions.

I take it that the Leibniz biconditionals are also part of the 'orthodox' view about modal reality, found in, for instance, Lewis and Stalnaker.[51] We are now in a position to state our second connection between width contingency and the structure of modal reality:

Width contingency requires possible failures of the Leibniz biconditionals.

What implications do the Leibniz biconditionals have for mathematical modalities? Firstly we can show that if something is mathematically possible then it is true at a mathematically possible world.[52]

Theorem 15 ($\mathsf{C}^{\blacksquare}$)**.** *Given* LB^t, $\blacklozenge p \leftrightarrow \exists w(\mathrm{World}\, w \wedge w \leq p \wedge \blacklozenge w)$

[51] They are explicitly postulated, or derived, in the theories of Williamson (2013), Fritz (MS), Goodsell and Yli-Vakkuri (MS).

[52] Unless otherwise stated, proofs of all numbered theorems and propositions to follow may be found in appendix A.

Since $\mathsf{C}^{\blacksquare}$ only adds to Classicism the assumption that $\blacksquare$ is an infinitely closed necessity, it is a quite general theorem of Classicism with the Leibniz biconditionals that for any infinitely closed modality, X, a proposition is X-possible iff it is true at an X-possible world. (It does not hold for necessities that are not infinitely closed. Supposing, again, that having chance 1 is a necessity, then one can have chance-possible propositions that are not true at any chance-possible world propositions. For instance, its chance-possible that our dart hits the dartboard, because it has non-zero chance. But each broadly possible world where it hits the dartboard has chance 0, since a broadly possible world will settle the exact point that that the dart lands.)

We can now prove that the Leibniz biconditionals imply the rigidity of each stage of sets.

Theorem 16 ($\mathsf{C}^{\blacksquare\in}$)**.** $\mathsf{LB}^{t\to t}$ *and* LB^{t} *imply that* V_α *is rigid for every ordinal* α*.*

There is a way of glossing this argument with quantification over 'possible sets', which is strictly speaking inaccurate but which nonetheless gives an intuition for what is going on. The idea is to find, for any possible set, x, a world property W that applies to just that set. From W we can define an actual set, y, containing just those actual things that would have belonged to the W set, if W had been instantiated. Now the members of y are all of lower rank, so we may assume for induction that the actual sets of that rank are in fact the only possible sets of that rank, so x and y have the same members, are identical, and thus that x actually exists. The proof in appendix A is essentially an attempt to make this informal idea precise without any illegitimate quantification.

As before, we can obtain as two straightforward corollaries from the rigidity of V_α, the determinacy of the continuum hypothesis and the necessity of uncountability (and so a refutation of Countabilism).

Corollary 17 ($\mathsf{C}^{\blacksquare\in}$)**.**

1. $\mathsf{LB}^{t} \wedge \mathsf{LB}^{e\to t} \to \blacksquare CH \vee \blacksquare\neg CH.$
2. $\mathsf{LB}^{t} \wedge \mathsf{LB}^{e\to t} \to \forall_e x(\text{Uncountable}\, x \to \blacksquare\, \text{Uncountable}\, x)$

Corollary 18 ($\mathsf{C}^{\blacksquare\in}\mathsf{LB}^{\sigma}$)**.**

1. $\blacksquare CH \vee \blacksquare\neg CH.$
2. $\forall_e x(\text{Uncountable}\, x \to \blacksquare\, \text{Uncountable}\, x)$

Observe that our proof rested not only on the existence of world propositions, but also on a slightly less familiar consequence of naïve use of the possible worlds framework — the existence of world properties. The propositional Leibniz biconditionals do not appear to entail the property Leibniz biconditionals. In light of this, I offer another route to the property Leibniz biconditionals using a strengthening of the axiom of choice. An ordinary second-order choice principle can be formulated by saying that the universe of individuals can be well-ordered. By necessitating this principle we ensure that there is a well-order of the universe at every possible world, although it might witnessed by 'new' well-orders—that is to say, a world w might entail that there is a global well-order, while there is no relation such that w entails that *it* is a global well-order. The strengthening of necessitated choice we will investigate is the idea that for each world, there is a relation which that world entails to be a well-order

Strong Modal Choice $\forall_t w(\text{World}\, w \to \exists_{e\to e\to t} R\, w \leq \text{WO}\, R)$

With this principle we can close the gap between the propositional and property Leibniz biconditionals.

Theorem 19 (Classicism)**.** *Strong Modal Choice and* LB^t *entail* $\mathsf{LB}^{\overline{\sigma}\to t}$*.*

It should be noted that there could be width contingentists who reject the necessity of the axiom of choice on the grounds that it, like the continuum hypothesis, is indeterminate or mathematically unsettled. This would, of course, be grounds to reject the stronger principle of De Re Modal Choice. However this is a minority view, and most mathematicians take the axiom of choice to be settled and in as good a standing as other principles of set theory. The necessity of choice is validated, for instance, in the modal logic of forcing, where $\Box$ is interpreted as meaning truth in all generic forcing extensions, since the truth value of the axiom of choice (unlike CH) is preserved in generic extensions.

7 Free logic

Our theory $\mathsf{C}^{\blacksquare}$—Classicism plus the claim that $\blacksquare$ is a necessity that is closed under infinitary consequence—has lead us to some striking results. First, Classicism, in virtue of being closed under classical quantificational logic and necessitation for broad necessity, proves the broad necessity of existence, NNE^e, and a closely related principle, CBF^e. Second, supplementing Classicism with the principles of $\mathsf{S5}$ for the broadest necessity lets us derive the non-contingency of the set theoretic universe up to a given stage given some modest assumptions of modal set-theory. Third, supplementing Classicism with the Leibniz biconditionals lets us do the same.

Could the lover of width contingency restore orthodoxy in the second and third respects, by rejecting it in the first respect? That is, could they retain $\mathsf{S5}$ and the Leibniz biconditionals by weakening quantificational logic and adopting instead a free logic for the quantifiers? Unlike in classical logic, it is not possible to derive the necessity of existence or the converse Barcan formula in free logic. Moreover, the Prior-Lemmon proof of the Barcan formula within $\mathsf{S5}$ is not sound in free logic. In short, things can come and go into existence freely once classical quantificational logic is weakened, giving us more options for making sense of mathematical contingency about which sets exist. We earlier stipulated that in this setting one should read the symbols $\forall_\sigma$ in terms of the *outer* quantifiers. But one might wonder whether it is possible to also reject outer quantifiers in the free logical setting, thereby avoiding these results?

Classicism individuates propositions, properties and relations by provable equivalence in classical higher-order logic. So in order to explore this idea, we should look into the parallel theory that individuates entities instead by provable equivalence in free logic. That is we weaken the quantificational axioms of H along the lines of a free logic and close under the rule of equivalence. Call this system *Free Classicism*, or FC—it is defined in appendix B. Within this framework one can provide definitions of broad necessity and other notions of section 3. Strengthening this system with the principles of $\mathsf{S5}$ and the Strong Leibniz Biconditionals yields the system being proposed, which we can call $\mathsf{FC5(SLB)}$. The reader can find the details in appendix B.

There is a vast literature on the topic of contingent existence in the framework of higher-order logic that I will not attempt to contribute to.[53] I will limit myself instead to a couple of local points about its application to mathematical contingency.

First recall that width contingency seems to require at least the possibility of new individuals, so that in a $\mathsf{S5}$ setting we must also reject the necessity of existence. For if there could have been new individuals, then, according to Brouwer's principle, there would have been a possibility (namely actuality) where those new individuals didn't exist. We must, furthermore, reject the necessity of existence not merely for concrete individuals but for mathematical objects like sets.

There are some general reasons to think that sets exist of mathematical necessity, that are quite independent of the issue of necessary existence for concrete objects and other sorts of mathematical objects. There is a pervasive—and I think independently attractive—idea that a set is determined by its members. This idea is articulated in various ways in contemporary philosophy — sometimes it is the idea that the existence of a set is completely grounded in the existence of its members, or that a set is 'nothing over and above' its members.[54] According to this idea, while a set could fail to exist at a world if one of its members fails to exist, if all of its members at that world exist, the set itself must exist. More generally, if a proposition (a world proposition or otherwise) entails the members of x exist it must also entail x exists:

$$\forall_e x(\mathrm{Set}\, x \rightarrow \forall_t p(\forall_e y \in x(p \leq \exists_e z.z = y) \rightarrow p \leq \exists_e z.z = x))$$

If the proposition p is tautologous, we can infer that if the members of a set necessarily exist, then so does the set

$$\forall_e x(\mathrm{Set}\, x \rightarrow \forall_e y \in x\Box(\exists_e z.z = y) \rightarrow \Box(\exists_e z.z = x))$$

We can prove the necessary existence of sets as follows. Suppose that there is a set, x, that doesn't necessarily exist. By the well-foundedness of membership, we may assume without loss of generality that x is a possible non-existent of minimal rank, so that all of its members necessarily exist. But then we have a contingently existing set whose members necessarily exist. Assuming the necessity of our principles about set existence, and the well-foundedness of membership, this reasoning can be necessitated so necessarily every set necessarily exists.

One might think that these sorts of thoughts are antithetical to various brands of potentialism about sets. For instance, the height potentialist will typically maintain

[53] See Fine (1977), Williamson (2013), Stalnaker (2012), Fritz and Goodman (2016), Fritz (2018a), Fritz (2018b).

[54] See Fine (1994). Roberts (2022) also articulates precisely the related idea that pluralities are nothing over and above their members. The idea that sets are grounded in their members is ubiquitous in the grounding literature: see, for instance, Fine (2012) section 4, or Berker (2018).

that there are some pluralities—e.g. the non-self-membered sets—which do not form a set but which could have done. Note, however, that our argument only applied to things which already form a set: if x is a set then it couldn't have gone out of existence without at least one of its members going of out existence. The potentialist picture is entirely consistent with this because the new set collating a previous non-set-sized plurality of things is not already a set.

There is a more elementary argument for the necessary existence of sets that specifically targets the width contingentist: that any set $x \subseteq V_\alpha$ necessarily exists, for any given α. The separation axiom, along with the assumption that the ZF axioms are mathematically necessary, ensures that for any condition, $A(y)$, it's necessary that there is a set containing all and only the individuals y such that $A(y)$. Now, for any set x we have the condition $y \in x$. So, it's mathematically necessary that there is a set containing all and only the y belonging to V_α such that $y \in x$, i.e. $\{y \in V_\alpha \mid y \in x\}$ necessarily exists. Now one might object that this fails to establish the necessary existence of x, because as soon as x fails to exist it has no members, and so $\{y \in V_\alpha \mid y \in x\}$ is the empty set. This would contradict the mathematical rigidity of sets, and so we may already want to insist that even non-existent sets contain traces of their members. But even if we grant the objection, the potentialist we are considering accepts a logic of $\mathsf{S5}$ for broad necessity, and believes in world propositions, and so can introduce the set as follows. First, let w be the true world proposition. Then we have:

> It's mathematically necessary that there is a set containing all and only the y belonging to V_α such that $w \leq (y \in x)$: $\{y \in V_\alpha \mid w \leq (y \in x)\}$.

In the context of $\mathsf{C5}$ $w \leq$ behaves essentially like an actuality operator, letting us talk about the actual membership conditions of x at any world, so that by separation x must exist at every world.

There are further moves that certain sorts of higher-order contingentists might make at this juncture: perhaps *the property of belonging to* x, $\lambda y.w \leq (y \in x)$, fails to exist whenever x fails to exist. But this move doesn't really help: in order to retain classical propositional logic, these contingentists must draw a distinction between the satisfaction conditions of an open formula and the predicate you obtain from it by λ-abstraction. The open formula $w \leq (y \in x)$ still lets us classify every set as either satisfying it, or not, and so the separation axiom lets us prove that that there is a set of sets of any given certain rank belonging to x.[55]

My second point relates to the fact that one can introduce 'outer-quantifiers' in Free Classicism and read the previous results in terms of those outer quantifiers. Now there are independent motivations for positing outer quantifiers. They let the contingentist meet various expressive challenges that appear to beset their view. They

[55] One could restrict the separation axiom to conditions specified by predicates, thus excluding conditions specified with open formulas in one variable. But at this point we are no longer embracing the spirit of separation. An open formula, $A(x)$, lets us classify each natural number in one of two ways. As we run through the numbers we may find, for instance that $A(0), A(1), \neg A(2), A(3), \neg A(4), \neg A(5), A(6), A(7) \ldots$, which in turn specifies a list of numbers $0, 1, 3, 6, 7 \ldots$ satisfying the formula $A(x)$. It would be mathematically revisionary to suggest that this list doesn't correspond to a subset of the natural numbers.

have introduction and elimination rules that pin down their inferential role uniquely (see Harris (1982)), so they could be introduced directly as new primitives. In some contexts, however, it is impossible to avoid outer quantifiers because they can be defined explicitly in terms of the contingentists quantifiers and modal operators—it turns out that S5 with the Leibniz biconditionals is one of these contexts.

For instance, if we have an actuality operator and I want to say that every possible individual is F I can say 'necessarily, everything is F in actuality'. The formula $\Box\forall_e x@Fx$ thus simulates "possibilist quantification" over all possible individuals provided at the actual world. While this paraphrase is materially adequate, this fact is, of course, highly contingent: had different things been F, that paraphrase would still evaluate with respect what is actually F and deliver incorrect results.[56] Kit Fine (Prior and Fine (1979) p144) thus paraphrases quantification over all possible Fs by saying 'the true world proposition w (whatever it might be) is such that necessarily everything is entailed by w to be F'.

It turns out that the assumptions built into FC5(SLB)—specifically the assumption that every possible proposition is entailed by a strong world—ensures that this quantifier behaves classically: see theorem 26 in appendix B. This means, among other things, that they satisfy the converse Barcan formula and prove the necessity of existence. As we have observed already, we need additional modal assumptions about the modal logic of $\Box$ to show that classical quantifiers satisfy the Barcan formula—the S5 principles—but these are built into FC5(SLB) as well. Indeed, not only is every theorem of Classicism derivable with respect to the outer quantifiers in FC5(SLB), but also the theorems we get by adding S5 and the Strong Leibniz Biconditionals to Classicism.

Theorem 20. FC5(SLB) *interprets* C5(SLB).

Thus, for every theorem of C5(SLB) there is a corresponding a theorem (under translation) of FC5(SLB).[57]

This theorem paves the way for applying the results in section 5 and 6 in a contingentist setting, provided that the non-logical assumptions from section 4 can also be defended on this interpretation. Although I am confident that it can be done, I will not attempt to defend these non-logical assumptions under this reinterpretation. I will instead side-step the issue and shift attention to some purely logical statements that imply that there isn't any mathematical contingency of the relevant sort.

[56] See the discussion in Williamson (2010) p685-686.

[57] A certain sort of higher-order contingentist may reject the Strong Leibniz Biconditionals on the grounds that existing propositions cannot be about non-existing individuals. On this picture strong worlds shouldn't exist, because they settle questions about merely possible individuals (see Stalnaker (2010)). In this case a different motivation and definition of the outer quantifiers is needed. I believe that this is possible using the results from work in progress with Cian Dorr, Peter Fritz and Ethan Russo, where the assumption of the Strong Leibniz Biconditionals can be weakened to the assumption that being true entails being entailed by a truth ($\lambda p.p \leq_{t\to t} \lambda p\exists_t q(q \leq p)$ — an assumption that we believe can be motivated even in a higher-order contingentist setting. It's also worth pointing out that there are internal pressures for propositional contingentists, like Stalnaker and Fine, to posit primitive outer quantifiers even if they cannot be defined, since they are necessary for even expressing the idea that propositions do not exist unless the individuals they are about exist (these worries are pressed in Fritz and Goodman (2016)). Thanks to an anonymous referee for pressing this concern.

In this paper we have concerned ourselves with the set-theoretic continuum hypothesis, which is stated in terms of the non-logical predicate $\in$. However, there is another purely logical claim that is closely related to the set-theoretic continuum hypothesis. Let's call it the higher-order continuum hypothesis. It is possible in higher-order logic to say that a property's extension is (i) countably infinite, (ii) that is has the size of the first uncountable infinity (there is a bijection between it and the well-orders-up-to-isomorphism on a countably infinite property) and (iii) has the size of the continuum (there is a bijection between it and subproperties-up-to-extension of a countably infinite property). We call these properties $\aleph_0$, $\aleph_1$ and Continuum. See Shapiro (1991) p105. Then we may formulate the continuum hypothesis as follows:

Higher-Order CH $\forall_{e\to t}X(\text{Continuum}\,X \leftrightarrow \aleph_1 X)$

Higher-order CH entails the set-theoretic continuum hypothesis, since if x is an uncountable set of real numbers, the property of belonging to x, $\lambda y.y \in x$, must be at least $\aleph_1$ sized, and at most continuum sized, and so Higher-Order CH implies it is continuum sized.

What does our free logician say about higher-order CH? Of course in Free Classicism, properties and relations, like sets, can exist contingently: what subproperties a countably infinite property has may exist contingently, and the relevant bijective relations could also fail to exist making it very plausible that one could construct models in which Higher-order CH is contingent. But suppose we consider yet another variant of the continuum hypothesis, now formulated using the classical outer quantifiers. Let us write A^* for the result of replacing each occurrence of the free quantifiers in A with the corresponding outer quantifier. We are now concerned with (Higher-Order CH)*.

One might reasonably ask what relation this sentence bears to the mathematical question of the continuum hypothesis. For that is formulated in familiar quantificational terms, whereas we have granted that the outer quantifiers may bear no relation to ordinary quantificational words, as they appear in ordinary English. I won't insist that we refer to (Higher-Order CH)* as a 'version of the continuum hypothesis'. However, the question of whether it is true or not is nonetheless something that can be raised and investigated in the pure language of higher-order logic. And, like the set-theoretic continuum hypothesis and its vanilla higher-order variant, it does not seem to be something we can settle using any mathematical or logical methods presently available to us. The reasons we have to think that Higher-order CH is indeterminate apparently extend to (Higher-Order CH)*.

The problem we are presented with is this. If we add to Classicism the principles of $\mathsf{S5}$ and the Leibniz biconditionals (or the Strong Leibniz Biconditionals) one can prove the following schema, stating that there is no broad contingency in things stated in purely logical terms:

No Pure Contingency $P \to \Box P$, where P is closed and contains no non-logical vocabulary.

If purely logical statements cannot be broadly contingent, they cannot be mathematically contingent either. The argument for this is due to Zach Goodsell.[58]

[58] A proof is presented in Bacon and Dorr (forthcoming).

Theorem 21 (Goodsell). C5(LB) *proves No Pure Contingency.*

But given theorem 21, every theorem of C5(LB) translates to a theorem of FC5(SLB) implying that (Higher-Order CH)* is not broadly contingent (and consequently is not indeterminate or mathematically contingent).

I will not insist that (Higher-Order CH)* and Higher-Order CH must stand or fall together, or that the methods necessary for settling either must be equally high-powered. Indeed, one method for settling these questions is to make contentious metaphysical posits that would imply their trivial truth or falsity, and so make an asymmetric treatment of their indeterminacy seem less *ad hoc*. One emerging line of thought in the width contingentist literature is the idea that any things whatsoever could have been the image of a function on the natural numbers (cf. the weaker thesis of Countabilism, that concerns sets). Perhaps, then, (Higher-Order CH)* is trivially true because in terms of the possibilist quantifiers all properties are countable*, and so Continuum* X and $\aleph_1^* X$ are vacuously coextensive. But contentious metaphysics can equally settle the plain higher-order continuum hypothesis. The assumption that there are in fact only countably many things settles it vacuously in exactly the same way, much as the thesis of nominalism would too.

It is also very unclear why we should care about the indeterminacy or contingency of the versions of CH formulated using the contingentist free-logical quantifiers. There will be some restrictions of the outer quantifiers by properties under which Higher-Order CH can have any combination of truth or falsity with contingency or necessity (this ought to be so, for instance, if there are infinitely many things in the outer sense of the quantifiers). The version of CH we have been calling Higher-Order CH corresponds to the contingency of CH under a restriction of the outer quantifiers by the property of *existence*, in a distinctively metaphysical and inflationary sense—a sense that is not pinned down by anything like inferential role in the way that the classical outer quantifiers are. It would be hard to convince mathematicians that *this* is the real question they should be focusing on, and it is far from obvious that mathematics is the appropriate methodology for settling it. According to the classical quantifiers, the truth of an existential *there are* F*s* can be inferred from a true instance, *a is* F, so that if F is itself is expressed using only logical and mathematical vocabulary, logico-mathematical methods can be used to settle the question of whether there are Fs. By contrast, the conditions under which an individual exists in the more demanding sense involves extra mathematical considerations. Consider, for instance, a debate about the existence conditions for material objects, such as whether a table could have existed without the matter that constitutes it. It's a hard question, and we shouldn't expect the questions to become easier when we shift attention to set existence. Certainly traditional mathematical methods are not equipped to answer these questions.

8 Conclusion

I have argued that certain kinds of set-theoretic contingency require surrendering two pieces of modal orthodoxy: that the broadest necessity has a logic of S5, and the

Leibniz biconditionals, connecting what is possible with what holds at some maximally specific possibility. Both of these modal doctrines deserve some scrutiny. The simplest kind of model of modal logic employs possible worlds, and treats the broadest necessity as quantifying unrestrictedly over all worlds in the model, so it is easy to see where the orthodoxy may have originated. But model theory alone does not make for a positive argument. We now know how to model modal logic without building in either of these assumptions.[59] One of these generalizations, possibility semantics—which replaces the complete worlds of possible world semantics with incomplete possibilities—was in fact implicit in Cohen's original papers introducing the forcing method of the independence.[60] Furthermore, there are several positions in higher-order metaphysics that require rejecting $\mathsf{S5}$ for the broadest necessity—the philosophical terrain here is still largely unexplored.[61] But before we can sign off on width contingency, we need some guarantee that there aren't any unforeseen inconsistencies in the view. A strong version of width contingency maintains, putting it informally, that all forcing extensions of the set-theoretic universe are mathematically possible—the principle I earlier called Forcing Possibilism. Indeed, I believe Forcing Possibilism to be consistent with our background theory:[62]

Conjecture 22. *Forcing Possibilism is consistent with* $\mathsf{C}^{\blacksquare\in}$.

Forcing Possibilism provides us with a particular view about how much mathematical contingency there is. It is natural to push this line of thought further, and ask if even more radical visions of mathematical contingency are consistent. For instance, could one posit mathematical possibilities corresponding not just to first-order models obtained by forcing but to arbitrary models satisfying the ZF axioms? If this were consistent it would represent a vision in which mathematical contingency is as widespread as possible: the axioms of ZF must express mathematical necessities, given that we have a principle to that effect, but any statement independent of ZF would express a mathematically contingent proposition. It turns out, however, that mathematical contingency cannot be this rampant. We were able to derive (corollary 10), in the minimal background theory $\mathsf{C}^{\in\blacksquare}$, that arithmetical statements are not mathematically contingent or indeterminate, even if they are independent of our favored axiomatic theories (as, for instance, their consistency statements are).[63]

[59]Sometimes it is argued that the broadest necessity must be modeled by a universal accessibility relation (see for instance Lewis (1986)). A similar argument can be made in the possibility framework. But this appeal to model theory is questionable, and ignores the possibility that which 'worlds' of the model represent genuine possibilities might itself be contingent, and so depend on what world you are evaluating at. For further discussion of these sorts arguments, see Bacon (2018a) §5.4.

[60]Work on possibility semantics for modal logic was initiated in Humberstone (1981), and has been continued more recently by Holliday and coauthors (see, for instance, Holliday (forthcoming)). Prior even to possible world semantics, we had the algebraic approach to modal logic, found in Bjarni Jonsson (1953), that makes no assumptions akin to the existence of possible worlds.

[61]Bacon (2023a) chapter 8, and Bacon and Dorr (forthcoming) section 2.4-2.6 overview some of the options here.

[62]I have some partial models of of Forcing Possibilism and $\mathsf{C}^{\blacksquare\in}$ for fragments of higher-order logic that seem to generalize. However the verification of this conjecture will have to wait until a future occasion.

[63]Corollary 10 is an analogue, in the present system, of Goodsell (2022) result concerning the determinacy of arithmetic; here Goodsell's assumption of Rigid Comprehension is not needed due to

Determining where the line between the set-theoretic statements that are necessary (determinate) and contingent (indeterminate) lies is thus non-trivial. We know that the former includes at least all the arithmetical statements, and the latter may include statements like the continuum hypothesis, but figuring out more about where this line lies seems to be an important avenue for further inquiry. For instance, it is a well-known fact that if the axiom of choice is true in a model it is true in all forcing extensions of it, so we might consider adding the axiom of choice to the list of claims that are mathematically necessary.[64]

References

Andrew Bacon. Can the classical logician avoid the revenge paradoxes? *Philosophical Review*, 124(3):299–352, 2015. doi: 10.1215/00318108-2895327.

Andrew Bacon. The broadest necessity. *Journal of Philosophical Logic*, 47(5):733–783, 2018a. doi: 10.1007/s10992-017-9447-9.

Andrew Bacon. *Vagueness and Thought*. Oxford, England: Oxford University Press, 2018b.

Andrew Bacon. Logical combinatorialism. *Philosophical Review*, 129(4):537–589, 2020a. doi: 10.1215/00318108-8540944.

Andrew Bacon. Viii–vagueness at every order. *Proceedings of the Aristotelian Society*, 120(2):165–201, 2020b. doi: 10.1093/arisoc/aoaa011.

Andrew Bacon. *A Philosophical Introduction to Higher-Order Logics*. Routledge, 2023a.

Andrew Bacon. Zermelian extensibility, 2023b. Unpublished manuscript.

Andrew Bacon and Cian Dorr. Classicism. In Peter Fritz and Nicholas K. Jones, editors, *Higher-order Metaphysics*. Oxford University Press, forthcoming.

Andrew Bacon and Jin Zeng. A theory of necessities. *Journal of Philosophical Logic*, 51(1):151–199, 2022. doi: 10.1007/s10992-021-09617-5.

Selim Berker. The unity of grounding. *Mind*, 127(507):729–777, 2018. doi: 10.1093/mind/fzw069.

Alfred Tarski Bjarni Jonsson. Boolean algebras with operators. *Journal of Symbolic Logic*, 18(1):70–71, 1953. doi: 10.2307/2266339.

our assumption that sets have rigid membership conditions. Note that there is another form of possibilism which states that there are *broad* possibilities corresponding to every model of ZF—indeed it's consistent that anything consistent with Classicism is broadly possible; see the constructions in appendix E of Bacon and Dorr (forthcoming) and chapter 18 of Bacon (2023a).

[64]One can actually derive this claim—$\mathrm{AC} \to \blacksquare\,\mathrm{AC}$—in the system obtained by adding the *converse* of Forcing Possibilism to $\mathsf{C}^{\blacksquare\in}$. The converse states that only claims made true by forcing are mathematically possible; the possibility of AC failing then would imply the existence of a forcing condition that refutes AC, and so AC would in fact be false.

George Boole. *The Mathematical Analysis of Logic: Being an Essay Towards a Calculus of Deductive Reasoning*. Cambridge, England: Macmillan, Barclay & Macmillan, 1847.

Ethan Brauer. The modal logic of potential infinity: Branching versus convergent possibilities. *Erkenntnis*, pages 1–19, 2020. doi: 10.1007/s10670-020-00296-3.

David Builes and Jessica M. Wilson. In defense of countabilism. *Philosophical Studies*, 179(7):2199–2236, 2022. doi: 10.1007/s11098-021-01760-8.

M. J. Cresswell and G. E. Hughes. *A New Introduction to Modal Logic*. Routledge, 1996.

Cian Dorr. Propositions and counterpart theory. *Analysis*, 65(3):210–218, 2005. doi: 10.1111/j.1467-8284.2005.00551.x.

Cian Dorr. Of numbers and electrons. *Proceedings of the Aristotelian Society*, 110 (2pt2):133–181, 2010. doi: 10.1111/j.1467-9264.2010.00282.x.

Cian Dorr, John Hawthorne, and Juhani Yli-Vakkuri. *The Bounds of Possibility: Puzzles of Modal Variation*. Oxford: Oxford University Press, 2021.

Hartry Field. Which undecidable mathematical sentences have determinate truth values. In H. G. Dales and Gianluigi Oliveri, editors, *Truth in Mathematics*, pages 291–310. Oxford University Press, Usa, 1998.

Hartry Field. A revenge-immune solution to the semantic paradoxes. *Journal of Philosophical Logic*, 32(2):139–177, 2003. doi: 10.1023/a:1023027808400.

Kit Fine. An ascending chain of s4 logics. *Theoria*, 40(2):110–116, 1974. doi: 10.1111/j.1755-2567.1974.tb00081.x.

Kit Fine. Properties, propositions and sets. *Journal of Philosophical Logic*, 6(1): 135–191, 1977. doi: 10.1007/bf00262054.

Kit Fine. Essence and modality. *Philosophical Perspectives*, 8(Logic and Language): 1–16, 1994. doi: 10.2307/2214160.

Kit Fine. Relatively unrestricted quantification. In Agustín Rayo and Gabriel Uzquiano, editors, *Absolute Generality*, pages 20–44. Oxford University Press, 2006.

Kit Fine. Guide to ground. In Fabrice Correia and Benjamin Schnieder, editors, *Metaphysical Grounding*, pages 37–80. Cambridge University Press, 2012.

Peter Fritz. Higher-order contingentism, part 2: Patterns of indistinguishability. *Journal of Philosophical Logic*, 47(3):407–418, 2018a. doi: 10.1007/s10992-017-9432-3.

Peter Fritz. Higher-order contingentism, part 3: Expressive limitations. *Journal of Philosophical Logic*, 47(4):649–671, 2018b. doi: 10.1007/s10992-017-9443-0.

Peter Fritz. From propositions to possible worlds, MS.

Peter Fritz and Jeremy Goodman. Higher-order contingentism, part 1: Closure and generation. *Journal of Philosophical Logic*, 45(6):645–695, 2016. doi: 10.1007/s10992-015-9388-0.

Zach Goodsell and Juhani Yli-Vakkuri. *Logical Foundations of Philosophy*. MS.

Zachary Goodsell. Arithmetic is determinate. *Journal of Philosophical Logic*, 51(1): 127–150, 2022. doi: 10.1007/s10992-021-09613-9.

Joel David Hamkins. A simple maximality principle. *Journal of Symbolic Logic*, 68 (2):527–550, 2003. doi: 10.2178/jsl/1052669062.

Joel David Hamkins. The set-theoretic multiverse. *Review of Symbolic Logic*, 5(3): 416–449, 2012. doi: 10.1017/s1755020311000359.

Joel David Hamkins and Øystein Linnebo. The modal logic of set-theoretic potentialism and the potentialist maximality principles. *Review of Symbolic Logic*, 15 (1):1–35, 2022. doi: 10.1017/s1755020318000242.

Joel David Hamkins, George Leibman, and Benedikt Löwe. Structural connections between a forcing class and its modal logic. *Isr. J. Math.*, 201:617–651, 2015.

J. H. Harris. What's so logical about the ?logical? axioms? *Studia Logica*, 41(2-3): 159–171, 1982. doi: 10.1007/BF00370342.

Geoffrey Hellman. *Mathematics Without Numbers: Towards a Modal-Structural Interpretation*. Oxford, England: Oxford University Press, 1989.

Wesley H. Holliday. Possibility semantics. In Melvin Fitting, editor, *Selected Topics from Contemporary Logics*. London: College Publications, forthcoming.

I. L. Humberstone. From worlds to possibilities. *Journal of Philosophical Logic*, 10 (3):313–339, 1981. doi: 10.1007/bf00293423.

Georg Kreisel. Informal rigour and completeness proofs. In Imre Lakatos, editor, *Problems in the Philosophy of Mathematics*, pages 138–157. North-Holland, 1967.

Saul A. Kripke. Identity and necessity. In Milton Karl Munitz, editor, *Identity and Individuation*, pages 135–164. New York: New York University Press, 1971.

Saul A. Kripke. *Naming and Necessity: Lectures Given to the Princeton University Philosophy Colloquium*. Cambridge, MA: Harvard University Press, 1980.

David Lewis. *On the Plurality of Worlds*. Wiley-Blackwell, 1986.

David K. Lewis. Counterpart theory and quantified modal logic. *Journal of Philosophy*, 65(5):113–126, 1968. doi: 10.2307/2024555.

Øystein Linnebo. The potential hierarchy of sets. *Review of Symbolic Logic*, 6(2): 205–228, 2013. doi: 10.1017/s1755020313000014.

Toby Meadows. Naive infinitism: The case for an inconsistency approach to infinite collections. *Notre Dame Journal of Formal Logic*, 56(1):191–212, 2015. doi: 10.1215/00294527-2835074.

Charles Parsons. *Mathematics in Philosophy: Selected Essays*, chapter Sets and Modality, pages 298–341. Cornell University Press, 1983.

Graham Priest. Hopes fade for saving truth. *Philosophy*, 85(1):109–140, 2010. doi: 10.1017/s0031819109990489.

A. N. Prior. Modality and quantification in s5. *Journal of Symbolic Logic*, 21(1): 60–62, 1956. doi: 10.2307/2268488.

A. N. Prior and Kit Fine. Times, worlds and selves. *Synthese*, 40(2):389–408, 1979.

Arthur N. Prior. *Past, Present and Future*. Oxford University Press, 1967.

Arthur N. Prior and Norman Prior. *Formal Logic*. Oxford University Press, 1955.

Alexander R. Pruss. Might all infinities be the same size? *Australasian Journal of Philosophy*, 98(3):604–617, 2020. doi: 10.1080/00048402.2019.1638949.

Sam Roberts. Pluralities as nothing over and above. *Journal of Philosophy*, 119(8): 405–424, 2022. doi: 10.5840/jphil2022119828.

Chris J. Scambler. Can all things be counted? *Journal of Philosophical Logic*, 50(5): 1079–1106, 2021. doi: 10.1007/s10992-021-09593-w.

Schiller Joe Scroggs. Extensions of the lewis system s5. *Journal of Symbolic Logic*, 16(2):112–120, 1951. doi: 10.2307/2266683.

Stewart Shapiro. *Intentional Mathematics*. Elsevier, 1985.

Stewart Shapiro. *Foundations Without Foundationalism: A Case for Second-Order Logic*. Oxford, England: Oxford University Press, 1991.

Robert Stalnaker. Possible worlds. *Noûs*, 10(1):65–75, 1976. doi: 10.2307/2214477.

Robert Stalnaker. Merely possible propositions. In Bob Hale and Aviv Hoffmann, editors, *Modality: Metaphysics, Logic, and Epistemology*, pages 21–32. Oxford University Press, 2010.

Robert Stalnaker. *Mere Possibilities: Metaphysical Foundations of Modal Semantics*. Princeton University Press, 2012.

J. P. Studd. The iterative conception of set: A (bi-)modal axiomatisation. *Journal of Philosophical Logic*, 42(5):1–29, 2013. doi: 10.1007/s10992-012-9245-3.

Gabriel Uzquiano. Varieties of indefinite extensibility. *Notre Dame Journal of Formal Logic*, 56(1):147–166, 2015. doi: 10.1215/00294527-2835056.

Jared Warren and Daniel Waxman. A metasemantic challenge for mathematical determinacy. *Synthese*, 197(2):477–495, 2020. doi: 10.1007/s11229-016-1266-y.

Timothy Williamson. Everything. *Philosophical Perspectives*, 17(1):415–465, 2003a. doi: 10.1111/j.1520-8583.2003.00017.x.

Timothy Williamson. Vagueness in reality. In Michael J. Loux and Dean W. Zimmerman, editors, *The Oxford Handbook of Metaphysics*. Oxford University Press, 2003b.

Timothy Williamson. Necessitism, contingentism, and plural quantification. *Mind*, 119(475):657–748, 2010. doi: 10.1093/mind/fzq042.

Timothy Williamson. *Modal Logic as Metaphysics*. Oxford, England: Oxford University Press, 2013.

Ernst Zermelo. On boundary numbers and domains of sets. new investigations in the foundations of set theory. In Heinz-Dieter Ebbinghaus and Akihiro Kanamori, editors, *Ernst Zermelo: Collected Works Vol I*, pages 401–429. Springer Verlag, 2010.

A Appendix: Proofs of Theorems

Theorem 5. ($\mathsf{C5}$)
$\forall X(\mathrm{Nec}_\infty X \to \mathsf{BF}^\sigma_X)$

Proof. $\mathsf{C5}$ contains the broad Barcan formula, $\mathsf{BF}^\sigma_\Box$.

Suppose that X is infinitely closed and that $\forall_\sigma x\, X(Fx)$. We want to show that $X(\forall_\sigma x Fx)$. Since X is infinitely closed, it suffices to show that anything entailing every X-necessary proposition also entails $\forall_\sigma x Fx$. Suppore r entails every X-necessary proposition. Since Fx is X-necessary for every x, $\forall_\sigma x.\Box(r \to Fx)$. By the broad Barcan formula, $\Box\forall_\sigma x(r \to Fx)$ and so $\Box(r \to \forall_\sigma x Fx)$. Thus r entails $\forall_\sigma x Fx$ as required. Since X is closed under entailment, $X(\forall_\sigma x Fx)$. $\Box$

Theorem 7. ($\mathsf{C}^{\blacksquare\in}$)
Given $\mathsf{BF}^e_\Box$ (for broad necessity), being of stage α (i.e. V_α) is rigid for every ordinal α.

Proof. As we have noted (theorem 5), $\mathsf{BF}^e_\Box$ for broad necessity implies the Barcan formula for $\blacksquare$, $\mathsf{BF}^e_\blacksquare$. Subsequent uses of the word ‘possibly’ and ‘necessarily’ in the proof refer to $\blacklozenge$ and $\blacksquare$.

The proof is by transfinite induction. V_0 is necessarily empty, and so vacuously rigid.

Suppose that α is an ordinal, and for each $\beta \in \alpha$, V_β is rigid. We want to show that V_α is rigid. Suppose $\blacklozenge\exists x(V_\alpha x \wedge Fx)$. We must show $\exists x(V_\alpha x \wedge \blacklozenge Fx)$.

The Barcan formula ensures there is an x such that $\blacklozenge(V_\alpha x \wedge Fx)$, but we have no guarantee that x is in fact V_α, or even if it is a set. Instead of directly showing x is a set, we'll define another set by separation containing the elements that would have belonged to x if x had been a set, and show that this is a V_α set that is possibly F.

$$x' := \{y \in \bigcup_{\beta\in\alpha} V_\beta \mid \blacksquare(\mathrm{Set}\, x \to y \in x)\}$$

We can now show that x' is identical x as follows. Given the mathematical necessity of Set Extensionality it suffices to show that necessarily if x is a set, x coextensive with x': $\blacksquare(\mathrm{Set}\, x \to \forall y.(y \in x \leftrightarrow y \in x')$. We break this up into two claims:

1. $\blacksquare(\mathrm{Set}\, x \to \forall y(y \in x' \to y \in x)$
2. $\blacksquare(\mathrm{Set}\, x \to \forall y(y \in x \to y \in x')$

We establish 1 first. From the definition of membership in x', we immediately have $\forall y \in x' \blacksquare(\mathrm{Set}\, x \to y \in x)$. Since Sets are Rigid, it follows that x'-restricted quantification satisfies BF, so we can infer $\blacksquare\forall y \in x'(\mathrm{Set}\, x \to y \in x)$. By applying first-order logic under the scope of $\blacksquare$, this is equivalent to 1.

To establish 2, it suffices to show $\forall y\blacksquare(\mathrm{Set}\, x \to (y \in x \to y \in x'))$ by the Barcan formula. Let y be an arbitrary individual. Now either $y \in x'$ or $y \notin x'$. Suppose the former. Then by the rigidity of set membership y is necessarily in x' and so $\blacksquare(\mathrm{Set}\, x \to (y \in x \to y \in x'))$ follows. Suppose, then, that $y \notin x'$. By the definition of

x' this would mean that $\blacklozenge(\mathrm{Set}\, x \wedge y \notin x)$. It follows that $\blacksquare(\mathrm{Set}\, x \to (y \in x \to y \in x'))$, for if this were false we'd have $\blacklozenge y \in x$ and by Sets are Rigid we could conclude $\blacksquare(\mathrm{Set}\, x \to y \in x)$ contradicting the previous line. (It is here that we must use the stronger version of Sets are Rigid outlined in footnote 36 if we are not assuming the convergence axiom.)

By the necessity of Extensionality, we have shown that $\blacksquare(\mathrm{Set}\, x \to x = x')$. $\blacklozenge(V_\alpha x \wedge Fx)$ thus entails $\blacklozenge Fx'$. And by construction x' is V_α so $\exists x(V_\alpha x \wedge \blacklozenge Fx)$ as required.

□

Lemma 23 ($\mathsf{C}^{\blacksquare\in}$). *Sets are mathematically necessarily distinct:* $\forall_e xy(\mathrm{Set}\, x \wedge \mathrm{Set}\, y \to x \neq y \to \blacksquare x \neq y)$

Proof. Suppose the claim is false for contradiction. Choose x to be $\in$-minimal such that x possibly identical to some set it is distinct from. Choose y to be $\in$-minimal such that it is distinct from, but possibly identical to x.

Since x and y are distinct we may suppose, without loss of generality, that there is some set z belonging to x but not belonging to y. By Set Rigidity, $\blacksquare z \in x$. So $\blacklozenge z \in y$, since $\blacklozenge x = y$. Since $\blacklozenge \exists z' \in y.z' = z$ it follows by Set Rigidity that $\exists z' \in y \blacklozenge z' = z$. Since x is an $\in$-minimal failure of the necessity of distinctness, z cannot be possibly identical to anything distinct from it. It follows that whatever member of y that is possibly identical to z is in fact identical to z, so that z is a member of y after all, a contradiction. □

Theorem 8. ($\mathsf{C}^{\blacksquare\in}$)
Suppose $A(\overline{x})$ is a first-order formula with free variables $\overline{x}$. If all the quantifiers in $A(\overline{x})$ are restricted to rigid properties of sets, then A is modally absolute with respect to the parameters $\overline{y}$.

Proof. By Set Rigidity, $x \in y$ is modally absolute, since if y is a set and $x \in y$ then by Set Rigidity x is necessarily in y. And if $x \notin y$ and y is a set, then by the necessity of distinctness of sets x could not be identical to a member of y. The necessity of identity and distinctness for sets ensures the modal absoluteness of $x = y$. Suppose A and B are modally absolute. If for any sequence of sets $\overline{x}$, $A(\overline{x})$ and $B(\overline{x}$, then the modal absoluteness of A and B ensures that $\blacksquare A(\overline{x})$ and $\blacksquare B(\overline{x})$ and so $\blacksquare(A(\overline{x}) \wedge B(\overline{x}))$. Similarly if $\neg(A(\overline{x}) \wedge B(\overline{x}))$ either $\neg A(\overline{x})$ or $\neg B(\overline{x})$ and so given the modal absoluteness of A and B we have either $\blacksquare\neg A(\overline{x})$ or $\blacksquare\neg B(\overline{x})$ and in either case $\blacksquare\neg(A \wedge B)$ as required. The disjunction case is a dualization of the above, and the negation case is trivial.

Now suppose $B(y\overline{x})$ is modally absolute, and $\lambda y.A(y\overline{x})$ is a rigid property of sets ($\lambda y.A(y\overline{x})$ entails Set). We will show the modal absoluteness of $\forall_e y(A(y\overline{x}) \to B(y\overline{x})$. Let $\overline{x}$ be a sequence of sets, and suppose $\forall_e y(A(y\overline{x}) \to B(y\overline{x})$. By the modal absoluteness of B we can conclude $\forall_e y(A(y\overline{x}) \to \blacksquare B(y\overline{x}))$, and by the rigidity of A we can get $\blacksquare\forall_e y(A(y\overline{x}) \to B(y\overline{x})$. On the other hand, if $\neg\forall_e y(A(y\overline{x}) \to B(y\overline{x})$ then for some set y, $(A(y\overline{x}) \wedge \neg B(y\overline{x})$. By the modal absoluteness of B, $\blacksquare\neg B(y\overline{x})$ and by the rigidity of A, $\blacksquare A(y\overline{x})$ so $\blacksquare\exists_e y(A(y\overline{x}) \wedge \neg B(y\overline{x}))$, as required. The existential case involves dualizing this argument. □

Theorem 9. ($\mathsf{C}^{\blacksquare\in}$)
Given the truth of the theorems of $\mathsf{C}^{\blacksquare\in}$, the following formulas are modally absolute.

1. *being an ordinal.*
2. *being a limit ordinal.*
3. *being the smallest limit ordinal, the successor of the smallest limit ordinal, the successor of the succcessor of the smallest limit ordinal...*

moreover, the properties in 3. are rigid.

Proof. α is an ordinal if and only if α is (i) transitive $\forall x \in \alpha \forall y \in x.y \in \alpha$) and (ii) linearly ordered by membership $(\forall x \in \alpha \forall y \in \alpha(x \neq y \rightarrow x \in y \vee y \in x)$. All the quantifiers in these definitions are restricted by conditions of the form $\in z$, which is rigid by Set Rigidity, and entails sethood (by the definition of Set as $\lambda y \exists x.y \in x$). Thus they are all modally absolute.

α is a limit ordinal if it is an ordinal and additionally $\forall x \in \alpha \exists y \in \alpha(x \in y)$ and $\exists x \in \alpha$. These have the same property. α is the smallest limit ordinal iff it is a limit ordinal, and for every $x \in \alpha$ x is not a limit ordinal. α is the successor of the smallest limit ordinal iff every member of α is either belongs to the smallest limit ordinal or is identical to it. Again, all quantifiers are restricted by membership to some set.

Finally we can show that the properties in 3 are rigid. Let ω be the set that is actually the smallest limit ordinal. By the modal absoluteness, ω is necessarily the smallest limit ordinal, and uniquely so, since is a theorem of ZF that if two sets are the smallest limit ordinal they are identical. Suppose it is possible that something is the smallest limit ordinal is also F. Then it is possible that ω is F, and thus there is an actual smallest limit ordinal, ω, which is possibly F. Similar strategies apply to the other properties listed in 3. □

Theorem 12. ($\mathsf{C5}^{\blacksquare\in}$)

1. $\blacksquare CH \vee \blacksquare \neg CH$.
2. $\forall_e x(\text{Uncountable}\, x \rightarrow \blacksquare\, \text{Uncountable}\, x)$

Proof. Let $V_{\omega+2}y$ be the property 'λy.for some set α, α is the successor of the successor of the smallest limit ordinal, and y is V_α'. Using the results above, it is easily seen that this property is rigid.

The continuum hypothesis can be formulated in such a way that all quantifiers are restricted by the predicate $V_{\omega+2}$. Since this predicate is rigid, CH is modally absolute: $CH \rightarrow \blacksquare CH$ and $\neg CH \rightarrow \blacksquare CH$. This establishes 1.

Let x be an uncountable set, and suppose that α is an ordinal such that $x \in V_\alpha$. Then the claim that $x \in V_\alpha$ and is uncountable is equivalent to the claim that $x \in V_\alpha$ and there is no set of ordered pairs belonging to $V_{\alpha+3}$ that is an injective function from the smallest limit ordinal to x. All of the quantifiers in this claim are similarly restricted to rigid properties.

□

Theorem 15. ($\mathsf{C}^{\blacksquare}$)
Given LB^t, $\blacklozenge p \leftrightarrow \exists w(\text{World}\, w \wedge w \leq p \wedge \blacklozenge w)$

Proof. Mathematical Necessity states that anything entailed by the $\blacksquare$-necessities must be itself $\blacksquare$-necessary. So any $\blacklozenge$-possibility is such that its negation is not entailed by the $\blacksquare$-necessities.

Thus if $\blacklozenge p$, $\blacksquare \not\leq \neg p$. That is, for some r such that $\forall q(\blacksquare q \to r \leq q)$, $r \not\leq \neg p$. This means $\Diamond(r \wedge p)$, so by LB^t, there is a world proposition w that entail $r \wedge p$. We finally can see that w must be $\blacklozenge$-possible. For if not, then $\blacksquare\neg w$, and since r entails every $\blacksquare$-necessity, $r \leq \neg w$. But since $w \leq r$, $w \leq \neg w$, contradicting the assumption that w is a world.

The right-to-left direction is obvious. □

Theorem 16. ($\mathsf{C}^{\blacksquare\in}$)
$\mathsf{LB}^{t\to t}$ *and* LB^t *imply that* V_α *is rigid for every ordinal* α.

Proof. The proof is by transfinite induction. V_0 is necessarily empty, and so vacuously rigid.

Suppose that α is an ordinal, and for each $\beta \in \alpha$, V_β is rigid. We want to show that V_α is rigid. Suppose $\blacklozenge\exists x(V_\alpha x \wedge Fx)$. We must show $\exists x(V_\alpha x \wedge \blacklozenge Fx)$.

Since $\lambda x(V_\alpha x \wedge Fx)$ is broadly possibly instantiated, it follows by the Leibniz Biconditionals, $\mathsf{LB}^{e\to t}$, that there is a world property W that that entails it, and by proposition 15 it will be a world property that is mathematically possibly instantiated.[65] We can use this world property to define the actual member of V_α that's possibly F explicitly:

$$x' := \{y \in \bigcup_{\beta\in\alpha} V_\beta \mid \blacksquare\forall x(Wx \to y \in x)\}$$

Roughly W singles out a merely possible set. x' is the set of ys in V_α that would have belonged to the merely possible object picked out by W if it had existed. We can now show that x' is identical to the merely possible W: i.e. we show $\blacksquare\forall x(Wx \to x = x')$. Given the mathematical necessity of Set Extensionality and the mathematical possibility of W it suffices to show that necessarily whatever is W is coextensive with x': $\blacksquare\forall x(Wx \to \forall y.(y \in x \leftrightarrow y \in x')$. We break this up into two claims:

1. $\blacksquare\forall x(Wx \to \forall y(y \in x' \to y \in x)$
2. $\blacksquare\forall x(Wx \to \forall y(y \in x \to y \in x')$

We establish 1 first. From the definition of membership in x', we immediately have $\forall y \in x' \blacksquare\forall x(Wx \to y \in x)$. Since Sets are Rigid, it follows that x'-restricted quantification satisfies BF, so we can infer $\blacksquare\forall y \in x'\forall x(Wx \to y \in x)$. By applying first-order logic under the scope of $\blacksquare$, this is equivalent to 1.

To establish 2, we first show $\forall\beta \in \alpha\forall y(V_\beta y \to \blacksquare\forall x(Wx \to (y \in x \to y \in x'))$. Let $\beta \in \alpha$ and let y be an arbitrary set of rank β. Now either $y \in x'$ or $y \notin x'$.

[65] $\blacklozenge\exists x(V_\alpha x \wedge Fx)$ implies by theorem 15 that there is a mathematically possible world proposition $w \leq \exists x(V_\alpha x \wedge Fx)$, and since $\Diamond\exists x(w \wedge V_\alpha x \wedge Fx)$ there is a world property W entailing $\lambda x(w \wedge V_\alpha x \wedge Fx)$.

Suppose the former. Then by the rigidity of set membership y is necessarily in x' and so $\blacksquare\forall x(Wx \to (y \in x \to y \in x'))$ follows. Suppose then that $y \notin x'$. By the condition for belonging to x', this means that W doesn't entail the property of containing y. Since W is a world property, it must entail the property of not belonging to y, and thus must also mathematically necessitate it: $\blacksquare\forall x(Wx \to y \notin x)$. So this means $\blacksquare\forall x(Wx \to (y \in x \to y \in x'))$, by applying some straightforward logic under the $\blacksquare$ (namely that $y \notin x$ entails $y \in x \to y \in x'$).

This completes the argument that $\forall\beta \in \alpha\forall y(V_\beta y \to \blacksquare\forall x(Wx \to (y \in x \to y \in x'))$. By the inductive hypothesis, V_β is rigid, and so we can infer $\forall\beta \in \alpha\blacksquare\forall y(V_\beta y \to \forall x(Wx \to (y \in x \to y \in x'))$. Since α is a set and sets are rigid, we can also infer $\blacksquare(\forall\beta \in \alpha\forall y(V_\beta y \to \forall x(Wx \to (y \in x \to y \in x'))$. Thus $\blacksquare\forall x(Wx \to \forall y(y \in x \to \exists\beta \in \alpha.V_\beta y \to y \in x')))$ applying first-order logic under $\blacksquare$. Recall that necessarily whatever the W set is, it's V_α: thus, necessarily, whatever the W set is, if y belongs to it, y is in V_β for some $\beta \in \alpha$ (by the definition of V_α). That is we have (a) $\blacksquare\forall x(Wx \to V_\alpha x)$, (b) $\blacksquare\forall x(V_\alpha x \wedge y \in x \to \exists\beta \in \alpha.V_\beta y)$ (by definition of the V relation and the mathematical necessity of ZF). So putting this together $\blacksquare\forall x(Wx \to \forall y(y \in x \to y \in x')))$ as required.

Since W mathematically necessitates being identical to x' ($\blacksquare\forall x(Wx \to x = x')$, and W is mathematically possible, it follows that $\blacklozenge Wx'$. Finally, since W entails F it follows that $\blacklozenge Fx'$. By construction $V_\alpha x'$ so $\exists x(V_\alpha x \wedge \blacklozenge Fx)$ as required. $\square$

Theorem 18. ($\mathsf{C}^{\blacksquare\in}\mathsf{LB}^\sigma$)

1. $\blacksquare CH \vee \blacksquare\neg CH$.
2. $\forall_e x(\text{Uncountable}\, x \to \blacksquare\, \text{Uncountable}\, x)$

Theorem 19. (C)
De Re Modal Choice and LB^t *entail* $\mathsf{LB}^{\overline{\sigma}\to t}$.

Proof. We show $\mathsf{LB}^{e\to t}$, since that is the instance required for theorem 16, however the proof generalizes trivially.

Suppose that $\Diamond\exists xFx$. By LB^t, there is a world proposition w such that $w \leq_t \exists xFx$. Let R be a relation which is necessarily a well-order, and consider the property of being the R minimal F while w is true: $W := \lambda x(w \wedge \text{Min}\, RFx)$ where $\text{Min} = \lambda RFx(Fx \wedge \forall y(Fy \to Rxy \vee x = y))$. Clearly W entails F. Let G be another property. Since there is at most one minimal F of a well-order, we know that $\Box(\text{WO}\, R \to \forall x(\text{Min}\, RFx \to Gx) \vee \forall x(\text{Min}\, RFx \to \neg Gx))$, and since $\Box\, \text{WO}\, R$, $\Box(\forall x(\text{Min}\, RFx \to Gx) \vee \forall x(\text{Min}\, RFx \to \neg Gx))$. Since w settles every question it either entails every R-minimal F is G, or that it's not, $\Box(w \to \forall x(\text{Min}\, RFx \to Gx)) \vee \Box(w \to \forall x(\text{Min}\, RFx \to \neg Gx))$. Rearranging a little and appealing to the definition of W this is $\Box\forall x(Wx \to Gx) \vee \Box\forall x(Wx \to \neg Gx)$

$\square$

B Appendix: Free Logic

In this appendix we provide the necessary background for the results discussed in section 7.

Free logic replaces the law of universal instantiation with its universal closure, $\forall_\sigma y(\forall_\sigma x\, Fx \to Fy)$. We must then also add the principle that universal quantification distributes over conditionals. We of course, may apply the analogous substitutions at other types.

Free Instantiation $\forall_\sigma y(\forall_\sigma x\, Fx \to Fy)$ provided y is not free in F.

Quantifier Normality $\forall_\sigma x(A \to B) \to (\forall_\sigma xA \to \forall_\sigma xB)$

The remaining principles of H—Gen, and the laws governing the truth-functional connectives and λ—remain the same. Let FH, 'free higher-order logic', be the result of making these substitutions to H, ad Free Classicism, FC, the result of closing FH under the rule of equivalence.

Because the logic of the quantifiers in Free Classicism is weaker than Classicism, notions we defined using the quantification over all necessities—entailment, broad necessity, world, etc—may behave in undesirable ways. For example, a natural quantificational definition of property entailment in Free Classicism, $\Box\forall_e x\Box(Fx \to Gx)$, is consistent with pathological situations where F entails G, a is F but a is not G.[66] However, in Classicism many of the notions that we defined in terms of the classical quantifiers can be given equivalent definitions in terms of identity, and because the logic of identity in Free Classicism is classical, we can recover the desired behaviour by using the identity-theoretic definitions instead. For instance, there is a long tradition in logic, tracing back to George Boole, of defining entailment in terms of identity. For properties F and G, F entails G when the property conjunction of F with G (i.e. $\lambda x(Fx \wedge Gx)$) just is F.[67] The pathological situation mentioned above cannot arise, for if F entails G and then $F = (\lambda x.Fx \wedge Gx)$. So by Leibniz's law $Fa \to (\lambda x.Fx \wedge Gx)a$, and thus $Fa \to Ga$ by β and propositional logic.

Proposition 24. *In Classicism, the following identities are derivable:*

1. $\Box = \lambda p.p =_t \top$
2. $\leq_{\overline{\sigma}} = \lambda RS(R \wedge_{\overline{\sigma}} S =_{\overline{\sigma}\to t} R)$
3. $\mathrm{SWorld} = \lambda w((w \neq_t \bot) \wedge (\lambda p(w \leq p \vee w \leq \neg p) =_{t\to t} \lambda p.\top))$

Proofs of 1 and 2 may be found in Bacon (2023a) p149. The first conjunct of the RHS of 3, $w \neq_t \bot$, is equivalent to $\Diamond w$ by 1, and the second conjunct is equivalent to $\Box\forall_t p(\top \to (w \leq p \vee w \leq \neg p))$ by 2, and thus to $\Box\forall_t p(w \leq p \vee w \leq \neg p)$.

Perhaps it is possible to augment Free Classicism with further principles that would rule out these pathological situations, but we will avoid the need for any further assumptions by adopting the identity theoretic definitions of these three notions listed in proposition 24 as our official ones when working in Free Classicism.

[66] Models of this will interpret a with an individual that does not belong to the domain of quantification at any world. It is quite easy to generate an extensional model of Free Classicism in which $\forall_e x(Fx \to Gx)$ (and thus $\Box\forall_e x\Box(Fx \to Gx)$), Fa and $\neg Ga$ are all true.

[67] Boole (1847), p20 project Gutenberg.

We can now define a possibilist quantifier along the lines of Fine's definition discussed in section 3:

$$\Pi_\sigma := \lambda F \exists_t w(\text{SWorld}\, w \wedge w \wedge (\lambda x.w \leq_\sigma F)$$

where F has type $\sigma \to t$ and x has type σ. $\Pi_e F$ means that, when w is the true strong world proposition, the vacuous property of being such that w entails F. We have replaced Fine's $\Box\forall_\sigma x\Box(w \to Fx)$—the potentially ill-behaved notion of entailment mentioned above—with the corresponding identity theoretic entailment for reasons detailed above.

We are now in a position to formulate the orthodox possible worlds metaphysics within Free Classicism. We can do this by adding to FC the Strong Leibniz Conditionals and the B schema and closing under the rule of equivalence as well as the background logical rules, remembering, of course, that SWorld, $\leq$, $\Diamond$, etc are now given in identity theoretic terms.

SLB^t $\Diamond A \leftrightarrow \exists_t w(\text{SWorld}\, w \wedge w \leq A)$

B $A \to \Box\Diamond A$

We will call the result $\mathsf{FC5(SLB)}$. Note that because necessitated quantificational claims are weak in Free Classicism, merely adding the necessitations of the universal closures of these principles to Free Classicism would fail to deliver identities that one could obtain from the result of closing under the rule of equivalence. We could acheive the same effect as closing under the rule of equivalence by adding a pair of identities to Free Classicism. The claim that *to be possible is to be true at some possible world*, and the claim that *to be true* entails *to be necessarily possible.*

SLB^t_λ $\Diamond =_{t\to t} \lambda p(\exists_t w(\text{SWorld}\, w \wedge w \leq p)$

B_λ $\lambda p.p \leq_{t\to t} \lambda p.\Box\Diamond p$

$\mathsf{FC5}$ and $\mathsf{FC(SLB)}$ stand for the result adding, in the same way, only one of these principles.

Lemma 25. $\mathsf{FC5(SLB)}$ *contains* $A \to \exists w(\text{SWorld}\, w \wedge w \wedge w \leq_t A)$*.*

Proof. First we show $\text{SWorld}\, w \to \Box\, \text{SWorld}\, w \wedge \Box(\exists_t p.w = p)$. Since $\text{SWorld}\, w$ is the conjunction of a distinctness claim and an identity claim, the necessity of the first conjunct follows from the necessity of distinctness and the necessity of identity both of which are well-known theorems of $\mathsf{S5}$ with the classical axioms of identity.[68] Using SLB, and the fact that w is necessarily possible, $\Box\exists_t v(\text{SWorld}\, v \wedge v \leq w)$. It's also necessary that for any strong world $v \leq w$, $w \leq v$. For w is necessarily a strong world, and so must entail v or $\neg v$ for any strong world $v \leq w$, and it couldn't entail $\neg v$ since otherwise $v \leq \neg v$ by the transitivity of entailment, contradicting the fact that v is possible. So necessarily, any strong world entailing w is identical to w, thus $\Box\exists_t v(\text{SWorld}\, v \wedge v =_t w)$.

[68] For the necessity of identity see Kripke (1971), for the necessity of distinctness see Prior and Prior (1955), pp. 206–7.

Now we argue that every strong world, w, entails (i) w, (ii) that w is an existent strong world, and (iii) $A \to (w \leq A)$. (i) is trivial, (ii) is established above. For (iii), $\lambda p.w \leq_{t\to t} \lambda p.(w \wedge (p \to (\lambda p \top)p)$ since $p \to \top$ is a tautology. And since $\lambda p.\top =_{t\to t} \lambda p(w \leq p \vee w \leq \neg p)$ (since w is a strong world) we have $\lambda p.w \leq \lambda p(w \wedge (p \to (w \leq p \vee w \leq \neg p)))$. We also have $\lambda p.w \leq \lambda p(w \wedge p \to w \not\leq \neg p)$ since $w \wedge p \to w \not\leq \neg p$ is a theorem of Free Classicism. Since operator entailment is closed under propositional logic, $\lambda p.w \leq \lambda p(p \to w \leq p)$. Apply both these operators to A and using β we get w and $A \to w \leq A$, and since the former operator entails the latter, $w \leq (A \to w \leq A)$.

Putting (i),(ii) and (iii) together, we have that for every strong world, w, $w \leq (w \wedge \text{SWorld}\, w \exists_t p(p = w) \wedge (A \to w \leq A))$. Using the fact that entailment is closed under free logic we get $w \leq (A \to \exists_t w(w \wedge \text{SWorld}\, w \wedge w \leq A)))$. Since every strong world entails $A \to \exists_t w(w \wedge \text{SWorld}\, w \wedge w \leq A))$ we can infer $\Box(A \to \exists_t w(w \wedge \text{SWorld}\, w \wedge w \leq A)))$ by SLB.

$\square$

Theorem 26. $\mathsf{FC5(SLB)}$ *interprets* $\mathsf{C5(SLB)}$

Proof. We map each term M of $\mathcal{L}$ to M^*, the result of substituting each free quantifier $\forall_\sigma$ with Π_σ. We wish to show that whenever A is a theorem of Classicism, A^* is a theorem of Free Classicism$^+$.

Each tautology, instance of B, and instance of $\beta\eta$ are mapped to tautologies instances of B or instances of $\beta\eta$. Uses of modus ponens and the rule of equivalence are similarly mapped to themselves. It remains to show that UI* and SLB^* are theorems of $\mathsf{FC5(SLB)}$, and, for Gen, that if $(A \to B)^*$ is a theorem of Free Classicism$^+$, so is $(A \to \forall x B)^*$.

Let's begin with UI. We will show generally that $\Pi_\sigma F \to Fa$. Suppose $\Pi_\sigma F$, so that there is some truth, p, such that $\lambda x(p \wedge Fx) =_{\sigma\to t} \lambda x.p$. Want to show Fa. $(\lambda x.p)a =_t p$ by β, and since p is true, we can conclude $(\lambda x.p)a$. By the above identity, $\lambda x(p \wedge Fx)a$, so $p \wedge Fa$, and finally, Fa as required.

For the right-to-left direction of SLB^* we show the dualized contrapositive version. We will suppose that $\Pi_t w(\text{SWorld}\, w \to w \leq A)$ and show $\Box A$. Expanding the definition of Π, the true strong world, v, is such that $\lambda w.v \leq \lambda w.(\text{SWorld}\, w \to w \leq A)$. Applying $\forall_t$ to both sides we see that the claim that everything is such that v (i.e. $\forall_t p.v)$) entails that every strong world is entails A (i.e $\forall_t w(\text{SWorld}\, w \to w \leq A))$. Since v is true, everything is such that v, and so every strong world entails A. By SLB, $\Box A$. For the converse of SLB^* suppose $\Sigma_t w(\text{SWorld}\, w \wedge w \leq A)$—i.e. $\lambda w.v \not\leq \lambda w(Sw \to w \not\leq A)$ where v is a true strong world. We want to show $\Diamond A$. It suffices to show $\exists_t u(\text{SWorld}\, u \wedge u \leq A)$. Suppose for contradiction that $\forall_t u(\text{SWorld}\, u \to u \not\leq A)$. By lemma there is a strong world v that is true and entails $\forall_t u(\text{SWorld}\, u \to u \not\leq A)$, delivering also the corresponding entailment between vacuous operators: $\lambda w.v \leq \lambda w \forall u(\text{SWorld}\, u \to u \not\leq A)$. Since being a strong world entails existence, we have $\lambda w.v \leq \lambda w(\text{SWorld}\, w \to \exists_t r.r = w)$. Since the right-hand-sides of entailments are closed under free logical consequences, we have $\lambda w.v \leq \lambda w(\text{SWorld}\, w \wedge \exists_t r.r = w \to w \not\leq A)$ and so $\lambda w.v \leq \lambda w(\text{SWorld}\, w \to w \not\leq A)$. This contradicts our assumption.

For Gen it suffices to show that whenever we have a proof of $A \to B$ where x is not free in B there is also a proof of $A \to \Pi_\sigma x B$. Since we can prove $A \to B$, we can prove $(\lambda x(A \to B))y \leftrightarrow (\lambda x.\top)y$ using β and so by the rule of equivalence we then have $\lambda x(A \to B) = \lambda x.\top$.

Now we will show that $A \to \exists w(w \wedge \mathrm{SWorld}\, w \wedge \lambda x.w \leq \lambda x.B$. Suppose A, and let w be the true strong world entailing A (appealing to lemma 25). So $w \wedge \neg A =_t \bot$. Clearly $\lambda x.w \leq \lambda x(A \to B)$ since $\lambda x.w \leq \lambda x.\top$.

$\lambda x(w \wedge (A \to B)) =_{\sigma \to t} \lambda x.w$. The left-hand-side is $\lambda x.((w \wedge \neg A) \vee (w \wedge B))$ using Boolean equivalences that can be obtained from the Rule of Equivalence. Since x isn't free in A and $w \wedge \neg A = \bot$ we can infer the the left-hand-side is $\lambda x.(w \wedge B)$ by Leibniz's law and Boolean equivalences. So $\lambda x(w \wedge B) = \lambda x.w$ as required. $\square$